SOLVABILITY OF SOME INTEGRO-DIFFERENTIAL EQUATIONS WITH THE DOUBLE SCALE ANOMALOUS DIFFUSION IN HIGHER DIMENSIONS

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Abstract: The article is devoted to the studies of the existence of solutions of an integro-differential equation in the case of the double scale anomalous diffusion with the sum of the two negative Laplacians raised to two distinct fractional powers in $\mathbb{R}^d, \ d = 4, 5$. The proof of the existence of solutions is based on a fixed point technique. Solvability conditions for the non-Fredholm elliptic operators in unbounded domains are used.

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1. Introduction

The present article deals with the studies of the existence of the stationary solutions of the following nonlocal integro-differential equation

$$\frac{\partial u}{\partial t} = -D[(-\Delta)^{s_1} + (-\Delta)^{s_2}]u + \int_{\mathbb{R}^d} K(x-y)g(u(y,t))dy + f(x) \qquad (1.1)$$

with $d = 4, 5, \ 0 < s_1 < s_2 < 1$ and $\frac{3}{2} - \frac{d}{4} < s_2 < 1$. The problems of this kind are relevant to the cell population dynamics. The results of the article are derived in these particular ranges of the values of the parameters s_1 and s_2 in the powers of the negative Laplace operators, which is based on the solvability of the linear Poisson type equation (4.1) and the applicability of the Sobolev inequality for the fractional Laplacian (1.4). The solvability of the equation analogous to (1.1) involving a single fractional Laplacian in the diffusion term was discussed in [29]. The space variable x here corresponds to the cell genotype, $u(x,t)$ denotes the cell density as a function of their genotype and time. The right side of our equation describes the evolution of the cell density by virtue of the cell proliferation, mutations and cell influx. The double scale anomalous diffusion term in such context is corresponding to the change of genotype due to the small random mutations, and the integral production term describes large mutations. The function $g(u)$ stands for the rate of the cell birth depending on u (density dependent proliferation), and the kernel $K(x-y)$ designates the proportion of the newly born cells, which change their genotype from y to x. It is assumed here that it depends on the distance between the genotypes. The last term in the right side of (1.1) denotes the influx or efflux of cells for different genotypes.

The fractional Laplacian describes a particular case of the anomalous diffusion actively considered in the context of different applications in plasma physics and turbulence [8], [24], surface diffusion [19], [22], semiconductors [23] and so on. The anomalous diffusion can be described as a random process of the particle motion characterized by the probability density distribution of the jump length. The moments of this density distribution are finite in the case of the normal diffusion, but this is not the case for the anomalous diffusion. The asymptotic behavior at the infinity of the probability density function determines the value of the power of the negative Laplace operator (see [20]). In the present work we will consider the case of $0 < s_1 < s_2 < 1$, $\frac{3}{2} - \frac{d}{4} < s_2 < 1$ with $d = 4, 5$. The solvability of the integro-differential equation in the case of the standard Laplace operator in the diffusion term was discussed in [31]. The necessary conditions of the preservation of the nonnegativity of the solutions of a system of parabolic equations in the case of the double scale anomalous diffusion were derived in [14]. The article [16] deals with the simultaneous inversion for the fractional exponents in the space-time fractional diffusion equation.

Let us set $D = 1$ and establish the existence of solutions of the problem

$$-[(-\Delta)^{s_1} + (-\Delta)^{s_2}]u + \int_{\mathbb{R}^d} K(x-y)g(u(y))dy + f(x) = 0 \tag{1.2}$$

with $0 < s_1 < s_2 < 1$, $\frac{3}{2} - \frac{d}{4} < s_2 < 1$ and $d = 4, 5$. We address the situation when the linear part of such operator fails to satisfy the Fredholm property. As a consequence, the conventional methods of the nonlinear analysis may not be applicable. We use the solvability conditions for the non-Fredholm operators along with the method of contraction mappings.

Consider the equation

$$-\Delta u + V(x)u - au = f \tag{1.3}$$

with $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either zero identically or tends to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of problem (1.3) contains the origin. As a consequence, this operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work is devoted to the studies of the certain properties of the operators of this kind. Note that the elliptic problems with non-Fredholm operators were treated actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [3], [4], [5], [6], [7]. The non-Fredholm Schrödinger type operators were studied with the methods of the spectral and the scattering theory in [25], [30]. The nonlinear non-Fredholm elliptic problems were considered in [13], [14], [29], [31], [32]. The important applications to the theory of the reaction-diffusion equations were developed in [10], [11]. Fredholm structures, topological invariants and applications were covered in [12]. The articles [15] and [21] are crucial for the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of the second order and of the operators of this kind on $\mathbb{R}^N$. The operators without the Fredholm property arise also when studying the wave systems with an infinite number of localized traveling waves (see [1]). The standing lattice solitons in the discrete NLS equation with saturation were considered in [2]. Particularly, when $a = 0$ the operator A is Fredholm in certain properly chosen weighted spaces (see [3], [4], [5], [6], [7]). However, the case of $a \neq 0$ is considerably different and the approach developed in these works cannot be used. The front propagation problems with the anomalous diffusion were studied actively in recent years (see e.g. [26], [27]).

We set $K(x) = \varepsilon \mathcal{K}(x)$, where $\varepsilon \geq 0$ and suppose that the conditions below are fulfilled.

Assumption 1.1. *Consider* $0 < s_1 < s_2 < 1$ *and* $\frac{3}{2} - \frac{d}{4} < s_2 < 1$ *with* $d = 4, 5$. *Let* $f(x) : \mathbb{R}^d \to \mathbb{R}$ *be nontrivial, so that* $f(x) \in L^1(\mathbb{R}^d)$ *and* $(-\Delta)^{\frac{3}{2} - s_2} f(x) \in L^2(\mathbb{R}^d)$. *We assume also that* $\mathcal{K}(x) : \mathbb{R}^d \to \mathbb{R}$ *and* $\mathcal{K}(x) \in L^1(\mathbb{R}^d)$. *Additionally,* $(-\Delta)^{\frac{3}{2} - s_2} \mathcal{K}(x) \in L^2(\mathbb{R}^d)$, *so that*

$$Q := \left\| (-\Delta)^{\frac{3}{2} - s_2} \mathcal{K}(x) \right\|_{L^2(\mathbb{R}^d)} > 0.$$

Let us choose the space dimensions $d = 4, 5$. This is relevant to the solvability conditions for the linear Poisson type equation (4.1) formulated in Lemma 4.1 below. From the perspective of the applications, the space dimensions are not limited to $d = 4, 5$ since the space variable corresponds to the cell genotype but not to the usual physical space. We use the Sobolev inequality for the fractional Laplacian (see Lemma 2.2 of [17], also [18])

$$\|f(x)\|_{L^{\frac{2d}{d-6+4s_2}}(\mathbb{R}^d)} \le c_{s_2,d}\|(-\Delta)^{\frac{3}{2}-s_2}f(x)\|_{L^2(\mathbb{R}^d)}, \quad \frac{3}{2}-\frac{d}{4} < s_2 < 1, \tag{1.4}$$

where $d = 4, 5$ along with Assumption 1.1 above and the standard interpolation argument. This yields

$$f(x) \in L^2(\mathbb{R}^d) \tag{1.5}$$

as well. For the technical purposes, we use the Sobolev space

$$H^{2s_2}(\mathbb{R}^d) := \big\{u(x) : \mathbb{R}^d \to \mathbb{R} \mid u(x) \in L^2(\mathbb{R}^d),\ (-\Delta)^{s_2}u \in L^2(\mathbb{R}^d)\big\}$$

with $\frac{3}{2}-\frac{4}{4} < s_2 < 1$ and $d = 4, 5$. It is equipped with the norm

$$\|u\|^2_{H^{2s_2}(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \big\|(-\Delta)^{s_2}u\big\|^2_{L^2(\mathbb{R}^d)}. \tag{1.6}$$

By means of the standard Sobolev embedding in dimensions $d = 4, 5$, we have

$$\|u\|_{L^\infty(\mathbb{R}^d)} \le c_e\|u\|_{H^3(\mathbb{R}^d)}, \tag{1.7}$$

where $c_e > 0$ is the constant of the embedding. Here

$$\|u\|^2_{H^3(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \big\|(-\Delta)^{\frac{3}{2}}u\big\|^2_{L^2(\mathbb{R}^d)}. \tag{1.8}$$

When the nonnegative parameter $\varepsilon = 0$, we obtain the linear Poisson type equation (4.1). By virtue of Lemma 4.1 below along with Assumption 1.1, problem (4.1) has a unique solution

$$u_0(x) \in H^{2s_2}(\mathbb{R}^d), \quad 0 < s_1 < s_2 < 1, \quad \frac{3}{2}-\frac{d}{4} < s_2 < 1, \quad d = 4, 5$$

so that no orthogonality conditions are required. By means of Assumption 1.1,

$$[(-\Delta)^{\frac{3}{2}-s_2+s_1} + (-\Delta)^{\frac{3}{2}}]u_0(x) = (-\Delta)^{\frac{3}{2}-s_2}f(x) \in L^2(\mathbb{R}^d). \tag{1.9}$$

It can be easily deduced from (1.9) using the standard Fourier transform (2.1) that $(-\Delta)^{\frac{3}{2}}u_0(x) \in L^2(\mathbb{R}^d)$. By virtue of the definition of the norm (1.8) we obtain for the unique solution of linear equation (4.1) that $u_0(x) \in H^3(\mathbb{R}^d)$.
We look for the resulting solution of nonlinear problem (1.2) as

$$u(x) = u_0(x) + u_p(x). \tag{1.10}$$

Clearly, we arrive at the perturbative equation

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]u_p(x) = \varepsilon \int_{\mathbb{R}^d} \mathcal{K}(x-y)g(u_0(y)+u_p(y))dy \tag{1.11}$$

with $0 < s_1 < s_2 < 1, \ \frac{3}{2} - \frac{d}{4} < s_2 < 1, \ d = 4, 5$. Let us introduce a closed ball in the Sobolev space as

$$B_\rho := \{u(x) \in H^3(\mathbb{R}^d) \mid \|u\|_{H^3(\mathbb{R}^d)} \le \rho\}, \quad 0 < \rho \le 1. \tag{1.12}$$

We seek the solution of equation (1.11) as the fixed point of the auxiliary nonlinear problem

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]u(x) = \varepsilon \int_{\mathbb{R}^d} \mathcal{K}(x-y) g(u_0(y) + v(y)) dy, \quad d = 4, 5, \tag{1.13}$$

where $0 < s_1 < s_2 < 1, \ \frac{3}{2} - \frac{d}{4} < s_2 < 1, \ d = 4, 5$ in ball (1.12). For a given function $v(y)$ this is an equation with respect to $u(x)$. The left side of (1.13) contains the operator without the Fredholm property

$$l := (-\Delta)^{s_1} + (-\Delta)^{s_2} : H^{2s_2}(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \tag{1.14}$$

which is defined via the spectral calculus. This is the pseudo-differential operator with symbol $|p|^{2s_1} + |p|^{2s_2}$, namely

$$lu(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (|p|^{2s_1} + |p|^{2s_2}) \widehat{u}(p) e^{ipx} dp, \quad u(x) \in H^{2s_2}(\mathbb{R}^d),$$

where the standard Fourier transform is defined in (2.1). The essential spectrum of (1.14) fills the nonnegative semi-axis $[0, +\infty)$. Thus, such operator has no bounded inverse. The similar situation appeared in articles [31] and [32]. But as distinct from the present case, the problems discussed there required the orthogonality relations. The fixed point technique was used in [28] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator contained in the nonlinear problem there had the Fredholm property (see Assumption 1 of [28], also [9]).
We introduce the interval on the real line

$$I := \big[- c_e \|u_0\|_{H^3(\mathbb{R}^d)} - c_e, \ c_e \|u_0\|_{H^3(\mathbb{R}^d)} + c_e \big], \quad d = 4, 5 \tag{1.15}$$

along with the closed ball in the space of $C_2(I)$ functions, namely

$$D_M := \{g(z) \in C_2(I) \mid \|g\|_{C_2(I)} \le M\}, \quad M > 0. \tag{1.16}$$

The norm involved in (1.16)

$$\|g\|_{C_2(I)} := \|g\|_{C(I)} + \|g'\|_{C(I)} + \|g''\|_{C(I)}, \tag{1.17}$$

where $\|g\|_{C(I)} := \max_{z \in I} |g(z)|$.

Let us impose the following technical conditions on the nonlinear part of problem (1.2). It will vanish at the origin along with its first derivative. From the point of view of the biological applications, $g(z)$ can be, for instance the quadratic function describing the cell-cell interaction.

Assumption 1.2. *Let $g(z) : \mathbb{R} \to \mathbb{R}$, so that $g(0) = 0$ and $g'(0) = 0$. We also assume that $g(z) \in D_M$ and it does not vanish identically on the interval I.*

We introduce the operator T_g, such that $u = T_g v$, where u is a solution of problem (1.13). Our first main proposition is as follows.

Theorem 1.3. *Let Assumptions 1.1 and 1.2 hold. Then for every $\rho \in (0, 1]$ equation (1.13) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all*

$$0 < \varepsilon \leq$$

$$\leq \frac{\rho}{2M(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2 \Big[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)} (\|u_0\|_{H^3(\mathbb{R}^d)}+1)^{\frac{8s_2}{d}-2} d}{(2\pi)^{4s_2}(d-4s_2)} \Big(\frac{|S^d|}{16 s_2}\Big)^{\frac{4s_2}{d}} + \frac{Q^2}{4}\Big]^{\frac{1}{2}}}. \tag{1.18}$$

The unique fixed point $u_p(x)$ of this map T_g is the only solution of problem (1.11) in B_ρ.

Here and below S^d denotes the unit sphere in the space of $d = 4, 5$ dimensions centered at the origin and $|S^d|$ stands for its Lebesgue measure.
Obviuosly, the resulting solution of equation (1.2) given by (1.10) will be nontrivial since the source term $f(x)$ is nontrivial and $g(z)$ vanishes at the origin as assumed. We have the following elementary statement.

Lemma 1.4. *For $R \in (0, +\infty)$ and $d = 4, 5$ consider the function*

$$\varphi(R) := \alpha R^{d-4s_2} + \frac{1}{R^{4s_2}}, \quad \frac{3}{2} - \frac{d}{4} < s_2 < 1, \quad \alpha > 0.$$

It achieves the minimal value at $R^* := \left(\frac{4s_2}{\alpha(d-4s_2)}\right)^{\frac{1}{d}}$*, which is given by*

$$\varphi(R^*) = \left(\frac{\alpha}{4s_2}\right)^{\frac{4s_2}{d}} \frac{d}{(d-4s_2)^{\frac{d-4s_2}{d}}}.$$

Our second main result is devoted to the continuity of the resulting solution of equation (1.2) given by (1.10) with respect to the nonlinear function g.

Theorem 1.5. *Let $j = 1, 2$, the assumptions of Theorem 1.3 hold, so that $u_{p,j}(x)$ is the unique fixed point of the map $T_{g_j} : B_\rho \to B_\rho$, which is a strict contraction for* *all the values of* ε*, which satisfy (1.18) and the resulting solution of equation (1.2) with* $g(z) = g_j(z)$ *is given by*

$$u_j(x) = u_0(x) + u_{p,j}(x). \tag{1.19}$$

Then for all the values of ε *satisfying inequality (1.18), the estimate*

$$\|u_1 - u_2\|_{H^3(\mathbb{R}^d)} \leq \frac{\varepsilon}{1-\varepsilon\sigma}(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2 \times$$

$$\times \left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^{\frac{8s_2}{d}-2}|S^d|^{\frac{4s_2}{d}}}{(16s_2)^{\frac{4s_2}{d}}(2\pi)^{4s_2}} \frac{d}{d-4s_2} + \frac{Q^2}{4} \right]^{\frac{1}{2}} \|g_1 - g_2\|_{C_2(I)}. \tag{1.20}$$

is valid.

Note that σ is defined in formula (3.1) below. We proceed to the proof of our first main statement.

2. The existence of the perturbed solution

Proof of Theorem 1.3. We choose an arbitrary $v(x) \in B_\rho$ and denote the term contained in the integral expression in the right side of equation (1.13) as

$$G(x) := g(u_0(x) + v(x)).$$

Let us use the standard Fourier transform

$$\widehat{\phi}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \phi(x) e^{-ipx} dx, \quad d = 4, 5. \tag{2.1}$$

Clearly, the upper bound

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|\phi(x)\|_{L^1(\mathbb{R}^d)}. \tag{2.2}$$

holds. Let us apply (2.1) to both sides of equation (1.13). This yields

$$\widehat{u}(p) = \varepsilon(2\pi)^{\frac{d}{2}} \frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{|p|^{2s_1} + |p|^{2s_2}}$$

with $0 < s_1 < s_2 < 1, \ \frac{3}{2} - \frac{d}{4} < s_2 < 1, \ d = 4, 5$. Thus, for the norm we have

$$\|u\|^2_{L^2(\mathbb{R}^d)} = (2\pi)^d \varepsilon^2 \int_{\mathbb{R}^d} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{[|p|^{2s_1} + |p|^{2s_2}]^2} dp. \tag{2.3}$$

As distinct from the earlier articles [31] and [32] with the standard Laplace operator in the diffusion term, here we do not try to control the norm

$$\left\| \frac{\widehat{\mathcal{K}}(p)}{|p|^{2s_1}+|p|^{2s_2}} \right\|_{L^\infty(\mathbb{R}^d)}.$$

Instead, let us estimate the right side of (2.3) by virtue of the analog of inequality (2.2) applied to functions $\mathcal{K}$ and G with $R>0$ as

$$(2\pi)^d \varepsilon^2 \int_{\mathbb{R}^d} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{[|p|^{2s_1}+|p|^{2s_2}]^2} dp \leq$$

$$\leq (2\pi)^d \varepsilon^2 \int_{|p|\leq R} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{|p|^{4s_2}} dp + (2\pi)^d \varepsilon^2 \int_{|p|>R} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{|p|^{4s_2}} dp \leq$$

$$\leq \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)} \left\{ \frac{1}{(2\pi)^d} \|G(x)\|^2_{L^1(\mathbb{R}^d)} |S^d| \frac{R^{d-4s_2}}{d-4s_2} + \frac{1}{R^{4s_2}} \|G(x)\|^2_{L^2(\mathbb{R}^d)} \right\}. \tag{2.4}$$

Since $v(x) \in B_\rho$, the inequality

$$\|u_0+v\|_{L^2(\mathbb{R}^d)} \leq \|u_0\|_{H^3(\mathbb{R}^d)} + 1$$

holds. Sobolev embedding (1.7) gives us

$$|u_0+v| \leq c_e(\|u_0\|_{H^3(\mathbb{R}^d)} + 1).$$

Obviously,

$$G(x) = \int_0^{u_0+v} g'(z)dz,$$

such that

$$|G(x)| \leq \text{sup}_{z\in I}|g'(z)||u_0+v| \leq M|u_0+v|,$$

where the interval I defined in (1.15). Thus,

$$\|G(x)\|_{L^2(\mathbb{R}^d)} \leq M\|u_0+v\|_{L^2(\mathbb{R}^d)} \leq M(\|u_0\|_{H^3(\mathbb{R}^d)} + 1).$$

Evidently,

$$G(x) = \int_0^{u_0+v} dy \Big[\int_0^y g''(z)dz \Big].$$

This yields

$$|G(x)| \leq \frac{1}{2}\text{sup}_{z\in I}|g''(z)||u_0+v|^2 \leq \frac{M}{2}|u_0+v|^2,$$

so that

$$\|G(x)\|_{L^1(\mathbb{R}^d)} \leq \frac{M}{2}\|u_0+v\|^2_{L^2(\mathbb{R}^d)} \leq \frac{M}{2}(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2. \tag{2.5}$$

Hence, we arrive at the upper bound for the right side of (2.4) given by

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}M^2(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2\left\{\frac{(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2|S^d|R^{d-4s_2}}{4(2\pi)^d(d-4s_2)}+\frac{1}{R^{4s_2}}\right\},$$

where $R\in(0,+\infty)$. Let us recall Lemma 1.4 to obtain the minimal value of the expression above. Thus, $\|u\|^2_{L^2(\mathbb{R}^d)}\le$

$$\le\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}M^2(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^{2+\frac{8s_2}{d}}\left(\frac{|S^d|}{16s_2}\right)^{\frac{4s_2}{d}}\frac{d}{(2\pi)^{4s_2}(d-4s_2)}. \qquad (2.6)$$

Clearly, by virtue of (1.13) we have

$$[(-\Delta)^{\frac{3}{2}-s_2+s_1}+(-\Delta)^{\frac{3}{2}}]u(x)=\varepsilon(-\Delta)^{\frac{3}{2}-s_2}\int_{\mathbb{R}^d}\mathcal{K}(x-y)G(y)dy.$$

Let us use the standard Fourier transform (2.1), the analog of inequality (2.2) applied to function G and (2.5). Hence,

$$\|(-\Delta)^{\frac{3}{2}}u\|^2_{L^2(\mathbb{R}^d)}\le\varepsilon^2\|G\|^2_{L^1(\mathbb{R}^d)}Q^2\le\varepsilon^2\frac{M^2}{4}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^4Q^2. \qquad (2.7)$$

By means of the definition of the norm (1.8) along with estimates (2.6) and (2.7) we derive that

$$\|u\|_{H^3(\mathbb{R}^d)}\le\varepsilon(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2M\times$$

$$\times\left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^{\frac{8s_2}{d}-2}d}{(2\pi)^{4s_2}(d-4s_2)}\left(\frac{|S^d|}{16s_2}\right)^{\frac{4s_2}{d}}+\frac{Q^2}{4}\right]^{\frac{1}{2}}\le\rho \qquad (2.8)$$

for all the values of the parameter ε, satisfying (1.18). Therefore, $u(x)\in B_\rho$ as well.

Let us suppose that for a certain $v(x)\in B_\rho$ there exist two solutions $u_{1,2}(x)\in B_\rho$ of equation (1.13). Obviously, the difference function $w(x):=u_1(x)-u_2(x)\in H^3(\mathbb{R}^d)$ solves the homogeneous problem

$$[(-\Delta)^{s_1}+(-\Delta)^{s_2}]w=0.$$

Since the operator $l:H^{2s_2}(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ introduced in (1.14) does not possess any nontrivial zero modes, $w(x)$ will vanish in $\mathbb{R}^d$. Thus, equation (1.13) defines a map $T_g:B_\rho\to B_\rho$ for all the values of ε, which satisfy inequality (1.18).

Let us establish that under the stated assumptions this map is a strict contraction. We choose arbitrarily $v_{1,2}(x)\in B_\rho$. The argument above implies that $u_{1,2}:=T_g v_{1,2}\in B_\rho$ as well for ε satisfying (1.18). By means of equation (1.13), we have

$$[(-\Delta)^{s_1}+(-\Delta)^{s_2}]u_1(x)=\varepsilon\int_{\mathbb{R}^d}\mathcal{K}(x-y)g(u_0(y)+v_1(y))dy, \qquad (2.9)$$

$$[(-\Delta)^{s_1}+(-\Delta)^{s_2}]u_2(x)=\varepsilon\int_{\mathbb{R}^d}\mathcal{K}(x-y)g(u_0(y)+v_2(y))dy, \tag{2.10}$$

where $0<s_1<s_2<1,\ \dfrac{3}{2}-\dfrac{d}{4}<s_2<1,\ d=4,5$. Let us introduce

$$G_1(x):=g(u_0(x)+v_1(x)), \quad G_2(x):=g(u_0(x)+v_2(x)).$$

We apply the standard Fourier transform (2.1) to both sides of problems (2.9) and (2.10). This yields

$$\widehat{u_1}(p)=\varepsilon(2\pi)^{\frac{d}{2}}\frac{\widehat{\mathcal{K}}(p)\widehat{G_1}(p)}{|p|^{2s_1}+|p|^{2s_2}}, \quad \widehat{u_2}(p)=\varepsilon(2\pi)^{\frac{d}{2}}\frac{\widehat{\mathcal{K}}(p)\widehat{G_2}(p)}{|p|^{2s_1}+|p|^{2s_2}},$$

such that

$$\|u_1-u_2\|^2_{L^2(\mathbb{R}^d)}=\varepsilon^2(2\pi)^d\int_{\mathbb{R}^d}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G_1}(p)-\widehat{G_2}(p)|^2}{[|p|^{2s_1}+|p|^{2s_2}]^2}dp. \tag{2.11}$$

The right side of (2.11) can be easily bounded from above by virtue of inequality (2.2) as

$$\varepsilon^2(2\pi)^d\int_{|p|\le R}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G_1}(p)-\widehat{G_2}(p)|^2}{|p|^{4s_2}}dp+$$

$$+\varepsilon^2(2\pi)^d\int_{|p|>R}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G_1}(p)-\widehat{G_2}(p)|^2}{|p|^{4s_2}}dp\le$$

$$\le\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}\left\{\frac{|S^d|}{(2\pi)^d}\|G_1(x)-G_2(x)\|^2_{L^1(\mathbb{R}^d)}\frac{R^{d-4s_2}}{d-4s_2}+\frac{\|G_1(x)-G_2(x)\|^2_{L^2(\mathbb{R}^d)}}{R^{4s_2}}\right\},$$

where $R\in(0,+\infty)$. Clearly,

$$G_1(x)-G_2(x)=\int_{u_0+v_2}^{u_0+v_1}g'(z)dz.$$

Hence,

$$|G_1(x)-G_2(x)|\le \text{sup}_{z\in I}|g'(z)||v_1(x)-v_2(x)|\le M|v_1(x)-v_2(x)|,$$

so that

$$\|G_1(x)-G_2(x)\|_{L^2(\mathbb{R}^d)}\le M\|v_1-v_2\|_{L^2(\mathbb{R}^d)}\le M\|v_1-v_2\|_{H^3(\mathbb{R}^d)}.$$

Let us use the identity

$$G_1(x)-G_2(x)=\int_{u_0+v_2}^{u_0+v_1}dy\Big[\int_0^y g''(z)dz\Big].$$

Obviously, $G_1(x)-G_2(x)$ can be easily estimated from above in the absolute value by

$$\frac{1}{2}\text{sup}_{z\in I}|g''(z)||(v_1-v_2)(2u_0+v_1+v_2)|\le\frac{M}{2}|(v_1-v_2)(2u_0+v_1+v_2)|.$$

Using the Schwarz inequality, we obtain the upper bound for the norm

$$\|G_1(x)-G_2(x)\|_{L^1(\mathbb{R}^d)}\le\frac{M}{2}\|v_1-v_2\|_{L^2(\mathbb{R}^d)}\|2u_0+v_1+v_2\|_{L^2(\mathbb{R}^d)}\le$$

$$\le M\|v_1-v_2\|_{H^3(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1). \tag{2.12}$$

Thus, we arrive at the estimate from above for $\|u_1(x)-u_2(x)\|^2_{L^2(\mathbb{R}^d)}$ given by

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}M^2\|v_1-v_2\|^2_{H^3(\mathbb{R}^d)}\Big\{\frac{|S^d|}{(2\pi)^d}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2\frac{R^{d-4s_2}}{d-4s_2}+\frac{1}{R^{4s_2}}\Big\}.$$

By means of Lemma 1.4 we minimize the expression above over $R\in(0,+\infty)$. Hence,

$$\|u_1(x)-u_2(x)\|^2_{L^2(\mathbb{R}^d)}\le$$

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}M^2\|v_1-v_2\|^2_{H^3(\mathbb{R}^d)}\frac{|S^d|^{\frac{4s_2}{d}}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^{\frac{8s_2}{d}}}{(2\pi)^{4s_2}(4s_2)^{\frac{4s_2}{d}}}\frac{d}{d-4s_2}. \tag{2.13}$$

Let us use formulas (2.9) and (2.10) to obtain that

$$[(-\Delta)^{\frac{3}{2}-s_2+s_1}+(-\Delta)^{\frac{3}{2}}](u_1-u_2)(x)=\varepsilon(-\Delta)^{\frac{3}{2}-s_2}\int_{\mathbb{R}^d}\mathcal{K}(x-y)[G_1(y)-G_2(y)]dy.$$

By virtue of the standard Fourier transform (2.1) along with bounds (2.2) and (2.12), we derive

$$\|(-\Delta)^{\frac{3}{2}}(u_1-u_2)\|^2_{L^2(\mathbb{R}^d)}\le\varepsilon^2Q^2\|G_1-G_2\|^2_{L^1(\mathbb{R}^d)}\le$$

$$\le\varepsilon^2Q^2M^2\|v_1-v_2\|^2_{H^3(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2. \tag{2.14}$$

By means of (2.13) and (2.14), the norm $\|u_1-u_2\|_{H^3(\mathbb{R}^d)}$ can be estimated from above by the expression

$$\varepsilon M(\|u_0\|_{H^3(\mathbb{R}^d)}+1)\times$$

$$\times\Bigg\{\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}|S^d|^{\frac{4s_2}{d}}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^{\frac{8s_2}{d}-2}}{(2\pi)^{4s_2}(4s_2)^{\frac{4s_2}{d}}}\frac{d}{d-4s_2}+Q^2\Bigg\}^{\frac{1}{2}}\times$$

$$\times\|v_1-v_2\|_{H^3(\mathbb{R}^d)}. \tag{2.15}$$

It follows easily from (1.18) that the constant in the right side of (2.15) is less than one. Thus, the map $T_g: B_\rho\to B_\rho$ defined by equation (1.13) is a strict contraction for all the values of ε satisfying inequality (1.18). Its unique fixed point $u_p(x)$ is the only solution of problem (1.11) in the ball B_ρ. By virtue of (2.8), we have $\|u_p(x)\|_{H^3(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$. The resulting $u(x) \in H^3(\mathbb{R}^d)$ given by formula (1.10) solves equation (1.2). ■

Let us turn our attention to establishing the validity of the second main statement of the article.

3. The continuity of the resulting solution

Proof of Theorem 1.5. Evidently, for all the values of the parameter ε, which satisfy (1.18), we have

$$u_{p,1} = T_{g_1} u_{p,1}, \quad u_{p,2} = T_{g_2} u_{p,2}.$$

Hence,

$$u_{p,1} - u_{p,2} = T_{g_1} u_{p,1} - T_{g_1} u_{p,2} + T_{g_1} u_{p,2} - T_{g_2} u_{p,2}.$$

Clearly,

$$\|u_{p,1} - u_{p,2}\|_{H^3(\mathbb{R}^d)} \le \|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^3(\mathbb{R}^d)} + \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^3(\mathbb{R}^d)}.$$

By means of upper bound (2.15), we have

$$\|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^3(\mathbb{R}^d)} \le \varepsilon\sigma \|u_{p,1} - u_{p,2}\|_{H^3(\mathbb{R}^d)}.$$

Note that $\varepsilon\sigma < 1$ because the map $T_{g_1} : B_\rho \to B_\rho$ is a strict contraction under the given conditions. Here the positive constant

$$\sigma := M(\|u_0\|_{H^3(\mathbb{R}^d)} + 1) \times$$

$$\times \left\{ \frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)} |S^d|^{\frac{4s_2}{d}} (\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^{\frac{8s_2}{d} - 2}}{(2\pi)^{4s_2} (4s_2)^{\frac{4s_2}{d}}} \frac{d}{d - 4s_2} + Q^2 \right\}^{\frac{1}{2}}. \tag{3.1}$$

Therefore,

$$(1 - \varepsilon\sigma) \|u_{p,1} - u_{p,2}\|_{H^3(\mathbb{R}^d)} \le \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^3(\mathbb{R}^d)}. \tag{3.2}$$

Note that for our fixed point $T_{g_2} u_{p,2} = u_{p,2}$. We introduce $\xi(x) := T_{g_1} u_{p,2}$. Evidently,

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]\xi(x) = \varepsilon \int_{\mathbb{R}^d} \mathcal{K}(x - y) g_1(u_0(y) + u_{p,2}(y)) dy, \tag{3.3}$$

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}] u_{p,2}(x) = \varepsilon \int_{\mathbb{R}^d} \mathcal{K}(x - y) g_2(u_0(y) + u_{p,2}(y)) dy, \tag{3.4}$$

where $0 < s_1 < s_2 < 1, \ \frac{3}{2} - \frac{d}{4} < s_2 < 1, \ d = 4, 5$. Let us denote

$$G_{1,2}(x) := g_1(u_0(x) + u_{p,2}(x)), \quad G_{2,2}(x) := g_2(u_0(x) + u_{p,2}(x)).$$

We apply the standard Fourier transform (2.1) to both sides of equations (3.3) and (3.4). This yields

$$\widehat{\xi}(p) = \varepsilon(2\pi)^{\frac{d}{2}} \frac{\widehat{\mathcal{K}}(p)\widehat{G_{1,2}}(p)}{|p|^{2s_1} + |p|^{2s_2}}, \quad \widehat{u_{p,2}}(p) = \varepsilon(2\pi)^{\frac{d}{2}} \frac{\widehat{\mathcal{K}}(p)\widehat{G_{2,2}}(p)}{|p|^{2s_1} + |p|^{2s_2}}.$$

Obviously,

$$\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R}^d)} = \varepsilon^2 (2\pi)^d \int_{\mathbb{R}^d} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G_{1,2}}(p) - \widehat{G_{2,2}}(p)|^2}{[|p|^{2s_1} + |p|^{2s_2}]^2} dp. \tag{3.5}$$

Let us estimate the right side of (3.5) using inequality (2.2). Hence,

$$\varepsilon^2 (2\pi)^d \int_{|p| \le R} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G_{1,2}}(p) - \widehat{G_{2,2}}(p)|^2}{|p|^{4s_2}} dp +$$

$$+ \varepsilon^2 (2\pi)^d \int_{|p| > R} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G_{1,2}}(p) - \widehat{G_{2,2}}(p)|^2}{|p|^{4s_2}} dp \le$$

$$\le \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)} \left\{ \frac{|S^d|}{(2\pi)^d} \|G_{1,2} - G_{2,2}\|^2_{L^1(\mathbb{R}^d)} \frac{R^{d-4s_2}}{d - 4s_2} + \frac{\|G_{1,2} - G_{2,2}\|^2_{L^2(\mathbb{R}^d)}}{R^{4s_2}} \right\}$$

with $R \in (0, +\infty)$. Clearly, the equality

$$G_{1,2}(x) - G_{2,2}(x) = \int_0^{u_0(x) + u_{p,2}(x)} [g_1'(z) - g_2'(z)] dz$$

holds, such that

$$|G_{1,2}(x) - G_{2,2}(x)| \le \sup_{z \in I} |g_1'(z) - g_2'(z)| |u_0(x) + u_{p,2}(x)| \le$$

$$\le \|g_1 - g_2\|_{C_2(I)} |u_0(x) + u_{p,2}(x)|.$$

Thus,

$$\|G_{1,2} - G_{2,2}\|_{L^2(\mathbb{R}^d)} \le \|g_1 - g_2\|_{C_2(I)} \|u_0 + u_{p,2}\|_{L^2(\mathbb{R}^d)} \le$$

$$\le \|g_1 - g_2\|_{C_2(I)} (\|u_0\|_{H^3(\mathbb{R}^d)} + 1).$$

Let us use another identity

$$G_{1,2}(x) - G_{2,2}(x) = \int_0^{u_0(x) + u_{p,2}(x)} dy \Big[\int_0^y (g_1''(z) - g_2''(z)) dz \Big].$$

Evidently,

$$|G_{1,2}(x) - G_{2,2}(x)| \le \frac{1}{2}\sup_{z\in I}|g_1''(z) - g_2''(z)||u_0(x) + u_{p,2}(x)|^2 \le$$

$$\le \frac{1}{2}\|g_1 - g_2\|_{C_2(I)}|u_0(x) + u_{p,2}(x)|^2,$$

so that

$$\|G_{1,2} - G_{2,2}\|_{L^1(\mathbb{R}^d)} \le \frac{1}{2}\|g_1 - g_2\|_{C_2(I)}\|u_0 + u_{p,2}\|^2_{L^2(\mathbb{R}^d)} \le$$

$$\le \frac{1}{2}\|g_1 - g_2\|_{C_2(I)}(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2. \tag{3.6}$$

This gives us the upper bound for the norm $\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R}^d)}$ as

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2\|g_1 - g_2\|^2_{C_2(I)}\times$$

$$\times\Big[\frac{|S^d|(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2}{4(2\pi)^d}\frac{R^{d-4s_2}}{d-4s_2} + \frac{1}{R^{4s_2}}\Big]. \tag{3.7}$$

Expression (3.7) can be trivially minimized over $R \in (0, +\infty)$ using Lemma 1.4, such that $\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R}^d)} \le$

$$\le \varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^{2+\frac{8s_2}{d}}\|g_1 - g_2\|^2_{C_2(I)}\frac{|S^d|^{\frac{4s_2}{d}}}{(16s_2)^{\frac{4s_2}{d}}(2\pi)^{4s_2}}\frac{d}{d-4s_2}.$$

By means of formulas (3.3) and (3.4), we obtain

$$[(-\Delta)^{\frac{3}{2}-s_2+s_1} + (-\Delta)^{\frac{3}{2}}]\xi(x) = \varepsilon(-\Delta)^{\frac{3}{2}-s_2}\int_{\mathbb{R}^d}\mathcal{K}(x-y)G_{1,2}(y)dy,$$

$$[(-\Delta)^{\frac{3}{2}-s_2+s_1} + (-\Delta)^{\frac{3}{2}}]u_{p,2}(x) = \varepsilon(-\Delta)^{\frac{3}{2}-s_2}\int_{\mathbb{R}^d}\mathcal{K}(x-y)G_{2,2}(y)dy,$$

such that

$$[(-\Delta)^{\frac{3}{2}-s_2+s_1} + (-\Delta)^{\frac{3}{2}}](\xi(x) - u_{p,2}(x)) =$$

$$= \varepsilon(-\Delta)^{\frac{3}{2}-s_2}\int_{\mathbb{R}^d}\mathcal{K}(x-y)[G_{1,2}(y) - G_{2,2}(y)]dy.$$

Let us use the standard Fourier transform (2.1) along with inequalities (2.2) and (3.6) to establish that

$$\|(-\Delta)^{\frac{3}{2}}[\xi(x) - u_{p,2}(x)]\|^2_{L^2(\mathbb{R}^d)} \le$$

$$\le \varepsilon^2\|G_{1,2} - G_{2,2}\|^2_{L^1(\mathbb{R}^d)}Q^2 \le \frac{\varepsilon^2Q^2}{4}(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^4\|g_1 - g_2\|^2_{C_2(I)}.$$

Hence,

$$\|\xi(x)-u_{p,2}(x)\|_{H^3(\mathbb{R}^d)} \leq \varepsilon\|g_1-g_2\|_{C_2(I)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2\times$$

$$\times\left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^{\frac{8s_2}{d}-2}|S^d|^{\frac{4s_2}{d}}}{(16s_2)^{\frac{4s_2}{d}}(2\pi)^{4s_2}}\frac{d}{d-4s_2}+\frac{Q^2}{4}\right]^{\frac{1}{2}}.$$

By virtue of bound (3.2), we have

$$\|u_{p,1}-u_{p,2}\|_{H^3(\mathbb{R}^d)} \leq \frac{\varepsilon}{1-\varepsilon\sigma}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2\times$$

$$\times\left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^{\frac{8s_2}{d}-2}|S^d|^{\frac{4s_2}{d}}}{(16s_2)^{\frac{4s_2}{d}}(2\pi)^{4s_2}}\frac{d}{d-4s_2}+\frac{Q^2}{4}\right]^{\frac{1}{2}}\|g_1-g_2\|_{C_2(I)}.$$

We complete the proof of the theorem by using formula (1.19). ■

4. Auxiliary results

Let us derive the solvability conditions for the linear Poisson type equation with a square integrable right side in the case of the double scale anomalous diffusion

$$[(-\Delta)^{s_1}+(-\Delta)^{s_2}]u=f(x), \quad x\in\mathbb{R}^d, \quad d=4,5, \quad 0<s_1<s_2<1. \tag{4.1}$$

The auxiliary statement below is easily established by applying the standard Fourier transform (2.1) to both sides of problem (4.1).

Lemma 4.1. *Let* $0<s_1<s_2<1$, $f(x):\mathbb{R}^d\to\mathbb{R}$, $d=4,5$ *and* $f(x)\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. *Then problem (4.1) possesses a unique solution* $u(x)\in H^{2s_2}(\mathbb{R}^d)$.

Proof. It can be easily verified that if $u(x)\in L^2(\mathbb{R}^d)$ is a solution of equation (4.1) with a square integrable right side, it will belong to $H^{2s_2}(\mathbb{R}^d)$ as well. For that purpose, we apply the standard Fourier transform (2.1) to both sides of (4.1). This yields

$$(|p|^{2s_1}+|p|^{2s_2})\widehat{u}(p)=\widehat{f}(p)\in L^2(\mathbb{R}^d),$$

so that

$$\int_{\mathbb{R}^d}[|p|^{2s_1}+|p|^{2s_2}]^2|\widehat{u}(p)|^2dp<\infty.$$

Let us use the simple identity

$$\|(-\Delta)^{s_2}u\|^2_{L^2(\mathbb{R}^d)}=\int_{\mathbb{R}^d}|p|^{4s_2}|\widehat{u}(p)|^2dp<\infty.$$

Hence, we have obtain that $(-\Delta)^{s_2}u \in L^2(\mathbb{R}^d)$. By virtue of the definition of the norm (1.6), we have $u(x) \in H^{2s_2}(\mathbb{R}^d)$ as well.
To demonstrate the uniqueness of solutions for our equation, we suppose that problem (4.1) admits two solutions $u_{1,2}(x) \in H^{2s_2}(\mathbb{R}^d)$. Then their difference $w(x) := u_1(x) - u_2(x) \in H^{2s_2}(\mathbb{R}^d)$ solves the homogeneous equation

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]w = 0.$$

Because the operator $l : H^{2s_2}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined in (1.14) does not have any nontrivial zero modes, $w(x)$ will vanish in $\mathbb{R}^d$.
We apply the standard Fourier transform (2.1) to both sides of equation (4.1). This gives us

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}}\chi_{\{|p|\le 1\}} + \frac{\widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}}\chi_{\{|p|>1\}}. \tag{4.2}$$

Here and below χ_A will stand for the characteristic function of a set $A \subseteq \mathbb{R}^d$. Obviously, the second term in the right side of (4.2) can be bounded from above in the absolute value by $\frac{|\widehat{f}(p)|}{2} \in L^2(\mathbb{R}^d)$ via the one of our assumptions.
Clearly, the first term in the right side of (4.2) can be estimated from above in the absolute value by

$$\frac{\|f(x)\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}|p|^{2s_2}}\chi_{\{|p|\le 1\}} \tag{4.3}$$

by means of inequality (2.2). It can be trivially checked that the term (4.3) with $d = 4, 5$ and $0 < s_2 < 1$ belongs to $L^2(\mathbb{R}^d)$. ■

Let us note that by proving the lemma above we demonstrate the solvability of problem (4.1) in $H^{2s_2}(\mathbb{R}^d),\ d = 4, 5$ for all the values of the powers of the fractional Laplacians $0 < s_1 < s_2 < 1$, so that no orthogonality relations are required for the right side $f(x)$. This is similar to the situation when the Poisson type equation is considered with a single fractional Laplacian in the spaces of the same dimensions (see Theorem 1.1 of [33], also [29]). The solvability of the problem similar to (4.1) involving a scalar potential was discussed in [13].

We write down the correspoding sequence of the approximate equations related to problem (4.1) with $n \in \mathbb{N}$ as

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]u_n = f_n(x), \quad x \in \mathbb{R}^d, \quad d = 4, 5, \quad 0 < s_1 < s_2 < 1. \tag{4.4}$$

The right sides of (4.4) converge to the right side of (4.1) as $n \to \infty$. Let us establish that under the certain technical conditions each equation (4.4) admits a unique solution $u_n(x) \in H^{2s_2}(\mathbb{R}^d)$, limiting problem (4.1) possesses a unique solution $u(x) \in H^{2s_2}(\mathbb{R}^d)$ and $u_n(x) \to u(x)$ in $H^{2s_2}(\mathbb{R}^d)$ as $n \to \infty$. This is the so called *solvability in the sense of sequences* for equation (4.1). The final proposition of the article is as follows.

Lemma 4.2. *Let $n \in \mathbb{N},\ 0 < s_1 < s_2 < 1,\ f_n(x) : \mathbb{R}^d \to \mathbb{R},\ d = 4, 5$ and $f_n(x) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, such that $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^d)$ and $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Then problems (4.1) and (4.4) have unique solutions $u(x) \in H^{2s_2}(\mathbb{R}^d)$ and $u_n(x) \in H^{2s_2}(\mathbb{R}^d)$ respectively, so that $u_n(x) \to u(x)$ in $H^{2s_2}(\mathbb{R}^d)$ as $n \to \infty$.*

Proof. By means of the result of Lemma 4.1 above, equations (4.1) and (4.4) admit unique solutions $u(x) \in H^{2s_2}(\mathbb{R}^d)$ and $u_n(x) \in H^{2s_2}(\mathbb{R}^d),\ n \in \mathbb{N}$ respectively.
Let us suppose that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. It can be trivially checked that $u_n(x) \to u(x)$ in $H^{2s_2}(\mathbb{R}^d)$ as $n \to \infty$ as well. Indeed, by virtue of (4.4) and (4.1)

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}](u_n(x) - u(x)) = f_n(x) - f(x).$$

Let us use the standard Fourier transform (2.1) to derive that

$$\|(-\Delta)^{s_2}(u_n(x) - u(x))\|_{L^2(\mathbb{R}^d)} \le \|f_n(x) - f(x)\|_{L^2(\mathbb{R}^d)} \to 0, \quad n \to \infty$$

as we assume. Norm definition (1.6) yields $u_n(x) \to u(x)$ in $H^{2s_2}(\mathbb{R}^d)$ as $n \to \infty$.
We apply (2.1) to both sides of equations (4.1) and (4.4) and arrive at

$$\widehat{u_n}(p) - \widehat{u}(p) = \frac{\widehat{f_n}(p) - \widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| \le 1\}} + \frac{\widehat{f_n}(p) - \widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| > 1\}}. \tag{4.5}$$

Evidently, the second term in the right side of (4.5) can be estimated from above in the absolute value by $\dfrac{|\widehat{f_n}(p) - \widehat{f}(p)|}{2}$. Thus,

$$\left\| \frac{\widehat{f_n}(p) - \widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| > 1\}} \right\|_{L^2(\mathbb{R}^d)} \le \frac{1}{2} \|f_n(x) - f(x)\|_{L^2(\mathbb{R}^d)} \to 0, \quad n \to \infty$$

due to the one of our assumptions.
Clearly, the first term in the right side of (4.5) can be bounded from above in the absolute value using (2.2) by

$$\frac{\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}} |p|^{2s_2}} \chi_{\{|p| \le 1\}}.$$

Hence,

$$\left\| \frac{\widehat{f_n}(p) - \widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| \le 1\}} \right\|_{L^2(\mathbb{R}^d)} \le \frac{\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}} \sqrt{\frac{|S^d|}{d - 4s_2}} \to 0, \quad n \to \infty$$

as assumed. Therefore,

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}^d) \quad as \quad n \to \infty$$

for $0 < s_1 < s_2 < 1$ and $d = 4, 5$, which completes the proof of the lemma. ■

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