ON THE SOLVABILITY OF SOME SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH TRANSPORT AND CONCENTRATED SOURCES

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Abstract: The article is devoted to the existence of solutions of a system of integro-differential equations involving the drift terms in the case of the normal diffusion and the influx/efflux terms proportional to the Dirac delta function. The proof of the existence of solutions is based on a fixed point technique. We use the solvability conditions for the non- Fredholm elliptic operators in unbounded domains. We emphasize that the study of the systems is more difficult than of the scalar case and requires to overcome more cumbersome technicalities.

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1. Introduction

In the present work we deal with the existence of stationary solutions of the following system of nonlocal reaction-diffusion equations

$$\frac{\partial u_m}{\partial t} = D_m \frac{\partial^2 u_m}{\partial x^2} + b_m \frac{\partial u_m}{\partial x} + \int_{-\infty}^{\infty} K_m(x-y) g_m(w(y)u(y,t))dy + \alpha_m \delta(x), \quad (1.1)$$

where the constants $b_m, \ \alpha_m \in \mathbb{R}, \ 1 \leq m \leq N$ are nontrivial and $w(x)$ involved in system (1.1) is a cut-off function. The conditions on it will be formulated below.

The systems of this kind are relevant to the cell population dynamics. The space variable x here is correspondent to the cell genotype, $u_m(x,t)$ denote the cell density distributions for the various groups of cells as the functions of their genotype and time,

$$u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

The right side of the system of equations (1.1) describes the evolution of the cell densities by means of the cell proliferation, mutations and cell influx or efflux. The diffusion terms here correspond to the change of genotype by virtue of the small random mutations, and the nonlocal production terms describe the large mutations. The functions $g_m(w(x)u(x,t))$ stand for the rates of cell birth depending on u, w (density dependent proliferation). The kernels $K_m(x-y)$ express the proportions of newly born cells, which change their genotypes from y to x. We assume that they depend on the distance between the genotypes. The last term in the right side of each equation of our system, which is proportional to the Dirac delta function designates the influx/efflux of cells for different genotypes. The solvability of the single integro-differential equation analogous to (1.1) but without the transport term was covered in [35]. The similar equation in one dimension in the case of the standard negative Laplace operator raised to the power $0 < s < \dfrac{1}{4}$ in the diffusion term involving the drift term was treated in [42]. But in [42] there was an assumption that the influx/efflux term $f(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Therefore, in the present article we deal with the more singular case. In neuroscience, the integro-differential problems are used to describe the nonlocal interaction of neurons (see [9] and the references therein).

Let us set all $D_m = 1$ and demonstrate the existence of solutions of the system of equations

$$\frac{d^2u_m}{dx^2} + b_m\frac{du_m}{dx} + \int_{-\infty}^{\infty} K_m(x-y)g_m(w(y)u(y))dy + \alpha_m\delta(x) = 0, \tag{1.2}$$

where $1 \le m \le N$. We consider the situation when the linear parts of the operators involved in system (1.2) fail to satisfy the Fredholm property. Consequently, the conventional methods of the nonlinear analysis may not be applicable. Let us use the solvability conditions for the non-Fredholm operators along with the method of the contraction mappings.

Consider the equation

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ either vanishes in the whole space or converges to 0 at infinity. If $a \ge 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of problem (1.3) contains the origin. Consequently, such operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. In the present work we deal with the studies of the certain properties of the operators of this kind. Note that the elliptic problems containing the non-Fredholm operators were considered actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [4], [5], [6], [7], [8]. The Schrödinger type operators without the Fredholm property were studied with the methods of the spectral and the scattering theory in [17], [31], [36], [37]. Fredholm structures, topological invariants and their applications were covered in [13]. The article [14] is devoted to the finite and infinite dimensional attractors for the evolution problems of mathematical physics. The large time behavior of the solutions of a class of fourth-order parabolic equations defined on unbounded domains using the Kolmogorov ε-entropy as a measure was considered in [15]. The attractor for a nonlinear reaction-diffusion system in an unbounded domain in the space of three dimensions was discussed in [23]. The works [25] and [30] are dedicated to the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of second order and of the operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in the second-order quasilinear elliptic systems of equations were discussed in [26]. The Laplace operator with drift from the point of view of the non-Fredholm operators was considered in [39] and the linearized Cahn-Hilliard problems in [33] and [40]. Nonlinear non-Fredholm elliptic equations were considered in [16], [18], [19], [20], [21], [22], [32], [38], [41], [42]. The important applications to the theory of reaction-diffusion problems were developed in [11], [12]. The operators without the Fredholm property arise when studying wave systems with an infinite number of localized traveling waves as well (see [2]). The work [3] deals with the standing lattice solitons in the discrete NLS equation with saturation. In particular, when $a = 0$ our operator A is Fredholm in some properly chosen weighted spaces (see [4], [5], [6], [7], [8]). However, the case of $a \neq 0$ is considerably different and the approach developed in these articles cannot be applied. The existence, stability and bifurcations of the solutions of the nonlinear partial differential equations involving the Dirac delta function type potentials were studied extensively in [1], [24], [27], [28].

We set $K_m(x) = \varepsilon_m \mathcal{K}_m(x)$, where $\varepsilon_m \geq 0$ and introduce

$$\varepsilon := \max_{1 \leq m \leq N} \varepsilon_m. \tag{1.4}$$

Suppose all the nonnegative parameters ε_m vanish. Then we obtain the linear Poisson type equations

$$-\frac{d^2 u_m}{dx^2} - b_m \frac{du_m}{dx} = \alpha_m \delta(x), \quad 1 \leq m \leq N \tag{1.5}$$

with the constants $b_m,\ \alpha_m \in \mathbb{R}$ and $b_m,\ \alpha_m \neq 0$. It can be easily verified that each problem (1.5) has a continuous solution, which is trivial on the negative semi-axis, namely

$$u_{0,m}(x) := \begin{cases} \frac{\alpha_m}{b_m}(e^{-b_m x} - 1), & x \geq 0 \\ 0, & x < 0, \end{cases} \tag{1.6}$$

where $1 \le m \le N$, such that

$$u_0(x) := (u_{0,1}(x), u_{0,2}(x), ..., u_{0,N}(x))^T. \tag{1.7}$$

Evidently, each $u_{0,m}(x)$ is not contained in $H^1(\mathbb{R})$. It is bounded if $b_m > 0$ and it is unbounded for $b_m < 0$. We recall the similar case discussed in [35]. The solution of the corresponding Poisson equation without the transport term used there was proportional to the ramp function. It was not bounded and it was not contained in $H^1(\mathbb{R})$. In the article [42] the authors were dealing with the Poisson type equation with the fractional Laplacian and the drift term. Its bounded solution belonged to $H^1(\mathbb{R})$. Let us suppose that the following conditions are satisfied.

Assumption 1.1. *Let* $1 \le m \le N$, $\mathcal{K}_m(x) : \mathbb{R} \to \mathbb{R}$ *are nontrivial, such that* $\mathcal{K}_m(x),\ x\mathcal{K}_m(x) \in L^1(\mathbb{R})$ *and orthogonality conditions (4.2) hold. Suppose also that the cut-off function* $w(x) : \mathbb{R} \to \mathbb{R}$ *is such that* $w(x)u_{0,m}(x)$ *do not vanish identically on the real line and* $w(x)u_{0,m}(x) \in H^1(\mathbb{R})$. *Moreover,* $w(x) \in H^1(\mathbb{R})$ *and for* $b_m,\ \alpha_m \in \mathbb{R},\ b_m,\ \alpha_m \neq 0$ *the estimate from above*

$$\|w(x)u_0(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le 1 \tag{1.8}$$

is valid.

It can be easily verified that $w(x) = e^{-2|b||x|},\ x \in \mathbb{R}$, where $|b| = \max_{1\le m\le N}|b_m|$ satisfies the conditions above, so that it can be used as our cut-off function. Note that in the argument of [42] such cut-off was not needed due to the more regular behaviour of the solution of the corresponding Poisson type equation. In the article we choose the space dimension $d = 1$, which is related to the solvability of the linear Poisson type equations (1.5) considered above. From the point of view of the applications, the space dimension is not limited to $d = 1$, since the space variable is correspondent to the cell genotype but not to the usual physical space. Let us use the Sobolev space

$$H^1(\mathbb{R}) := \Big\{\phi(x) : \mathbb{R} \to \mathbb{R} \mid \phi(x) \in L^2(\mathbb{R}),\ \frac{d\phi}{dx} \in L^2(\mathbb{R})\Big\}.$$

It is equipped with the norm

$$\|\phi\|^2_{H^1(\mathbb{R})} := \|\phi\|^2_{L^2(\mathbb{R})} + \Big\|\frac{d\phi}{dx}\Big\|^2_{L^2(\mathbb{R})}. \tag{1.9}$$

Evidently, by means of the standard Fourier transform (2.1), this norm can be written as

$$\|\phi\|^2_{H^1(\mathbb{R})} = \|\widehat{\phi}(p)\|^2_{L^2(\mathbb{R})} + \|p\widehat{\phi}(p)\|^2_{L^2(\mathbb{R})}. \tag{1.10}$$

For a vector function

$$u(x) = (u_1(x), u_2(x), ..., u_N(x))^T,$$

we will use the norm

$$\|u\|^2_{H^1(\mathbb{R},\mathbb{R}^N)} := \|u\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} + \sum_{m=1}^{N} \left\| \frac{du_m}{dx} \right\|^2_{L^2(\mathbb{R})}, \tag{1.11}$$

where

$$\|u\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} := \sum_{m=1}^{N} \|u_m\|^2_{L^2(\mathbb{R})}.$$

The Sobolev inequality in one dimension (see e.g. Sect 8.5 of [29]) yields

$$\|\phi(x)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2}} \|\phi(x)\|_{H^1(\mathbb{R})}. \tag{1.12}$$

Let us seek the resulting solution of the nonlinear system of equations (1.2) as

$$u(x) = u_0(x) + u_p(x), \tag{1.13}$$

where

$$u_p(x) := (u_{p,1}(x), u_{p,2}(x), ..., u_{p,N}(x))^T$$

and $u_0(x)$ is defined in (1.7). Clearly, we obtain the perturbative system

$$-\frac{d^2 u_{p,m}(x)}{dx^2} - b_m \frac{du_{p,m}(x)}{dx} =$$

$$= \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(w(y)[u_0(y) + u_p(y)]) dy, \tag{1.14}$$

where $1 \le m \le N$. Let us use a closed ball in the Sobolev space

$$B_\rho := \{u(x) \in H^1(\mathbb{R}, \mathbb{R}^N) \mid \|u\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le \rho\}, \quad 0 < \rho \le 1. \tag{1.15}$$

We look for the solution of the system of equations (1.14) as the fixed point of the auxiliary nonlinear system of equations

$$-\frac{d^2 u_m(x)}{dx^2} - b_m \frac{du_m(x)}{dx} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(w(y)[u_0(y) + v(y)]) dy, \tag{1.16}$$

with $1 \le m \le N$ in ball (1.15). For a given vector function $v(y)$ this is a system of equations with respect to $u(x)$. The equations of (1.16) in their left sides contain the operators

$$L_{b_m} = -\frac{d^2}{dx^2} - b_m \frac{d}{dx}, \quad 1 \le m \le N, \tag{1.17}$$

which act on $L^2(\mathbb{R})$. By virtue of the standard Fourier transform, it can be easily derived that the essential spectrum of L_{b_m} is

$$\lambda_{b_m}(p) = p^2 - i b_m p, \quad p \in \mathbb{R}, \quad 1 \le m \le N. \tag{1.18}$$

Clearly, each (1.18) contains the origin, so that L_{b_m} fails to satisfy the Fredholm property. This operator has no bounded inverse. The similar situation in the context of the integro-differential equations occurred in articles [38] and [41] as well. The equations studied there also required the application of the orthogonality conditions. The contraction argument was used in [34] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed. But the Schrödinger operator involved in the nonlinear problem there satisfied the Fredholm property (see Assumption 1 of [34], also [10]). Let us introduce the closed ball in the space of N dimensions as

$$I := \left\{ z \in \mathbb{R}^N \mid |z|_{\mathbb{R}^N} \le \frac{1}{\sqrt{2}} + \frac{1}{2}\|w(x)\|_{H^1(\mathbb{R})} \right\}, \tag{1.19}$$

along with the closed ball in the space of $C^1(I, \mathbb{R}^N)$ vector functions, namely $D_M :=$

$$\{g(z) := (g_1(z), g_2(z), ..., g_N(z)) \in C^1(I, \mathbb{R}^N) \mid \|g\|_{C^1(I,\mathbb{R}^N)} \le M\}, \tag{1.20}$$

where $M > 0$. In this context the norms

$$\|g\|_{C^1(I,\mathbb{R}^N)} := \sum_{m=1}^{N} \|g_m\|_{C^1(I)}, \tag{1.21}$$

$$\|g_m\|_{C^1(I)} := \|g_m\|_{C(I)} + \sum_{n=1}^{N} \left\| \frac{\partial g_m}{\partial z_n} \right\|_{C(I)}, \tag{1.22}$$

where $\|g_m\|_{C(I)} := \max_{z\in I}|g_m(z)|$. From the point of view of the biological applications, the rates of the cell birth functions are nonlinear and are trivial at the origin.

Assumption 1.2. *Let $1 \le m \le N$. We suppose that $g_m(z) : \mathbb{R}^N \to \mathbb{R}$, such that $g_m(0) = 0$. Let us also assume that $g(z) \in D_M$ and it does not vanish identically in the ball I.*

We recall the earlier work [42]. The function $g(z)$ there was assumed to be twice continuously differentiable on the corresponding interval I. Let us use the following positive technical quantities

$$Q_m := \max\left\{ \left\| \frac{\widehat{\mathcal{K}_m}(p)}{p^2 - ib_m p} \right\|_{L^\infty(\mathbb{R})}, \quad \left\| \frac{\widehat{\mathcal{K}_m}(p)}{p - ib_m} \right\|_{L^\infty(\mathbb{R})} \right\}, \quad 1 \le m \le N \tag{1.23}$$

with the constants $b_m \in \mathbb{R},\ b_m \neq 0$, so that

$$Q := \max_{1\le m\le N} Q_m. \tag{1.24}$$

Evidently, Q is finite under the assumptions of Theorem 1.3 by virtue of the result of Lemma 4.1 below.

Let us introduce the operator T_g, such that $u = T_g v$, where u is a solution of the system of equations (1.16). Our first main proposition is as follows.

Theorem 1.3. *Let Assumptions 1.1 and 1.2 hold. Then for every $\rho \in (0, 1]$ system (1.16) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all*

$$0 < \varepsilon \leq \frac{\rho}{2\sqrt{\pi}QM(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})})}. \tag{1.25}$$

The unique fixed point $u_p(x)$ of this map T_g is the only solution of the system of equations (1.14) in B_ρ.

Obviously, the cumulative solution of system (1.2) given by formula (1.13) will be nontrivial on the real line since $g_m(0) = 0,\ \alpha_m \neq 0,\ 1 \leq m \leq N$ as assumed.

Our second main statement is about the continuity of the resulting solution of the system of equations (1.2) given by (1.13) with respect to the nonlinear vector function g. Let us define the following positive technical quantity

$$\sigma := \sqrt{2\pi}QM\|w(x)\|_{H^1(\mathbb{R})}. \tag{1.26}$$

Theorem 1.4. *Let $j = 1, 2$ and suppose that the conditions of Theorem 1.3 hold, such that $u_{p,j}(x)$ is the unique fixed point of the map $T_{g_j} : B_\rho \to B_\rho$, which is a strict contraction for all the values of ε, which satisfy (1.25) and the cumulative solution of system (1.2) with $g(z) = g_j(z)$ is given by*

$$u_j(x) = u_0(x) + u_{p,j}(x). \tag{1.27}$$

Then for all ε satisfying bound (1.25), the estimate from above

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq$$

$$\leq \frac{2\sqrt{\pi}\varepsilon Q(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})})}{1 - \varepsilon\sigma}\|g_1(z) - g_2(z)\|_{C^1(I,\mathbb{R}^N)} \tag{1.28}$$

is valid.

We proceed to the proof of the first main result.

2. The existence of the perturbed solution

Proof of Theorem 1.3. Let us choose arbitrarily $v(x) \in B_\rho$. The terms involved in the integral expressions in the right side of system (1.16) are designated as

$$G_m(x) := g_m(w(x)[u_0(x) + v(x)]), \quad 1 \leq m \leq N.$$

We use the standard Fourier transform

$$\widehat{\phi}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{-ipx} dx, \quad p \in \mathbb{R}, \tag{2.1}$$

such that the upper bound

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}} \|\phi(x)\|_{L^1(\mathbb{R})} \tag{2.2}$$

holds. Let us apply (2.1) to both sides of the system of equations (1.16). This yields

$$\widehat{u_m}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)\widehat{G_m}(p)}{p^2 - ib_m p}, \quad p\widehat{u_m}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)\widehat{G_m}(p)}{p - ib_m},$$

where $1 \le m \le N$. Clearly,

$$|\widehat{u_m}(p)| \le \varepsilon \sqrt{2\pi} Q |\widehat{G_m}(p)|, \quad |p\widehat{u_m}(p)| \le \varepsilon \sqrt{2\pi} Q |\widehat{G_m}(p)|, \quad 1 \le m \le N. \tag{2.3}$$

Here Q is defined in (1.24). It is finite by means of Lemma 4.1 under the given conditions. By virtue of (1.10) and (1.11) along with inequalities (2.3), we easily derive the estimate from above for the norm as

$$\|u(x)\|^2_{H^1(\mathbb{R},\mathbb{R}^N)} \le 4\pi\varepsilon^2 Q^2 \sum_{m=1}^{N} \|G_m(x)\|^2_{L^2(\mathbb{R})}. \tag{2.4}$$

It can be trivially checked that for $v(x) \in B_\rho$, we have

$$|w(x)[u_0(x) + v(x)]|_{\mathbb{R}^N} \le \frac{1}{\sqrt{2}} + \frac{1}{2}\|w(x)\|_{H^1(\mathbb{R})}. \tag{2.5}$$

Obviously, the left side of (2.5) can be easily bounded from above using the triangle inequality by

$$|w(x)u_0(x)|_{\mathbb{R}^N} + |w(x)v(x)|_{\mathbb{R}^N}. \tag{2.6}$$

Let us recall inequalities (1.12) and (1.8) to deal with the first term in (2.6). Thus,

$$|w(x)u_0(x)|_{\mathbb{R}^N} \le \sqrt{\sum_{m=1}^{N} \|w(x)u_{0,m}(x)\|^2_{L^\infty(\mathbb{R})}} \le$$

$$\le \frac{1}{\sqrt{2}} \|w(x)u_0(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le \frac{1}{\sqrt{2}}.$$

We use (1.12) to obtain the upper bound on the second term in (2.6), so that

$$\sqrt{\sum_{m=1}^{N} |w(x)v_m(x)|^2} \le \sqrt{\sum_{m=1}^{N} \|w(x)\|^2_{L^\infty(\mathbb{R})} \|v_m(x)\|^2_{L^\infty(\mathbb{R})}} \le$$

$$\leq \frac{1}{2}\|w(x)\|_{H^1(\mathbb{R})}\sqrt{\sum_{m=1}^{N}\|v_m(x)\|^2_{H^1(\mathbb{R})}} = \frac{1}{2}\|w(x)\|_{H^1(\mathbb{R})}\|v(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq$$

$$\leq \frac{1}{2}\|w(x)\|_{H^1(\mathbb{R})}.$$

Therefore, (2.5) holds. Similarly, for $v(x) \in B_\rho$

$$\|w(x)[u_0(x) + v(x)]\|_{L^2(\mathbb{R},\mathbb{R}^N)} \leq 1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}. \tag{2.7}$$

Evidently, the left side of (2.7) can be trivially bounded from above by means of the triangle inequality by

$$\|w(x)u_0(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)} + \|w(x)v(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)}. \tag{2.8}$$

For the first term in (2.8), we have by virtue of (1.8) along with (1.11) that

$$\|w(x)u_0(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)} \leq \|w(x)u_0(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq 1.$$

For the second term in (2.8), we use (1.12), which yields

$$\|w(x)v(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)} \leq \|w(x)\|_{L^\infty(\mathbb{R})}\sqrt{\sum_{m=1}^{N}\|v_m(x)\|^2_{L^2(\mathbb{R})}} =$$

$$= \|w(x)\|_{L^\infty(\mathbb{R})}\|v(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)} \leq \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\|v(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq$$

$$\leq \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}.$$

Thus, (2.7) is valid. By means of Assumption 1.2, we can write

$$G_m(x) = \int_0^1 \nabla g_m(tw(x)[u_0(x) + v(x)]).w(x)[u_0(x) + v(x)]dt, \quad 1 \leq m \leq N.$$

Here and further down the dot denotes the scalar product of the two vectors in our space of N dimensions. Clearly, by virtue of (2.5)

$$|G_m(x)| \leq \sup_{z\in I}|\nabla g_m(z)|_{\mathbb{R}^N}|w(x)[u_0(x) + v(x)]|_{\mathbb{R}^N}, \quad 1 \leq m \leq N,$$

where the ball I is defined in (1.19). Let us use inequality (2.7). We arrive at

$$\|G_m(x)\|^2_{L^2(\mathbb{R})} \leq [\sup_{z\in I}|\nabla g_m(z)|_{\mathbb{R}^N}]^2\|w(x)[u_0(x) + v(x)]\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \leq$$

$$\leq [\sup_{z\in I}|\nabla g_m(z)|_{\mathbb{R}^N}]^2\Big(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big)^2, \quad 1 \leq m \leq N. \tag{2.9}$$

Estimates (2.4) and (2.9) yield

$$\|u(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le 2\sqrt{\pi}\varepsilon QM\Big(1+\frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big) \le \rho \tag{2.10}$$

for all the values of our parameter ε, satisfying condition (1.25), so that $u(x) \in B_\rho$ as well.

Let us suppose that for some $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of system of (1.16). Obviously, their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$ solves the homogeneous system of equations

$$-\frac{d^2 w_m(x)}{dx^2} - b_m \frac{dw_m(x)}{dx} = 0, \quad 1 \le m \le N.$$

But each operator L_{b_m} (see (1.17)) considered on $L^2(\mathbb{R})$ does not have any nontrivial zero modes. Thus, $w(x)$ vanishes identically on the real line. Therefore, system (1.16) defines a map $T_g : B_\rho \to B_\rho$ for all the values of ε satisfying inequality (1.25).

We establish that under the given conditions such map is a strict contraction. Let us choose arbitrarily $v_{1,2}(x) \in B_\rho$. The argument above gives us that $u_{1,2} := T_g v_{1,2} \in B_\rho$ as well for ε, which satisfies (1.25). By means of (1.16), we have for $1 \le m \le N$

$$-\frac{d^2 u_{1,m}(x)}{dx^2} - b_m \frac{du_{1,m}(x)}{dx} =$$

$$= \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(w(y)[u_0(y) + v_1(y)])dy, \tag{2.11}$$

$$-\frac{d^2 u_{2,m}(x)}{dx^2} - b_m \frac{du_{2,m}(x)}{dx} =$$

$$= \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(w(y)[u_0(y) + v_2(y)])dy. \tag{2.12}$$

Let us define

$$G_{1,m}(x) := g_m(w(x)[u_0(x) + v_1(x)]), \quad G_{2,m}(x) := g_m(w(x)[u_0(x) + v_2(x)]),$$

where $1 \le m \le N$. We apply the standard Fourier transform (2.1) to both sides of systems (2.11) and (2.12) and obtain

$$\widehat{u_{1,m}}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)\widehat{G_{1,m}}(p)}{p^2 - ib_m p}, \quad \widehat{u_{2,m}}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)\widehat{G_{2,m}}(p)}{p^2 - ib_m p},$$

such that

$$\widehat{u_{1,m}}(p) - \widehat{u_{2,m}}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)[\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)]}{p^2 - ib_m p},$$

$$p[\widehat{u_{1,m}}(p) - \widehat{u_{2,m}}(p)] = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)[\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)]}{p - ib_m}, \quad 1 \leq m \leq N.$$

Therefore, the upper bounds

$$|\widehat{u_{1,m}}(p) - \widehat{u_{2,m}}(p)| \leq \varepsilon \sqrt{2\pi} Q |\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)|,$$

$$|p[\widehat{u_{1,m}}(p) - \widehat{u_{2,m}}(p)]| \leq \varepsilon \sqrt{2\pi} Q |\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)|, \quad 1 \leq m \leq N$$

hold. This allows us to derive the estimate from above on the norm by means of (1.10) and (1.11), namely $\|u_1(x) - u_2(x)\|^2_{H^1(\mathbb{R}, \mathbb{R}^N)} =$

$$= \sum_{m=1}^{N} \Big\{ \int_{-\infty}^{\infty} |\widehat{u_{1,m}}(p) - \widehat{u_{2,m}}(p)|^2 dp + \int_{-\infty}^{\infty} |p(\widehat{u_{1,m}}(p) - \widehat{u_{2,m}}(p))|^2 dp \Big\} \leq$$

$$\leq 4\pi \varepsilon^2 Q^2 \sum_{m=1}^{N} \|G_{1,m}(x) - G_{2,m}(x)\|^2_{L^2(\mathbb{R})}.$$

Hence,

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R}, \mathbb{R}^N)} \leq 2\sqrt{\pi} \varepsilon Q \sqrt{\sum_{m=1}^{N} \|G_{1,m}(x) - G_{2,m}(x)\|^2_{L^2(\mathbb{R})}}. \tag{2.13}$$

Evidently, we can express $G_{1,m}(x) - G_{2,m}(x) =$

$$= \int_0^1 \nabla g_m(w(x)[u_0(x) + tv_1(x) + (1-t)v_2(x)]).w(x)[v_1(x) - v_2(x)]dt,$$

with $1 \leq m \leq N$. Clearly, for $t \in [0, 1]$ we have

$$\|tv_1(x) + (1-t)v_2(x)\|_{H^1(\mathbb{R}, \mathbb{R}^N)} \leq t\|v_1(x)\|_{H^1(\mathbb{R}, \mathbb{R}^N)} + (1-t)\|v_2(x)\|_{H^1(\mathbb{R}, \mathbb{R}^N)} \leq$$

$$\leq \rho.$$

Thus, $tv_1(x) + (1-t)v_2(x) \in B_\rho$. According to inequality (2.5),

$$|w(x)[u_0(x) + tv_1(x) + (1-t)v_2(x)]|_{\mathbb{R}^N} \leq \frac{1}{\sqrt{2}} + \frac{1}{2}\|w(x)\|_{H^1(\mathbb{R})}.$$

Then we estimate

$$|G_{1,m}(x) - G_{2,m}(x)| \leq \sup_{z \in I} |\nabla g_m(z)|_{\mathbb{R}^N} |w(x)(v_1(x) - v_2(x))|_{\mathbb{R}^N} \leq$$

$$\leq \|g_m\|_{C^1(I)} |w(x)(v_1(x) - v_2(x))|_{\mathbb{R}^N}, \quad 1 \leq m \leq N,$$

with the ball I introduced in (1.19). Let us obtain the upper bound for the norm via (1.12) as

$$\|G_{1,m}(x)-G_{2,m}(x)\|^2_{L^2(\mathbb{R})}\leq \frac{1}{2}\|g_m\|^2_{C^1(I)}\|w(x)\|^2_{H^1(\mathbb{R}}\|v_1(x)-v_2(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)}\leq$$

$$\leq \frac{1}{2}\|g_m\|^2_{C^1(I)}\|w(x)\|^2_{H^1(\mathbb{R})}\|v_1(x)-v_2(x)\|^2_{H^1(\mathbb{R},\mathbb{R}^N)}, \quad 1\leq m\leq N. \tag{2.14}$$

By means of (2.13) and (2.14) along with Assumption 1.2, we arrive at

$$\|u_1(x)-u_2(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)}\leq$$

$$\leq \sqrt{2\pi}\varepsilon QM\|w(x)\|_{H^1(\mathbb{R})}\|v_1(x)-v_2(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)}. \tag{2.15}$$

Clearly,

$$\frac{\rho}{2\sqrt{\pi}QM\Big(1+\frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big)}<\frac{1}{\sqrt{2\pi}QM\|w(x)\|_{H^1(\mathbb{R})}}.$$

Therefore, by virtue of (1.25) for our parameter ε we have that

$$0<\varepsilon<\frac{1}{\sqrt{2\pi}QM\|w(x)\|_{H^1(\mathbb{R})}},$$

such that the constant in the right side of bound (2.15) is less than one. Hence, the map $T_g : B_\rho \to B_\rho$ defined by system (1.16) is a strict contraction for all the values of ε, which satisfy inequality (1.25). Its unique fixed point $u_p(x)$ is the only solution of the system of equations (1.14) in the ball B_ρ. Using (2.10), we obtain that $\|u_p(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)}\to 0$ as $\varepsilon\to 0$. The cumulative $u(x)$ given by formula (1.13) is a solution of system (1.2). ∎

Let us turn our attention to establishing the validity of the second main proposition of our article.

3. The continuity of the resulting solution

Proof of Theorem 1.4. Evidently, for all the values of our parameter ε satisfying (1.25), we have

$$u_{p,1}=T_{g_1}u_{p,1}, \quad u_{p,2}=T_{g_2}u_{p,2},$$

so that

$$u_{p,1}-u_{p,2}=T_{g_1}u_{p,1}-T_{g_1}u_{p,2}+T_{g_1}u_{p,2}-T_{g_2}u_{p,2}.$$

Thus,

$$\|u_{p,1}-u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}\leq \|T_{g_1}u_{p,1}-T_{g_1}u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}+\|T_{g_1}u_{p,2}-T_{g_2}u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}.$$

By virtue of bound (2.15), we obtain

$$\|T_{g_1}u_{p,1} - T_{g_1}u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq \varepsilon\sigma\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)},$$

with σ introduced in (1.26). Obviously, $\varepsilon\sigma < 1$, because the map $T_{g_1} : B_\rho \to B_\rho$ is a strict contraction under the given conditions. Hence,

$$(1 - \varepsilon\sigma)\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq \|T_{g_1}u_{p,2} - T_{g_2}u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}. \tag{3.1}$$

Clearly, for the fixed point $T_{g_2}u_{p,2} = u_{p,2}$. Let us define $\eta(x) := T_{g_1}u_{p,2}$. Therefore, for $1 \leq m \leq N$, we have

$$-\frac{d^2\eta_m(x)}{dx^2} - b_m\frac{d\eta_m(x)}{dx} =$$

$$= \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y)g_{1,m}(w(y)[u_0(y) + u_{p,2}(y)])dy, \tag{3.2}$$

$$-\frac{d^2u_{p,2,m}(x)}{dx^2} - b_m\frac{du_{p,2,m}(x)}{dx} =$$

$$= \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y)g_{2,m}(w(y)[u_0(y) + u_{p,2}(y)])dy. \tag{3.3}$$

Let us introduce

$$G_{1,2,m}(x) := g_{1,m}(w(x)[u_0(x) + u_{p,2}(x)]),$$

$$G_{2,2,m}(x) := g_{2,m}(w(x)[u_0(x) + u_{p,2}(x)]), \quad 1 \leq m \leq N.$$

We apply the standard Fourier transform (2.1) to both sides of the systems of equations (3.2) and (3.3). This yields

$$\widehat{\eta_m}(p) = \varepsilon_m\sqrt{2\pi}\frac{\widehat{\mathcal{K}_m}(p)\widehat{G_{1,2,m}}(p)}{p^2 - ib_mp}, \quad \widehat{u_{p,2,m}}(p) = \varepsilon_m\sqrt{2\pi}\frac{\widehat{\mathcal{K}_m}(p)\widehat{G_{2,2,m}}(p)}{p^2 - ib_mp},$$

where $1 \leq m \leq N$, such that

$$\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p) = \varepsilon_m\sqrt{2\pi}\frac{\widehat{\mathcal{K}_m}(p)}{p^2 - ib_mp}[\widehat{G_{1,2,m}}(p) - \widehat{G_{2,2,m}}(p)],$$

$$p[\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)] = \varepsilon_m\sqrt{2\pi}\frac{\widehat{\mathcal{K}_m}(p)}{p - ib_m}[\widehat{G_{1,2,m}}(p) - \widehat{G_{2,2,m}}(p)].$$

Evidently, the upper bounds

$$|\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)| \leq \varepsilon\sqrt{2\pi}Q|\widehat{G_{1,2,m}}(p) - \widehat{G_{2,2,m}}(p)|, \tag{3.4}$$

$$|p[\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)]| \leq \varepsilon\sqrt{2\pi}Q|\widehat{G_{1,2,m}}(p) - \widehat{G_{2,2,m}}(p)| \tag{3.5}$$

hold with $1 \le m \le N$. By means of (3.4), we have

$$\|\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)|^2 dp \le$$

$$\le 2\pi\varepsilon^2 Q^2 \|G_{1,2,m}(x) - G_{2,2,m}(x)\|^2_{L^2(\mathbb{R})}, \quad 1 \le m \le N. \tag{3.6}$$

Similarly, by virtue of inequality (3.5) we obtain

$$\|p[\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)]\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |p[\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)]|^2 dp \le$$

$$\le 2\pi\varepsilon^2 Q^2 \|G_{1,2,m}(x) - G_{2,2,m}(x)\|^2_{L^2(\mathbb{R})}, \quad 1 \le m \le N. \tag{3.7}$$

Formulas (1.9), (1.10), (1.11), (3.6) and (3.7) give us that

$$\|\eta(x) - u_{p,2}(x)\|^2_{H^1(\mathbb{R},\mathbb{R}^N)} =$$

$$= \sum_{m=1}^{N} \{\|\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)\|^2_{L^2(\mathbb{R})} + \|p[\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)]\|^2_{L^2(\mathbb{R})}\} \le$$

$$\le 4\pi\varepsilon^2 Q^2 \sum_{m=1}^{N} \|G_{1,2,m}(x) - G_{2,2,m}(x)\|^2_{L^2(\mathbb{R})}.$$

Hence,

$$\|\eta(x) - u_{p,2}(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le 2\sqrt{\pi}\varepsilon Q \sqrt{\sum_{m=1}^{N} \|G_{1,2,m}(x) - G_{2,2,m}(x)\|^2_{L^2(\mathbb{R})}}. \tag{3.8}$$

Clearly, for $1 \le m \le N$ we can write $G_{1,2,m}(x) - G_{2,2,m}(x) =$

$$= \int_0^1 \nabla[g_{1,m} - g_{2,m}](tw(x)[u_0(x) + u_{p,2}(x)]).w(x)[u_0(x) + u_{p,2}(x)]dt.$$

Let us use inequality (2.5). Thus,

$$|tw(x)[u_0(x) + u_{p,2}(x)]|_{\mathbb{R}^N} \le \frac{1}{\sqrt{2}} + \frac{1}{2}\|w(x)\|_{H^1(\mathbb{R})}, \quad t \in [0,1].$$

We easily estimate that $|G_{1,2,m}(x) - G_{2,2,m}(x)| \le$

$$\le \sup_{z\in I} |\nabla[g_{1,m} - g_{2,m}](z)|_{\mathbb{R}^N} |w(x)[u_0(x) + u_{p,2}(x)]|_{\mathbb{R}^N} \le$$

$$\le \|g_{1,m} - g_{2,m}\|_{C^1(I)} |w(x)[u_0(x) + u_{p,2}(x)]|_{\mathbb{R}^N}, \quad 1 \le m \le N,$$

where the ball I is introduced in (1.19). This allows us to obtain the upper bound on the norm using (2.7) as

$$\|G_{1,2,m}(x) - G_{2,2,m}(x)\|^2_{L^2(\mathbb{R})} \le$$

$$\leq \|g_{1,m} - g_{2,m}\|^2_{C^1(I)} \|w(x)[u_0(x) + u_{p,2}(x)]\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \leq$$

$$\leq \|g_{1,m} - g_{2,m}\|^2_{C^1(I)} \Big(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big)^2, \quad 1 \leq m \leq N. \tag{3.9}$$

By virtue of estimates (3.8) and (3.9), we derive that $\|\eta(x) - u_{p,2}(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq$

$$\leq 2\sqrt{\pi}\varepsilon Q\Big(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big)\|g_1(z) - g_2(z)\|_{C^1(I,\mathbb{R}^N)}. \tag{3.10}$$

Inequalities (3.1) and (3.10) imply that $\|u_{p,1}(x) - u_{p,2}(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq$

$$\leq \frac{2\sqrt{\pi}\varepsilon Q\Big(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big)}{1 - \varepsilon\sigma}\|g_1(z) - g_2(z)\|_{C^1(I,\mathbb{R}^N)}. \tag{3.11}$$

By means of (1.27) along with upper bound (3.11), estimate (1.28) is valid. ∎

4. Auxiliary results

Let us recall the quantities $Q_m, \ 1 \leq m \leq N$ defined in (1.23) and obtain the conditions under which they are finite. We denote the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx, \tag{4.1}$$

with a slight abuse of notations when the functions contained in (4.1) do not belong to $L^2(\mathbb{R})$, like for example the ones involved in orthogonality conditions (4.2) of Lemma 4.1. Indeed, if $f(x) \in L^1(\mathbb{R})$ and $g(x)$ is bounded, then the integral in the right side of (4.1) makes sense. The proof of Lemma 4.1 was partially presented in the second part of the first lemma of the Appendix of [20]. We provide it here for the convenience of the readers.

Lemma 4.1. *Let $1 \leq m \leq N$, the constants $b_m \in \mathbb{R}, \ b_m \neq 0$, the functions $\mathcal{K}_m(x) : \mathbb{R} \to \mathbb{R}$ do not vanish identically on the real line, such that $\mathcal{K}_m(x), \ x\mathcal{K}_m(x) \in L^1(\mathbb{R})$. Then $Q_m < \infty$ if and only if the orthogonality conditions*

$$(\mathcal{K}_m(x), 1)_{L^2(\mathbb{R})} = 0 \tag{4.2}$$

hold.

Proof. It can be verified using inequality (2.2) that $\dfrac{\widehat{\mathcal{K}_m}(p)}{p - ib_m}$ is bounded. Indeed, we easily obtain

$$\left|\frac{\widehat{\mathcal{K}_m}(p)}{p - ib_m}\right| = \frac{|\widehat{\mathcal{K}_m}(p)|}{\sqrt{p^2 + b_m^2}} \leq \frac{1}{\sqrt{2\pi}|b_m|}\|\mathcal{K}_m(x)\|_{L^1(\mathbb{R})} < \infty$$

via our conditions. Clearly, if the drift constant b_m here is trivial, we will be dealing with a more singular situation. Let us express

$$\widehat{\mathcal{K}_m}(p) = \widehat{\mathcal{K}_m}(0) + \int_0^p \frac{d\widehat{\mathcal{K}_m}(q)}{dq}dq, \quad 1 \le m \le N,$$

such that

$$\frac{\widehat{\mathcal{K}_m}(p)}{p^2 - ib_m p} = \frac{\widehat{\mathcal{K}_m}(0)}{p^2 - ib_m p} + \frac{\int_0^p \frac{d\widehat{\mathcal{K}_m}(q)}{dq}dq}{p^2 - ib_m p}, \quad 1 \le m \le N. \tag{4.3}$$

From the definition of the standard Fourier transform (2.1) we easily deduce that

$$\left|\frac{d\widehat{\mathcal{K}_m}(p)}{dp}\right| \le \frac{1}{\sqrt{2\pi}}\|x\mathcal{K}_m(x)\|_{L^1(\mathbb{R})}, \quad 1 \le m \le N.$$

Thus,

$$\left|\frac{\int_0^p \frac{d\widehat{\mathcal{K}_m}(q)}{dq}dq}{p^2 - ib_m p}\right| \le \frac{1}{\sqrt{2\pi}|b_m|}\|x\mathcal{K}_m(x)\|_{L^1(\mathbb{R})} < \infty, \quad 1 \le m \le N$$

as assumed. By means of definition (2.1),

$$\widehat{\mathcal{K}_m}(0) = \frac{1}{\sqrt{2\pi}}(\mathcal{K}_m(x), 1)_{L^2(\mathbb{R})}, \quad 1 \le m \le N.$$

Hence, the first term in the right side of (4.3) equals to

$$\frac{(\mathcal{K}_m(x), 1)_{L^2(\mathbb{R})}}{\sqrt{2\pi}(p^2 - ib_m p)}, \quad 1 \le m \le N. \tag{4.4}$$

Evidently, each quantity (4.4) is bounded if and only if orthogonality conditions (4.2) hold. ■

Let us note that as distinct from the similar proposition without a drift term given in [35], the statement of Lemma 4.1 above uses only a single orthogonality relation (4.2) for each $1 \le m \le N$ and the argument of the proof is simpler. The argument of [42] does not rely on the orthogonality conditions at all.

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