On the solvability of some systems of Fredholm integro-differential equations with mixed diffusion in a square

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Abstract. We establish the existence in the sense of sequences of solutions for a certain system of integro-differential equations in a square in two dimensions with periodic boundary conditions involving the normal diffusion in one direction and the superdiffusion in the other direction in a constrained subspace of H^2 for the vector functions via the fixed point technique. The system of elliptic equations contains a second order differential operator, which satisfies the Fredholm property. It is demonstrated that, under the reasonable technical conditions, the convergence in the appropriate function spaces of the integral kernels implies the existence and convergence in $H_c^2(\Omega, \mathbb{R}^N)$ of the solutions. We generalize the results derived in our previous article [18] for the analogical system studied in the whole $\mathbb{R}^2$ which involved non-Fredholm operators. Let us emphasize that the study of the systems is more complicated than of the scalar case and requires to overcome more cumbersome technicalities.

Keywords: solvability conditions, Fredholm operators, integro-differential systems, mixed diffusion
AMS subject classification: 35P30, 47F05, 47G20

1 Introduction

We recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. As a consequence, the problem $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. Such properties of the Fredholm operators are widely used in many methods of the linear and nonlinear analysis.

The elliptic problems considered in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, the proper ellipticity and the Shapiro-Lopatinskii conditions are fulfilled (see e.g. [2], [9], [27], [31]). This is the main result of the theory of the linear elliptic equations. In the case of the unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For example, the Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ does not satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.
For the linear elliptic problems in the unbounded domains the Fredholm property is satisfied if and only if, in addition to the conditions given above, the limiting operators are invertible (see [32]). In certain simple cases, the limiting operators can be constructed explicitly. For instance, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have the limits at the infinities,

$$a_{\pm} = \lim_{x \to \pm\infty} a(x), \quad b_{\pm} = \lim_{x \to \pm\infty} b(x), \quad c_{\pm} = \lim_{x \to \pm\infty} c(x),$$

the limiting operators are:

$$L_{\pm}u = a_{\pm}u'' + b_{\pm}u' + c_{\pm}u.$$

Since the coefficients here are constants, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ does not have the Fredholm property, can be found explicitly via the standard Fourier transform, such that:

$$\lambda_{\pm}(\xi) = -a_{\pm}\xi^2 + b_{\pm}i\xi + c_{\pm}, \quad \xi \in \mathbb{R}.$$

The limiting operators are invertible if and only if the essential spectrum does not contain the origin.
For the general elliptic equations, the analogous assertions are valid. The Fredholm property is satisfied if the origin does not belong to the essential spectrum or if the limiting operators are invertible. However, such conditions may not be written explicitly.
For the non-Fredholm operators the usual solvability relations may not be applicable and in a general situation the solvability conditions are not known. But there are some classes of operators for which the solvability relations were derived recently. Let us illustrate them with the following example. Consider the problem

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, $d \in \mathbb{N}$, where a is a positive constant. The operator L here coincides with its limiting operators. The corresponding homogeneous equation has a nonzero bounded solution, such that the Fredholm property is not satisfied. However, since the operator contained in (1.1) has the constant coefficients, we can apply the standard Fourier transform to obtain the solution explicitly. The solvability conditions can be formulated as follows. If $f(x) \in L^2(\mathbb{R}^d)$

and $xf(x) \in L^1(\mathbb{R}^d)$, then there exists a unique solution of this problem in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see Lemmas 5 and 6 of [40]). Here $S^d_{\sqrt{a}}$ stands for the sphere in $\mathbb{R}^d$ of radius $\sqrt{a}$ centered at the origin. Thus, despite the fact that the Fredholm property is not satisfied for the operator, the solvability conditions are formulated analogously. Clearly, such similarity is only formal since the range of the operator is not closed.
In the case of the operator involving a potential function,

$$Lu \equiv \Delta u + a(x)u = f,$$

the standard Fourier transform is not applicable directly. Nevertheless, the solvability relations in $\mathbb{R}^3$ can be derived by a rather sophisticated application of the theory of the self-adjoint Schrödinger type operators (see [36]). Similarly to the constant coefficient case, the solvability conditions are written in terms of the orthogonality to the solutions of the adjoint homogeneous problem. There are several other examples of the linear elliptic non-Fredholm operators for which the solvability relations can be obtained (see [13], [15], [32], [33], [34], [35], [38], [39], [40]).
The solvability relations play a crucial role in the analysis of the nonlinear elliptic problems. When the operators without the Fredholm property are involved, in spite of the certain progress in the understanding of the linear equations, there exist only few examples where the nonlinear non- Fredholm operators were analyzed (see [7], [8], [12], [14], [15], [16], [17], [18], [37], [40], [41], [42], [43]). Fredholm structures, topological invariants and their applications were considered in [9]. The article [10] is devoted to the finite and infinite dimensional attractors for evolution equations of mathematical physics. The large time behavior of solutions of a class of fourth-order parabolic equations defined on unbounded domains using the Kolmogorov ε-entropy as a measure was studied in [11]. The attractor for a nonlinear reaction-diffusion system in an unbounded domain in $\mathbb{R}^3$ was investigated in [20]. The works [23] and [29] are crucial for the understanding of the Fredholm and properness properties of quasilinear elliptic systems of second order and of operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in second-order quasilinear elliptic systems were discussed in [24]. A local bifurcation theorem for C^1-Fredholm maps was established in [21]. In [22] the authors develop a degree theory for C^2-Fredholm mappings of zero index between Banach spaces. Standing lattice solitons in the discrete NLS equation with saturation were covered in [1]. The present work deals with another class of the stationary nonlinear systems of equations, for which the Fredholm property is satisfied:

$$\frac{\partial^2 u_k}{\partial x_1^2} - \sqrt{-\frac{\partial^2}{\partial x_2^2}}u_k + \int_\Omega G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy = 0 \tag{1.2}$$

with $1 \leq k \leq N$, $N \geq 2$, $x = (x_1, x_2) \in \Omega$, $y = (y_1, y_2) \in \Omega$ and the square $\Omega := [0, 2\pi] \times [0, 2\pi]$ with the periodic boundary conditions specified further down. Here and below the vector function

$$u := (u_1, u_2, ..., u_N)^T \in \mathbb{R}^N. \tag{1.3}$$

We generalize the results obtained for the analogous system in the whole $\mathbb{R}^2$ studied in [18]. Thus, it involved the non-Fredholm operators. For the solvability of the single equations of this kind see [14] and [19]. The novelty of such works is that in the diffusion terms we add the free Laplacian in the x_1 variable to the negative Laplace operator in x_2 raised to a fractional power $0 < s_k < 1$, $1 \le k \le N$, $N \ge 2$ and defined via the spectral calculus. As distinct from the analogical system of equations discussed in [18], in the present article we restrict our attention to $s_k = \dfrac{1}{2}$ for all k. The models of this type are new. They are not well understood, especially in the context of the nonlocal reaction-diffusion equations. The difficulty we have to overcome is that such problems become anisotropic and it is more complicated to derive the desired estimates when working with them. In the population dynamics in the Mathematical Biology the integro-differential equations describe the models with the intra-specific competition and nonlocal consumption of resources (see e.g. [3], [4]). It is crucial to consider the problems of this type from the point of view of the understanding of the spread of the viral infections, since many countries have to deal with the pandemics. We use the explicit form of the solvability conditions and establish the existence of solutions of our nonlinear system. In the case of the standard Laplacian in the diffusion terms, the system of equations analogous to (1.2) was covered in [43] (see also [37]) in the whole space and on a finite interval with the periodic boundary conditions. The solvability of the integro-differential problems involving in the diffusion terms only the negative Laplacian raised to a fractional power was actively studied in recent years in the context of the anomalous diffusion (see e.g. [17], [41], [42]). The anomalous diffusion can be described as a random process of the particle motion characterized by the probability density distribution of the jump length. The moments of this density distribution are finite in the case of the normal diffusion, but this is not the case for the anomalous diffusion. The asymptotic behavior at the infinity of the probability density function determines the value of the power of the Laplacian (see [28]). In [30] the authors discuss the mixed local-nonlocal semi-linear elliptic problems driven by the superposition of Brownian and Levy processes and establish the L^∞ boundedness of any weak solution. The work [6] deals with a new type of mixed local and nonlocal equations under the Neumann conditions. The spectral properties associated to a weighted eigenvalue problem are considered and a global estimate for subsolutions is presented.

2 Formulation of the results

The technical assumptions of the present work will be analogical to the ones of [19], adapted to the work with vector functions. Performing the analysis in the Sobolev spaces for the vector functions is more difficult. For the nonlinear part of the system of equations (1.2) the following regularity conditions will be valid. Here $x = (x_1, x_2) \in \Omega$.

Assumption 2.1. *Let* $1 \le k \le N$. *Functions* $F_k(u, x) : \mathbb{R}^N \times \Omega \to \mathbb{R}$ *are satisfying the*

Caratheodory condition (see [26]), such that

$$\sqrt{\sum_{k=1}^{N} F_k^2(u,x)} \le K|u|_{\mathbb{R}^N} + h(x) \quad for \quad u \in \mathbb{R}^N, \ x \in \Omega \tag{2.1}$$

with a constant $K > 0$ and $h(x) : \Omega \to \mathbb{R}^+$, $h(x) \in L^2(\Omega)$. Moreover, they are Lipschitz continuous functions, so that for any $u^{(1),(2)} \in \mathbb{R}^N, \ x \in \Omega$:

$$\sqrt{\sum_{k=1}^{N} (F_k(u^{(1)},x) - F_k(u^{(2)},x))^2} \le L|u^{(1)} - u^{(2)}|_{\mathbb{R}^N} \tag{2.2}$$

with a constant $L > 0$. Furthermore, for $1 \le k \le N$

$$F_k(u,0,x_2) = F_k(u,2\pi,x_2) \quad for \quad u \in \mathbb{R}^N, \quad 0 \le x_2 \le 2\pi$$

and

$$F_k(u,x_1,0) = F_k(u,x_1,2\pi) \quad for \quad u \in \mathbb{R}^N, \quad 0 \le x_1 \le 2\pi.$$

Here and further down the norm of a vector function given by (1.3) is:

$$|u|_{\mathbb{R}^N} := \sqrt{\sum_{k=1}^{N} u_k^2}.$$

The solvability of a local elliptic equation in a bounded domain in $\mathbb{R}^N$ was covered in [5]. The nonlinear function there was allowed to have a sublinear growth. To demonstrate the existence of solutions of (1.2), we will use the auxiliary system with $1 \le k \le N, \ N \ge 2, \ x = (x_1,x_2) \in \Omega, \ y = (y_1,y_2) \in \Omega$, namely

$$-\frac{\partial^2 u_k}{\partial x_1^2} + \sqrt{-\frac{\partial^2}{\partial x_2^2}}u_k = \int_{\Omega} G_k(x-y)F_k(v_1(y),v_2(y),...,v_N(y),y)dy. \tag{2.3}$$

We denote

$$(f_1(x_1,x_2), f_2(x_1,x_2))_{L^2(\Omega)} := \int_0^{2\pi}\int_0^{2\pi} f_1(x_1,x_2)\bar{f}_2(x_1,x_2)dx_1dx_2. \tag{2.4}$$

Let us use the Sobolev space

$$H^2(\Omega) := \{\phi(x_1,x_2) : \Omega \to \mathbb{R} \mid \phi(x_1,x_2), \ \Delta\phi(x_1,x_2) \in L^2(\Omega), \ \phi(0,x_2) = \phi(2\pi,x_2),$$

$$\frac{\partial \phi}{\partial x_1}(0,x_2) = \frac{\partial \phi}{\partial x_1}(2\pi,x_2) \ for \ 0 \le x_2 \le 2\pi,$$

$$\phi(x_1,0)=\phi(x_1,2\pi),\ \frac{\partial\phi}{\partial x_2}(x_1,0)=\frac{\partial\phi}{\partial x_2}(x_1,2\pi)\ for\ 0\le x_1\le 2\pi\}.$$

Here and further down the cumulative Laplace operator $\Delta:=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$. We introduce the following auxiliary constrained subspace

$$H_0^2(\Omega):=\{\phi(x_1,x_2)\in H^2(\Omega)\mid (\phi(x_1,x_2),1)_{L^2(\Omega)}=0\}. \tag{2.5}$$

Evidently, (2.5) is a Hilbert space as well (see e.g. Chapter 2.1 of [25]). It is equipped with the norm

$$\|\phi\|^2_{H_0^2(\Omega)}:=\|\phi\|^2_{L^2(\Omega)}+\|\Delta\phi\|^2_{L^2(\Omega)}. \tag{2.6}$$

The resulting space used to establish the existence of solutions $u(x):\Omega\to\mathbb{R}^N$ of system (2.3) will be the direct sum of the spaces

$$H_c^2(\Omega,\mathbb{R}^N):=\oplus_{k=1}^N H_0^2(\Omega). \tag{2.7}$$

The corresponding Sobolev norm of a vector function is given by

$$\|u\|^2_{H_c^2(\Omega,\mathbb{R}^N)}:=\sum_{k=1}^N\{\|u_k\|^2_{L^2(\Omega)}+\|\Delta u_k\|^2_{L^2(\Omega)}\}, \tag{2.8}$$

where $u(x):\Omega\to\mathbb{R}^N$. Let us also use the norm

$$\|u\|^2_{L^2(\Omega,\mathbb{R}^N)}:=\sum_{k=1}^N\|u_k\|^2_{L^2(\Omega)}.$$

By means of Assumption 2.1 above, we do not consider the higher powers of the nonlinearities than the first one. This is restrictive from the point of view of the applications. But this guarantees that our nonlinear vector function is a bounded and continuous map from $L^2(\Omega,\mathbb{R}^N)$ to $L^2(\Omega,\mathbb{R}^N)$. The system of equations (2.3) involves the operator

$$L_r:=-\frac{\partial^2}{\partial x_1^2}+\sqrt{-\frac{\partial^2}{\partial x_2^2}}:\quad H_0^2(\Omega)\to L^2(\Omega). \tag{2.9}$$

Its eigenvalues are given by

$$\lambda_{r,n_1,n_2}:=n_1^2+|n_2|,\quad (n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}. \tag{2.10}$$

The corresponding eigenfunctions are:

$$\frac{e^{in_1x_1}}{\sqrt{2\pi}}\frac{e^{in_2x_2}}{\sqrt{2\pi}},\quad (n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}. \tag{2.11}$$

Clearly, (2.9) is a Fredholm operator and its kernel is trivial. In the present work we will establish that under the reasonable technical assumptions system (2.3) defines a map $T_r : H_c^2(\Omega, \mathbb{R}^N) \to H_c^2(\Omega, \mathbb{R}^N)$, which is a strict contraction.

Theorem 2.2. *Let $N \geq 2$, $1 \leq k \leq N$, Assumption 2.1 holds, the functions $G_k(x_1, x_2) : \Omega \to \mathbb{R}$, such that $G_k(0, x_2) = G_k(2\pi, x_2)$ with $0 \leq x_2 \leq 2\pi$ and $G_k(x_1, 0) = G_k(x_1, 2\pi)$ with $0 \leq x_1 \leq 2\pi$. Furthermore, $G_k(x_1, x_2) \in C(\Omega)$ and $\dfrac{\partial G_k(x_1, x_2)}{\partial x_2} \in L^1(\Omega)$. We also assume that orthogonality conditions (4.7) hold for $1 \leq k \leq N$ and that $2\sqrt{2}\pi \mathcal{N}_r L < 1$.*
Then the map $T_r v = u$ on $H_c^2(\Omega, \mathbb{R}^N)$ defined by system (2.3) has a unique fixed point v_r, which is the only solution of problem (1.2) in $H_c^2(\Omega, \mathbb{R}^N)$.
This fixed point v_r is nontrivial provided the Fourier coefficients $G_{k,n_1,n_2} F_k(0, x)_{n_1,n_2} \neq 0$ for a certain $1 \leq k \leq N$ and some $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$.

Related to the system of equations (1.2) in the square Ω, we study the sequence of the approximate systems

$$\frac{\partial^2 u_k^{(m)}}{\partial x_1^2} - \sqrt{-\frac{\partial^2}{\partial x_2^2}} u_k^{(m)} + \int_\Omega G_{k,m}(x - y) F_k(u_1^{(m)}(y), u_2^{(m)}(y), ..., u_N^{(m)}(y), y) dy = 0, \tag{2.12}$$

with $1 \leq k \leq N$, $N \geq 2$, $m \in \mathbb{N}$, $x = (x_1, x_2) \in \Omega$, $y = (y_1, y_2) \in \Omega$. Each sequence of kernels $\{G_{k,m}(x)\}_{m=1}^\infty$ tends to $G_k(x)$ as $m \to \infty$ in the function spaces listed below. We demonstrate that, under the appropriate technical conditions, each of systems (2.12) has a unique solution $u^{(m)}(x) \in H_c^2(\Omega, \mathbb{R}^N)$, limiting system of equations (1.2) admits a unique solution $u(x) \in H_c^2(\Omega, \mathbb{R}^N)$, and $u^{(m)}(x) \to u(x)$ in $H_c^2(\Omega, \mathbb{R}^N)$ as $m \to \infty$. This is the so-called *existence of solutions in the sense of sequences*. In this case, the solvability relations can be formulated for the iterated kernels $G_{k,m}$. They yield the convergence of the kernels in terms of the Fourier transforms (see the Appendix) and, as a consequence, the convergence of the solutions (Theorem 2.3 below). The similar ideas in the context of the standard Schrödinger type operators were exploited in [13], [15], [35]. Our second main proposition is as follows.

Theorem 2.3. *Let $m \in \mathbb{N}$, $N \geq 2$, $1 \leq k \leq N$, Assumption 2.1 holds, the functions $G_{k,m}(x_1, x_2) : \Omega \to \mathbb{R}$ are such that $G_{k,m}(0, x_2) = G_{k,m}(2\pi, x_2)$ with $0 \leq x_2 \leq 2\pi$ and $G_{k,m}(x_1, 0) = G_{k,m}(x_1, 2\pi)$ with $0 \leq x_1 \leq 2\pi$. Moreover,*

$$G_{k,m}(x_1, x_2) \in C(\Omega), \quad G_{k,m}(x_1, x_2) \to G_k(x_1, x_2) \quad in \quad C(\Omega) \quad as \quad m \to \infty.$$

In addition to that,

$$\frac{\partial G_{k,m}(x_1, x_2)}{\partial x_2} \in L^1(\Omega), \quad \frac{\partial G_{k,m}(x_1, x_2)}{\partial x_2} \to \frac{\partial G_k(x_1, x_2)}{\partial x_2} \quad in \quad L^1(\Omega) \quad as \quad m \to \infty.$$

We also assume that for each $1 \leq k \leq N$, $m \in \mathbb{N}$ orthogonality condition (4.11) is valid. Finally, we suppose that (4.12) holds for each $m \in \mathbb{N}$ with a certain fixed $0 < \varepsilon < 1$.

Then each system of equations (2.12) admits a unique solution $u^{(m)}(x) \in H_c^2(\Omega, \mathbb{R}^N)$, limiting system (1.2) has a unique solution $u(x) \in H_c^2(\Omega, \mathbb{R}^N)$, such that $u^{(m)}(x) \to u(x)$ in $H_c^2(\Omega, \mathbb{R}^N)$ as $m \to \infty$.

The unique solution $u^{(m)}(x)$ of each system of equations (2.12) does not vanish identically in Ω provided the Fourier coefficients $G_{k,m,n_1,n_2} F_k(0,x)_{n_1,n_2} \neq 0$ for a certain $1 \leq k \leq N$ and some pair $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$. Analogously, the unique solution $u(x)$ of limiting system (1.2) is nontrivial in Ω if $G_{k,n_1,n_2} F_k(0,x)_{n_1,n_2} \neq 0$ for some $1 \leq k \leq N$ and a certain pair $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$.

Remark 2.4. *In the present article we deal with the real valued vector functions by virtue of the conditions imposed on $F_k(u, x_1, x_2)$, $G_{k,m}(x_1, x_2)$ and $G_k(x_1, x_2)$ involved in the integral terms of the approximate and limiting systems of equations discussed in the work.*

Remark 2.5. *The significance of Theorem 2.3 stated above is the continuous dependence of the solutions with respect to the integral kernels.*

3 Proofs Of The Main Results

Proof of Theorem 2.2. Let us first suppose that for a certain $v(x) \in H_c^2(\Omega, \mathbb{R}^N)$ there exist two solutions $u^{(1)(2)}(x) \in H_c^2(\Omega, \mathbb{R}^N)$ of the system of equations (2.3). Then their difference $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H_c^2(\Omega, \mathbb{R}^N)$ will satisfy the homogeneous system

$$-\frac{\partial^2 w_k}{\partial x_1^2} + \sqrt{-\frac{\partial^2}{\partial x_2^2}} w_k = 0, \quad 1 \leq k \leq N.$$

Clearly, the operator $L_r : H_0^2(\Omega) \to L^2(\Omega)$ defined in (2.9) does not have any nontrivial zero modes. Thus, the vector function $w(x)$ vanishes identically in the square Ω.
We choose an arbitrarily $v(x) \in H_c^2(\Omega, \mathbb{R}^N)$. Let us apply the Fourier transform (4.1) to both sides of system (2.3). This gives us for $1 \leq k \leq N,\ N \geq 2,\ (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$ that

$$u_{k,n_1,n_2} = 2\pi \frac{G_{k,n_1,n_2} f_{k,n_1,n_2}}{n_1^2 + |n_2|}, \quad (n_1^2 + n_2^2) u_{k,n_1,n_2} = 2\pi \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2} f_{k,n_1,n_2}}{n_1^2 + |n_2|} \tag{3.1}$$

Here $f_{k,n_1,n_2} := F_k(v(x), x)_{n_1,n_2}$. Evidently, we have the estimates from above

$$|u_{k,n_1,n_2}| \leq 2\pi \mathcal{N}_{r,k} |f_{k,n_1,n_2}|, \quad |(n_1^2 + n_2^2) u_{k,n_1,n_2}| \leq 2\pi \mathcal{N}_{r,k} |f_{k,n_1,n_2}|$$

with $1 \leq k \leq N,\ (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$. Obviously, all $\mathcal{N}_{r,k} < \infty$ due to the result of Lemma 4.1 of the Appendix under the given conditions. This enables us to obtain the upper bound on the norm as

$$\|u\|^2_{H_c^2(\Omega, \mathbb{R}^N)} =$$

$$= \sum_{k=1}^{N} \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |u_{k,n_1,n_2}|^2 + \sum_{k=1}^{N} \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |(n_1^2+n_2^2)u_{k,n_1,n_2}|^2 \leq$$

$$\leq 8\pi^2 \sum_{k=1}^{N} \mathcal{N}_{r,k}^2 \|F_k(v(x),x)\|_{L^2(\Omega)}^2. \tag{3.2}$$

Let us recall inequality (2.1) of Assumption 2.1. Hence, the right side of (3.2) is finite for $v(x) \in L^2(\Omega, \mathbb{R}^N)$. Thus, for any $v(x) \in H_c^2(\Omega, \mathbb{R}^N)$ there exists a unique solution $u(x) \in H_c^2(\Omega, \mathbb{R}^N)$ of system (2.3), such that its Fourier image is given by (3.1). Therefore, the map $T_r : H_c^2(\Omega, \mathbb{R}^N) \to H_c^2(\Omega, \mathbb{R}^N)$ is well defined. This allows us to choose arbitrarily the vector functions $v^{(1),(2)}(x) \in H_c^2(\Omega, \mathbb{R}^N)$, such that their images $u^{(1),(2)} := T_r v^{(1),(2)} \in H_c^2(\Omega, \mathbb{R}^N)$. By means of (2.3), we have for $1 \leq k \leq N,\ N \geq 2,\ x = (x_1, x_2) \in \Omega,\ y = (y_1, y_2) \in \Omega$

$$-\frac{\partial^2 u_k^{(1)}}{\partial x_1^2} + \sqrt{-\frac{\partial^2}{\partial x_2^2}}\, u_k^{(1)} = \int_\Omega G_k(x-y) F_k(v_1^{(1)}(y), v_2^{(1)}(y), ..., v_N^{(1)}(y), y) dy, \tag{3.3}$$

$$-\frac{\partial^2 u_k^{(2)}}{\partial x_1^2} + \sqrt{-\frac{\partial^2}{\partial x_2^2}}\, u_k^{(2)} = \int_\Omega G_k(x-y) F_k(v_1^{(2)}(y), v_2^{(2)}(y), ..., v_N^{(2)}(y), y) dy. \tag{3.4}$$

Let us apply the Fourier transform (4.1) to both sides of the equations of systems (3.3), (3.4). This gives us for $1 \leq k \leq N,\ (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$

$$u_{k,n_1,n_2}^{(1)} = 2\pi \frac{G_{k,n_1,n_2} f_{k,n_1,n_2}^{(1)}}{n_1^2 + |n_2|}, \quad u_{k,n_1,n_2}^{(2)} = 2\pi \frac{G_{k,n_1,n_2} f_{k,n_1,n_2}^{(2)}}{n_1^2 + |n_2|}, \tag{3.5}$$

$$(n_1^2+n_2^2) u_{k,n_1,n_2}^{(1)} = 2\pi \frac{(n_1^2+n_2^2) G_{k,n_1,n_2} f_{k,n_1,n_2}^{(1)}}{n_1^2 + |n_2|}, \tag{3.6}$$

$$(n_1^2+n_2^2) u_{k,n_1,n_2}^{(2)} = 2\pi \frac{(n_1^2+n_2^2) G_{k,n_1,n_2} f_{k,n_1,n_2}^{(2)}}{n_1^2 + |n_2|}. \tag{3.7}$$

Here $f_{k,n_1,n_2}^{(1)}$ and $f_{k,n_1,n_2}^{(2)}$ stand for the images of $F_k(v^{(1)}(x), x)$ and $F_k(v^{(2)}(x), x)$ respectively under transform (4.1). By virtue of (3.5), (3.6) and (3.7), we derive the inequalities

$$|u_{k,n_1,n_2}^{(1)} - u_{k,n_1,n_2}^{(2)}| \leq 2\pi \mathcal{N}_{r,k} |f_{k,n_1,n_2}^{(1)} - f_{k,n_1,n_2}^{(2)}|,$$

$$|(n_1^2+n_2^2)[u_{k,n_1,n_2}^{(1)} - u_{k,n_1,n_2}^{(2)}]| \leq 2\pi \mathcal{N}_{r,k} |f_{k,n_1,n_2}^{(1)} - f_{k,n_1,n_2}^{(2)}|$$

with $1 \leq k \leq N,\ (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$. Hence,

$$\|u^{(1)} - u^{(2)}\|_{H_c^2(\Omega, \mathbb{R}^N)}^2 =$$

$$= \sum_{k=1}^{N} \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |u_{k,n_1,n_2}^{(1)} - u_{k,n_1,n_2}^{(2)}|^2 + \sum_{k=1}^{N} \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |(n_1^2+n_2^2)[u_{k,n_1,n_2}^{(1)} - u_{k,n_1,n_2}^{(2)}]|^2 \leq$$

$$\leq 8\pi^2 \mathcal{N}_r^2 \sum_{k=1}^{N} \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(\Omega)},$$

with $\mathcal{N}_r$ defined in (4.6). We recall condition (2.2) of Assumption 2.1. Thus,

$$\|T_r v^{(1)} - T_r v^{(2)}\|_{H_c^2(\Omega, \mathbb{R}^N)} \leq 2\sqrt{2}\pi \mathcal{N}_r L \|v^{(1)} - v^{(2)}\|_{H_c^2(\Omega, \mathbb{R}^N)}. \tag{3.8}$$

The constant in the right side of (3.8) is less than one as we assume. Therefore, by means of the Fixed Point Theorem, there exists a unique vector function $v_r \in H_c^2(\Omega, \mathbb{R}^N)$, such that $T_r v_r = v_r$. This is the only solution of the system of equations (1.2) in $H_c^2(\Omega, \mathbb{R}^N)$. Let us suppose that $v_r(x)$ vanishes identically in Ω. This will contradict to the given condition that the Fourier coefficients $G_{k,n_1,n_2} F_k(0, x)_{n_1,n_2} \neq 0$ for some $1 \leq k \leq N$ and a certain pair $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$. ■

Let us proceed to establishing the solvability in the sense of sequences for our system of integro-differential equations in the square Ω.

Proof of Theorem 2.3. By virtue of the result of Theorem 2.2 above, each system (2.12) possesses a unique solution $u^{(m)}(x) \in H_c^2(\Omega, \mathbb{R}^N)$, $m \in \mathbb{N}$. Limiting system of equations (1.2) admits a unique solution $u(x) \in H_c^2(\Omega, \mathbb{R}^N)$ by means of Lemma 4.2 below along with Theorem 2.2. Let us apply the Fourier transform (4.1) to both sides of systems (1.2) and (2.12). Hence, for $1 \leq k \leq N$, $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$ and $m \in \mathbb{N}$, we obtain

$$u_{k,n_1,n_2} = 2\pi \frac{G_{k,n_1,n_2} \varphi_{k,n_1,n_2}}{n_1^2 + |n_2|}, \quad (n_1^2 + n_2^2) u_{k,n_1,n_2} = 2\pi \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2} \varphi_{k,n_1,n_2}}{n_1^2 + |n_2|}, \tag{3.9}$$

$$u^{(m)}_{k,n_1,n_2} = 2\pi \frac{G_{k,m,n_1,n_2} \varphi^{(m)}_{k,n_1,n_2}}{n_1^2 + |n_2|}, \quad (n_1^2 + n_2^2) u^{(m)}_{k,n_1,n_2} = 2\pi \frac{(n_1^2 + n_2^2) G_{k,m,n_1,n_2} \varphi^{(m)}_{k,n_1,n_2}}{n_1^2 + |n_2|}. \tag{3.10}$$

In formulas (3.9) and (3.10) above φ_{k,n_1,n_2} and $\varphi^{(m)}_{k,n_1,n_2}$ stand for the Fourier images of $F_k(u(x), x)$ and $F_k(u^{(m)}(x), x)$ respectively under transform (4.1). Clearly,

$$|u^{(m)}_{k,n_1,n_2} - u_{k,n_1,n_2}| \leq 2\pi \left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} |\varphi_{k,n_1,n_2}| +$$

$$+ 2\pi \left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} |\varphi^{(m)}_{k,n_1,n_2} - \varphi_{k,n_1,n_2}|,$$

such that

$$\|u_k^{(m)} - u_k\|_{L^2(\Omega)} \leq 2\pi \left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \|F_k(u(x), x)\|_{L^2(\Omega)} +$$

$$+ 2\pi \left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \|F_k(u^{(m)}(x), x) - F_k(u(x), x)\|_{L^2(\Omega)}.$$

We recall bound (2.2) of Assumption 2.1. Hence,

$$\sqrt{\sum_{k=1}^{N}\|F_k(u^{(m)}(x),x)-F_k(u(x),x)\|_{L^2(\Omega)}^2} \le L\|u^{(m)}(x)-u(x)\|_{L^2(\Omega,\mathbb{R}^N)}. \tag{3.11}$$

Thus, using (4.9) and (4.10), we derive

$$\|u^{(m)}(x)-u(x)\|_{L^2(\Omega,\mathbb{R}^N)}^2 \le$$

$$\le 8\pi^2\sum_{k=1}^{N}\left\|\frac{G_{k,m,n_1,n_2}}{n_1^2+|n_2|}-\frac{G_{k,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}^2\|F_k(u(x),x)\|_{L^2(\Omega)}^2+$$

$$+8\pi^2\Big[\mathcal{N}_r^{(m)}\Big]^2L^2\|u^{(m)}(x)-u(x)\|_{L^2(\Omega,\mathbb{R}^N)}^2.$$

By means of (4.12), we obtain

$$\|u^{(m)}(x)-u(x)\|_{L^2(\Omega,\mathbb{R}^N)}^2 \le \frac{8\pi^2}{\varepsilon(2-\varepsilon)}\sum_{k=1}^{N}\left\|\frac{G_{k,m,n_1,n_2}}{n_1^2+|n_2|}-\frac{G_{k,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}^2\|F_k(u(x),x)\|_{L^2(\Omega)}^2.$$

Inequality (2.1) of Assumption 2.1 yields that all $F_k(u(x),x)\in L^2(\Omega)$ for $u(x)\in H_c^2(\Omega,\mathbb{R}^N)$. Hence, under the given conditions

$$u^{(m)}(x)\to u(x), \quad m\to\infty \tag{3.12}$$

in $L^2(\Omega,\mathbb{R}^N)$ via the result of Lemma 4.2 of the Appendix. Formulas (3.9) and (3.10) yield

$$|(n_1^2+n_2^2)u_{k,n_1,n_2}^{(m)}-(n_1^2+n_2^2)u_{k,n_1,n_2}| \le 2\pi\left\|\frac{(n_1^2+n_2^2)G_{k,m,n_1,n_2}}{n_1^2+|n_2|}-\frac{(n_1^2+n_2^2)G_{k,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}|\varphi_{k,n_1,n_2}|+$$

$$+2\pi\left\|\frac{(n_1^2+n_2^2)G_{k,m,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}|\varphi_{k,n_1,n_2}^{(m)}-\varphi_{k,n_1,n_2}|,$$

such that

$$\|\Delta u_k^{(m)}(x)-\Delta u_k(x)\|_{L^2(\Omega)} \le 2\pi\left\|\frac{(n_1^2+n_2^2)G_{k,m,n_1,n_2}}{n_1^2+|n_2|}-\frac{(n_1^2+n_2^2)G_{k,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}\times$$

$$\times\|F_k(u(x),x)\|_{L^2(\Omega)}+2\pi\left\|\frac{(n_1^2+n_2^2)G_{k,m,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}\|F_k(u^{(m)}(x),x)-F(u(x),x)\|_{L^2(\Omega)}.$$

Inequality (3.11) enables us to derive the estimate

$$\|\Delta u_k^{(m)}(x)-\Delta u_k(x)\|_{L^2(\Omega)} \le 2\pi\left\|\frac{(n_1^2+n_2^2)G_{k,m,n_1,n_2}}{n_1^2+|n_2|}-\frac{(n_1^2+n_2^2)G_{k,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}\times$$

$$\times \|F_k(u(x),x)\|_{L^2(\Omega)} + 2\pi \left\| \frac{(n_1^2+n_2^2)G_{k,m,n_1,n_2}}{n_1^2+|n_2|} \right\|_{l^\infty} L\|u^{(m)}(x)-u(x)\|_{L^2(\Omega,\mathbb{R}^N)}.$$

Let us recall the result of Lemma 4.2 of the Appendix along with statement (3.12). We obtain that $\Delta u_k^{(m)}(x) \to \Delta u_k(x)$ in $L^2(\Omega)$ as $m \to \infty$ for $1 \le k \le N$. By virtue of definition (2.8) of the norm, we have $u^{(m)}(x) \to u(x)$ in $H_c^2(\Omega, \mathbb{R}^N)$ as $m \to \infty$.
Let us suppose the solution $u^{(m)}(x)$ of system (2.12) discussed above vanishes identically in the square Ω for some $m \in \mathbb{N}$. This will contradict to the given condition that the Fourier coefficients $G_{k,m,n_1,n_2}F_k(0,x)_{n_1,n_2} \neq 0$ for some $1 \le k \le N$ and a certain pair $(n_1,n_2) \in \mathbb{Z}\times\mathbb{Z}$. The similar argument is valid for the solution $u(x)$ of the limiting problem (1.2). ■

4 Appendix

Let the function $G_k(x_1,x_2) : \Omega \to \mathbb{R}$, such that $G_k(0,x_2) = G_k(2\pi,x_2)$ with $0 \le x_2 \le 2\pi$ and $G_k(x_1,0) = G_k(x_1,2\pi)$ with $0 \le x_1 \le 2\pi$. Its Fourier image on the square equals to

$$G_{k,n_1,n_2} := \int_0^{2\pi}\int_0^{2\pi} G_k(x_1,x_2)\frac{e^{-in_1x_1}}{\sqrt{2\pi}}\frac{e^{-in_2x_2}}{\sqrt{2\pi}}dx_1dx_2, \quad (n_1,n_2) \in \mathbb{Z}\times\mathbb{Z}, \tag{4.1}$$

such that

$$G_k(x_1,x_2) = \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} G_{k,n_1,n_2}\frac{e^{in_1x_1}}{\sqrt{2\pi}}\frac{e^{in_2x_2}}{\sqrt{2\pi}}, \quad (x_1,x_2) \in \Omega.$$

Obviously, the upper bound

$$\|G_{k,n_1,n_2}\|_{l^\infty} \le \frac{1}{2\pi}\|G_k(x_1,x_2)\|_{L^1(\Omega)} \tag{4.2}$$

is valid with $\|G_{k,n_1,n_2}\|_{l^\infty} := \sup_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}}|G_{k,n_1,n_2}|$. Evidently, (4.2) yields

$$\|n_2G_{k,n_1,n_2}\|_{l^\infty} \le \frac{1}{2\pi}\left\|\frac{\partial G_k(x_1,x_2)}{\partial x_2}\right\|_{L^1(\Omega)}. \tag{4.3}$$

Furthermore, for a function continuous in the square Ω, the inequality

$$\|G_k(x_1,x_2)\|_{L^1(\Omega)} \le \|G_k(x_1,x_2)\|_{C(\Omega)}(2\pi)^2, \tag{4.4}$$

is valid with $\|G_k(x_1,x_2)\|_{C(\Omega)} := \max_{(x_1,x_2)\in\Omega}|G_k(x_1,x_2)|$. Let us introduce the following technical quantities

$$\mathcal{N}_{r,k} := \max\left\{\left\|\frac{G_{k,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}, \quad \left\|\frac{(n_1^2+n_2^2)G_{k,n_1,n_2}}{n_1^2+|n_2|}\right\|_{l^\infty}\right\}. \tag{4.5}$$

with $1 \le k \le N$, $N \ge 2$. Under the assumptions of Lemma 4.1 below, all the expressions (4.5) will be finite. Thus,

$$\mathcal{N}_r := \max_{1\le k\le N}\mathcal{N}_{r,k} < \infty. \tag{4.6}$$

The auxiliary statements below are the adaptations of the ones used in [19] to study the single integro-differential problem with mixed diffusion, analogical to the system of equations (1.2). Let us provide them for the convenience of the readers.

Lemma 4.1. *Let $N \geq 2,\ 1 \leq k \leq N$, the functions $G_k(x_1, x_2) : \Omega \to \mathbb{R}$, so that $G_k(0, x_2) = G_k(2\pi, x_2)$ with $0 \leq x_2 \leq 2\pi$ and $G_k(x_1, 0) = G_k(x_1, 2\pi)$ with $0 \leq x_1 \leq 2\pi$. Moreover, $G_k(x_1, x_2) \in C(\Omega)$ and $\dfrac{\partial G_k(x_1, x_2)}{\partial x_2} \in L^1(\Omega)$. Then $\mathcal{N}_{r,k} < \infty$ if and only if*

$$(G_k(x_1, x_2), 1)_{L^2(\Omega)} = 0. \tag{4.7}$$

Proof. Let us first demonstrate that under the given conditions $\dfrac{(n_1^2 + n_2^2)G_{k,n_1,n_2}}{n_1^2 + |n_2|} \in l^\infty$. Clearly, by means of (4.2) and (4.4),

$$\left|\frac{n_1^2 G_{k,n_1,n_2}}{n_1^2 + |n_2|}\right| \leq \|G_{k,n_1,n_2}\|_{l^\infty} \leq 2\pi \|G_k(x_1, x_2)\|_{C(\Omega)} < \infty$$

as we assume. By virtue of (4.3), we have

$$\left|\frac{n_2^2 G_{k,n_1,n_2}}{n_1^2 + |n_2|}\right| \leq \|n_2 G_{k,n_1,n_2}\|_{l^\infty} \leq \frac{1}{2\pi}\left\|\frac{\partial G_k(x_1, x_2)}{\partial x_2}\right\|_{L^1(\Omega)} < \infty$$

as assumed. Thus, $\dfrac{(n_1^2 + n_2^2)G_{k,n_1,n_2}}{n_1^2 + |n_2|}$ is bounded. We can write

$$\frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} = \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|}\chi_{\{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}\ |\ n_1=n_2=0\}} + \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|}\chi_{\{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}\ |\ n_1=n_2=0\}^c}. \tag{4.8}$$

Here and further down χ_A will stand for the characteristic function of a set $A \subseteq \mathbb{Z} \times \mathbb{Z}$ and A^c will denote the complement of A. Obviously, the second term in the right side of (4.8) can be estimated from above in the absolute value by means of (4.2) along with (4.4) by

$$|G_{k,n_1,n_2}| \leq 2\pi \|G_k(x_1, x_2)\|_{C(\Omega)} < \infty$$

via the one of our assumptions. Evidently, the first term in the right side of (4.8) is bounded if and only if $G_{k,0,0}$ vanishes. This is equivalent to orthogonality relation (4.7). ■

Let us note that the proof of the lemma above uses only on a single orthogonality condition for each $1 \leq k \leq N,\ N \geq 2$, as distinct from the analogical case in the whole $\mathbb{R}^2$ considered in [18].

For the purpose of the studies of the systems of equations (2.12), we will use the auxiliary expressions

$$\mathcal{N}_{r,k}^{(m)} := \max\left\{\left\|\frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|}\right\|_{l^\infty}, \quad \left\|\frac{(n_1^2 + n_2^2)G_{k,m,n_1,n_2}}{n_1^2 + |n_2|}\right\|_{l^\infty}\right\}, \quad m \in \mathbb{N}, \tag{4.9}$$

where $1 \leq k \leq N,\ N \geq 2$. Under the conditions of Lemma 4.2 below, all expressions (4.9) will be finite. This will enable us to introduce

$$\mathcal{N}_r^{(m)} := \max_{1\leq k\leq N} \mathcal{N}_{r,k}^{(m)}, \quad m \in \mathbb{N}. \tag{4.10}$$

The final statement of the article is as follows.

Lemma 4.2. *Let $m \in \mathbb{N},\ N \geq 2,\ 1 \leq k \leq N$, the functions $G_{k,m}(x_1, x_2) : \Omega \to \mathbb{R}$, such that $G_{k,m}(0, x_2) = G_{k,m}(2\pi, x_2)$ with $0 \leq x_2 \leq 2\pi$ and $G_{k,m}(x_1, 0) = G_{k,m}(x_1, 2\pi)$ with $0 \leq x_1 \leq 2\pi$. Furthermore,*

$$G_{k,m}(x_1, x_2) \in C(\Omega), \quad G_{k,m}(x_1, x_2) \to G_k(x_1, x_2) \quad in \quad C(\Omega) \quad as \quad m \to \infty.$$

Additionally,

$$\frac{\partial G_{k,m}(x_1, x_2)}{\partial x_2} \in L^1(\Omega), \quad \frac{\partial G_{k,m}(x_1, x_2)}{\partial x_2} \to \frac{\partial G_k(x_1, x_2)}{\partial x_2} \quad in \quad L^1(\Omega) \quad as \quad m \to \infty.$$

We also suppose that for all $1 \leq k \leq N,\ m \in \mathbb{N}$

$$(G_{k,m}(x_1, x_2), 1)_{L^2(\Omega)} = 0 \tag{4.11}$$

holds. Finally, we assume that

$$2\sqrt{2}\pi \mathcal{N}_r^{(m)} L \leq 1 - \varepsilon \tag{4.12}$$

is valid for each $m \in \mathbb{N}$ with some fixed $0 < \varepsilon < 1$.

Then, for all $1 \leq k \leq N$, we have

$$\frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \to \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|}, \quad m \to \infty, \tag{4.13}$$

$$\frac{(n_1^2 + n_2^2) G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \to \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2}}{n_1^2 + |n_2|}, \quad m \to \infty \tag{4.14}$$

in l^∞, such that

$$\left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \to \left\| \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}, \quad m \to \infty, \tag{4.15}$$

$$\left\| \frac{(n_1^2 + n_2^2) G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \to \left\| \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}, \quad m \to \infty. \tag{4.16}$$

Moreover,

$$2\sqrt{2}\pi \mathcal{N}_r L \leq 1 - \varepsilon \tag{4.17}$$

holds.

Proof. Evidently, under the stated assumptions all $\mathcal{N}_{r,k}^{(m)}$ are finite by virtue of the result of Lemma 4.1 above, such that $\mathcal{N}_r^{(m)} < \infty,\ m \in \mathbb{N}$.

It can be easily verified that the limiting kernels are periodic functions as well. Indeed, for $0 \leq x_2 \leq 2\pi$, we obtain

$$|G_k(0, x_2) - G_k(2\pi, x_2)| \leq |G_{k,m}(0, x_2) - G_k(0, x_2)| + |G_{k,m}(2\pi, x_2) - G_k(2\pi, x_2)| \leq$$

$$\leq 2\|G_{k,m}(x_1, x_2) - G_k(x_1, x_2)\|_{C(\Omega)} \to 0, \quad m \to \infty$$

due to our assumptions. Hence,

$$G_k(0, x_2) = G_k(2\pi, x_2) \quad for \quad 0 \leq x_2 \leq 2\pi$$

with $1 \leq k \leq N$. Analogously, for $0 \leq x_1 \leq 2\pi$

$$|G_k(x_1, 0) - G_k(x_1, 2\pi)| \leq |G_{k,m}(x_1, 0) - G_k(x_1, 0)| + |G_{k,m}(x_1, 2\pi) - G_k(x_1, 2\pi)| \leq$$

$$\leq 2\|G_{k,m}(x_1, x_2) - G_k(x_1, x_2)\|_{C(\Omega)} \to 0, \quad m \to \infty$$

as we assume. Thus,

$$G_k(x_1, 0) = G_k(x_1, 2\pi) \quad for \quad 0 \leq x_1 \leq 2\pi,$$

where $1 \leq k \leq N$. Let us demonstrate that the limiting orthogonality relations

$$(G_k(x_1, x_2), 1)_{L^2(\Omega)} = 0, \quad 1 \leq k \leq N \tag{4.18}$$

are valid. With the help of (4.11), we derive

$$|(G_k(x_1, x_2), 1)_{L^2(\Omega)}| = |(G_k(x_1, x_2), 1)_{L^2(\Omega)} - (G_{k,m}(x_1, x_2), 1)_{L^2(\Omega)}| \leq$$

$$\leq \|G_{k,m}(x_1, x_2) - G_k(x_1, x_2)\|_{C(\Omega)}(2\pi)^2 \to 0, \quad m \to \infty$$

via the one of our assumptions, such that (4.18) holds.
Hence, by means of Lemma 4.1, all $\mathcal{N}_{r,k}$ are finite, such that $\mathcal{N}_r < \infty$ as well.
Let us recall orthogonality conditions (4.18) and (4.11) along with the definition of the Fourier transform (4.1). Clearly, we have

$$G_{k,0,0} = 0, \quad G_{k,m,0,0} = 0, \quad 1 \leq k \leq N, \quad m \in \mathbb{N}.$$

Then by virtue of bounds (4.2) and (4.4), we arrive at

$$\left\|\frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|}\right\|_{l^\infty} \leq 2\pi\|G_{k,m}(x_1, x_2) - G_k(x_1, x_2)\|_{C(\Omega)} \to 0, \quad m \to \infty$$

as we assume, so that (4.13) is valid. Note that (4.15) is an immediate consequence of (4.13) due to the standard triangle inequality.
Obviously, the estimate

$$\left|\frac{(n_1^2 + n_2^2)G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{(n_1^2 + n_2^2)G_{k,n_1,n_2}}{n_1^2 + |n_2|}\right| \leq$$

$$\leq \|G_{k,m,n_1,n_2} - G_{k,n_1,n_2}\|_{l^\infty} + \|n_2[G_{k,m,n_1,n_2} - G_{k,n_1,n_2}]\|_{l^\infty}$$

holds. Using formulas (4.2), (4.3) and (4.4), we derive the upper bound

$$\left\| \frac{(n_1^2+n_2^2)G_{k,m,n_1,n_2}}{n_1^2+|n_2|} - \frac{(n_1^2+n_2^2)G_{k,n_1,n_2}}{n_1^2+|n_2|} \right\|_{l^\infty} \leq 2\pi \|G_{k,m}(x_1,x_2) - G_k(x_1,x_2)\|_{C(\Omega)} +$$

$$+ \frac{1}{2\pi} \left\| \frac{\partial G_{k,m}(x_1,x_2)}{\partial x_2} - \frac{\partial G_k(x_1,x_2)}{\partial x_2} \right\|_{L^1(\Omega)} \to 0, \quad m \to \infty$$

by means of our assumptions. Hence, (4.14) is valid. Let us use the standard triangle inequality to establish that (4.16) follows easily from (4.14).
An easy limiting argument relying on (4.5), (4.6), (4.9), (4.10), (4.12), (4.15) and (4.16) yields (4.17). ■

Remark 4.3. *The existence in the sense of sequences of the solutions of the system of equations (1.2) involving the transport terms will be covered in the following article.*

Acknowledgement

V. V. is grateful to Israel Michael Sigal for the partial support by the NSERC grant NA 7901.

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