A Double Schwarzschild Solution

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Abstract

A two-body vacuum solution of Einstein's equations is derived from which the traditional single-body Schwarzschild vacuum solution is shown to emerge. The necessary conditions suggest a role for distant Cosmological matter.

1 Derivation

1.1 Single body vacuum solution

With $G_N = 1$ the line-element in spherical Schwarzschild coordinates of the Schwarzschild vacuum solution of Einstein's equations is [1,2]

$$\mathrm{d}s^2 = (1 - 2m/r_\mathrm{s})\,\mathrm{d}t^2 - (1 - 2m/r_\mathrm{s})^{-1}\mathrm{d}r_\mathrm{s}^2 - r_\mathrm{s}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{1}$$

The original form of Schwarzschild's vacuum solution [1] (English translation [2]), first communicated by letter to Einstein in 1915, was subsequently recast by Droste [3] and Hilbert [4] into the form (1) that has since become known as the Schwarzschild solution. See Abrams [5, 6], Antoci [7, 8] and Mol [9] for interesting discussions of the history of the two forms.

The singularity in (1) at $r_\mathrm{s} = 0$ is a curvature singularity, and of physical significance therefore. $r_\mathrm{s} = 2m$ is an event horizon because the coordinate speed of light vanishes there. The curvature at $r_\mathrm{s} = 2m$ is zero however, and the apparent singularity there can be removed with an appropriate coordinate choice, including Kruskal–Szekeres, Lemaître, and harmonic coordinates. It turns out to be convenient however to express the double Schwarzschild solution in isotropic coordinates, though these do not extend inside the event horizon. They are related to the coordinates in (1) by the transformation

$$r_\mathrm{s} = r\left(1 + \frac{m}{2r}\right)^2 \tag{2}$$

through which one obtains the isotropic representation of the Schwarzschild line-element

$$\mathrm{d}s^2 = \left(\frac{1 - m/2r}{1 + m/2r}\right)^2 \mathrm{d}t^2 - (1 + m/2r)^4\mathrm{d}\mathbf{x}^2 \tag{3}$$

[1] We use r_s for the Schwarzschild radial coordinate, leaving r and $\mathbf{x}$ for the isotropic coordinates.

[2] The weak-field limit of the Schwarzschild solution is the Newtonian gravitational potential of a point mass. From that perspective $\phi = -m/4\pi|r_\mathrm{s}|$ and therefore $\nabla^2\phi = -m\delta(r_\mathrm{s})/2\pi r_\mathrm{s}^2$, and so 'vacuum' pertains to everywhere but $r_\mathrm{s} = 0$. GR permits extension of the coordinate chart through $r_\mathrm{s} = 0$ however, with the result that $\nabla^2\phi = 0$ there. Even so, mass-energy is expected to be found at the terminus of geodesics in the extended chart which therefore is understood to be excluded from the scope of 'vacuum' used in this document.

where

$$\mathrm{d}\mathbf{x}^2 \equiv \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{4}$$

1.2 Constraints on a more general vacuum solution

We seek a generalization of (4) subject to the constraint it remains a static vacuum solution outside of any static mass singularity or singularities. Given the structure of (4) we could try

$$\mathrm{d}s^2 = \phi(\mathbf{x})\,\mathrm{d}t^2 - \chi(\mathbf{x})\,\mathrm{d}\mathbf{x}^2 \tag{5}$$

though without loss of generality it turns out to be more convenient to write this instead as

$$\mathrm{d}s^2 = \left(\frac{\phi(\mathbf{x})}{\chi(\mathbf{x})}\right)^2 \mathrm{d}t^2 - \chi^4(\mathbf{x})\,\mathrm{d}\mathbf{x}^2. \tag{6}$$

A vacuum solution requires

$$\forall \mu,\nu:\ \ T_{\mu\nu} = 0 \Leftrightarrow G_{\mu\nu} = 0 \tag{7}$$

which here implies constraints on ϕ and χ. Upon substitution of the metric corresponding to (6) into Einstein's equations, up to an overall constant the diagonal terms are [3]

$$G_{00} = -4\frac{\phi^2}{\chi^7}\nabla^2\chi \tag{8}$$

and

$$G_{ii} = \frac{1}{\phi\chi}\left(\phi\nabla^2\chi + \chi\nabla^2\phi - 2\nabla\phi\cdot\nabla\chi - \phi\chi_{,ii} - \chi\phi_{,ii} + 6\phi_{,i}\chi_{,i}\right). \tag{9}$$

For the off-diagonal terms the space-time components vanish $G_{0i} = G_{i0} = 0$, whilst the space-space components are

$$G_{ij} = \frac{1}{\phi\chi}\left(3\phi_{,i}\chi_{,j} + 3\chi_{,i}\phi_{,j} - \phi\chi_{,ij} - \chi\phi_{,ij}\right)\ \forall\ \ i \neq j. \tag{10}$$

Summing the diagonal space-space terms in (9) immediately gives

$$\sum_{i=1}^{3} G_{ii} = 2\frac{\nabla^2\chi}{\chi} + 2\frac{\nabla^2\phi}{\phi}. \tag{11}$$

It follows from (8) and (11) that a necessary condition for the satisfaction of (7) is ϕ and χ satisfy Laplace's equation

$$\nabla^2\phi = \nabla^2\chi = 0 \tag{12}$$

which result motivated the choice of parameterization in (6) versus that in (5). If (12) is satisfied then (9) reduces to

$$G_{ii} = \frac{1}{\phi\chi}\left(6\phi_{,i}\chi_{,i} - \phi\chi_{,ii} - \chi\phi_{,ii} - 2\nabla\phi\cdot\nabla\chi\right). \tag{13}$$

[3] The gradient and Laplacian derivatives are understood to be exclusively with respect to contra-variant coordinates:

$$\nabla\phi\cdot\nabla\chi = \sum_{i=1}^{3}\phi_{,i}\chi_{,i}\ ,\quad \nabla^2\phi = \sum_{i=1}^{3}\phi_{,ii}\,.$$

1.3 Point source solutions

Attention is restricted to solutions of (12) corresponding to point sources

$$\phi(\mathbf{x}) = \sum_{k=1}^{n} \phi_j(\mathbf{x})\ ,\ \ \chi(\mathbf{x}) = \sum_{k=1}^{n} \chi_j(\mathbf{x}) \tag{14}$$

where

$$\phi_k(\mathbf{x}) = \frac{p_k}{|\mathbf{x}-\mathbf{a}_k|}\ ,\ \ \chi_k(\mathbf{x}) = \frac{q_k}{|\mathbf{x}-\mathbf{a}_k|} \tag{15}$$

where the p_k, q_k and $\mathbf{a}_k \equiv (a_k, b_k, c_k)$ are constant degrees of freedom to be adjusted for satisfaction for all i, j of $G_{i,j} = 0$. The number of sources n has yet to be determined. Writing the space-space terms as $G_{i,j} = \hat{\mathrm{G}}_{i,j}(\phi(\mathbf{x}), \chi(\mathbf{x}))$ then for $G_{i,j}$ to vanish requires

$$\sum_{k,l=1}^{n} \hat{\mathrm{G}}_{i,j}(\phi_k(\mathbf{x}), \chi_l(\mathbf{x})) = 0\ \ \forall\ i, j\,. \tag{16}$$

The $G_{i,j}$ vanish completely for the particular case that $\phi = \chi = 1/r$: substitution in (9) gives

$$\hat{\mathrm{G}}_{i,i}\left(\frac{1}{r}, \frac{1}{r}\right) = r^2\left(-2\frac{\mathbf{x}}{r^3}\cdot\frac{\mathbf{x}}{r^3} + 2\frac{1}{r}\frac{\partial}{\partial x_i}\frac{x_i}{r^3} + 6\frac{x_i^2}{r^2}\right) = 0\,, \tag{17}$$

and substitution in (10) gives

$$\hat{\mathrm{G}}_{i,j}\left(\frac{1}{r}, \frac{1}{r}\right) = r^2\left(6\frac{x_i}{r^3}\frac{x_j}{r^3} + 2\frac{1}{r}\frac{\partial}{\partial x_i}\frac{x_j}{r^3}\right) = 0\,. \tag{18}$$

Given the lack of an explicit dependence of the $G_{i,j}$ on the absolute position it follows that the $\phi_k = \chi_l$ 'diagonal' terms in (16) already vanish

$$\hat{\mathrm{G}}_{i,j}(\phi_k(\mathbf{x}), \phi_k(\mathbf{x})) = \hat{\mathrm{G}}_{i,j}(\chi_k(\mathbf{x}), \chi_k(\mathbf{x})) = 0\,. \tag{19}$$

Noticing also that the $G_{i,j}$ are symmetric in ϕ and χ

$$\hat{\mathrm{G}}_{i,j}(\phi, \chi) = \hat{\mathrm{G}}_{i,j}(\chi, \phi) \tag{20}$$

it follows that for (16) to be satisfied it remains only to require

$$\sum_{\substack{k,l=1\\ k\neq l}}^{n} \left\{\hat{\mathrm{G}}_{i,j}(\phi_k(\mathbf{x}), \chi_l(\mathbf{x})) + \hat{\mathrm{G}}_{i,j}(\phi_l(\mathbf{x}), \chi_k(\mathbf{x}))\right\} = 0\,. \tag{21}$$

For each k the p_k and q_k can be considered as amplitudes of functionally independent contributions to $\phi(\mathbf{x})$ and $\chi(\mathbf{x})$ respectively because, for finite n, there is no danger of over-determination / degeneracy over all $\mathbf{x}$ in (15). Satisfaction of (21) therefore requires

$$p_k q_l + p_l q_k = 0\ \ \forall\ k \neq l\,. \tag{22}$$

It remains to determine the n for which (22) can be satisfied. Clearly it is automatically satisfied when $n = 1$ because in that case there are no combinations for which $k \neq l$. It is also satisfied when $n = 2$ provided

$$q_1 = \lambda p_1\ ,\ \ q_2 = -\lambda p_2 \tag{23}$$

for any λ. Using these in (14) and (15) gives

$$\phi(\mathbf{x}) = \frac{p_1}{|\mathbf{x}-\mathbf{a}_1|} - \frac{p_2}{|\mathbf{x}-\mathbf{a}_2|}\ ,\ \ \chi(\mathbf{x}) = \frac{p_1}{|\mathbf{x}-\mathbf{a}_1|} + \frac{p_2}{|\mathbf{x}-\mathbf{a}_2|}\,. \tag{24}$$

When considering $n > 2$ we note first of all that if any particular p_k vanishes then (22) requires that the corresponding q_k must vanish also. But this possibility can be ignored because it effectively reduces n. In the end this means (22) can be regarded as a set of linear homogeneous equations in the q_k, given the p_k. With this in mind, when $n = 3$ (22) can be written

$$\begin{bmatrix} p_2 & p_1 & 0 \\ p_2 & 0 & p_1 \\ 0 & p_3 & p_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{25}$$

For solutions to exist the determinant must vanish, which means that at least one of the p_k is zero, which in this context reduces the effective number of degrees of freedom from 3 back to 2. The same applies to all n greater than 2. Hence $n = 1$ and $n = 2$ are the only possibilities.[4]

Returning to (6), it follows that the most general static isotropic vacuum metric is a two-body metric due to two massive bodies

$$\mathrm{d}s^2 = \left(\frac{\dfrac{m_\mathrm{a}}{2\,|\mathbf{x}-\mathbf{a}|} - \dfrac{m_\mathrm{b}}{2\,|\mathbf{x}-\mathbf{b}|}}{\dfrac{m_\mathrm{a}}{2\,|\mathbf{x}-\mathbf{a}|} + \dfrac{m_\mathrm{b}}{2\,|\mathbf{x}-\mathbf{b}|}} \right)^2 \mathrm{d}t^2 - \left(\frac{m_\mathrm{a}}{2\,|\mathbf{x}-\mathbf{a}|} + \frac{m_\mathrm{b}}{2\,|\mathbf{x}-\mathbf{b}|} \right)^4 \mathrm{d}\mathbf{x}^2 \tag{26}$$

where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, and we have set $p_1 = m_\mathrm{a}/2$ and $p_2 = m_\mathrm{b}/2$ in units in which $G_N = c = 1$ to accord with (3). The case $n = 1$ can be regarded as subsumed in (26) as either $m_\mathrm{a} = 0$ or $m_\mathrm{b} = 0$.[5]

2 Discussion

2.1 Sign of the masses

There is no loss of generality in the particular sign choice in (26) unless and until it is to be understood that $m_\mathrm{a} = |m_\mathrm{a}|$ and $m_\mathrm{b} = |m_\mathrm{b}|$. Assuming so, (26) remains a vacuum line-element under independent negation of the masses, i.e. $m_\mathrm{a} \equiv |m_\mathrm{a}| \to -m_\mathrm{a} \equiv -|m_\mathrm{a}|$ and / or $m_\mathrm{b} \equiv |m_\mathrm{b}| \to -m_\mathrm{b} \equiv -|m_\mathrm{b}|$, though the physical meaning changes when one but not both masses are negated. The signs in (26) were chosen to facilitate comparison with the one-body isotropic line-element (3). They are consistent also with the Schwarzschild line-element (1) through the transformation (33) when $\gamma = 1$, the conditions for which are discussed in Section 2.3. In Section 2.5 however we make a different sign choice appropriate to the full two-body system.

2.2 Asymptotic flatness

Spacetime is expected to asymptote to Minksowski spacetime far from a localized mass [10]. At $m = 0$ Eq. (3) becomes the Minkowski metric, which can be regarded therefore as the *background* to which the standard isotropic metric asymptotes as $r \to \infty$. Minkowski spacetime emerges from Eq. (26) in the same limit though in a different manner. When $n = 1$ in (14), Eq. (26) reduces to

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(\frac{\lambda}{r}\right)^4 \mathrm{d}\mathbf{x}^2\,. \tag{27}$$

[4] Here we have no reason to presume assume the p_k, q_k are other than ordinary commuting scalars.

[5] The factors of 1/2 are retained to facilitate comparison with the one-body form of the isotropic line element.

Here $\lambda = m/2$ where m is whichever of the two masses is not zero. Eq. (27) is the Minkowski line-element with an inverted radius and signifies Minkowski spacetime therefore. Specifically $\{t, r, \theta, \phi\} = \left\{t', \lambda^2/r', \theta', \phi'\right\}$ gives

$$\mathrm{d}\mathbf{x}^2 = \left(\frac{\lambda}{r'}\right)^4 \mathrm{d}\mathbf{x}'^2 \implies \mathrm{d}s^2 = \mathrm{d}t'^2 - \left(\frac{\lambda}{r}\right)^4 \left(\frac{\lambda}{r'}\right)^4 \mathrm{d}\mathbf{x}'^2 = \mathrm{d}t'^2 - \mathrm{d}\mathbf{x}'^2 . \tag{28}$$

Hence (27) and (28) are physically equivalent through the transformation $r \to \lambda^2/r$ everywhere except at the origin. The difference is important. Because it derives from (26) one presumes the origin $\mathbf{x} = \mathbf{0}$ in (27) hosts a mass singularity, and so the Minkowski spacetime of (28) is generated by a lone mass, though without an intrinsic background - Minkowski or otherwise.

Minkowski spacetime is likewise present at sufficiently large distances from both masses because in that case both nominally local masses will appear to coalesce to a single mass, to which the preceding applies. Specifically, at sufficiently large r (26) reduces to $\mathrm{d}s^2 = \mathrm{d}t'^2 - \mathrm{d}\mathbf{x}'^2$ through the transformations

$$r' = (m_\mathrm{a} + m_\mathrm{b})^2 / 4r\,, \quad t' = t\,(m_\mathrm{a} - m_\mathrm{b}) / (m_\mathrm{a} + m_\mathrm{b})\,. \tag{29}$$

2.3 One mass very distant

Consider now a limit of (26) in which one of the two masses $m_\mathrm{a} = M$ say, is very distant compared to the radius of volume of interest that contains the other 'local' mass $m_\mathrm{b} = m$ say. Without loss of generality the coordinate system can be centered on the local mass by choosing $\mathbf{b} = \mathbf{0}$. Defining $R = |\mathbf{a}|$, the expansion to second order of $1/|\mathbf{x} - \mathbf{a}|$ is

$$\frac{1}{|\mathbf{x} - \mathbf{a}|} = \frac{1}{R}\left(1 + \frac{r}{R}\cos\theta + O\left(\left(\frac{r}{R}\right)^2\right)\right) \tag{30}$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{a}$. If the mass M is sufficiently far away that $r/R << 1$ then only the first term in (30) need be retained, whereupon (26) becomes

$$\mathrm{d}s^2 = \left(\frac{\gamma - m/2r}{\gamma + m/2r}\right)^2 \mathrm{d}t^2 - (\gamma + m/2r)^4 \mathrm{d}\mathbf{x}^2 \tag{31}$$

where

$$\gamma \equiv M/2R\,. \tag{32}$$

Note that because the first term in (30) is insensitive to the angle between $\mathbf{x}$ and $\mathbf{a}$, to leading order all masses on a sphere of radius R are equivalent. Hence though (31) derives from a two-body metric the second body could be a massive shell at large R with total mass M but whose mass-distribution is otherwise arbitrary.[6]

A spherically symmetric form of (31) can be obtained with the radial transformation

$$r_\mathrm{s} = r\left(\gamma + \frac{m}{2r}\right)^2 \tag{33}$$

which results in a Schwarzschild-like line element

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{\gamma r_\mathrm{s}}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{\gamma r_\mathrm{s}}\right)^{-1} \mathrm{d}r_\mathrm{s}^2 - r_\mathrm{s}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{34}$$

[6] Possibly the contribution from the next term washes out, in which case the leading term could be a better approximation than it might initially appear.

Exact correspondence between (31) and the standard isotropic form (3), and likewise between (34) and the standard Schwarzschild solution (1), is apparently possible by re-scaling the local mass and coordinates as

$$m' = \gamma m\,, \quad r' = \gamma^2 r \tag{35}$$

in which case (31) becomes

$$\mathrm{d}s^2 = \left(\frac{1 - m'/2r'}{1 + m'/2r'}\right)^2 \mathrm{d}t^2 - (1 + m'/2r')^4 \mathrm{d}\mathbf{x}'^2 \tag{36}$$

and likewise for the Schwarzschild form.

2.4 Cosmological interpretation

The prescription (35) does not respect the symmetry of the line-element (26) with respect to the exchange of m_a and m_b. One might ask why the same re-scaling should not also be applied to M and R. But doing so results in (31), though now in the primed masses and coordinates, and so removes γ as an effectively degree of freedom with which to recover the standard isotropic form (3) from (31). Recovery of the standard Schwarzschild solution from the two-body solution through (35) though possible is not completely satisfactory therefore. Eqs. (3) and (31) can be reconciled without selective scaling of the mass and coordinates if perhaps $\gamma = 1$.

A very approximate estimate of the cosmological contribution to γ can be obtained by supposing that M stands for the total mass within the current Hubble volume of radius $R = 1/H_0$. In that case, and restoring G_N, (32) is

$$\gamma = \frac{G_N M}{2R} \approx \frac{2\pi G_N \rho_{\mathrm{m},0}}{3H_0^2} = \frac{\Omega_\mathrm{m}}{4} \tag{37}$$

where Ω_m is the cosmological mass density contribution to the Friedmann equation. The sign difference $\gamma - m/2r$ in the numerator of g_{00} in (31) can be attributed to the fact that r in that line-element is exterior to the local mass and interior to the shell formed by distant matter.

The recent PDG estimate for Ω_m is 0.315, implying therefore $\gamma \approx 0.08$. Though this is an order of magnitude less than required it is close enough to warrant a more careful analysis, taking account of the very rough approximations made here.[7] (It is hoped to give a more accurate and detailed account of the cosmological contribution in a future report.)

Far from the singularity the traditional Schwarzschild solution (1) is asymptotic to Minkowski spacetime. And it is asymptotically spatially flat because it is constant in time. An asymptote to Minkowski spacetime could have been accommodated at the outset by including a constant term in the decomposition of ϕ, χ in (14). But an explicit background plus a pole due to a single mass-singularity exhausts the available degrees of freedom in the expansion (14) that can solve (27); there are no two-body vacuum solutions that include, additionally, a Minkowski background. The outcome would have been the traditional isotropic form of the Schwarzschild solution. Anticipating a more accurate derivation of $\gamma \approx 1$, the two body vacuum metric permits a novel re-interpretation of the traditional Schwarzschild

[7] The approximations requiring revision include the restriction of contributions to M from the current Hubble volume, and the implicit presumption that those contributions can be regarded as located on shell at the Hubble radius. It turns out however that additional considerations are necessary in order to perform a more accurate calculation. With those revisions the cosmological contribution can be shown to come within 10% of the required value.

solution in which these asymptotes are established instead by distant matter on a Cosmological scale. Such an interpretation would leave no role for a background spacetime independent of the metric established by the local mass and the distant shell.[8]

2.5 Two local masses

Here we briefly consider use of the two-body metric as the source of geodesics of a test mass when both masses are present locally. Whereas Section 2.4 was concerned with the emergence of the traditional isotropic metric from the two-body metric, here the masses should contribute with the same sign. Accordingly, if both masses are local then (26) should be replaced with

$$\mathrm{d}s^2 = \left(\frac{\dfrac{m_\mathrm{a}}{2\,|\mathbf{x}-\mathbf{a}|} + \dfrac{m_\mathrm{b}}{2\,|\mathbf{x}-\mathbf{b}|}}{\dfrac{m_\mathrm{a}}{2\,|\mathbf{x}-\mathbf{a}|} - \dfrac{m_\mathrm{b}}{2\,|\mathbf{x}-\mathbf{b}|}} \right)^2 \mathrm{d}t^2 - \left(\frac{m_\mathrm{a}}{2\,|\mathbf{x}-\mathbf{a}|} - \frac{m_\mathrm{b}}{2\,|\mathbf{x}-\mathbf{b}|} \right)^4 \mathrm{d}\mathbf{x}^2 \tag{38}$$

with $m_\mathrm{a} = |m_\mathrm{a}|$ and $m_\mathrm{b} = |m_\mathrm{b}|$ understood.

The Schwarzschild and isotropic vacuum metrics seemingly lend themselves to the interpretation that a mass exerts its influence *in* a Minkowski background, because that is what remains when the mass is set to zero. Though this point of view is qualified in Section 2.4 (which proposes that the Minkowski background depends on the presence of distant masses) it remains locally valid, regardless. By contrast the two-body spacetime is asymptotic to a Minkowski spacetime which cannot similarly be considered a background, i.e. in which the two masses exert their influence. The two-body metric cannot accommodate such a background *and* the influence of two local masses because it is already exhausted by a single mass plus background (the latter due to distant masses). Relatedly, the line-element (26) is un-physical when both mass are set to zero. Eq. (26) can still be used as the source of geodesics of a test mass when both bodies are present locally, but its utility is then restricted to the case that the influence of distant matter can be ignored. Possibly this will be so only for geodesics inside the event horizons of one or perhaps both of the larger masses. In any case the metric experienced by a test mass in a coordinate system that is stationary relative to a rotating binary system can be obtained by transforming (38) to rotate at the angular frequency of the two-mass system about its barycenter. The relevant manipulations are not given here, but may be presented in a future publication.[9]

2.6 Vacuum Dipole

We consider momentarily the case that one of the local masses is a negative (mass) point source, though there is no physical evidence of such. Then $m_\mathrm{a} = |m_\mathrm{a}|$ and $m_\mathrm{b} = -|m_\mathrm{b}|$ say in (38), so that (26) is recovered, but now with $m_\mathrm{a} = |m_\mathrm{a}|$ and $m_\mathrm{b} = |m_\mathrm{b}|$ unambiguously. Borrowing from (30), at large r we have

$$\frac{1}{|\mathbf{x}-\mathbf{a}|} \approx \frac{1}{r} + \frac{\mathbf{x}\cdot\mathbf{a}}{r^3}\,. \tag{39}$$

[8] This is reminiscent of the direct particle interaction formulation of electrodynamics, in which the electromagnetic action can be expressed entirely in terms of pair-wise interactions between particles - also without reference to a background spacetime.

[9] It is implicit here that the rotational motion is a consequence of each of the masses in the binary system following the geodesics generated by the other mass. And also that the sense and angular frequency of a passive rotational transformation of the coordinate system should be such as to appear to induce that motion in the two masses.

In the particular case that the two masses are equal, $m_{\mathrm{a}} = m_{\mathrm{b}} = m = |m|$ say, the $1/r$ contributions to the numerator in g_{00} cancel, with the result that the leading term falls off more rapidly than the leading term in the denominator. Eq. (26) then becomes

$$\mathrm{d}s^2 \to \left((\mathbf{x}\cdot(\mathbf{a}-\mathbf{b}))^2\,\mathrm{d}t^2 - m^2\mathrm{d}\mathbf{x}^2\right)/r^4\,. \tag{40}$$

Inversion of the radius but with $r \to r' = m/r$ then gives

$$\mathrm{d}s^2 = \left(\hat{\mathbf{x}}'\cdot\boldsymbol{\mu}\right)^2\mathrm{d}t^2 - \mathrm{d}\mathbf{x}'^2 = \mu^2\cos^2\theta\,\mathrm{d}t^2 - \mathrm{d}\mathbf{x}'^2 \tag{41}$$

where $\boldsymbol{\mu} = m(\mathbf{a}-\mathbf{b})$ is a mass dipole, $\hat{\mathbf{x}}' \equiv \mathbf{x}'/r'$, and θ is the azimuth angle when the dipole is oriented along the z' direction. It is easily verified that the Einstein tensor for the metric implied by the line-element (41) vanishes, confirming that (41) is a vacuum line-element of a mass dipole. Clearly there is no Minkowski limit in this particular case, and so a (singular) exception to the claim of asymptotic flatness in Section 2.2 that otherwise pertains at large distances.

3 Summary

We have shown there exists a two-body generalization of the isotropic form of the Schwarzschild metric that is also a vacuum solution of Einstein's equations. Under stated conditions we show this two-body metric is the most general static isotropic vacuum solution. Investigation of the conditions under which the two-body metric reduces to the standard isotropic vacuum metric appears to require a novel re-interpretation of the latter in which its Minkowski background is of Cosmological origin.

Conflict of interest

The authors declare that they have no conflict of interest.

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