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Horizons, spin coefficients and gravitational entropy

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Abstract The evolution of the universe is in accordance with the second law of thermodynamics, and following Penrose's hypothesis of describing cosmological evolution in the form of gravitational entropy via the Weyl curvature, there have been many proposals describing gravitational entropy using different formalisms. In this paper, we will use the spin coefficients formalism to gravitational entropy and understand its implications in terms of spacetimes with a horizon using a recent proposal [1]. Our particular interest is in the motivation towards horizons and the gravitational entropy for a given spacetime with a horizon. We will also look at a comparison with the CET proposal and the Weyl invariant based proposal in general, with emphasis on the formalism adopted and the results obtained for certain spacetimes.

Keywords Cosmology, gravitational entropy, Newman-Penrose formalism

1 Introduction

The gravitational entropy propsoal is based on Penrose's hypothesis [18] that the Weyl curvature can be used to define a geometric contribution towards the entropy of the universe. The conditions on gravitational entropy are that (1) it must necessarily be monotonically increasing, following the second law of thermodynamics, (2) it must account for the structure formations that result in anisotropies, and (3) it must reduce to the usual Hawking-Bekenstein entropy in the case of black holes. The proposals use the Weyl curvature to formulate the gravitational entropy. This was first formulated by [3], who used the equivalence of the gravitational entropy and the holographic entropy as a base and defined a surface integral to define the gravitational entropy. However, this model has a problem with the de Sitter spacetime, since the entropy of such cosmologies is defined based on the cosmological horizon, and as it was shown by Gibbons and Hawking [4], the entropy of such models reduces to a form resembling the Hawking-Bekenstein relation, while the Rudjord and Gron proposal results in a vanishing Weyl curvature. The Clifton, Ellis and Tavakol proposal [2] uses the notion of tetrads to define the gravitational analogs of thermodynamic parameters, and this allows us to define the gravitational entropy as the integral composed of the gravitational energy density analog and the temperature. While this only holds for Petrov type **D** or **N** spacetimes, this proposal is quite consistent with most of the models, but as it was shown by Gregoris *et al* in 2020 [5], the CET proposal was applied for a spacetime that resulted in a non-increasing gravitational entropy, which is an important condition we originally wished to be consistent with the proposal. A recent paper by Gregoris and Ong [1] adopts the Newman-Penrose formalism [7] to define the gravitational entropy, based on the spin coefficients for the spacetime in consideration. In this paper, we will use the formulation of gravitational entropy suggested by Gregoris And Ong and understand the case of static and spherically symmetric black hole solutions. A discussion on cosmologies will also be provided in this paper.

The description of gravitational entropy was first provided by Rudjord and Gron, who started by considering the equivalence between the Hawking-Bekenstein relation in the case of the gravitational entropy of black hole solutions. The Hawking-Bekenstein relation shows that the entropy of black holes is proportional to the area by a factor of 4 (in the units $c = k_B = \hbar = G = 1$).

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The aspect of gravitational entropy in the case of black hole solutions is interesting for two reasons – firstly, black holes posses the maximal entropy configuration for any physical state of their area, with the entropy being only related to the area by a factor of 4 rather than a large fractional exponent in the case of stars. Secondly, since the contribution of entropy even in the case of vacuum solutions would be non-zero for black holes (following the generalised second law of thermodynamics as given by Bekenstein), a gravitational description would explain the non-zero value of the black hole entropy. For black holes, the gravitational entropy is given by [2,6]

$$S_g = S_{HB} \tag{1}$$

Where S_{HB} is the holographic entropy. Using this, the first proposal was that the gravitational entropy could be written in terms of a surface integral on the horizon:

$$S_g = k_s \int_{\Sigma} \Psi \mathbf{e}_r \cdot d\Sigma \tag{2}$$

Where Ψ is a scalar built of the Weyl invariant defined by the "square" of the Weyl tensor, $W = C_{abcd}C^{abcd}$. However, while this is the most elementary form of the scalar, this would render the proposal inconsistent in cases of isotropic singularities Wainwright and Anderson [8]. We therefore consider a factor of the Kretschmann invariant along with the Weyl invaraint. From this and the divergence theorem, we can define a volume integral to define the entropy density

$$s_d = k_s |\nabla \cdot \Psi|$$

Where the absolute brackets prevent a negative value of the entropy density. We can solve for the case of the Schwarzschild solution, where for our chosen form of the scalar Ψ we would have a maximal entropy configuration. This would result in a consistent result, where the entropy would be equal to the Hawking-Bekenstein entropy by adopting the above formulation.

However, there appears a problem when we consider the case of cosmological horizons, where the entropy in terms of the Weyl curvature in the case of the de Sitter spacetime is zero even though the entropy of the cosmology is non-zero, which is given by the Gibbons-Hawking entropy. This would resemble the Hawking-Bekenstein entropy – however, instead of a black hole horizon, we would instead consider the cosmological horizon.

The proposal we will consider in this paper requires, in fact, a horizon to define gravitational entropy. This particular point is our interest in this paper, and we will explore different forms of spacetimes with a horizon and the description of entropy in such cases.

2 Tetrads and formalism

The recent proposal describes gravitational entropy in a similar format as that of the CET proposal, which uses the gravitational analogs of thermodynamic variables to describe gravitational entropy:

$$T_g dS_g = dU_g + p_g dV \tag{3}$$

Where T_g, S_g, U_g and p_g are the gravitational analogs of temperature, entropy, internal energy and pressure[1]. In order to find the gravitational entropy, we built a tetrad, using which we find the gravitational terms ρ_g and T_g, which would give us the volume integral that gives the gravitational entropy. The tetrad is built of two real null vectors and a complex null vector and its conjugate. In terms of spacelike and timelike vectors (x^a, y^a, z^a, u^a), the tetrad can be written as the following:

$$\begin{aligned} l^a &= \tfrac{1}{\sqrt{2}}(u^a - z^a) \\ n^a &= \tfrac{1}{\sqrt{2}}(u^a + z^a) \\ m^a &= \tfrac{1}{\sqrt{2}}(x^a - iy^a) \\ \bar{m}^a &= \tfrac{1}{\sqrt{2}}(x^a + iy^a) \end{aligned}$$

In order to be able to understand the forms of the Weyl scalars we wish to solve for certain types of spacetimes, we will follow the Petrov classification, i.e. those spacetimes that have a non-vanishing Ψ_2 are Petrov type **D** and those that have a non-vanishing Ψ_4 are of type **N**, and **D** and **N** contain Petrov type **O** spacetimes as a subclass.

The Friedmann-Robertson metric defines a cosmology that is homoegeneous and isotropic, and is defined based on the scale factor $a(t)$ and the sectional curvature k:

$$ds^2 = -dt^2 + a^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \tag{4}$$

At the beginning of the universe, the anisotropies would have been far lesser than at later times, accounting for the lower entropy state at that time. Therefore, the universe would resemble a Friedmann-Robertson spacetime due to the homogeneous and isotropic nature of the model at times when $S \to 0$. Since g_{FRW} is conformally flat, i.e. $g_{\mu\nu} = \omega \eta_{\mu\nu}$, the Weyl tensor here would be vanishing, and therefore we expect the gravitational entropy to be zero at the initial singularity and monotonically increase, a gravitational version of the second law of thermodynamics, referring to the clumping of

[1] Note that these terms do not contribute to the energy-momentum tensor in the Einstein field equations in any form – these are more or less gravitational contributions and are defined in terms of geometric quantities and not matter field contributions.

matter due to gravitation. Since (M, g_{FRW}) is a Petrov **O** spacetime, the Weyl scalars vanish, and using the CET formulation (the gravitational energy density is related to the Weyl scalar Ψ_2 only by a constant factor), the gravitational entropy is given by [2,10]

$$S_g = \int \frac{\rho_g}{T_g} dV = 0 \quad \text{Petrov } \mathbf{O} \text{ spacetimes} \tag{5}$$

It is important here to note that this means that the gravitational entropy of an FLRW universe is zero, as we initially imposed as a condition for gravitational entropy proposals. However, since the gravitational entropy must also be non-zero and resemble the Gibbons-Hawking entropy for cosmological spacetimes with a horizon (for a de Sitter spacetime this is the conflicting point – the gravitational entropy is zero following the Weyl invariant proposal, which implied that the non-zero thermodynamic entropy was making the contribution towards the cosmology). Therefore, there is a *weak* and a *strong* aspect of gravitational entropy – the "weak" aspect being that the gravitational entropy must be zero following a direct use of the Weyl invariant, while the "strong" aspect defines the gravitational entropy of cosmologies in terms of the horizon, in acceptance with the Gibbons-Hawking entropy (refer to [13] to look at a description of Lemaitre-Tolman-Bondi spacetimes using the Weyl invariant).

Following an almost similar track, the spin coefficients proposal starts by considering a tetrad, then constructing the spin coefficients for the gravitational entropy. Using the spin coefficients, the resulting form of gravitational entropy can be found out. We will consider the example of the Schwarzschild solution (M, g_{sch}), with the metric defined as:

$$ds^2 = -dt^2 f(r) + dr^2 g(r) + r^2 d\Omega^2 \tag{6}$$

Where $f(r) = g(r)^{-1} = \left(1 - \frac{2GM}{c^2 r}\right)$. In this spacetime, we have a horizon at $r = R_{sch}$, where R_{sch} is the Schwarzschild radius. The spin coefficients in the Newman-Penrose formalism would be equal and of the form $\Delta W/\Psi_2$, where W is the Weyl tensor, and Ψ_2 is the non-zero Weyl scalar in **D** type spacetimes [7].

For this, we will work with the tetrad

$$\begin{aligned} l_a &= \tfrac{1}{\sqrt{2}}\left(\sqrt{f(r)}dt - \sqrt{g(r)}dr\right) \\ n_a &= \tfrac{1}{\sqrt{2}}\left(\sqrt{f(r)}dt + \sqrt{g(r)}dr\right) \\ m_a &= \tfrac{1}{\sqrt{2}}\left(rd\theta + ir\sin\theta d\phi\right) \\ \bar{m}_a &= \tfrac{1}{\sqrt{2}}\left(rd\theta - ir\sin\theta d\phi\right) \end{aligned}$$

Using this tetrad, we can find the respective Weyl scalar Ψ_2 and $\Delta W \equiv n^a \nabla_a C_{abcd}$. Using this, the integral for the gravitational entropy can be written as:

$$S_g = k_s \int_0^{r_{sch}} \int_\Sigma \left|\frac{\Delta W}{\Psi_2}\right| \frac{dV}{\sqrt{f(r)}} \tag{7}$$

Where we used $f(r) = g(r)^{-1}$ and dV is the volume element $dV = r^2 \sin\theta dr d\theta d\phi$. The integrand can be found out by the tetrad listed by recalling that the Weyl scalar Ψ_2 is of the form

$$\Psi_2 = C_{abcd} n^a m^b \bar{m}^c l^d$$

Using this, the integral would be simply

$$S_g = k_s \int_0^{R_{sch}} \int_\Sigma \left|\frac{3\sqrt{2(1-2Mr^{-1})}}{2r}\right| \frac{dV}{\sqrt{1-2Mr^{-1}}} \tag{8}$$

The integral reduces to $S \sim A$, in agreement with the reduction condition to the Hawking-Bekenstein relation. The factor k_s is discussed in section IIB of [1].

The approach used here requires the existence of a horizon as the upper limit in the integral to define the entropy of the spacetime. This has many important aspects in terms of cosmology, where the horizon is replaced by a cosmological horizon. In the case of the Rudjord and Gron proposal, there was a complication in the case of the de Sitter spacetime. In this spacetime, the entropy is given by

$$S_{dS} = \frac{A_{dS}}{4} \tag{9}$$

Where A_{dS} is the area of the de Sitter horizon, $r_{dS} = \sqrt{3/\Lambda}$, where Λ is the cosmological constant. The de Sitter spacetime can be represented as a case of $M = 0$ and $\Lambda > 0$ under the Schwarzschild de Sitter spacetime. Due to this, the gravitational entropy proposal would be vanishing, since it is directly based on the Weyl invariant. However, by the Gregoris and Ong proposal, the spin coefficients make the contribution to the integral, and therefore the cosmological horizon can be set as the upper limit on the integral, allowing us to define the gravitational entropy in the form of (14).

We will now consider the case of the Schwarzschild solution, where we will implement the above mentioned approach following Gregoris and Ong, and see that the holographic entropy is obtained from the gravitational entropy, which is the form we wish the gravitational entropy of black hole solutions to reduce to. We will further look at the cosmological implications of the spin coefficients approach, and compare some particular models with the CET formulation.

3 Schwarzschild spacetime gravitational entropy

The Schwarzschild spacetime is defined as a pair (M, g) such that g is static and spherically symmetric, and the energy-momentum tensor is vanishing. The metric in this spacetime is given by

$$ds^2 = -dt^2\left(1 - \frac{2GM}{c^2 r}\right) + dr^2\left(1 - \frac{2GM}{c^2 r}\right)^{-1} + r^2 d\Omega^2 \tag{10}$$

We can identify the tetrad for this spacetime directly, which would be of the form (we have set $\left(1 - \frac{2GM}{c^2 r}\right) = f(r)$ for convenience):

$$\begin{aligned} l_\mu &= \tfrac{1}{2}\left(\sqrt{f(r)} - \tfrac{dr}{\sqrt{f(r)}}\right) \\ n_\mu &= \tfrac{1}{\sqrt{2}}\left(\sqrt{f(r)} + \tfrac{dr}{\sqrt{f(r)}}\right) \\ m_\mu &= \tfrac{1}{\sqrt{2}}(r d\theta + i r \sin\theta d\phi) \\ \bar{m}_\mu &= \tfrac{1}{\sqrt{2}}(r d\theta - i r \sin\theta d\phi) \end{aligned}$$

In the Schwarzschild case, the black hole is perfectly static and spherically symmetric[2], and the terms μ and ρ (the spin coefficients) are defined in the usual way. Note that the spin coefficients μ and ρ are equal to each other in the case of a static spacetime (such as the LRS Bianchi type I spacetime), and therefore we can generalise the approach towards forms of $f(r)$ that also include angular momentum Ω. The proposal considers the gravitational entropy to be given by the integral

$$S_g = k \int_0^{r_H} \int_\Sigma \left| \frac{\Delta W}{\Psi_2} \right| dV \tag{11}$$

Where $\Delta \equiv n^\mu \nabla_\mu$ and dV is the volume element. In this approach, we will consider the non-vanishing Weyl scalar Ψ_2 and find out the integral above. We start by recalling that the Weyl tensor is simply the form

$$\Psi_2 = C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta$$

Using this, we can find out the components for the integral in terms of the derivatives of $f(r)$ with respect to the radial coordinate r. The integral can then be found out by noting that the volume element would be

$$dV = \frac{1}{\sqrt{f(r)}} r^2 \sin\theta dr d\theta d\phi$$

Solving the integral by identifying the upper limit on the integral would yield us the usual Hawking-Bekenstein entropy, which is an important condition in the consistency of gravitational entropy proposals in the case of black holes. In the Schwarzschild case, r_H is merely the Schwarzschild radius, and the integral seen above reduces to the proper entropy without any *ad hoc* operations such as the removal of the singularity at $r = 0$ by defining a sphere of radius r and removing this from the integral to prevent divergence.

Higher order theories of black holes can also be described by the gravitational entropy proposal described above. The metric of a Schwarzschild black hole in $D = 5$ (called the Tangherlini metric) would resemble the usual Schwarzschild metric with an additional coordinate ω:

$$ds^2 = -dt^2\left(1 - \frac{2GM}{c^2 r}\right) + dr^2\left(1 - \frac{2GM}{c^2 r}\right)^{-1} + r^2 d\Omega_5^2 \tag{12}$$

In this case, we consider the Weyl tensor directly rather than constructing the scalar Ψ_2 and the m^μ terms have additional components. The volume element would be of the form $dV = \frac{1}{\sqrt{f(r)}} r^3 \sin^2\theta \sin\phi dr d\theta d\phi d\omega$, and the integral would reduce to the Hawking-Bekenstein formula in $D = 5$.

The proposal can also be considered in the case of Kerr and Reissner-Nordstrom black holes by following the above mentioned approach, only with setting the appropriate parameters non-zero and identifying the nature of the spacetime. In the case of a Kerr black hole, clearly the black hole is not spherically symmetric due to angular momentum – however, this is based on the identification of the type of metric it is with regard to the case where the $|g_{00}|$ component is equal to the $|g_{11}|$ component (where $||$ denote the absolute brackets). In this case, we can see that the gravitational entropy is based on an extra factor (the angle θ) [1] and has a constraint to be satisfied in order to prove to be a viable gravitational entropy reducing to the Hawking-Bekenstein entropy.

Interestingly, in the case of this proposal, one of the requirements is that the upper limit is necessarily a horizon. In the case of the Weyl invariant based proposal, there were three points that limited the proposal's consistency – first, we required the removal of a spherical element around the singularity to prevent a divergence of the integral. Second, we required curvature invariant corrections to the considered form of Ψ in (2) due to isotropic singularities, meaning that the gravitational entropy was based on a form of Ψ that requires corrections to including the Ricci scalar or the Kretschmann scalar. Third, the proposal predicts that the gravitational entropy would vanish in the case of the

[2]As opposed to those metrics that define black holes with an angular momentum which causes the black holes to be static but not spherically symmetric, such as Kerr and Kerr-Newmann black holes.

de Sitter spacetime, indicating either that the proposal requires additional corrections or a non-gravitational contribution of the entropy – this particularly has to do with the explicit nature of the gravitational entropy's dependence on the Weyl invariant, whereas in our case, the proposal only requires the spin coefficients and the Weyl scalar to define the gravitational entropy. Curvature invariant corrections, therefore, are no longer a requirement in our integral. Secondly, since the integral specified in (7) is defined in terms of the horizon $r_H =$ (for the Schwarzschild spacetime this is r_{sch}), our only requirement is that the integral's upper limit be a horizon – selecting some $r = k$ would not yield the gravitational entropy, since the solution contains a horizon within which the entropy is specified. Due to this, there are several key features in this proposal that allow us to define spacetimes with a horizon consistently, which can be measured by looking at the corresponding horizon-dependent entropy expression (for black holes this would be the Hawking-Bekenstein entropy, while for cosmological spacetimes this would be the Gibbons-Hawking entropy).

It would then be important to compare the CET proposal with the new proposal in terms of cosmological horizons. Since these gravitational entropy proposals also form the basis for the Weyl curvature hypothesis, it is important to look at the behaviour of cosmological spacetimes and their gravitational entropy variation with time. The Weyl curvature hypothesis can be loosely defined as [18][3]:

Definition 1 For a given spacetime (M, g) for which a gravitational entropy S_g is attributed, the variation $\dot{S}_g$ must always be non-negative in order to account for the structure formations and the increasing thermodynamic entropy.

One of the many reasons this is of particular interest is because one of the primary reasons the gravitational entropy landscape was introduced was specifically to understand how different models follow WCH. It is also important to note that we don't require the Weyl tensor to be a direct quantification of the gravitational entropy such as the case of Weyl invariant based proposal – we only a weak condition on the Weyl curvature, i.e. we require the Weyl curvature to be zero in the case of conformally flat spacetimes, accounting for spatial homogeneity and isotropicity. While this has been investigated in several papers, the results of the CET proposal and the results using the spin coefficients proposal can be compared to look at the factors that contribute to the determination of the monotonically increasing gravitational entropy. We will illustrate this by considering a spacetime with multiple scale factors (referred to as a Bianchi type I spacetime) with local rotational symmetry to look at the factors adding up to the variation being non-negative.

[3]Again, here we are not defining S_g in terms of the Weyl tensor directly.

4 Comparing to the CET proposal

The CET proposal uses the tetrad formalism in a similar way in some sense, but has no relation to the spin coefficients in that there is no role of μ, ρ or ΔW determining the gravitational entropy. However, the CET proposal uses the gravitational analogs defined in (3) to define the gravitational entropy S_g. Noting that the CET proposal works only for Petrov type **D** or **N** (i.e. the scalars Ψ_2 and Ψ_4 determine the proposal), since type **O** spacetimes are a class under **D** and **N**, we can consider the case of the FLRW spacetime to look at the CET proposal. The FLRW metric is defined by (14), and the tetrad in this case can be identified as:

$$\begin{aligned} l_\mu &= \tfrac{1}{\sqrt{2}}\left(dt - \tfrac{a(t)}{\sqrt{1-kr^2}}dr\right) \\ n_\mu &= \tfrac{1}{\sqrt{2}}\left(dt + \tfrac{a(t)}{\sqrt{1-kr^2}}dr\right) \\ m_\mu &= \tfrac{1}{\sqrt{2}}(ra(t)d\theta + ira(t)\sin\theta d\phi) \\ \bar{m}_\mu &= \tfrac{1}{\sqrt{2}}(ra(t)d\theta - ira(t)\sin\theta d\phi) \end{aligned}$$

It is trivial to see that the Weyl tensor in this spacetime vanishes since this spacetime is conformally flat. However, following the CET proposal, we define the gravitational entropy as the integral

$$S_g = \int \frac{\rho_g}{T_g} dV \tag{13}$$

Since ρ_g would be a function of the Weyl scalar Ψ_2 and $T_g \neq 0$, the integral would reduce to zero:

$$S_g = \int \frac{\rho_g}{T_g} dV \longrightarrow \rho_g = 0 \Longleftrightarrow S_g = 0$$

Which is what we expected, since the Weyl curvature must be zero, which would imply that the gravitational entropy is zero. However, considering cosmological spacetimes with a horizon require a non-zero gravitational entropy. In the case of the spin coefficients approach, we can simply start by considering the tetrad specified above and use it to define the integral (11) in terms of the spin coefficients μ and ρ in the integrand. Using this approach, the FLRW model gravitational entropy will be defined in a similar format to that of the Schwarzschild case, only with the spin coefficients in consideration and a cosmological horizon. Observing that the FLRW model will asymptotically approach a de Sitter spacetime, it is then necessary to look at the gravitational entropy in this perspective.

The integral for the FLRW model with a horizon r_H would reduce to (refer to Appendix B):

$$S_g = \pi \left(a(t) r_H\right)^2 \equiv \pi R_H^2 \tag{14}$$

The de Sitter spacetime would then have an entropy defined by the Gibbons-Hawking formula (9), which would have to be the entropy we wish the above gravitational entropy to reduce to. As seen before in section 2, the horizon of a de Sitter spacetime is at $r_{dS} = \sqrt{3/\Lambda}$, and therefore we require $r_H = r_{dS}$, which would be the upper bound for the integral above. This is therefore consistent with the expectation of the entropy to be of a gravitational contribution rather than a purely thermodynamic contribution.

Other types of spacetimes are also of interest, particularly those that are homogeneous and anisotropic. The metric for the Bianchi type I spacetime is generally described as

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2 dz^2 \tag{15}$$

Where $a_1(t)$, $a_2(t)$ and $a_3(t)$ are scale factors of the model. Spacetimes with local rotational symmetry (designated LRS spacetimes) can be considered as (for a more detailed description of LRS spacetimes, the interested reader is directed to [9])

Definition 2 A given spacetime (M, g) is said to be an LRS spacetime if there is a tetrad $(l, n, m, \bar{m})$ such that the spin coefficients κ, π, σ, τ, λ and ν are equal to zero, implying rotational invariance. These spacetimes are assumed to be of Petrov type **D**, and therefore the only non-vanishing Weyl scalar is Ψ_2, or $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ among other conditions.

The FLRW metric is a homogeneous and isotropic model with a scale factor $a(t)$. The Bianchi type I spacetime is defined in terms of multiple scale factors for each spatial component as seen above. Imposing the locally rotationally symmetric condition, which would give us the metric [10]

$$ds^2 - dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_2^2(t)dz^2 \tag{16}$$

The tetrad in this spacetime would be given by the following vectors

$$\begin{aligned} u_\mu &= (1,\ 0,\ 0,\ 0) \\ z_\mu &= (0,\ a_1(t),\ 0,\ 0) \\ x_\mu &= (0,\ 0,\ a_2(t),\ ia_2(t)) \\ y_\mu &= (0,\ 0,\ a_2(t),\ -ia_2(t)) \end{aligned}$$

Using this tetrad, we can find out the gravitational entropy via the CET proposal. Using this approach, the gravitational entropy for this spacetime would be dependent on the scale factors $a_1(t)$ and $a_2(t)$. Clearly, in order to investigate the Weyl curvature hypothesis it is necessary to take into consideration of the additional contributions that may act as constraints on the gravitational entropy. Following this, the derivative of S_g with respect to t would be non-negative only in certain conditions where the scale factors are non-negative or their combinations are non-negative, which may also contribute to the spin coefficients approach since the scale factors determine the scalar Ψ_2.

An example of how the CET proposal compares to the approach via spin coefficients can be taken in the case of Lemaitre-Tolman-Bondi (LTB) spacetimes, which describe an inhomogeneous cosmology. For this, the metric is of the form

$$ds^2 = -dt^2 + \frac{R'^2(r,t)}{1-kr^2} + R^2(r,t)d\Omega^2 \tag{17}$$

Where $R'(r,t) \equiv \frac{\partial R(r,t)}{\partial r}$. In the CET framework, both ρ_g and T_g are determined in terms of $R(r,t)$. The tetrad for this spacetime is:

$$\begin{aligned} l_\mu &= \tfrac{1}{\sqrt{2}}\left(dt - \tfrac{R'(r,t)}{\sqrt{\sqrt{1-kr^2}}}\right) \\ n_\mu &= \tfrac{1}{\sqrt{2}}\left(dt + \tfrac{R'(r,t)}{\sqrt{\sqrt{1-kr^2}}}\right) \\ m_\mu &= \tfrac{1}{\sqrt{2}}(R(r,t)d\theta + iR(r,t)\sin\theta d\phi) \\ \bar{m}_\mu &= \tfrac{1}{\sqrt{2}}(R(r,t)d\theta - iR(r,t)\sin\theta d\phi) \end{aligned}$$

In the previously seen case of the Schwarzschild spacetime in section 3, the spin coefficients we considered were of the form $\mu = \rho$ being proportional to the directional derivative $\Delta W/\Psi_2$ (refer to Appendix A for a note on the proportionality between the spin coefficient ρ and the directional derivatives in terms of Petrov type **D** spacetimes.). In cases when this is not so, we consider $\mu + \rho$ form to find out the gravitational entropy. The proportionality arises in several cases, and the equivalence of μ and ρ is found in several spacetimes. For instance, in the case of a Kerr-NUT AdS spacetime, the spin coefficients μ and ρ are equal to each other and ρ is proportional to a Cartan invariant. In fact, this is the motivation towards considering $\Delta W/\Psi_2$ forms, and why the spin coefficient ρ is of particular interest in terms of understanding the horizon. This is because ρ has to be related to the fact that this would be a trapped surface, and therefore non-expanding and $\rho = 0$. It is therefore of particular interest to notice the nature of the spin coefficients, and the combined $\mu + \rho$ integrand allows us to define gravitational entropy in cases where the $\Delta W/\Psi_2$ form is not the integrand. This specifically has been applied to the FLRW spacetime, which resulted in the result (14). In general, it is of interest to understand those metrics that are Schwarzschild-like[4].

If we wish to impose the condition of the Weyl invariant also being in accordance with the Weyl curvature hypothesis and not merely of interest in terms of the holographic entropy, we can choose to define the case of the above spacetime via the curvature invariant $C_{abcd}C^{abcd}$ [13]. However, if we adopt the case of $R(r,t)$ being the areal radius $R_H \equiv a(t)r$, the Weyl tensor vanishes under certain conditions that are based on the factor r – this is in fact the natural case when the considered model is an FLRW spacetime. In the case of observing a divergence for the LTB spacetime (17) without any settings reducing to the FLRW spacetime, the natural case of $R(r,t) \to 0$ would yield a divergence at the initial singularity, which is in conflict with the statement of the WCH, which would require curvature invariant corrections such as including the prefactor $R_{ab}R^{ab}$ or $R_{abcd}R^{abcd}$. However, the CET proposal achieves this without requiring curvature invariant corrections to the gravitational entropy. Since both the effective energy density ρ_g and temperature T_g increase, the evolution from the initial FLRW state would be guided by an increasing gravitational entropy, in accordance with the WCH. The spin coefficients method from can be used to directly identify the gravitational entropy in terms of the radius $R_H = \pi R^2(r,t)$. The nature of the formalism in particular spacetimes such as the LRS Bianchi I spacetime, Lemaitre-Tolman-Bondi spacetime and other forms of homogeneous and anisotropic (or inhomogeneous solutions such as the LTB case), etc. will be explored in future works.

Remarks

The landscape of gravitational entropy can be divided into two distinct problems – the first being that of corresponding to a monotonically increasing description of entropy using the Weyl curvature, as Penrose hypothesised. The second part of the landscape concerns in particular the specific cosmology in consideration. This can be illustrated by looking at the case of the de Sitter spacetime. In such a spacetime, under solely the Weyl invariant based proposal, the Weyl tensor vanishes, which indicates that the gravitational entropy in this spacetime is zero. However, clearly this is not so, following [4], and therefore the contribution towards entropy must be either purely based on thermodynamic parameters or the formulation considered must be modified. In the case of the FLRW model, the Weyl curvature is zero – however, as seen in section 4, the gravitational entropy can be defined in terms of the spin coefficients, which would allow us to quantify the gravitational entropy in terms of the cosmology rather than being a direct measurement in terms of the Weyl tensor. Therefore, the gravitational entropy proposal has two aspects – one concerning the Weyl curvature, which Penrose had not stated to be a direct measure of gravitational entropy, and secondly, that concerning the cosmological horizon. It is of interest to understand how the proposal works for other types of spacetimes that contain a cosmological horizon. It is yet to be determined how different spin coefficients correspond to different physical parameters, such as the spin coefficient ρ. In particular, it is of interested also to see the relation between the CET proposal and the spin coefficients approach. If the proposals are related, the link between the invariant ΔW and Ψ_2 as seen in section 2 and the effective energy density and temperature ρ_g and T_g can be found out. This was in fact mentioned in [1], however whether there is a relation between the spin coefficients μ and ρ and the gravitational terms is yet to be found out. However, since the gravitational entropy of the FLRW universe is zero following the CET proposal while the gravitational entropy is non-zero and in terms of the Gibbons-Hawking entropy following the spin coefficients approach, this hints that there is a difference between the proposals, which will be studied in later works. The proposal for solutions, particularly those involving homogeneous and anisotropic cosmological models and the determination of the variation of the gravitational entropy to understand the contribution of different parameters to the consistency with the Weyl curvature hypothesis will also be studied in later works.

[4]Here by "Schwarzschild-like" solutions we mean those spacetimes (M,g) where the metric is of the form $g_{\mu\nu} = diag(-f(r,t), g(r,t), P(r,t)r^2, P(r,t)r^2\sin^2\theta)$. The case of the Lemaitre-Tolman-Bondi solution above is such a particular case. The explicit nature of the gravitational entropy of such spacetimes in a general format can be seen by using the $\mu+\rho$ formulation as seen above.

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Appendix A

The interest of static but spherically asymmetric spacetimes in terms of gravitational entropy can be found also in terms of the invariants that build the spin coefficients. for instance, the example of the Kerr metric can be considered for a motivation towards the employment of $\mu = \rho$ terms. The tetrad for the Kerr metric in $D = 4$ can be found out, using which we can compute the Weyl scalars Ψ_0, Ψ_1, Ψ_2, Ψ_3 and Ψ_4. In this case, the only non-zero Weyl scalar would be Ψ_2, indicating that the spacetime we are working in is a Petrov type **D** spacetime. For this, the spin coefficients would be of the form $\mu = \rho$, and further, the spin coefficient would be of the form

$$\rho = \frac{\Delta\Psi_2}{3\Psi_2} \tag{.1}$$

This indicates that the spin coefficient is proportional to ΔW, where by W we adopt the notation convention specified in [1], where by W we mean the Weyl tensor C_{abcd} or the Weyl scalar Ψ_2 for Petrov **D** spacetimes. This approach can be used for any Petrov **D** spacetime that is a vacuum solution, and the corresponding derivatives are related to the spin coefficients and the corresponding Weyl scalar. A detailed description of this for several spacetimes is given in [11].

Appendix B

In the case of spacetimes where the proportionality $\rho \propto \Delta\Psi_2/\Psi_2$ cannot be used (as illustrated in section 4 in the case of the FLRW spacetime), for instance in the case of spacetimes with $\Psi_2 = 0$, we can use the spin coefficients μ and ρ to define the gravitational entropy. This is done by defining the integral

$$S_g = k_s \int_0^{r_H} \int_{\Sigma} \left| \frac{\mu + \rho}{2} \right| dV \tag{.2}$$

This would reduce to the form (14) as seen above, where the spin coefficients would be defined in terms of the constant k and the scale factor $a(t)$. Note that the sectional curvature k does not make any contribution to the gravitational entropy since this cancels out due to the inclusion of the $1/(\sqrt{1 - kr^2})$ factor in the volume element (refer to section III C of [1]).

For the LTB spacetime (17), we use the above integral to solve for the general case of the metric rather than define the parameter $R(r, t)$. This metric in itself is a particular case of Schwarzschild-like spacetimes defined by the metric

$$ds^2 = -dt^2 f(r, t) + dr^2 g(r, t) + R^2 d\Omega^2 \tag{.3}$$

We can obtain the gravitational entropy by first finding the tetrad $l, n, m, \bar{m}$ and finding the spin coefficients μ and ρ. This can be directly used to find the gravitational entropy using the respective volume element, which will yield the usual Hawking-Bekenstein entropy, with the upper limit of the integral being the horizon defined in terms of the parameter $R(r, t)$, which implies that the areal radius R_H is a function of r and t.

This algorithm is of interest in determining the gravitational entropy of different types of cosmologies. Particularly, those cosmologies that have multiple scale factors have different properties to that of homogeneous and isotropic spacetimes. Such spacetimes fall under many categories, and Bianchi spacetimes are a part of such cosmologies. The case of the LRS Bianchi I spacetime considered in section 4 has additional factors contributing to the gravitational entropy as per the CET proposal. In further works, we will look at the variation of gravitational entropy with time to account for the Weyl curvature hypothesis.