Black hole spectra from Vaz's quantum gravitational collapse

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Abstract

In 2014, in a famous paper Hawking strongly criticized the firewall paradox by claiming that it violates the equivalence principle and breaks the CPT invariance of quantum gravity. He proposed that the final result of the gravitational collapse should not be an event horizon, but an apparent horizon instead. On the other hand, Hawking did not give a mechanism for how this could work. In the same year, Vaz endorsed Hawking's proposal in a quantum gravitational model of dust collapse by winning the Second Prize in the 2014 Gravity Research Foundation Essay Competition. He indeed showed that continued collapse to a singularity can only be obtained if one combines two independent and entire solutions of the Wheeler-DeWitt equation. Vaz's interpretation of the paradox was in terms of simply forbidding such a combination. This leads naturally to matter condensing on the apparent horizon during quantum collapse. In that way, an entirely new framework for black holes (BHs) has emerged.

In this letter we derive the BH mass and energy spectra via a Schrodinger-like approach permitting, by further supporting Vaz's conclusions that instead of a spacetime singularity covered by an event horizon, the final result of the gravitational collapse is an essentially quantum object, an extremely compact "dark star". This "gravitational atom" is held up not by any degeneracy pressure but by quantum gravity in the same way that ordinary atoms are sustained by quantum mechanics. Finally, by evoking the generalized uncertainty principle, the maximum value of the density of Vaz's shell will be estimated.

Via a Gedankenexperiment, Almheiri et. al. (AMPS) [1] claimed that the three fundamental assumptions underlying BH Complementarity, that are [2, 3]

1. Hawking radiation is pure;
2. Effective field theory is valid outside a stretched horizon;
3. infalling observers encounter nothing unusual as they cross the horizon;

cannot be simultaneously consistent. In particular, AMPS proposed that assumption 3. should be false and the infalling observer burns up at the horizon (the horizon should act as a "firewall"). Hawking strongly criticized the MPS paradox by claiming that it violates the equivalence principle and breaks the CPT invariance of quantum gravity [4]. He proposed that the final result of the gravitational collapse should not be an event horizon, but an apparent horizon instead [4], but without giving a physical mechanism for how this could work. Vaz endorsed Hawking's proposal in a quantum gravitational model of dust collapse by winning the Second Prize in the 2014 Gravity Research Foundation Essay Competition [3]. In Vaz's approach, continued collapse to a singularity can only be obtained if one combines two independent and entire solutions of the Wheeler-DeWitt equation. Vaz's interpretation of the paradox was in terms of simply forbidding such a combination [3]. The natural result was matter condensing on the apparent horizon during quantum collapse. In that way, an entirely new BH framework has emerged [3].

In this letter the BH mass and energy spectra via a Schrodinger-like approach will be obtained. This further supports Vaz's conclusions that instead of a spacetime singularity covered by an event horizon, the final result of the gravitational collapse is an essentially quantum object, an extremely compact "dark star". This "gravitational atom" is held up not by any degeneracy pressure but by quantum gravity in the same way that ordinary atoms are sustained by quantum mechanics.

Let us start to recall Vaz's result. Vaz realized a quantum approach to the spherical collapse of inhomogeneous dust in AdS of dimension $d = n + 2$ [3], which is described by the LeMaitre-Tolman-Bondi family of metrics [5–7]. Via Dirac quantization of the constraints leading to a Wheeler-DeWitt equation, Vaz found two independent solutions in terms of shell wave functions supported everywhere in spacetime [3] (hereafter Planck units will be used, i.e. $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$)

$$\psi_i = \psi_i^{(1)} + A_i \psi_i^{(2)}, \tag{1}$$

where $\psi_i^{(1)}$ represents dust shells condensing to the apparent horizon on both sides of it and $\psi_i^{(2)}$ represents dust shells move away from the apparent horizon on either side of it where the exterior, outgoing wave is suppressed by the Boltzmann factor at the Hawking temperature for the shell, see [3] for details. One has to stress that there is nothing within the theory that suggests a value for A_i [3]. In addition, further input should be needed to determine these amplitudes

[3]. If $0 < |A_i| \leq 1$, then the dust will ultimately pass through the horizon via a continued collapse arriving at a central singularity [3]. Consequently, an event horizon will form, with emission of thermal radiation in the exterior [3]. In order to avoid the AMPS firewall paradox, $|A_i|$ must vanish [3]. Then, $\psi_i^{(1)}$ alone results to be the complete description of the quantum collapse [3]. But the meaning of $\psi_i^{(1)}$ is that each shell will condense to the apparent horizon, by stopping the gravitational collapse [3]. Thus, there should be no tunneling into the exterior and no AMPS firewall paradox [3]. Each shell converges to the apparent horizon and a "dark star" forms [3].

Now, one can find the BH mass and energy spectra of this "gravitational atom" via a Schrodinger-like approach. One starts to observe that, if both the shells described by $\psi_i^{(1)}$converge to the apparent horizon by forming a "dark star", by assuming absence of rotations and of dissipation during the collapse, such a final object will be a spherical symmetric shell. In that case, a "dark star" having mass M will be subjected to the classical potential

$$V = -\frac{M^2}{2R}, \tag{2}$$

which is indeed the self-interaction gravitational potential of a spherical massive shell, where R is its radius [8]. In the current case, R is nothing else than the gravitational radius, which, in a quantum framework, is subjected to quantum fluctuations, due also to the potential absorption of external particles [10]. On the other hand, Eq. (2) represents also the potential of a two-particle system composed of two identical masses M gravitationally interacting with a relative position $2R$. Thus, the spherical shell is physically equivalent to a two-particle system of two identical masses, but, clearly, as the BH mass M does not double, one has to consider the two identical masses M as being fictitious and representing the real physical shell. Let us recall the general problem of a two-particle system where the particles have different masses [11]. This is a 6-dimensional problem which can be splitted into two 3-dimensional problems, that of a static or free particle, and that of a particle in a static potential if the sole interaction which is felt by the particles is their mutual interaction depending only on their relative position [11]. One denotes by m_1, m_2 the masses of the particles, by d_1, d_2 their positions and by $\overrightarrow{p}_1$, $\overrightarrow{p}_2$ the respective momenta. Being $\overrightarrow{d}_1 - \overrightarrow{d}_2$ their relative position, the Hamiltonian of the system reads [11]

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(\overrightarrow{d}_1 - \overrightarrow{d}_2). \tag{3}$$

One sets [11]:

$$\begin{aligned} m_T &= m_1 + m_2, \qquad & \overrightarrow{D} &= \tfrac{m_1 \overrightarrow{d}_1 + m_2 \overrightarrow{d}_2}{m_1 + m_2}, \qquad & \overrightarrow{p}_T &= \overrightarrow{p}_1 + \overrightarrow{p}_2, \\ m &= \tfrac{m_1 m_2}{m_1 + m_2} & \overrightarrow{d} &= \overrightarrow{d}_1 - \overrightarrow{d}_2 & \overrightarrow{p} &= \tfrac{m_1 \overrightarrow{p}_1 + m_2 \overrightarrow{p}_2}{m_1 + m_2}. \end{aligned} \tag{4}$$

The change of variables of Eq. (4) is a canonical transformation because it conserves the Poisson brackets [11]. According to the change of variables of Eq.

(4), the motion of the two particles is interpreted as being the motion of two fictitious particles: i) the *center of mass*, having position $\overrightarrow{D}$, total mass m_T and total momentum $\overrightarrow{p}_T$ and, ii) the *relative particle* (which is the particle associated with the relative motion), having position $\overrightarrow{d}$, mass m, called r*educed mass*, and momentum $\overrightarrow{p}$ [11]. The Hamiltonian of Eq. (3) considered as a function of the new variables of Eq. (4) becomes [11]:

$$H = \frac{p_T^2}{2m_T} + \frac{p^2}{2m} + V(\overrightarrow{d}). \tag{5}$$

The new variables obey the same commutation relations as if they should represent two particles of positions $\overrightarrow{D}$ and $\overrightarrow{d}$ and momenta $\overrightarrow{p}_T$ and $\overrightarrow{p}$ respectively [11]. The Hamiltonian of Eq. (5) can be considered as being the sum of two terms [11]:

$$H_T = \frac{p_T^2}{2m_T}, \tag{6}$$

and

$$H_m = \frac{p^2}{2m} + V(\overrightarrow{d}). \tag{7}$$

The term of Eq. (6) depends only on the variables of the center of mass, while the term of Eq. (7) depends only on the variables of the relative particle. Thus, the Schrodinger equation in the representation $\overrightarrow{D}$, $\overrightarrow{d}$ is [11]:

$$\left[\left(-\frac{1}{2m_T}\triangle_D\right) + \left(-\frac{1}{2m}\triangle_d + V(d)\right)\right]\Psi\left(D,\, d\right) = E\Psi\left(D,\, d\right), \tag{8}$$

being $\triangle_{\overrightarrow{D}}$ and $\triangle_{\overrightarrow{d}}$ the Laplacians relative to the coordinates $\overrightarrow{D}$ and $\overrightarrow{d}$ respectively. Now, one observes that the reduced mass of the previously introduced two-particle system composed of two identical masses M is

$$m = \frac{M * M}{M + M} = \frac{M}{2} \tag{9}$$

In that case, by recalling that in Schwarzschild coordinates the BH center of mass coincides with the origin of the coordinate system and with the replacements

$$d \rightarrow R, \tag{10}$$

the Schrodinger equation (8) becomes

$$\left(-\frac{1}{2m}\triangle_{2R} + V(2R)\right)\Psi\left(2R\right) = E\Psi\left(2R\right). \tag{11}$$

Setting

$$r \equiv \frac{R}{2}, \tag{12}$$

the potential of Eq. (2) becomes

$$V = -\frac{m^2}{r}, \tag{13}$$

and the Schrodinger equation (11) reads

$$-\frac{1}{2m}\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r}\right) + V\Psi = E\Psi. \tag{14}$$

The Schrodinger equation (14) is formally identical to the traditional Schrodinger equation of the s states ($l = 0$) of the hydrogen atom which obeys to the Coulombian potential [11]

$$V(r) = -\frac{e^2}{r}. \tag{15}$$

In the potential of Eq. (13) the squared electron charge e^2 is replaced by the squared reduced mass m^2. Thus, Eq. (14) can be interpreted as the Schrodinger equation of a particle, the "electron", which interacts with a central field, the "nucleus". On the other hand, this is only a mathematical artifact because the real nature of the quantum BH is in terms of Vaz's shell. For the bound states ($E < 0$) the energy spectrum is

$$E_n = -\frac{m^5}{2n^2}. \tag{16}$$

Hence, in order to completely solve the problem, one must find the relationship between the reduced mass and the total energy of Vaz's shell. One observes that, in the case of absence of rotations and of dissipation during the collapse, the final result of the LeMaitre-Tolman-Bondi gravitational collapse is exactly the same of the Oppenheimer and Snyder gravitational collapse, that is the Schwarzschild BH [12–14]. Thus, in order to find the relationship between the reduced mass and the total energy one can use the Oppenheimer and Snyder framework. In that case, the internal solution is given by the well known Friedmann-Lemaitre-Robertson-Walker line-element, which, by using spherical coordinates and comoving time is [14]

$$ds^2 = d\tau^2 - a^2(\tau)\left(\frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right). \tag{17}$$

Considering the Einstein field equation [14]

$$G_{\mu\nu} = -8\pi T_{\mu\nu} \tag{18}$$

and assuming zero pressure one gets the following relations [15]

$$\begin{aligned} \dot{a}^2 &= \tfrac{8}{3}\pi a^2\rho - 1 \\ \ddot{a} &= -\tfrac{4}{3}\pi a\rho \end{aligned} \tag{19}$$

being $\dot{a}=\frac{da}{d\tau}$. In order to have consistency, one obtains [16]

$$\frac{d\rho}{da}=-\frac{3\rho}{a}. \tag{20}$$

The last equation is integrated as

$$\rho=\frac{C}{a^3}. \tag{21}$$

The integration constant C is obtained via the initial conditions as [14]

$$C=\frac{3a_0}{8\pi}. \tag{22}$$

Eq. (21) can be rewritten as

$$\rho=\frac{3a_0}{8\pi a^3}. \tag{23}$$

One also recalls that the standard Einstein - Hilbert Lagrangian is [14]

$$L_{EH}=\frac{\sqrt{-g}R}{16\pi}. \tag{24}$$

If one uses the Friedmann-Lemaitre-Robertson-Walker line-element (17) one gets

$$L_{FLRW}=\dot{a}^2+\frac{8}{3}\pi a^2\rho. \tag{25}$$

The Lagrangian (25) can be rescaled as

$$L=\frac{M\dot{a}^2}{2}+\frac{4}{3}M\pi a^2\rho, \tag{26}$$

which can be put as

$$L=\frac{M\dot{a}^2}{2}-V(a), \tag{27}$$

where

$$V(a)\equiv-\frac{4}{3}M\pi a^2\rho. \tag{28}$$

If one inserts Eq. (23) into Eq. (28) one gets

$$V(a)=-\frac{Ma_0}{2a}. \tag{29}$$

One also finds the energy function associated to the Lagrangian as

$$E=\frac{\partial L}{\partial\dot{a}}\dot{a}-L. \tag{30}$$

Now, if one inserts Eq. (26) in Eq. (30) and uses the first of Eqs. (19) one gets

$$E=-\frac{M}{2}. \tag{31}$$

A final correction is needed. Let us clarify the reason. One compares Eq. (13) with the analogous potential energy of the hydrogen atom, i.e. Eq. (15). Eqs. (13) and (15) are almost formally identical. In fact, one recognizes a fundamental difference. In Eq. (15) the charge of the electron is constant for all the hydrogen atom's energy levels. Instead, in Eq. (13) the BH mass changes during the jumps from an energy level to another because of the emissions of quanta and the absorptions of external particles. The BH mass indeed decreases for emissions and increases for absorptions. Hence, one needs also to consider such a BH dynamical behavior. A good way to realize this is via the introduction of the *BH effective mass* M_E [10], which is the average of the BH initial and final masses which are involved in a quantum transition. M_E indeed represents the BH mass *during* the BH expansion (contraction), which is triggered by an absorption (emission) of a particle [10]. The rigorous definition of the BH effective mass is [10]

$$M_E \equiv M \pm \frac{\omega}{2}, \tag{32}$$

being ω the mass-energy of the absorbed (emitted) particle (the sign plus concerns absorptions, the sign minus concerns emissions). Hence, one sees that introducing the BH effective mass in the BH dynamical framework is very intuitive. But, for the sake of mathematical rigor, that introduction has been completely justified via Hawking's periodicity argument [10]. One chooses the positive sign in Eq. (32) if one considers the BH formation in terms of absorptions of external particles in a quantum framework. Therefore, Eq. (32) reads

$$M_E \equiv M + \frac{\omega}{2}. \tag{33}$$

In a quantum framework one wants to obtain the energy eigenvalues as being absorptions starting from the BH formation, that is from the BH having null mass, where with "the BH having null mass" one means the situation of the gravitational collapse before the formation of the first apparent horizon. This implies that one must replace $M \to 0$ and $\omega \to M$ in Eq. (33). Thus, one gets

$$M_E = \frac{M}{2} = m, \tag{34}$$

and Eq. (31) becomes

$$E = -\frac{m}{2}. \tag{35}$$

By inserting this last equation in Eq. (16), a bit of algebra permits to obtain the energy spectrum

$$E_n = -\frac{1}{2}\sqrt{n}, \tag{36}$$

and the corresponding mass spectrum

$$M_n = 2\sqrt{n}. \tag{37}$$

It is also important to estimate the maximum value of the density of Vaz's shell. By considering the shell's mathematical description of Eqs. (13) and (14) in terms of a quantum system composed by a fictitious particle, the "electron", which interacts through a quantum gravitational interaction with a central field, the "nucleus", the Born rule [17] and the Copenhagen interpretation of quantum mechanics [18] imply that the position of the "electron" cannot be exactly localized, but it is only possible to get the probability density of finding the "electron" at a given point which is, in turn, proportional to the square of the magnitude of the wavefunction of the "electron" at that point. Being the system "electron-nucleus" only a fictitious representation of the physical quantum shell, this implies that one cannot exactly localize the position of the quantum shell via the oscillating gravitational radius, and must, in turn, use an average radius. The average radius of Vaz's shell is given by the shell's expected radial distance [11]

$$\bar{R}_n = \frac{3}{2} M_n = 3\sqrt{n}. \tag{38}$$

If one evokes the generalized uncertainty principle [19], which guarantees that the shell must have a physical thickness, at least of the order of the Planck length, one can compute the minimum volume of Vaz's shell (in Planck units) as the difference between the volume of the sphere having radius $3\sqrt{n} + \frac{1}{2}$ and the volume of the sphere having radius $3\sqrt{n} - \frac{1}{2}$. Thus, one gets:

$$\begin{aligned} V_{min} &= \tfrac{4}{3}\pi \left[\left(3\sqrt{n} + \tfrac{1}{2}\right)^3 - \left(3\sqrt{n} - \tfrac{1}{2}\right)^3 \right] \\ &= \tfrac{4}{3}\pi \left(27n + \tfrac{1}{4}\right) = 36\pi n + \tfrac{\pi}{3}. \end{aligned} \tag{39}$$

On the other hand, the mass spectrum of the shell is given by Eq. (37). Hence, one obtains the maximum value of the density of the quantum shell as

$$\rho_{max} = \frac{2\sqrt{n}}{36\pi n + \frac{\pi}{3}}. \tag{40}$$

The maximum density decreases with increasing n, as one intuitively expects. Thus, the maximum density corresponds to the ground state of Vaz's shell, that, for $n = 1$, is a density of

$$\rho_{max}(n = 1) = \frac{2}{36\pi + \frac{\pi}{3}} \simeq 0.0175, \tag{41}$$

in Planck units. By recalling that the Planck density is roughly 10^{93} grams per cubic centimetre in standard units, one gets a value of

$$\rho_{max}(n = 1) \simeq 1.752 * 10^{91} \, grams \; per \; cubic \; centimetre \tag{42}$$

for the density of the ground state of Vaz's shell in standard units, which is very high but about two order of magnitude less than the Planck density. For large n Eq. (40) is well approximated by

$$\rho_{max} \simeq \frac{1}{18\pi\sqrt{n}}. \tag{43}$$

For a BH having mass of the order of 10 solar masses Eq. (37) gives

$$\sqrt{n} = \frac{10M_{\odot}}{2} = 5M_{\odot} \sim \frac{10^{34}\, grams}{M_p} \sim 5 * 10^{38}. \tag{44}$$

being $M_{\odot} \sim 2*10^{33}\, grams$ the solar mass and $M_p \sim 2*10^{-5}\, grams$ the Planck mass. By inserting the result of Eq. (44) in Eq. (43) one gets

$$\rho_{max}(10M_{\odot}) \sim \frac{1}{5 * 18\pi * 10^{38}} \sim 3.5 * 10^{-41} \tag{45}$$

in Planck units and, being the Planck density roughly 10^{93} grams per cubic centimetre in standard units, one finds a value of

$$\rho_{max}(10M_{\odot}) \sim 3.5 * 10^{52}\, grams\, per\, cubic\, centimetre. \tag{46}$$

Summarizing, in this letter the mass and energy spectra of Vaz's quantum shell have been obtained via a Schrodinger-like approach permitting, by further supporting Vaz's conclusions, that instead of a spacetime singularity covered by an event horizon, the final result of the gravitational collapse is an essentially quantum object, an extremely compact "dark star". This "gravitational atom" is held up not by any degeneracy pressure but by quantum gravity in the same way that ordinary atoms are sustained by quantum mechanics. By evoking the generalized uncertainty principle, the maximum value of the density of Vaz's shell has been estimated.

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