ON THE SOLVABILITY OF SOME SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH CONCENTRATED SOURCES

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Abstract: The article is devoted to the existence of solutions of a system of integro-differential equations in the case of the normal diffusion and the influx/efflux terms proportional to the Dirac delta function. The proof of the existence of solutions relies on a fixed point technique. We use the solvability conditions for the non-Fredholm elliptic operators in unbounded domains.

AMS Subject Classification: 35J05, 35P30, 47F05
Key words: integro-differential systems, Dirac delta function, non-Fredholm operators, Sobolev spaces

1. Introduction

The present work deals with the existence of stationary solutions of the following system of nonlocal reaction-diffusion equations

$$\frac{\partial u_m}{\partial t} = D_m \frac{\partial^2 u_m}{\partial x^2} + \int_{-\infty}^{\infty} K_m(x-y) g_m(w(y)u(y,t))dy + \alpha_m \delta(x), \tag{1.1}$$

with $\alpha_m \in \mathbb{R},\ \alpha_m \neq 0,\ 1 \leq m \leq N$. We use $w(x)$ in system (1.1) as a cut-off function. The assumptions on it will be stated further down. The problems of this kind are relevant to the cell population dynamics. The space variable x here corresponds to the cell genotype, $u_m(x,t)$ stand for the cell density distributions for the various groups of cells as the functions of their genotype and time,

$$u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

The right side of our system of equations describes the evolution of cell densities due to the cell proliferation, mutations and cell influx or efflux. The diffusion terms here correspond to the change of genotype by means of the small random mutations, and the nonlocal production terms describe the large mutations. The functions $g_m(w(x)u(x,t))$ designate the rates of cell birth depending on u, w (density dependent proliferation). The kernels $K_m(x-y)$ express the proportions of newly born cells changing their genotypes from y to x. Let us assume that they depend on the distance between the genotypes. The last term in the right side of each equation of our system, which is proportional to the Dirac delta function stands for the influx/efflux of cells for different genotypes. The solvability of the single integro-differential equation analogous to (1.1) was discussed in [33]. The similar system of equations in one dimension in the case of the standard negative Laplacian raised to the power $0 < s < \dfrac{1}{4}$ in the diffusion term was studied in [40]. But in [40] it was assumed that the influx/efflux terms $f_m(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus, in the present work we treat the more singular case. In neuroscience, the integro-differential problems describe the nonlocal interaction of neurons (see [9] and the references therein).
We set all $D_m = 1$ and establish the existence of solutions of the system

$$\frac{d^2 u_m}{dx^2} + \int_{-\infty}^{\infty} K_m(x-y) g_m(w(y)u(y))dy + \alpha_m \delta(x) = 0, \quad 1 \le m \le N. \tag{1.2}$$

Let us deal with the situation when the linear part of such operator does not satisfy the Fredholm property. As a consequence, the conventional methods of the nonlinear analysis may not be applicable. We use the solvability conditions for the non-Fredholm operators along with the method of the contraction mappings.
Consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either trivial in the whole space or tends to 0 at infinity. If $a \ge 0$, the essential spectrum of the operator $A : E \to F$, which corresponds to the left side of equation (1.3) contains the origin. As a consequence, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present article deals with the studies of the certain properties of the operators of this kind. Note that the elliptic equations involving the non-Fredholm operators were treated actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [4], [5], [6], [7], [8]. The Schrödinger type operators without the Fredholm property were studied with the methods of the spectral and the scattering theory in [17], [30], [34], [35]. Fredholm structures, topological invariants and their applications were covered in [13]. The work [14] deals with the finite and infinite dimensional attractors for the evolution equations of mathematical physics.

The large time behavior of the solutions of a class of fourth-order parabolic problems defined on unbounded domains using the Kolmogorov ε-entropy as a measure was studied in [15]. The attractor for a nonlinear reaction-diffusion system in an unbounded domain in the space of three dimensions was discussed in [22]. The articles [24] and [29] are devoted to the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of second order and of the operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in the second-order quasilinear elliptic systems were discussed in [25]. The Laplace operator with drift from the perspective of the operators without the Fredholm property was considered in [37] and linearized Cahn-Hilliard equations in [31] and [38]. Nonlinear non-Fredholm elliptic problems were treated in [16], [18], [19], [20], [21], [36], [39], [40]. The significant applications to the theory of reaction-diffusion equations were developed in [11], [12]. The operators without the Fredholm property arise when studying wave systems with an infinite number of localized traveling waves as well (see [2]). The article [3] is devoted to the standing lattice solitons in the discrete NLS equation with saturation. In particular, when $a = 0$ our operator A is Fredholm in some properly chosen weighted spaces (see [4], [5], [6], [7], [8]). However, the case of $a \neq 0$ is significantly different and the method developed in these works cannot be applied. The existence, stability and bifurcations of the solutions of the nonlinear partial differential equations containing the Dirac delta function type potentials were studied actively in [1], [23], [26], [27].

Let us set $K_m(x) = \varepsilon_m \mathcal{K}_m(x)$ with $\varepsilon_m \geq 0$, so that

$$\varepsilon := \max_{1 \leq m \leq N} \varepsilon_m. \tag{1.4}$$

Suppose all the nonnegative parameters ε_m are trivial. Then we arrive at the linear Poisson equations

$$-\frac{d^2 u_m}{dx^2} = \alpha_m \delta(x), \quad 1 \leq m \leq N. \tag{1.5}$$

We will use the ramp function

$$R(x) := \begin{cases} x, & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{1.6}$$

Clearly, the solution of each equation (1.5), which is trivial at the negative infinity is

$$-\alpha_m R(x), \quad 1 \leq m \leq N, \tag{1.7}$$

so that

$$u_0(x) := (-\alpha_1 R(x), -\alpha_2 R(x), ..., -\alpha_N R(x))^T. \tag{1.8}$$

As distinct from the case covered in [40], such solutions (1.7) are not bounded and do not belong to $H^1(\mathbb{R})$. We suppose that the conditions below are fulfilled.

Assumption 1.1. *Let* $1 \leq m \leq N$, $\mathcal{K}_m(x) : \mathbb{R} \to \mathbb{R}$ *are nontrivial, so that* $\mathcal{K}_m(x),\ x^2\mathcal{K}_m(x) \in L^1(\mathbb{R})$ *and orthogonality conditions (4.2) are valid. Suppose also that the cut-off function* $w(x) : \mathbb{R} \to \mathbb{R}$ *is such that* $w(x)R(x)$ *does not vanish identically on the real line and* $w(x)R(x) \in H^1(\mathbb{R})$. *Furthermore,* $w(x) \in H^1(\mathbb{R})$ *and for* $\alpha_m \in \mathbb{R},\ \alpha_m \neq 0$ *the upper bound*

$$|\alpha|_{\mathbb{R}^N} \leq \frac{1}{\|w(x)R(x)\|_{H^1(\mathbb{R})}} \tag{1.9}$$

holds.

Here and further down $\alpha := (\alpha_1, \alpha_2, ..., \alpha_N)^T$ and $|.|_{\mathbb{R}^N}$ will denote the length of a vector in $\mathbb{R}^N$. It can be trivially checked that $w(x) = e^{-|x|},\ x \in \mathbb{R}$ satisfies the assumptions above and therefore it can be used as our cut-off function. Let us recall that in the argument of [40] we did not need to use such cut-off due to the more regular behaviour of the solutions of the Poisson type equations. In the work we choose the space dimension $d = 1$, which is related to the solvability of the linear Poisson equations (1.5) discussed above. From the perspective of the applications, the space dimension is not restricted to $d = 1$ because the space variable corresponds to the cell genotype but not to the usual physical space. We use the Sobolev space

$$H^1(\mathbb{R}) := \Big\{\phi(x) : \mathbb{R} \to \mathbb{R} \mid \phi(x) \in L^2(\mathbb{R}),\ \frac{d\phi}{dx} \in L^2(\mathbb{R})\Big\},$$

equipped with the norm

$$\|\phi\|^2_{H^1(\mathbb{R})} := \|\phi\|^2_{L^2(\mathbb{R})} + \Big\|\frac{d\phi}{dx}\Big\|^2_{L^2(\mathbb{R})}. \tag{1.10}$$

Obviously, by virtue of the standard Fourier transform (2.1), such norm can be expressed as

$$\|\phi\|^2_{H^1(\mathbb{R})} = \|\widehat{\phi}(p)\|^2_{L^2(\mathbb{R})} + \|p\widehat{\phi}(p)\|^2_{L^2(\mathbb{R})}. \tag{1.11}$$

For a vector function

$$u(x) = (u_1(x), u_2(x), ..., u_N(x))^T,$$

we will use the norm

$$\|u\|^2_{H^1(\mathbb{R},\mathbb{R}^N)} := \|u\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} + \sum_{m=1}^{N} \Big\|\frac{du_m}{dx}\Big\|^2_{L^2(\mathbb{R})} \tag{1.12}$$

with

$$\|u\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} := \sum_{m=1}^{N} \|u_m\|^2_{L^2(\mathbb{R})}.$$

The Sobolev inequality on the real line (see e.g. Sect 8.5 of [28]) gives us

$$\|\phi(x)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}}\|\phi(x)\|_{H^1(\mathbb{R})}. \tag{1.13}$$

Let us use the algebraic property of our Sobolev space. For any $\phi_1(x),\ \phi_2(x) \in H^1(\mathbb{R})$

$$\|\phi_1(x)\phi_2(x)\|_{H^1(\mathbb{R})} \leq c_a\|\phi_1(x)\|_{H^1(\mathbb{R})}\|\phi_2(x)\|_{H^1(\mathbb{R})}, \tag{1.14}$$

where $c_a > 0$ is a constant. Inequality (1.14) can be trivially derived, for example via (1.13). We look for the resulting solution of the nonlinear system of equations (1.2) as

$$u(x) = u_0(x) + u_p(x), \tag{1.15}$$

where

$$u_p(x) := (u_{p,1}(x), u_{p,2}(x), ..., u_{p,N}(x))^T$$

and $u_0(x)$ is given by (1.8). Evidently, we arrive at the perturbative system

$$-\frac{d^2u_{p,m}(x)}{dx^2} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y)g_m(w(y)[u_0(y) + u_p(y)])dy, \tag{1.16}$$

with $1 \leq m \leq N$. Let us use a closed ball in our Sobolev space

$$B_\rho := \{u(x) \in H^1(\mathbb{R}, \mathbb{R}^N) \mid \|u\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{1.17}$$

We seek the solution of the system of equations (1.16) as the fixed point of the auxiliary nonlinear system

$$-\frac{d^2u_m(x)}{dx^2} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y)g_m(w(y)[u_0(y) + v(y)])dy, \tag{1.18}$$

where $1 \leq m \leq N$ in ball (1.17). For a given vector function $v(y)$ this is a system of equations with respect to $u(x)$. Each equation of (1.18) in its left side contains the operator $-\frac{d^2}{dx^2}$ acting on $L^2(\mathbb{R})$, which does not satisfy the Fredholm property. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Thus, such operator does not have a bounded inverse. The similar situation in the context of the integro-differential equations occurred also in works [36] and [39]. The problems considered there also required the application of the orthogonality relations. The contraction argument was used in [32] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear problem there satisfied the Fredholm property (see Assumption 1 of [32], also [10]). Let us define the closed ball in the space of N dimensions as

$$I := \left\{z \in \mathbb{R}^N \mid |z|_{\mathbb{R}^N} \leq \frac{1}{\sqrt{2}} + \frac{c_a}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\right\}, \tag{1.19}$$

along with the closed ball in the space of $C^1(I, \mathbb{R}^N)$ vector functions, namely $D_M :=$

$$\{g(z) := (g_1(z), g_2(z), ..., g_N(z)) \in C^1(I, \mathbb{R}^N) \mid \|g\|_{C^1(I,\mathbb{R}^N)} \le M\}, \tag{1.20}$$

with $M > 0$. In such context the norms

$$\|g\|_{C^1(I,\mathbb{R}^N)} := \sum_{m=1}^{N} \|g_m\|_{C^1(I)}, \tag{1.21}$$

$$\|g_m\|_{C^1(I)} := \|g_m\|_{C(I)} + \sum_{n=1}^{N} \Big\|\frac{\partial g_m}{\partial z_n}\Big\|_{C(I)}, \tag{1.22}$$

with $\|g_m\|_{C(I)} := \max_{z\in I}|g_m(z)|$. From the perspective of the biological applications, the rates of the cell birth functions are nonlinear and are trivial at the origin.

Assumption 1.2. *Let $1 \le m \le N$. Suppose that $g_m(z) : \mathbb{R}^N \to \mathbb{R}$, so that $g_m(0) = 0$. We also assume that $g(z) \in D_M$ and it does not vanish identically in the ball I.*

Note that in [40] the vector function $g(z)$ was supposed to be twice continuously differentiable in the corresponding closed ball I. We will use the following positive auxiliary expressions

$$Q_m := \max\left\{\left\|\frac{\widehat{\mathcal{K}_m}(p)}{p^2}\right\|_{L^\infty(\mathbb{R})}, \quad \left\|\frac{\widehat{\mathcal{K}_m}(p)}{p}\right\|_{L^\infty(\mathbb{R})}\right\}, \quad 1 \le m \le N. \tag{1.23}$$

Let

$$Q := \max_{1\le m\le N} Q_m. \tag{1.24}$$

Clearly, $Q < \infty$ under the conditions of Theorem 1.3 below by means of the result of Lemma 4.1.

We introduce the operator T_g, such that $u = T_g v$, where u is a solution of the system of equations (1.18). Our first main result is as follows.

Theorem 1.3. *Let Assumptions 1.1 and 1.2 hold. Then for every $\rho \in (0, 1]$ the system of equations (1.18) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all*

$$0 < \varepsilon \le \frac{\rho}{2\sqrt{\pi}QM(1 + c_a\|w(x)\|_{H^1(\mathbb{R})})}. \tag{1.25}$$

The unique fixed point $u_p(x)$ of this map T_g is the only solution of system (1.16) in B_ρ.

Clearly, the resulting solution of the system of equations (1.2) given by (1.15) will not vanish identically on the real line because $g_m(0) = 0, \ \alpha_m \neq 0, \ 1 \leq m \leq N$ due to our assumptions.

Our second main proposition is about the continuity of the cumulative solution of the system of equations (1.2) given by formula (1.15) with respect to the nonlinear vector function g. We introduce the following positive, technical expression

$$\sigma := 2\sqrt{\pi} Q M c_a \|w(x)\|_{H^1(\mathbb{R})}. \tag{1.26}$$

Theorem 1.4. *Let $j = 1, 2$ and suppose that the assumumptions of Theorem 1.3 are valid, so that $u_{p,j}(x)$ is the unique fixed point of the map $T_{g_j} : B_\rho \to B_\rho$, which is a strict contraction for all the values of ε satisfying bound (1.25) and the resulting solution of the system of equations (1.2) with $g(z) = g_j(z)$ is given by*

$$u_j(x) = u_0(x) + u_{p,j}(x). \tag{1.27}$$

Then for all ε satisfying inequality (1.25), the estimate from above

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R}, \mathbb{R}^N)} \leq$$

$$\leq \frac{2\sqrt{\pi}\varepsilon Q(1 + c_a\|w(x)\|_{H^1(\mathbb{R})})}{1 - \varepsilon\sigma} \|g_1(z) - g_2(z)\|_{C^1(I, \mathbb{R}^N)} \tag{1.28}$$

holds.

Let us proceed to the proof of our first main proposition.

2. The existence of the perturbed solution

Proof of Theorem 1.3. We choose arbitrarily $v(x) \in B_\rho$. The term contained in the integral expression in the right side of the system of equations (1.18) is denoted as

$$G_m(x) := g_m(w(x)[u_0(x) + v(x)]), \quad 1 \leq m \leq N.$$

Let us use the standard Fourier transform, which is given by

$$\widehat{\phi}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{-ipx} dx, \quad p \in \mathbb{R}. \tag{2.1}$$

Evidently, the inequality

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}} \|\phi(x)\|_{L^1(\mathbb{R})} \tag{2.2}$$

is valid. We apply (2.1) to both sides of system (1.18). This gives us

$$\widehat{u_m}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)\widehat{G_m}(p)}{p^2}, \quad p\widehat{u_m}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)\widehat{G_m}(p)}{p},$$

with $1 \le m \le N$. Thus,

$$|\widehat{u_m}(p)| \le \varepsilon\sqrt{2\pi} Q |\widehat{G_m}(p)|, \quad |p\widehat{u_m}(p)| \le \varepsilon\sqrt{2\pi} Q |\widehat{G_m}(p)|, \quad 1 \le m \le N. \tag{2.3}$$

Here Q is given by (1.24). It is finite by virtue of Lemma 4.1 below under our assumptions. By means of formulas (1.11) and (1.12) along with estimates (2.3) we trivially obtain the upper bound for the norm as

$$\|u(x)\|^2_{H^1(\mathbb{R},\mathbb{R}^N)} \le 4\pi\varepsilon^2 Q^2 \sum_{m=1}^{N} \|G_m(x)\|^2_{L^2(\mathbb{R})}. \tag{2.4}$$

It can be easily established that for $v(x) \in B_\rho$, we have

$$|w(x)[u_0(x) + v(x)]|_{\mathbb{R}^N} \le \frac{1}{\sqrt{2}}\{1 + c_a \|w(x)\|_{H^1(\mathbb{R})}\}. \tag{2.5}$$

Indeed, the left side of (2.5) can be trivially estimated from above via the triangle inequality by

$$|w(x)u_0(x)|_{\mathbb{R}^N} + |w(x)v(x)|_{\mathbb{R}^N}. \tag{2.6}$$

To treat the first term in (2.6), we recall inequalities (1.13) and (1.9). Hence,

$$|w(x)R(x)||\alpha|_{\mathbb{R}^N} \le \|w(x)R(x)\|_{L^\infty(\mathbb{R})}|\alpha|_{\mathbb{R}^N} \le$$

$$\le \frac{1}{\sqrt{2}}\|w(x)R(x)\|_{H^1(\mathbb{R})}|\alpha|_{\mathbb{R}^N} \le \frac{1}{\sqrt{2}}.$$

To derive the upper bound on the second term in (2.6), we use (1.13) and (1.14), such that

$$\sqrt{\sum_{m=1}^{N} |w(x)v_m(x)|^2} \le \sqrt{\sum_{m=1}^{N} \|w(x)v_m(x)\|^2_{L^\infty(\mathbb{R})}} \le$$

$$\le \sqrt{\sum_{m=1}^{N} \frac{1}{2}\|w(x)v_m(x)\|^2_{H^1(\mathbb{R})}} \le \frac{c_a}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\sqrt{\sum_{m=1}^{N}\|v_m(x)\|^2_{H^1(\mathbb{R})}} =$$

$$= \frac{c_a}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\|v(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le \frac{c_a}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}.$$

Thus, (2.5) is valid. Analogously, for $v(x) \in B_\rho$

$$\|w(x)[u_0(x) + v(x)]\|_{L^2(\mathbb{R},\mathbb{R}^N)} \le 1 + c_a\|w(x)\|_{H^1(\mathbb{R})}. \tag{2.7}$$

Clearly, the left side of (2.7) can be easily estimated from above by virtue of the triangle inequality by

$$\|w(x)u_0(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)} + \|w(x)v(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)}. \tag{2.8}$$

For the first term in (2.8), we derive using (1.9) that

$$|\alpha|_{\mathbb{R}^N}\|w(x)R(x)\|_{L^2(\mathbb{R})} \le |\alpha|_{\mathbb{R}^N}\|w(x)R(x)\|_{H^1(\mathbb{R})} \le 1.$$

For the second term in (2.8), we apply (1.14) to arrive at

$$\|w(x)v(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)} \le \|w(x)v(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le$$

$$\le c_a\|w(x)\|_{H^1(\mathbb{R})}\|v(x)\|_{H^1(\mathbb{R},\mathbb{R}^N))} \le c_a\|w(x)\|_{H^1(\mathbb{R})}.$$

Hence, (2.7) holds. Let us use Assumption 1.2 to express

$$G_m(x) = \int_0^1 \nabla g_m(tw(x)[u_0(x)+v(x)]).w(x)[u_0(x)+v(x)]dt, \quad 1 \le m \le N.$$

Here and below the dot stands for the scalar product of the two vectors in our space of N dimensions. Obviously, by means of (2.5)

$$|G_m(x)| \le \text{sup}_{z\in I}|\nabla g_m(z)|_{\mathbb{R}^N}|w(x)[u_0(x)+v(x)]|_{\mathbb{R}^N}, \quad 1 \le m \le N,$$

with the ball I defined in (1.19). Using inequality (2.7), we obtain

$$\|G_m(x)\|^2_{L^2(\mathbb{R})} \le [\text{sup}_{z\in I}|\nabla g_m(z)|_{\mathbb{R}^N}]^2\|w(x)[u_0(x)+v(x)]\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \le$$

$$\le [\text{sup}_{z\in I}|\nabla g_m(z)|_{\mathbb{R}^N}]^2(1+c_a\|w(x)\|_{H^1(\mathbb{R})})^2, \quad 1 \le m \le N. \tag{2.9}$$

Upper bounds (2.4) and (2.9) give us

$$\|u(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le 2\sqrt{\pi}\varepsilon QM(1+c_a\|w(x)\|_{H^1(\mathbb{R})}) \le \rho \tag{2.10}$$

for all the values of the parameter ε, which satisfy condition (1.25). Therefore, $u(x) \in B_\rho$ as well.

Suppose for a certain $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of the system of equations (1.18). Clearly, their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R},\mathbb{R}^N)$ satisfies the homogeneous system

$$-\frac{d^2}{dx^2}w_m(x) = 0, \quad 1 \le m \le N.$$

But the negative second derivative operator considered on the whole real line does not have any nontrivial square integrable zero modes. Therefore, $w(x)$ vanishes identically in $\mathbb{R}$. Hence, the system of equations (1.18) defines a map $T_g : B_\rho \to B_\rho$ for all the values of ε, which satisfy inequality (1.25).

Let us demonstrate that under the stated assumptions such map is a strict contraction. We choose arbitrarily $v_{1,2}(x) \in B_\rho$. The argument above yields that $u_{1,2} := T_g v_{1,2} \in B_\rho$ as well for ε satisfying bound (1.25). By virtue of (1.18), we have for $1 \le m \le N$

$$-\frac{d^2 u_{1,m}(x)}{dx^2} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(w(y)[u_0(y)+v_1(y)])dy, \tag{2.11}$$

$$-\frac{d^2 u_{2,m}(x)}{dx^2} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(w(y)[u_0(y)+v_2(y)])dy. \tag{2.12}$$

Let us introduce

$$G_{1,m}(x) := g_m(w(x)[u_0(x)+v_1(x)]), \quad G_{2,m}(x) := g_m(w(x)[u_0(x)+v_2(x)]),$$

with $1 \le m \le N$. We apply the standard Fourier transform (2.1) to both sides of the systems of equations (2.11) and (2.12). This gives us

$$\widehat{u_{1,m}}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)\widehat{G_{1,m}}(p)}{p^2}, \quad \widehat{u_{2,m}}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)\widehat{G_{2,m}}(p)}{p^2},$$

so that

$$\widehat{u_{1,m}}(p) - \widehat{u_{2,m}}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)[\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)]}{p^2},$$

$$p[\widehat{u_{1,m}}(p) - \widehat{u_{2,m}}(p)] = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)[\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)]}{p}, \quad 1 \le m \le N.$$

Thus, the estimates

$$|\widehat{u_{1,m}}(p) - \widehat{u_{2,m}}(p)| \le \varepsilon \sqrt{2\pi} Q |\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)|,$$

$$|p[\widehat{u_{1,m}}(p) - \widehat{u_{2,m}}(p)]| \le \varepsilon \sqrt{2\pi} Q |\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)|, \quad 1 \le m \le N$$

are valid. This enables us to derive the upper bound on the norm using (1.11) and (1.12) as $\|u_1(x) - u_2(x)\|^2_{H^1(\mathbb{R},\mathbb{R}^N)} =$

$$= \sum_{m=1}^{N} \left\{ \int_{-\infty}^{\infty} |\widehat{u_{1,m}}(p) - \widehat{u_{2,m}}(p)|^2 dp + \int_{-\infty}^{\infty} |p(\widehat{u_{1,m}}(p) - \widehat{u_{2,m}}(p))|^2 dp \right\} \le$$

$$\le 4\pi\varepsilon^2 Q^2 \sum_{m=1}^{N} \|G_{1,m}(x) - G_{2,m}(x)\|^2_{L^2(\mathbb{R})},$$

so that

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le 2\sqrt{\pi}\varepsilon Q \sqrt{\sum_{m=1}^{N} \|G_{1,m}(x) - G_{2,m}(x)\|^2_{L^2(\mathbb{R})}}. \tag{2.13}$$

Obviously, we can write for $1 \le m \le N$ that $G_{1,m}(x) - G_{2,m}(x) =$

$$= \int_0^1 \nabla g_m(w(x)[u_0(x) + tv_1(x) + (1-t)v_2(x)]).w(x)[v_1(x) - v_2(x)]dt.$$

Evidently, for $t \in [0,1]$

$$\|tv_1(x) + (1-t)v_2(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le t\|v_1(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} + (1-t)\|v_2(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le$$

$$\le \rho,$$

so that $tv_1(x) + (1-t)v_2(x) \in B_\rho$. Let us recall inequality (2.5). Hence,

$$|w(x)[u_0(x) + tv_1(x) + (1-t)v_2(x)]|_{\mathbb{R}^N} \le \frac{1}{\sqrt{2}}\{1 + c_a\|w(x)\|_{H^1(\mathbb{R})}\}.$$

Therefore, we arrive at

$$|G_{1,m}(x) - G_{2,m}(x)| \le \sup_{z\in I}|\nabla g_m(z)|_{\mathbb{R}^N}|w(x)(v_1(x) - v_2(x))|_{\mathbb{R}^N} \le$$

$$\le \|g_m\|_{C^1(I)}|w(x)(v_1(x) - v_2(x))|_{\mathbb{R}^N}, \quad 1 \le m \le N,$$

where the ball I is defined in (1.19). By means of (1.14),

$$\|G_{1,m}(x) - G_{2,m}(x)\|^2_{L^2(\mathbb{R})} \le \|g_m\|^2_{C^1(I)} \sum_{k=1}^N \|w(x)(v_{1,k}(x) - v_{2,k}(x))\|^2_{H^1(\mathbb{R})} \le$$

$$\le \|g_m\|^2_{C^1(I)} c_a^2 \|w(x)\|^2_{H^1(\mathbb{R})} \|v_1(x) - v_2(x)\|^2_{H^1(\mathbb{R},\mathbb{R}^N)}, \quad 1 \le m \le N. \tag{2.14}$$

By virtue of (2.13), (2.14) and Assumption 1.2, we obtain

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le$$

$$\le 2\sqrt{\pi}\varepsilon QMc_a\|w(x)\|_{H^1(\mathbb{R})}\|v_1(x) - v_2(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)}. \tag{2.15}$$

We have

$$\frac{\rho}{2\sqrt{\pi}QM(1 + c_a\|w(x)\|_{H^1(\mathbb{R})})} < \frac{1}{2\sqrt{\pi}QMc_a\|w(x)\|_{H^1(\mathbb{R})}}.$$

Hence, by means of (1.25) for our parameter ε we arrive at

$$0 < \varepsilon < \frac{1}{2\sqrt{\pi}QMc_a\|w(x)\|_{H^1(\mathbb{R})}}.$$

Thus, the constant in the right side of estimate (2.15) is less than one. This implies that the map $T_g : B_\rho \to B_\rho$ defined by the system of equations (1.18) is a strict contraction for all the values of the parameter ε satisfying inequality (1.25). Its unique fixed point $u_p(x)$ is the only solution of system (1.16) in the ball B_ρ. By virtue of (2.10), we have $\|u_p(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$. The resulting $u(x)$ given by formula (1.15) is a solution of the system of equations (1.2). ■

We turn our attention to demonstrating the validity of the second main statement of our work.

3. The continuity of the resulting solution

Proof of Theorem 1.4. Obviously, for all the values of the parameter ε, which satisfy bound (1.25), we have

$$u_{p,1} = T_{g_1} u_{p,1}, \quad u_{p,2} = T_{g_2} u_{p,2}.$$

Thus,

$$u_{p,1} - u_{p,2} = T_{g_1} u_{p,1} - T_{g_1} u_{p,2} + T_{g_1} u_{p,2} - T_{g_2} u_{p,2},$$

so that

$$\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le \|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} + \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}.$$

Using estimate (2.15), we derive

$$\|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le \varepsilon\sigma \|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)},$$

where σ is defined in (1.26). Clearly, $\varepsilon\sigma < 1$, since the map $T_{g_1} : B_\rho \to B_\rho$ is a strict contraction under our assumptions. Therefore,

$$(1 - \varepsilon\sigma)\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}. \tag{3.1}$$

Evidently, for our fixed point $T_{g_2} u_{p,2} = u_{p,2}$. Let us introduce $\eta(x) := T_{g_1} u_{p,2}$. Explicity, for $1 \le m \le N$, we have

$$-\frac{d^2\eta_m(x)}{dx^2} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_{1,m}(w(y)[u_0(y) + u_{p,2}(y)])dy, \tag{3.2}$$

$$-\frac{d^2 u_{p,2,m}(x)}{dx^2} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_{2,m}(w(y)[u_0(y) + u_{p,2}(y)])dy. \tag{3.3}$$

Let us define

$$G_{1,2,m}(x) := g_{1,m}(w(x)[u_0(x) + u_{p,2}(x)]),$$
$$G_{2,2,m}(x) := g_{2,m}(w(x)[u_0(x) + u_{p,2}(x)]), \quad 1 \le m \le N.$$

We apply the standard Fourier transform (2.1) to both sides of systems (3.2) and (3.3) above and arrive at

$$\widehat{\eta_m}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)\widehat{G_{1,2,m}}(p)}{p^2}, \quad \widehat{u_{p,2,m}}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)\widehat{G_{2,2,m}}(p)}{p^2},$$

with $1 \le m \le N$, so that

$$\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)}{p^2} [\widehat{G_{1,2,m}}(p) - \widehat{G_{2,2,m}}(p)],$$

$$p[\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)] = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)}{p} [\widehat{G_{1,2,m}}(p) - \widehat{G_{2,2,m}}(p)].$$

Obviously, the estimates from above

$$|\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)| \le \varepsilon \sqrt{2\pi} Q |\widehat{G_{1,2,m}}(p) - \widehat{G_{2,2,m}}(p)|, \tag{3.4}$$

$$|p[\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)]| \le \varepsilon \sqrt{2\pi} Q |\widehat{G_{1,2,m}}(p) - \widehat{G_{2,2,m}}(p)| \tag{3.5}$$

are valid for $1 \le m \le N$. Inequality (3.4) gives us

$$\|\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} |\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)|^2 dp \le$$

$$\le 2\pi\varepsilon^2 Q^2 \|G_{1,2,m}(x) - G_{2,2,m}(x)\|_{L^2(\mathbb{R})}^2, \quad 1 \le m \le N. \tag{3.6}$$

Analogously, by virtue of (3.5) we derive

$$\|p[\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)]\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} |p[\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)]|^2 dp \le$$

$$\le 2\pi\varepsilon^2 Q^2 \|G_{1,2,m}(x) - G_{2,2,m}(x)\|_{L^2(\mathbb{R})}^2, \quad 1 \le m \le N. \tag{3.7}$$

Formulas (1.10), (1.11), (1.12), (3.6) and (3.7) imply that

$$\|\eta(x) - u_{p,2}(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)}^2 =$$

$$= \sum_{m=1}^{N} \{\|\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)\|_{L^2(\mathbb{R})}^2 + \|p[\widehat{\eta_m}(p) - \widehat{u_{p,2,m}}(p)]\|_{L^2(\mathbb{R})}^2\} \le$$

$$\le 4\pi\varepsilon^2 Q^2 \sum_{m=1}^{N} \|G_{1,2,m}(x) - G_{2,2,m}(x)\|_{L^2(\mathbb{R})}^2,$$

so that

$$\|\eta(x) - u_{p,2}(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le 2\sqrt{\pi}\varepsilon Q \sqrt{\sum_{m=1}^{N} \|G_{1,2,m}(x) - G_{2,2,m}(x)\|_{L^2(\mathbb{R})}^2}. \tag{3.8}$$

Evidently, for $1 \le m \le N$ we have $G_{1,2,m}(x) - G_{2,2,m}(x) =$

$$= \int_0^1 \nabla[g_{1,m} - g_{2,m}](tw(x)[u_0(x) + u_{p,2}(x)]).w(x)[u_0(x) + u_{p,2}(x)]dt.$$

Let us recall inequality (2.5). Hence,

$$|tw(x)[u_0(x)+u_{p,2}(x)]|_{\mathbb{R}^N} \le \frac{1}{\sqrt{2}}\{1+c_a\|w(x)\|_{H^1(\mathbb{R})}\}, \quad t\in[0,1].$$

We easily obtain $|G_{1,2,m}(x)-G_{2,2,m}(x)| \le$

$$\le \sup_{z\in I}|\nabla[g_{1,m}-g_{2,m}](z)|_{\mathbb{R}^N}|w(x)[u_0(x)+u_{p,2}(x)]|_{\mathbb{R}^N} \le$$

$$\le \|g_{1,m}-g_{2,m}\|_{C^1(I)}|w(x)[u_0(x)+u_{p,2}(x)]|_{\mathbb{R}^N}, \quad 1\le m\le N,$$

with the ball I defined in (1.19). This enables us to derive the upper bound on the norm via (2.7) as

$$\|G_{1,2,m}(x)-G_{2,2,m}(x)\|^2_{L^2(\mathbb{R})} \le$$

$$\le \|g_{1,m}-g_{2,m}\|^2_{C^1(I)}\|w(x)[u_0(x)+u_{p,2}(x)]\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \le$$

$$\le \|g_{1,m}-g_{2,m}\|^2_{C^1(I)}(1+c_a\|w(x)\|_{H^1(\mathbb{R})})^2, \quad 1\le m\le N. \tag{3.9}$$

Using estimates (3.8) and (3.9), we arrive at $\|\eta(x)-u_{p,2}(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le$

$$\le 2\sqrt{\pi}\varepsilon Q(1+c_a\|w(x)\|_{H^1(\mathbb{R})})\|g_1(z)-g_2(z)\|_{C^1(I,\mathbb{R}^N)}. \tag{3.10}$$

By virtue of (3.1) and (3.10), the inequality $\|u_{p,1}(x)-u_{p,2}(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le$

$$\le \frac{2\sqrt{\pi}\varepsilon Q(1+c_a\|w(x)\|_{H^1(\mathbb{R})})}{1-\varepsilon\sigma}\|g_1(z)-g_2(z)\|_{C^1(I,\mathbb{R}^N)} \tag{3.11}$$

is valid. By means of (1.27) and (3.11), bound (1.28) holds. ■

4. Auxiliary results

Let us derive the conditions under which the expressions $Q_m,\ 1\le m\le N$ introduced in (1.23) are finite. We designate the inner product as

$$(f(x),g(x))_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx, \tag{4.1}$$

with a slight abuse of notations when the functions involved in (4.1) do not belong to $L^2(\mathbb{R})$, like for instance the ones present in the first orthogonality condition in (4.2) of Lemma 4.1 below. Indeed, if $f(x)\in L^1(\mathbb{R})$ and $g(x)\in L^\infty(\mathbb{R})$, then the integral in the right side of (4.1) is well defined. The technical lemma below is the adaptation of the one established in [33] for the purpose of the study of the single integro-differential equation with the influx/efflux term proportional to the delta function, analogous to system (1.1) (see also the part b) of Lemma A1 of [36]). Let us present it here for the convenience of the readers.

Lemma 4.1. *Let $1 \leq m \leq N$, the functions $\mathcal{K}_m(x) : \mathbb{R} \to \mathbb{R}$ do not vanish identically on the real line, so that $\mathcal{K}_m(x),\ x^2\mathcal{K}_m(x) \in L^1(\mathbb{R})$. Then Q_m is finite if and only if the orthogonality relations*

$$(\mathcal{K}_m(x), 1)_{L^2(\mathbb{R})} = 0, \quad (\mathcal{K}_m(x), x)_{L^2(\mathbb{R})} = 0 \tag{4.2}$$

are valid.

Proof. First we obtain the upper bound on the norm as

$$\|x\mathcal{K}_m(x)\|_{L^1(\mathbb{R})} = \int_{|x|\leq 1} |x||\mathcal{K}_m(x)|dx + \int_{|x|>1} |x||\mathcal{K}_m(x)|dx \leq$$

$$\leq \|\mathcal{K}_m(x)\|_{L^1(\mathbb{R})} + \|x^2\mathcal{K}_m(x)\|_{L^1(\mathbb{R})} < \infty, \quad 1 \leq m \leq N$$

due to our assumptions. Thus, the expression in the left side of the second orthogonality condition in (4.2) makes sense. Let us establish that if $\frac{\widehat{\mathcal{K}_m}(p)}{p^2} \in L^\infty(\mathbb{R})$, then $\frac{\widehat{\mathcal{K}_m}(p)}{p}$ is also bounded. Obviously,

$$\frac{\widehat{\mathcal{K}_m}(p)}{p} = \frac{\widehat{\mathcal{K}_m}(p)}{p}\chi_{\{|p|\leq 1\}} + \frac{\widehat{\mathcal{K}_m}(p)}{p}\chi_{\{|p|>1\}}, \quad 1 \leq m \leq N. \tag{4.3}$$

Here and further down χ_A will stand for the characteristic function of a set $A \subseteq \mathbb{R}$. Clearly, the second term in the right side of (4.3) can be estimated from above in the absolute value by virtue of (2.2) by

$$|\widehat{\mathcal{K}_m}(p)| \leq \frac{1}{\sqrt{2\pi}}\|\mathcal{K}_m(x)\|_{L^1(\mathbb{R})} < \infty, \quad 1 \leq m \leq N$$

as assumed. The first term in the right side of (4.3) can be easily bounded from above in the absolute value as

$$\left|\frac{\widehat{\mathcal{K}_m}(p)}{p^2}\right|\chi_{\{|p|\leq 1\}}|p| \leq \left\|\frac{\widehat{\mathcal{K}_m}(p)}{p^2}\right\|_{L^\infty(\mathbb{R})} < \infty, \quad 1 \leq m \leq N$$

due to our assumption. Thus, $\frac{\widehat{\mathcal{K}_m}(p)}{p} \in L^\infty(\mathbb{R})$ as well. Evidently,

$$\frac{\widehat{\mathcal{K}_m}(p)}{p^2} = \frac{\widehat{\mathcal{K}_m}(p)}{p^2}\chi_{\{|p|\leq 1\}} + \frac{\widehat{\mathcal{K}_m}(p)}{p^2}\chi_{\{|p|>1\}}, \quad 1 \leq m \leq N. \tag{4.4}$$

By virtue of (2.2), we arrive at

$$\left|\frac{\widehat{\mathcal{K}_m}(p)}{p^2}\chi_{\{|p|>1\}}\right| \leq |\widehat{\mathcal{K}_m}(p)| \leq \frac{1}{\sqrt{2\pi}}\|\mathcal{K}_m(x)\|_{L^1(\mathbb{R})} < \infty, \quad 1 \leq m \leq N$$

by means of the one of our assumptions. Clearly, we can write

$$\widehat{\mathcal{K}_m}(p) = \widehat{\mathcal{K}_m}(0) + p\frac{d\widehat{\mathcal{K}_m}}{dp}(0) + \int_0^p \left(\int_0^s \frac{d^2\widehat{\mathcal{K}_m}(q)}{dq^2} dq \right) ds, \quad 1 \le m \le N.$$

Hence, the first term in the right side of (4.4) equals to

$$\left[\frac{\widehat{\mathcal{K}_m}(0)}{p^2} + \frac{\frac{d\widehat{\mathcal{K}_m}}{dp}(0)}{p} + \frac{\int_0^p \left(\int_0^s \frac{d^2\widehat{\mathcal{K}_m}(q)}{dq^2} dq \right) ds}{p^2} \right] \chi_{\{|p| \le 1\}}, \quad 1 \le m \le N. \tag{4.5}$$

The definition of the standard Fourier transform (2.1) yields

$$\left| \frac{d^2\widehat{\mathcal{K}_m}(p)}{dp^2} \right| \le \frac{1}{\sqrt{2\pi}} \|x^2 \mathcal{K}_m(x)\|_{L^1(\mathbb{R})}, \quad 1 \le m \le N,$$

so that

$$\left| \frac{\int_0^p \left(\int_0^s \frac{d^2\widehat{\mathcal{K}_m}(q)}{dq^2} dq \right) ds}{p^2} \chi_{\{|p| \le 1\}} \right| \le \frac{1}{2\sqrt{2\pi}} \|x^2 \mathcal{K}_m(x)\|_{L^1(\mathbb{R})} < \infty, \quad 1 \le m \le N$$

via the one of our assumptions. By virtue of definition (2.1), we have

$$\widehat{\mathcal{K}_m}(0) = \frac{1}{\sqrt{2\pi}} (\mathcal{K}_m(x), 1)_{L^2(\mathbb{R})}, \quad \frac{d\widehat{\mathcal{K}_m}}{dp}(0) = -\frac{i}{\sqrt{2\pi}} (\mathcal{K}_m(x), x)_{L^2(\mathbb{R})}$$

with $1 \le m \le N$. This allows us to express the sum of the first two terms in (4.5) as

$$\left[\frac{(\mathcal{K}_m(x), 1)_{L^2(\mathbb{R})}}{\sqrt{2\pi} p^2} - i \frac{(\mathcal{K}_m(x), x)_{L^2(\mathbb{R})}}{\sqrt{2\pi} p} \right] \chi_{\{|p| \le 1\}}, \quad 1 \le m \le N. \tag{4.6}$$

Obviously, each expression (4.6) is bounded if and only if orthogonality relations (4.2) are valid. ■

Let us recall the earlier work [40]. As distinct from the present article, the argument there did not use the orthogonality conditions.

Acknowledgements

The second author has been supported by the RUDN University Strategic Academic Leadership Program.

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