Solvability of some Fredholm integro-differential equations with mixed diffusion in a square

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Abstract. We demonstrate the existence in the sense of sequences of solutions for some integro-differential type problems in a square in two dimensions with periodic boundary conditions containing the normal diffusion in one direction and the superdiffusion in the other direction in a constrained subspace of H^2 using the fixed point technique. The elliptic equation involves a second order differential operator satisfying the Fredholm property. It is established that, under the reasonable technical assumptions, the convergence in the appropriate function spaces of the integral kernels yields the existence and convergence in H_0^2 of the solutions. We generalize the results obtained in our preceding work [14] for the analogous equation considered in the whole $\mathbb{R}^2$ which contained a non-Fredholm operator.

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1 Introduction

Let us recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. Consequently, the equation $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. These properties of the Fredholm operators are broadly used in many methods of the linear and nonlinear analysis.
The elliptic equations considered in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, the proper ellipticity and the

Shapiro-Lopatinskii conditions are fulfilled (see e.g. [2], [9], [23], [27]). This is the main result of the theory of linear elliptic problems. In the case of unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For instance, the Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ fails to satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.
For the linear elliptic equations in the unbounded domains the Fredholm property is satisfied if and only if, in addition to the conditions mentioned above, the limiting operators are invertible (see [28]). In certain trivial cases, the limiting operators can be constructed explicitly. For example, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at infinity,

$$a_{\pm} = \lim_{x \to \pm\infty} a(x), \quad b_{\pm} = \lim_{x \to \pm\infty} b(x), \quad c_{\pm} = \lim_{x \to \pm\infty} c(x),$$

the limiting operators are:

$$L_{\pm}u = a_{\pm}u'' + b_{\pm}u' + c_{\pm}u.$$

Because the coefficients are constants, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ does not possess the Fredholm property, can be found explicitly using the standard Fourier transform, so that:

$$\lambda_{\pm}(\xi) = -a_{\pm}\xi^2 + b_{\pm}i\xi + c_{\pm}, \quad \xi \in \mathbb{R}.$$

The limiting operators are invertible if and only if the origin does not belong to the essential spectrum.
For the general elliptic problems, the analogical assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or when the limiting operators are invertible. However, these conditions may not be written explicitly.
For the non-Fredholm operators the usual solvability conditions may not be applicable and in a general case the solvability relations are unknown. However, there are certain classes of operators for which the solvability conditions were obtained recently. Let us illustrate them with the following example. Consider the equation

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, $d \in \mathbb{N}$, where a is a positive constant. The operator L here coincides with its limiting operators. The corresponding homogeneous problem has a nonzero bounded solution, so that the Fredholm property is not satisfied. However, since the operator contained in (1.1) has the constant coefficients, we can apply the standard Fourier transform to find the solution explicitly. The solvability relations can be formulated as follows. If $f(x) \in L^2(\mathbb{R}^d)$ and $xf(x) \in L^1(\mathbb{R}^d)$, then there exists a unique solution of this equation in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}} \right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see Lemmas 5 and 6 of [36]). Here $S^d_{\sqrt{a}}$ denotes the sphere in $\mathbb{R}^d$ of radius $\sqrt{a}$ centered at the origin. Hence, although the Fredholm property is not satisfied for the operator, the solvability relations are formulated similarly. Evidently, this similarity is only formal because the range of the operator is not closed.

In the situation of the operator with a potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

the standard Fourier transform is not directly applicable. Nevertheless, the solvability conditions in $\mathbb{R}^3$ can be obtained by a rather sophisticated application of the theory of the self-adjoint Schrödinger type operators (see [32]). As in the constant coefficient case, the solvability relations are expressed in terms of the orthogonality to the solutions of the adjoint homogeneous equation. There are several other examples of linear elliptic non-Fredholm operators for which the solvability conditions can be derived (see [13], [15], [28], [29], [30], [31], [34], [35], [36]).

Solvability conditions play a significant role in the analysis of the nonlinear elliptic equations. In the case of the operators without the Fredholm property involved, in spite of the certain progress in the understanding of the linear problems, there exist only few examples where the nonlinear non- Fredholm operators are analyzed (see [7], [8], [12], [14], [15], [16], [17], [33], [36], [37], [38], [39]). Fredholm structures, topological invariants and their applications were discussed in [9]. The work [10] deals with the finite and infinite dimensional attractors for evolution equations of mathematical physics. The large time behavior of solutions of a class of fourth-order parabolic equations defined on unbounded domains using the Kolmogorov ε-entropy as a measure was investigated in [11]. The attractor for a nonlinear reaction-diffusion system in an unbounded domain in $\mathbb{R}^3$ was studied in [18]. The articles [19] and [25] are important for the understanding of the Fredholm and properness properties of quasilinear elliptic systems of second order and of operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in second-order quasilinear elliptic systems were covered in [20]. Standing lattice solitons in the discrete NLS equation with saturation were discussed in [1]. The present article is devoted to another class of stationary nonlinear equations, for which the Fredholm property is satisfied:

$$\frac{\partial^2 u}{\partial x_1^2} - \sqrt{-\frac{\partial^2}{\partial x_2^2}}u + \int_\Omega G(x-y)F(u(y),y)dy = 0 \tag{1.2}$$

with $x = (x_1, x_2) \in \Omega,\ y = (y_1, y_2) \in \Omega$ and the square $\Omega := [0, 2\pi] \times [0, 2\pi]$ with the periodic boundary conditions imposed below. We generalize the results derived for the analogous equation in the whole $\mathbb{R}^2$ considered in [14]. Therefore, it contained the operator without the Fredholm property. The novelty of the works of this kind is that in the diffusion term we add the standard minus Laplacian in the x_1 variable with the negative Laplace operator in x_2 raised to a fractional power $0 < s < 1$ and defined via the spectral calculus. As distinct from the similar problem studied in [14], in the present work we restrict our attention to $s = \dfrac{1}{2}$.

The models of this kind are new. They are not well understood, especially in the context of the integro-differential equations. The difficulty we have to deal with is that such equation becomes anisotropic and it is more technical to obtain the desired estimates when working with it. In the population dynamics in the Mathematical Biology the integro-differential equations describe the models with the intra-specific competition and nonlocal consumption of resources (see e.g. [3], [4]). It is important to study the problems of this kind from the point of view of the understanding of the spread of the viral infections, because many countries have to deal with the pandemics. Let us use the explicit form of the solvability relations and demonstrate the existence of solutions of our nonlinear problem. In the situation of the standard Laplacian in the diffusion term, the equation analogical to (1.2) was studied in [33] and [39] in the whole space and on a finite interval with the periodic boundary conditions. The solvability of the integro-differential problems containing in the diffusion term only the negative Laplace operator raised to a fractional power was actively discussed in recent years in the context of the anomalous diffusion (see e.g. [17], [37], [38]). The anomalous diffusion can be described as a random process of the particle motion characterized by the probability density distribution of the jump length. The moments of this density distribution are finite in the situation of the normal diffusion, but this is not the case for the anomalous diffusion. The asymptotic behavior at the infinity of the probability density function determines the value of the power of the Laplacian (see [24]). In [26] the authors consider the mixed local-nonlocal semi-linear elliptic equations driven by the superposition of Brownian and Levy processes and show the L^{∞} boundedness of any weak solution. The article [6] deals with a new type of mixed local and nonlocal equation under the Neumann conditions. The spectral properties associated to a weighted eigenvalue problem are discussed and a global bound for subsolutions are presented.

2 Formulation of the results

For the nonlinear part of problem (1.2) the following regularity conditions will hold. Here $x=(x_1, x_2)\in \Omega$.

Assumption 2.1. *Function $F(u,x):\mathbb{R}\times\Omega\to\mathbb{R}$ is satisfying the Caratheodory condition (see [22]), so that*

$$|F(u,x)|\le k|u|+h(x) \quad for \quad u\in\mathbb{R},\ x\in\Omega \tag{2.1}$$

with a constant $k>0$ and $h(x):\Omega\to\mathbb{R}^{+}, \quad h(x)\in L^2(\Omega)$. Furthermore, it is a Lipschitz continuous function, such that

$$|F(u_1,x)-F(u_2,x)|\le l|u_1-u_2| \quad for \quad any \quad u_{1,2}\in\mathbb{R}, \quad x\in\Omega \tag{2.2}$$

with a constant $l>0$. Moreover,

$$F(u,0,x_2)=F(u,2\pi,x_2) \quad for \quad u\in\mathbb{R}, \quad 0\le x_2\le 2\pi$$

and

$$F(u,x_1,0)=F(u,x_1,2\pi) \quad for \quad u\in\mathbb{R}, \quad 0\le x_1\le 2\pi.$$

The solvability of a local elliptic problem in a bounded domain in $\mathbb{R}^N$ was studied in [5], where the nonlinear function was allowed to have a sublinear growth. In order to establish the existence of solutions of (1.2), we will use the auxiliary equation

$$-\frac{\partial^2 u}{\partial x_1^2}+\sqrt{-\frac{\partial^2}{\partial x_2^2}}u=\int_\Omega G(x-y)F(v(y),y)dy, \tag{2.3}$$

where $x=(x_1,x_2)\in\Omega$ and $y=(y_1,y_2)\in\Omega$. Let us denote

$$(f_1(x_1,x_2),f_2(x_1,x_2))_{L^2(\Omega)}:=\int_0^{2\pi}\int_0^{2\pi}f_1(x_1,x_2)\bar{f}_2(x_1,x_2)dx_1dx_2. \tag{2.4}$$

In the article we will work in the Sobolev space

$$H^2(\Omega):=\{u(x_1,x_2):\Omega\to\mathbb{R}\mid u(x_1,x_2),\ \Delta u(x_1,x_2)\in L^2(\Omega),\ u(0,x_2)=u(2\pi,x_2),$$
$$\frac{\partial u}{\partial x_1}(0,x_2)=\frac{\partial u}{\partial x_1}(2\pi,x_2)\ for\ 0\le x_2\le 2\pi,$$
$$u(x_1,0)=u(x_1,2\pi),\ \frac{\partial u}{\partial x_2}(x_1,0)=\frac{\partial u}{\partial x_2}(x_1,2\pi)\ for\ 0\le x_1\le 2\pi\}.$$

Here and below the cumulative Laplace operator $\Delta:=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$. We introduce the following auxiliary constrained subspace

$$H_0^2(\Omega):=\{u(x_1,x_2)\in H^2(\Omega)\mid (u(x_1,x_2),1)_{L^2(\Omega)}=0\}. \tag{2.5}$$

Clearly, (2.5) is a Hilbert space as well (see e.g. Chapter 2.1 of [21]). It is equipped with the norm

$$\|u\|^2_{H_0^2(\Omega)}:=\|u\|^2_{L^2(\Omega)}+\|\Delta u\|^2_{L^2(\Omega)}. \tag{2.6}$$

Equation (2.3) contains the operator

$$L_r:=-\frac{\partial^2}{\partial x_1^2}+\sqrt{-\frac{\partial^2}{\partial x_2^2}}: \quad H_0^2(\Omega)\to L^2(\Omega). \tag{2.7}$$

Its eigenvalues are

$$\lambda_{r,n_1,n_2}:=n_1^2+|n_2|, \quad (n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}. \tag{2.8}$$

The corresponding eigenfunctions are given by

$$\frac{e^{in_1x_1}}{\sqrt{2\pi}}\frac{e^{in_2x_2}}{\sqrt{2\pi}}, \quad (n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}. \tag{2.9}$$

Evidently, (2.7) is a Fredholm operator with a trivial kernel. In the present article we manage to demonstrate that under the reasonable technical conditions equation (2.3) defines a map $T_r : H_0^2(\Omega) \to H_0^2(\Omega)$, which is a strict contraction.

Theorem 2.2. *Let Assumption 2.1 hold, the function $G(x_1, x_2) : \Omega \to \mathbb{R}$, so that $G(0, x_2) = G(2\pi, x_2)$ for $0 \le x_2 \le 2\pi$ and $G(x_1, 0) = G(x_1, 2\pi)$ for $0 \le x_1 \le 2\pi$. Moreover, $G(x_1, x_2) \in C(\Omega)$ and $\dfrac{\partial G(x_1, x_2)}{\partial x_2} \in L^1(\Omega)$. Let us also assume that orthogonality relation (4.6) is valid and that $2\sqrt{2}\pi \mathcal{N}_r l < 1$.*
Then the map $T_r v = u$ on $H_0^2(\Omega)$ defined by equation (2.3) has a unique fixed point v_r, which is the only solution of problem (1.2) in $H_0^2(\Omega)$.
This fixed point v_r is nontrivial provided the Fourier coefficients $G_{n_1,n_2} F(0, x)_{n_1,n_2} \neq 0$ for some $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$.

Related to problem (1.2) in our square Ω, we consider the sequence of approximate equations with $m \in \mathbb{N}$

$$\frac{\partial^2 u_m}{\partial x_1^2} - \sqrt{-\frac{\partial^2}{\partial x_2^2}} u_m + \int_\Omega G_m(x - y) F(u_m(y), y) dy = 0, \tag{2.10}$$

where $x = (x_1, x_2) \in \Omega,\ y = (y_1, y_2) \in \Omega$. The sequence of kernels $\{G_m(x)\}_{m=1}^\infty$ converges to $G(x)$ as $m \to \infty$ in the function spaces specified below. Let us establish that, under the appropriate technical assumptions, each of problems (2.10) admits a unique solution $u_m(x) \in H_0^2(\Omega)$, limiting equation (1.2) has a unique solution $u(x) \in H_0^2(\Omega)$, and $u_m(x) \to u(x)$ in $H_0^2(\Omega)$ as $m \to \infty$. This is the so-called *existence of solutions in the sense of sequences*. In this situation, the solvability conditions can be formulated for the iterated kernels G_m. They give the convergence of the kernels in terms of the Fourier transforms (see the Appendix) and, consequently, the convergence of the solutions (Theorem 2.3 below). The analogous ideas in the context of the standard Schrödinger type operators were used in [31]. Our second main statement is as follows.

Theorem 2.3. *Let Assumption 2.1 hold, $m \in \mathbb{N}$, the functions $G_m(x_1, x_2) : \Omega \to \mathbb{R}$ are such that $G_m(0, x_2) = G_m(2\pi, x_2)$ for $0 \le x_2 \le 2\pi$ and $G_m(x_1, 0) = G_m(x_1, 2\pi)$ for $0 \le x_1 \le 2\pi$. Furthermore,*

$$G_m(x_1, x_2) \in C(\Omega), \quad G_m(x_1, x_2) \to G(x_1, x_2) \quad \text{in} \quad C(\Omega) \quad \text{as} \quad m \to \infty.$$

Similarly,

$$\frac{\partial G_m(x_1, x_2)}{\partial x_2} \in L^1(\Omega), \quad \frac{\partial G_m(x_1, x_2)}{\partial x_2} \to \frac{\partial G(x_1, x_2)}{\partial x_2} \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad m \to \infty.$$

Let us also suppose that for each $m \in \mathbb{N}$ orthogonality relation (4.9) holds. Finally, we assume that (4.10) is valid for each $m \in \mathbb{N}$ with some fixed $0 < \varepsilon < 1$.

Then each equation (2.10) possesses a unique solution $u_m(x) \in H_0^2(\Omega)$, limiting problem (1.2) admits a unique solution $u(x) \in H_0^2(\Omega)$ and $u_m(x) \to u(x)$ in $H_0^2(\Omega)$ as $m \to \infty$.

The unique solution $u_m(x)$ of each equation (2.10) is nontrivial provided the Fourier coefficients $G_{m,n_1,n_2}F(0,x)_{n_1,n_2} \neq 0$ for a certain pair $(n_1, n_2) \in \mathbb{Z}\times\mathbb{Z}$. Similarly, the unique solution $u(x)$ of limiting problem (1.2) does not vanish identically in Ω if $G_{n_1,n_2}F(0,x)_{n_1,n_2} \neq 0$ for some $(n_1, n_2) \in \mathbb{Z}\times\mathbb{Z}$.

Remark 2.4. *In the present work we deal with the real valued functions by means of the conditions imposed on $F(u, x_1, x_2)$, $G_m(x_1, x_2)$ and $G(x_1, x_2)$ contained in the nonlocal terms of the approximate and limiting equations considered in the article.*

Remark 2.5. *The significance of Theorem 2.3 above is the continuous dependence of the solution with respect to the integral kernel.*

3 Proofs Of The Main Results

Proof of Theorem 2.2. First we suppose that for some $v(x) \in H_0^2(\Omega)$ there exist two solutions $u_{1,2}(x) \in H_0^2(\Omega)$ of equation (2.3). Then their difference $w(x) := u_1(x) - u_2(x) \in H_0^2(\Omega)$ will solve the homogeneous problem

$$-\frac{\partial^2 w}{\partial x_1^2} + \sqrt{-\frac{\partial^2}{\partial x_2^2}}w = 0.$$

Since the operator $L_r : H_0^2(\Omega) \to L^2(\Omega)$ defined in (2.7) does not possess any nontrivial zero modes, the function $w(x)$ vanishes identically in Ω.
Let us choose arbitrarily $v(x) \in H_0^2(\Omega)$. We apply the Fourier transform (4.1) to both sides of (2.3). This yields for $(n_1, n_2) \in \mathbb{Z}\times\mathbb{Z}$ that

$$u_{n_1,n_2} = 2\pi\frac{G_{n_1,n_2}f_{n_1,n_2}}{n_1^2 + |n_2|}, \quad (n_1^2 + n_2^2)u_{n_1,n_2} = 2\pi\frac{(n_1^2 + n_2^2)G_{n_1,n_2}f_{n_1,n_2}}{n_1^2 + |n_2|}, \tag{3.1}$$

where $f_{n_1,n_2} := F(v(x), x)_{n_1,n_2}$. Obviously, we have the upper bounds

$$|u_{n_1,n_2}| \leq 2\pi\mathcal{N}_r|f_{n_1,n_2}|, \quad |(n_1^2 + n_2^2)u_{n_1,n_2}| \leq 2\pi\mathcal{N}_r|f_{n_1,n_2}|, \quad (n_1, n_2) \in \mathbb{Z}\times\mathbb{Z}.$$

We have $\mathcal{N}_r < \infty$ by virtue of the result of Lemma 4.1 of the Appendix under the stated assumptions. This allows us to derive the estimate from above on the norm

$$\|u\|^2_{H_0^2(\Omega)} =$$

$$= \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |u_{n_1,n_2}|^2 + \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |(n_1^2 + n_2^2)u_{n_1,n_2}|^2 \leq 8\pi^2\mathcal{N}_r^2\|F(v(x), x)\|^2_{L^2(\Omega)}. \tag{3.2}$$

The right side of (3.2) is finite by means of inequality (2.1) of Assumption 2.1 since $v(x) \in L^2(\Omega)$. Hence, for an arbitrary $v(x) \in H_0^2(\Omega)$ there exists a unique solution $u(x) \in H_0^2(\Omega)$ of equation (2.3), so that its Fourier image is given by (3.1). Therefore, the map $T_r : H_0^2(\Omega) \to H_0^2(\Omega)$ is well defined. This enables us to choose arbitrarily the functions $v_{1,2}(x) \in H_0^2(\Omega)$, so that their images $u_{1,2} := T_r v_{1,2} \in H_0^2(\Omega)$. Evidently, (2.3) gives us

$$-\frac{\partial^2 u_1}{\partial x_1^2} + \sqrt{-\frac{\partial^2}{\partial x_2^2}}u_1 = \int_\Omega G(x-y)F(v_1(y), y)dy, \tag{3.3}$$

$$-\frac{\partial^2 u_2}{\partial x_1^2} + \sqrt{-\frac{\partial^2}{\partial x_2^2}}u_2 = \int_\Omega G(x-y)F(v_2(y), y)dy. \tag{3.4}$$

We apply the Fourier transform (4.1) to both sides of the equations of system (3.3), (3.4) above, which yields

$$u_{1,n_1,n_2} = 2\pi \frac{G_{n_1,n_2} f_{1,n_1,n_2}}{n_1^2 + |n_2|}, \quad u_{2,n_1,n_2} = 2\pi \frac{G_{n_1,n_2} f_{2,n_1,n_2}}{n_1^2 + |n_2|}, \quad (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}. \tag{3.5}$$

Here f_{1,n_1,n_2} and f_{2,n_1,n_2} denote the images of $F(v_1(x), x)$ and $F(v_2(x), x)$ respectively under transform (4.1). Using (3.5), we obtain the estimates from above

$$|u_{1,n_1,n_2} - u_{2,n_1,n_2}| \le 2\pi \mathcal{N}_r |f_{1,n_1,n_2} - f_{2,n_1,n_2}|,$$

$$|(n_1^2 + n_2^2)[u_{1,n_1,n_2} - u_{2,n_1,n_2}]| \le 2\pi \mathcal{N}_r |f_{1,n_1,n_2} - f_{2,n_1,n_2}|.$$

Thus,

$$\|u_1 - u_2\|_{H_0^2(\Omega)}^2 = \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |u_{1,n_1,n_2} - u_{2,n_1,n_2}|^2 + \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |(n_1^2 + n_2^2)[u_{1,n_1,n_2} - u_{2,n_1,n_2}]|^2 \le$$

$$\le 8\pi^2 \mathcal{N}_r^2 \|F(v_1(x), x) - F(v_2(x), x)\|_{L^2(\Omega)}^2.$$

By means of condition (2.2), we arrive at

$$\|T_r v_1 - T_r v_2\|_{H_0^2(\Omega)} \le 2\sqrt{2}\pi \mathcal{N}_r l \|v_1 - v_2\|_{H_0^2(\Omega)}. \tag{3.6}$$

The constant in the right side of (3.6) is less than one as assumed. Therefores, by virtue of the Fixed Point Theorem, there exists a unique function $v_r \in H_0^2(\Omega)$, so that $T_r v_r = v_r$, which is the only solution of equation (1.2) in $H_0^2(\Omega)$. Suppose $v_r(x)$ vanishes identically in Ω. This will contradict to the given assumption that the Fourier coefficients $G_{n_1,n_2} F(0, x)_{n_1,n_2} \neq 0$ for a certain pair $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$. ■

We turn our attention to establishing the solvability in the sense of sequences for our integro-differential equation in Ω.

Proof of Theorem 2.3. By means of the result of Theorem 2.2 above, each equation (2.10) admits a unique solution $u_m(x) \in H_0^2(\Omega),\ m \in \mathbb{N}$. Limiting problem (1.2) possesses a unique solution $u(x) \in H_0^2(\Omega)$ by virtue of Lemma 4.2 below along with Theorem 2.2. We apply the

Fourier transform (4.1) to both sides of (1.2) and (2.10). This gives us for $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$ and $m \in \mathbb{N}$

$$u_{n_1,n_2} = 2\pi \frac{G_{n_1,n_2} f_{n_1,n_2}}{n_1^2 + |n_2|}, \quad (n_1^2 + n_2^2) u_{n_1,n_2} = 2\pi \frac{(n_1^2 + n_2^2) G_{n_1,n_2} f_{n_1,n_2}}{n_1^2 + |n_2|}, \tag{3.7}$$

$$u_{m,n_1,n_2} = 2\pi \frac{G_{m,n_1,n_2} f_{m,n_1,n_2}}{n_1^2 + |n_2|}, \quad (n_1^2 + n_2^2) u_{m,n_1,n_2} = 2\pi \frac{(n_1^2 + n_2^2) G_{m,n_1,n_2} f_{m,n_1,n_2}}{n_1^2 + |n_2|}. \tag{3.8}$$

Here f_{n_1,n_2} and f_{m,n_1,n_2} denote the Fourier images of $F(u(x), x)$ and $F(u_m(x), x)$ respectively under transform (4.1). Evidently,

$$|u_{m,n_1,n_2} - u_{n_1,n_2}| \le 2\pi \left\| \frac{G_{m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} |f_{n_1,n_2}| +$$

$$+ 2\pi \left\| \frac{G_{m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} |f_{m,n_1,n_2} - f_{n_1,n_2}|,$$

so that

$$\|u_m - u\|_{L^2(\Omega)} \le 2\pi \left\| \frac{G_{m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \|F(u(x), x)\|_{L^2(\Omega)} +$$

$$+ 2\pi \left\| \frac{G_{m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \|F(u_m(x), x) - F(u(x), x)\|_{L^2(\Omega)}.$$

Let us recall inequality (2.2) of Assumption 2.1. Thus,

$$\|F(u_m(x), x) - F(u(x), x)\|_{L^2(\Omega)} \le l \|u_m(x) - u(x)\|_{L^2(\Omega)}. \tag{3.9}$$

Hence, we obtain

$$\|u_m(x) - u(x)\|_{L^2(\Omega)} \left\{ 1 - 2\pi \left\| \frac{G_{m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} l \right\} \le$$

$$\le 2\pi \left\| \frac{G_{m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \|F(u(x), x)\|_{L^2(\Omega)}.$$

By virtue of (4.8) and (4.10), we have

$$\|u_m(x) - u(x)\|_{L^2(\Omega)} \le \frac{2\pi}{\varepsilon} \left\| \frac{G_{m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \|F(u(x), x)\|_{L^2(\Omega)}.$$

Let us recall upper bound (2.1) of Assumption 2.1. Thus, $F(u(x), x) \in L^2(\Omega)$ for $u(x) \in L^2(\Omega)$. Therefore, under the stated assumptions

$$u_m(x) \to u(x), \quad m \to \infty \tag{3.10}$$

in $L^2(\Omega)$ by means of the result of Lemma 4.2 of the Appendix. Using (3.7) and (3.8), we derive

$$|(n_1^2 + n_2^2) u_{m,n_1,n_2} - (n_1^2 + n_2^2) u_{n_1,n_2}| \le 2\pi \left\| \frac{(n_1^2 + n_2^2) G_{m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{(n_1^2 + n_2^2) G_{n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} |f_{n_1,n_2}| +$$

$$+2\pi\left\|\frac{(n_1^2+n_2^2)G_{m,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}|f_{m,n_1,n_2}-f_{n_1,n_2}|.$$

Hence,

$$\|\Delta u_m(x)-\Delta u(x)\|_{L^2(\Omega)}\leq 2\pi\left\|\frac{(n_1^2+n_2^2)G_{m,n_1,n_2}}{n_1^2+|n_2|}-\frac{(n_1^2+n_2^2)G_{n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}\|F(u(x),x)\|_{L^2(\Omega)}+$$

$$+2\pi\left\|\frac{(n_1^2+n_2^2)G_{m,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}\|F(u_m(x),x)-F(u(x),x)\|_{L^2(\Omega)}.$$

Upper bound (3.9) allows us to obtain the inequality

$$\|\Delta u_m(x)-\Delta u(x)\|_{L^2(\mathbb{R}^2)}\leq 2\pi\left\|\frac{(n_1^2+n_2^2)G_{m,n_1,n_2}}{n_1^2+|n_2|}-\frac{(n_1^2+n_2^2)G_{n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}\|F(u(x),x)\|_{L^2(\Omega)}+$$

$$+2\pi\left\|\frac{(n_1^2+n_2^2)G_{m,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}l\|u_m(x)-u(x)\|_{L^2(\Omega)}.$$

By virtue of the result of Lemma 4.2 of the Appendix along with (3.10), we arrive at $\Delta u_m(x)\to\Delta u(x)$ in $L^2(\Omega)$ as $m\to\infty$. Definition (2.6) of the norm implies that $u_m(x)\to u(x)$ in $H_0^2(\Omega)$ as $m\to\infty$.
Let us suppose the solution $u_m(x)$ of equation (2.10) considered above vanishes identically in Ω for some $m\in\mathbb{N}$. This will contradict to the stated assumption that the Fourier coefficients $G_{m,n_1,n_2}F(0,x)_{n_1,n_2}\neq 0$ for a certain pair $(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}$. The similar reasoning holds for the solution $u(x)$ of limiting problem (1.2). ∎

4 Appendix

Let the function $G(x_1,x_2):\Omega\to\mathbb{R}$, so that $G(0,x_2)=G(2\pi,x_2)$ for $0\leq x_2\leq 2\pi$ and $G(x_1,0)=G(x_1,2\pi)$ for $0\leq x_1\leq 2\pi$. Its Fourier transform on our square is given by

$$G_{n_1,n_2}:=\int_0^{2\pi}\int_0^{2\pi}G(x_1,x_2)\frac{e^{-in_1x_1}}{\sqrt{2\pi}}\frac{e^{-in_2x_2}}{\sqrt{2\pi}}dx_1dx_2,\quad (n_1,n_2)\in\mathbb{Z}\times\mathbb{Z},\tag{4.1}$$

so that

$$G(x_1,x_2)=\sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}}G_{n_1,n_2}\frac{e^{in_1x_1}}{\sqrt{2\pi}}\frac{e^{in_2x_2}}{\sqrt{2\pi}},\quad (x_1,x_2)\in\Omega.$$

Evidently, the estimate

$$\|G_{n_1,n_2}\|_{l^\infty}\leq\frac{1}{2\pi}\|G(x_1,x_2)\|_{L^1(\Omega)}\tag{4.2}$$

holds, where $\|G_{n_1,n_2}\|_{l^\infty}:=\sup_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}}|G_{n_1,n_2}|$. Clearly, (4.2) implies that

$$\|n_2G_{n_1,n_2}\|_{l^\infty}\leq\frac{1}{2\pi}\left\|\frac{\partial G(x_1,x_2)}{\partial x_2}\right\|_{L^1(\Omega)}.\tag{4.3}$$

Moreover, for a function continuous in Ω, we have

$$\|G(x_1,x_2)\|_{L^1(\Omega)} \le \|G(x_1,x_2)\|_{C(\Omega)}(2\pi)^2, \tag{4.4}$$

where $\|G(x_1,x_2)\|_{C(\Omega)} := \max_{(x_1,x_2)\in\Omega}|G(x_1,x_2)|$. For the technical purposes we define the following auxiliary expression

$$\mathcal{N}_r := max\left\{\left\|\frac{G_{n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}, \quad \left\|\frac{(n_1^2+n_2^2)G_{n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}\right\}. \tag{4.5}$$

Lemma 4.1. *Let the function $G(x_1,x_2):\Omega\to\mathbb{R}$, such that $G(0,x_2)=G(2\pi,x_2)$ for $0\le x_2\le 2\pi$ and $G(x_1,0)=G(x_1,2\pi)$ for $0\le x_1\le 2\pi$. Furthermore, $G(x_1,x_2)\in C(\Omega)$ and $\dfrac{\partial G(x_1,x_2)}{\partial x_2}\in L^1(\Omega)$. Then $\mathcal{N}_r<\infty$ if and only if the orthogonality condition*

$$(G(x_1,x_2),1)_{L^2(\Omega)} = 0 \tag{4.6}$$

holds.

Proof. First of all, it can be easily verified that under the given conditions $\dfrac{(n_1^2+n_2^2)G_{n_1,n_2}}{n_1^2+|n_2|}$ is bounded. Indeed, using (4.2) along with (4.4), we obtain that

$$\left|\frac{n_1^2 G_{n_1,n_2}}{n_1^2+|n_2|}\right| \le \|G_{n_1,n_2}\|_{l^\infty} \le 2\pi\|G(x_1,x_2)\|_{C(\Omega)} < \infty$$

as assumed. By means of (4.3) and the one of our assumptions,

$$\left|\frac{n_2^2 G_{n_1,n_2}}{n_1^2+|n_2|}\right| \le \|n_2 G_{n_1,n_2}\|_{l^\infty} \le \frac{1}{2\pi}\left\|\frac{\partial G(x_1,x_2)}{\partial x_2}\right\|_{L^1(\Omega)} < \infty.$$

Hence,

$$\frac{(n_1^2+n_2^2)G_{n_1,n_2}}{n_1^2+|n_2|} \in l^\infty$$

holds. Let us express

$$\frac{G_{n_1,n_2}}{n_1^2+|n_2|} = \frac{G_{n_1,n_2}}{n_1^2+|n_2|}\chi_{\{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}\ |\ n_1=n_2=0\}} + \frac{G_{n_1,n_2}}{n_1^2+|n_2|}\chi_{\{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}\ |\ n_1=n_2=0\}^c}. \tag{4.7}$$

Here and below χ_A will denote the characteristic function of a set $A\subseteq\mathbb{Z}\times\mathbb{Z}$ and A^c will stand for the complement of A. Evidently, the second term in the right side of (4.7) can be bounded from above in the absolute value by virtue of (4.2) and (4.4) by

$$|G_{n_1,n_2}| \le 2\pi\|G(x_1,x_2)\|_{C(\Omega)} < \infty$$

as assumed. Clearly, the first term in the right side of (4.7) is bounded if and only if $G_{0,0}=0$. This is equivalent to orthogonality condition (4.6). ∎

Note that the argument of the lemma above relies only on a single orthogonality relation, as distinct from the analogous situation in the whole $\mathbb{R}^2$ discussed in [14].

In order to study equations (2.10), we introduce the auxiliary quantities

$$\mathcal{N}_r^{(m)} := max\left\{\left\|\frac{G_{m,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}, \quad \left\|\frac{(n_1^2+n_2^2)G_{m,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}\right\}, \quad m \in \mathbb{N}. \tag{4.8}$$

Our final proposition is as follows.

Lemma 4.2. *Let $m \in \mathbb{N}$, the functions $G_m(x_1, x_2) : \Omega \to \mathbb{R}$, so that $G_m(0, x_2) = G_m(2\pi, x_2)$ for $0 \le x_2 \le 2\pi$ and $G_m(x_1, 0) = G_m(x_1, 2\pi)$ for $0 \le x_1 \le 2\pi$. Moreover,*

$$G_m(x_1, x_2) \in C(\Omega), \quad G_m(x_1, x_2) \to G(x_1, x_2) \quad in \quad C(\Omega) \quad as \quad m \to \infty.$$

Similarly,

$$\frac{\partial G_m(x_1, x_2)}{\partial x_2} \in L^1(\Omega), \quad \frac{\partial G_m(x_1, x_2)}{\partial x_2} \to \frac{\partial G(x_1, x_2)}{\partial x_2} \quad in \quad L^1(\Omega) \quad as \quad m \to \infty.$$

Let us also assume that for each $m \in \mathbb{N}$

$$(G_m(x_1, x_2), 1)_{L^2(\Omega)} = 0 \tag{4.9}$$

is valid. Finally, we suppose that

$$2\sqrt{2}\pi\mathcal{N}_r^{(m)} l \le 1 - \varepsilon \tag{4.10}$$

holds for each $m \in \mathbb{N}$ with a certain fixed $0 < \varepsilon < 1$.

Then

$$\frac{G_{m,n_1,n_2}}{n_1^2+|n_2|} \to \frac{G_{n_1,n_2}}{n_1^2+|n_2|}, \quad m \to \infty, \tag{4.11}$$

$$\frac{(n_1^2+n_2^2)G_{m,n_1,n_2}}{n_1^2+|n_2|} \to \frac{(n_1^2+n_2^2)G_{n_1,n_2}}{n_1^2+|n_2|}, \quad m \to \infty \tag{4.12}$$

in l^∞, so that

$$\left\|\frac{G_{m,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty} \to \left\|\frac{G_{n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}, \quad m \to \infty, \tag{4.13}$$

$$\left\|\frac{(n_1^2+n_2^2)G_{m,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty} \to \left\|\frac{(n_1^2+n_2^2)G_{n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}, \quad m \to \infty. \tag{4.14}$$

Furthermore,

$$2\sqrt{2}\pi\mathcal{N}_r l \le 1 - \varepsilon \tag{4.15}$$

is valid.

Proof. Clearly, under the given conditions $\mathcal{N}_r^{(m)} < \infty, \ m \in \mathbb{N}$ by means of the result of Lemma 4.1 above.
It can trivially checked that the limiting kernel is a periodic function as well. Indeed, for $0 \leq x_2 \leq 2\pi$, we have

$$|G(0, x_2) - G(2\pi, x_2)| \leq |G_m(0, x_2) - G(0, x_2)| + |G_m(2\pi, x_2) - G(2\pi, x_2)| \leq$$

$$\leq 2\|G_m(x_1, x_2) - G(x_1, x_2)\|_{C(\Omega)} \to 0, \quad m \to \infty$$

via our assumptions, so that

$$G(0, x_2) = G(2\pi, x_2) \quad for \quad 0 \leq x_2 \leq 2\pi.$$

Similarly, for $0 \leq x_1 \leq 2\pi$

$$|G(x_1, 0) - G(x_1, 2\pi)| \leq |G_m(x_1, 0) - G(x_1, 0)| + |G_m(x_1, 2\pi) - G(x_1, 2\pi)| \leq$$

$$\leq 2\|G_m(x_1, x_2) - G(x_1, x_2)\|_{C(\Omega)} \to 0, \quad m \to \infty$$

as assumed, such that

$$G(x_1, 0) = G(x_1, 2\pi) \quad for \quad 0 \leq x_1 \leq 2\pi.$$

Let us verify that the limiting orthogonality condition

$$(G(x_1, x_2), 1)_{L^2(\Omega)} = 0 \tag{4.16}$$

holds. Using (4.9), we obtain

$$|(G(x_1, x_2), 1)_{L^2(\Omega)}| = |(G(x_1, x_2) - G_m(x_1, x_2), 1)_{L^2(\Omega)}| \leq$$

$$\leq \|G_m(x_1, x_2) - G(x_1, x_2)\|_{C(\Omega)}(2\pi)^2 \to 0, \quad m \to \infty$$

due to the one of our assumptions, so that (4.16) is valid.
Therefore, by virtue of Lemma 4.1, we have $\mathcal{N}_r < \infty$.
Orthogonality relations (4.16) and (4.9) along with the definition of the Fourier transform (4.1) imply that

$$G_{0,0} = 0, \quad G_{m,0,0} = 0, \quad m \in \mathbb{N}.$$

Then by means of inequalities (4.2) and (4.4), we derive

$$\left\| \frac{G_{m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \leq 2\pi \|G_m(x_1, x_2) - G(x_1, x_2)\|_{C(\Omega)} \to 0, \quad m \to \infty$$

as assumed. Hence, (4.11) holds. Obviously, (4.13) easily follows from (4.11) via the standard triangle inequality.

Evidently, we can estimate

$$\left|\frac{(n_1^2+n_2^2)G_{m,n_1,n_2}}{n_1^2+|n_2|}-\frac{(n_1^2+n_2^2)G_{n_1,n_2}}{n_1^2+|n_2|}\right| \leq \|G_{m,n_1,n_2}-G_{n_1,n_2}\|_{l^\infty}+\|n_2[G_{m,n_1,n_2}-G_{n_1,n_2}]\|_{l^\infty}.$$

Let us use formulas (4.2), (4.3) and (4.4). This enables us to obtain the upper bound

$$\left\|\frac{(n_1^2+n_2^2)G_{m,n_1,n_2}}{n_1^2+|n_2|}-\frac{(n_1^2+n_2^2)G_{n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty} \leq 2\pi\|G_m(x_1,x_2)-G(x_1,x_2)\|_{C(\Omega)}+$$

$$+\frac{1}{2\pi}\left\|\frac{\partial G_m(x_1,x_2)}{\partial x_2}-\frac{\partial G(x_1,x_2)}{\partial x_2}\right\|_{L^1(\Omega)} \to 0, \quad m\to\infty$$

according to the given conditions. Thus, (4.12) is valid. We use the standard triangle inequality to demonstrate that (4.14) is an immediate consequence of (4.12).
A trivial limiting argument using (4.8), (4.10), (4.13) and (4.14) gives us (4.15). ■

Remark 4.3. *The existence in the sense of sequences of the solutions of problem (1.2) containing the drift term will be addressed in our consecutive work.*

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