DIFFUSIVE PROCESSES MODELED ON THE SPECTRAL FRACTIONAL LAPLACIAN WITH DIRICHLET AND NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. In this paper we provide a comprehensive approach to the spectral fractional heat equation that combines purely analytic and probabilistic perspectives.

Furthermore, we give two new results on the monotonicity properties of the spectral fractional heat diffusion with respect to the fractional parameter.

The first result deals with the spectral fractional heat kernel, evaluated at the initial singularity. The second result considers the probability for the corresponding stochastic process of being confined in a subregion of the domain.

In both results, the monotonicity property depends on the size of the first non-zero eigenvalue. The cases of homogeneous Dirichlet and Neumann boundary conditions are addressed in details.

1. Introduction

In this paper we consider the heat equation driven by the spectral fractional Laplacian, both with Dirichlet and Neumann data, and we obtain two original results concerning the monotonicity properties of the corresponding heat kernel with respect to the fractional parameter. In doing so, we revisit the analytic and probabilistic framework related to the fractional diffusion of spectral type, providing a unified and essentially self-contained setting.

To introduce these results at a colloquial level, we will denote by $r^s(t,x,y)$ the solution of the spectral fractional heat equation at time $t>0$ evaluated at a point $x\in\Omega$, with initial condition at time $t=0$ given by a Dirac delta function concentrated at a given point $y\in\Omega$.

For our purposes, Ω is an open, connected and bounded subset of $\mathbb{R}^n$ with smooth boundary, which allows us to expand functions in $L^2(\Omega)$ through a basis of eigenfunctions. The action of the fractional diffusive operator is therefore modelled through a power of the corresponding eigenvalues and the superscript s in the notation $r^s(t,x,y)$ refers to the fractional power $s\in(0,1]$ that we take into account. More specifically, the power $s=1$ would correspond to the classical Laplacian and thus to the classical heat kernel and the eigenfunctions and eigenvalues are considered here to be either with homogeneous Dirichlet boundary conditions (in which case the corresponding fractional diffusion kernel will be denoted by $r^s_D(t,x,x)$) or with homogeneous Neumann boundary conditions (in which case the corresponding fractional diffusion kernel will be denoted by $r^s_N(t,x,x)$).

The rigorous details related to these types of fractional diffusion will be presented in Sections 1.1 and 1.2, for the moment we stick to the notation $r^s(t,x,y)$ to represent the corresponding heat kernel, either with Dirichlet or Neumann condition. In this setting, our first original contribution focuses on the monotonicity of the function $r^s(t,x,x)$ with respect to the fractional parameter $s\in(0,1]$, for given $t\in(0,+\infty)$ and $x\in\Omega$.

This question has a concrete meaning since the function $r^s(t,x,x)$ measures the amount of heat remaining at time t precisely at the point corresponding to the initial heat singularity: roughly speaking, the larger this function is, the more persistent the "memory of the initial singularity" is,

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while the smaller this function is, the easier for the system to "discharge the initial singularity" away from its original location.

Interestingly, our result shows that the monotonicity of the spectral fractional heat kernel $r^s(t,x,x)$ with respect to the fractional parameter s depends on the geometry of the domain Ω encoded by the first non-zero eigenvalue. While the precise results will be provided in Theorems 1.10 and 1.22, we can anticipate their statements as follows:

Theorem 1.1. *If the first non-zero eigenvalue is bigger than or equal to* 1*, then the fractional heat kernel* $r^s(t,x,x)$ *is monotone decreasing in* s*.*

If instead the first non-zero eigenvalue is strictly less than 1*, then for every* $0 < s_0 < s_1 \leqslant 1$ *there exists some* $T > 0$ *such that for each* $t > T$

$$r^{s_0}(t,x,x) < r^{s_1}(t,x,x).$$

Additional comments relating the physical aspects of the fractional diffusion with the specific geometry of the domains considered will be provided after Theorems 1.10 and 1.22, once the formal mathematical setting has been fully developed.

The second original result that we present in this paper deals with the conditional probability for the stochastic process corresponding to the fractional heat equation to remain confined in a subregion Ω' of Ω. Roughly speaking, and for the moment surfing over the technical complications of a precise notation (which will be detailed in the forthcoming Sections 1.1 and 1.2), one could denote by Y_t the stochastic process corresponding to the spectral fractional diffusion (indices related to the boundary conditions can be added to be more specific, but here we aim at comprising both Dirichlet and Neumann data in a unified notation, for the sake of simplicity). Thus, one can also denote by $\mathbb{P}^s$ the probability notion related to such a process and one looks at the conditional probability for the stochastic process to lie in the subregion Ω' at time t, given that the initial location of the process was also in Ω'. Denoting this conditional probability by $\mathbb{P}^s(Y_t \in \Omega' | Y_0 \in \Omega')$, we obtain:

Theorem 1.2. *If the first non-zero eigenvalue is bigger than or equal to* 1*, then the conditional probability* $\mathbb{P}^s(Y_t \in \Omega' | Y_0 \in \Omega')$ *is monotone decreasing in* s*.*

If instead the first non-zero eigenvalue is strictly less than 1*, then for every* $0 < s_0 < s_1 \leqslant 1$ *there exists some* $T > 0$ *such that for each* $t > T$

$$\mathbb{P}^{s_0}(Y_t \in \Omega' | Y_0 \in \Omega') < \mathbb{P}^{s_1}(Y_t \in \Omega' | Y_0 \in \Omega').$$

A precise version of this result, accounting also for the specific boundary conditions, will be given in Theorems 1.11 and 1.23.

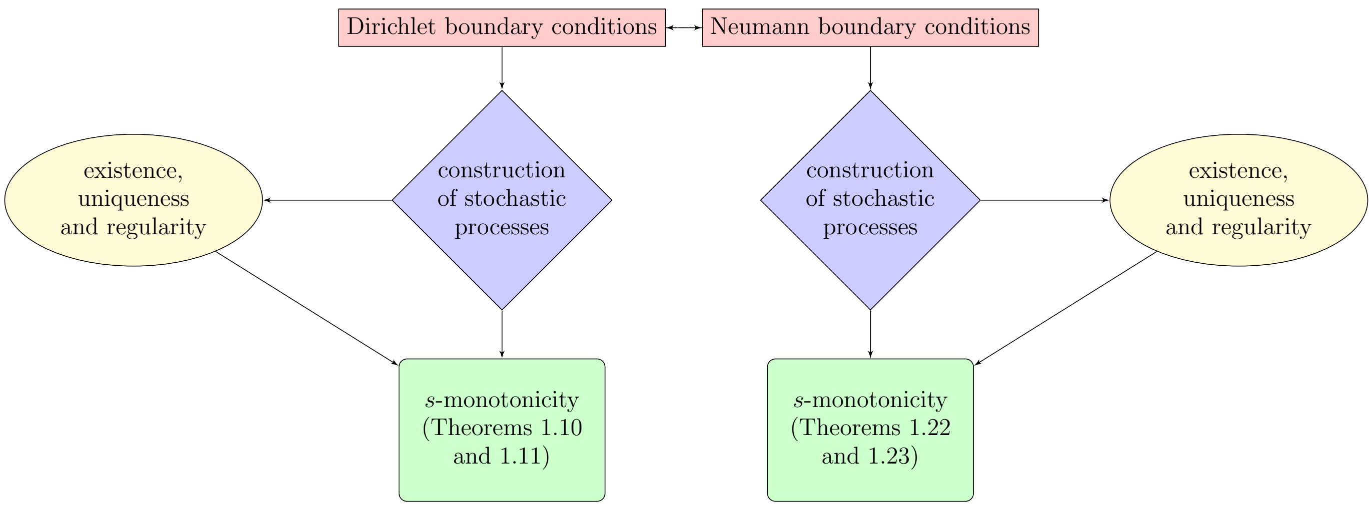
Theorems 1.1 and 1.2 (or, better to say, their rigorous formulation in Theorems 1.10, 1.11, 1.22 and 1.23) will be exploited in our forthcoming work [DGV] related to foraging models and hunting strategies in the study of animal behavior.

Moreover, giving a detailed formulation of these original results and providing thorough proofs of them offered us the occasion in this paper to provide an exhaustive analytic presentation of the spectral fractional Laplacian and its connections with the heat equation. Additionally, in this paper, the probabilistic methods are reviewed and compared to the analytic ones, providing a somewhat unified setting.

We also mention that the reason for which we focus here on the spectral version of the fractional Laplacian (rather than on other fractional possibilities appearing in the literature) consists mainly in its facility of dealing with different boundary conditions at the same time and in a rather natural way. For example, Dirichlet conditions for the integral (instead of spectral) fractional Laplacian need to be assigned in the whole complement of the domain, making practical computations sometimes complicated. Even more importantly, Neumann conditions for the integral fractional Laplacian appear at least in two different formulations, compare e.g. [FJ15] with [DROV17]: while one of these formulations has a clean geometric meaning, it appears to be often unfeasible from the point of view of variational methods; on the other hand, the other formulation does appear to be more amenable in terms of functional analysis and possesses a neat probabilistic interpretation (see [DV21a]), but its global setting as an integral prescription in the complement of the domain makes often explicit calculations quite unpractical.

The spectral formulation of the fractional problem presents instead the advantage of offering a natural way to encode different sorts of boundary conditions into a setting which is easy to introduce and structurally consistent with classical functional analysis. The price to pay for this convenience is however that the probabilistic setting related to the spectral fractional Laplacian is somewhat more sophisticated and possibly less transparent to non-specialists: to compensate for this disadvantage we try to offer in this paper a setting which is as accessible as possible to a broad community of readers.

To provide the reader with a glimpse of the conceptual structure of this paper, we give the following flowchart:



In further detail, this paper is organized as follows.

In Sections 1.1 and 1.2 we introduce the precise mathematical framework that will be used throughout the paper, and state the main results that will be proved. Here we define the Dirichlet and Neumann spectral fractional heat equation and we state the theorems regarding the existence, regularity and uniqueness of the solution to these systems of equations (see Theorems 1.8 and 1.20 respectively for the Dirichlet and Neumann case). Moreover, we provide a Maximum Principle for both the Dirichlet equation (Theorem 1.9) and Neumann equation (Theorem 1.21).

At the end of these sections we present the original results of this article. Specifically, in Theorem 1.10 (resp., Theorem 1.22), we give a precise statement of Theorem 1.1 for the Dirichlet case (resp., Neumann case). Analogously, in Theorem 1.11 (resp., 1.23), we rephrase Theorem 1.2 with a detailed probabilistic notation for the Dirichlet case (resp., Neumann case).

In Section 2.1 we provide a presentation of the stochastic process associated to the kernel r^s_D, which will be called subordinate killed Brownian motion, while in Section 2.2 we prove through analytical arguments that this kernel is the unique solution to the Dirichlet spectral fractional heat equation (1.10), giving thus a proof of Theorems 1.8 and 1.9. Finally, in Section 2.3 we prove Theorems 1.10 and 1.11.

Similarly, in Section 3.1 we introduce the main properties concerning the stochastic process associated with the Neumann kernel r^s_N, which will be called subordinate standard reflecting Brownian motion. Section 3.2, is instead devoted to an analytical approach to the Neumann spectral fractional heat equation (1.35). Here we prove Theorems 1.20 and 1.21, and therefore that the kernel r^s_N is the unique solution to the Neumann spectral fractional heat equation (1.35). The proofs of the original results for the Neumann case (Theorems 1.22 and 1.23) are given in detail in Section 3.3.

1.1. **The Dirichlet spectral fractional heat equation.**

To introduce the Dirichlet spectral fractional heat equation we need to define the *Dirichlet spectral fractional Laplacian*. We take a bounded, open and connected set $\Omega \subset \mathbb{R}^n$ with smooth boundary and an orthonormal basis $\{\phi_k\}_k$ of $L^2(\Omega)$ satisfying

$$\begin{cases} -\Delta\phi_k = \lambda_k\phi_k \ \text{ in } \ \Omega, \\ \phi_k \in H^1_0(\Omega), \end{cases} \tag{1.1}$$

where the values $0 < \lambda_1 \leqslant \lambda_2 < \lambda_3 \dots$ are the eigenvalues of the Laplace operator with homogeneous Dirichlet datum on the boundary of the domain (see for instance [Eva10, Section 6.5]). In particular, since Ω has smooth boundary, the eigenfunctions ϕ_k are all $C^\infty(\overline{\Omega})$ (namely, they are differentiable as many times as one wishes and their derivatives of any order are uniformly continuous in Ω).

Let us now consider a function $u(t,x)$, with $(t,x) \in (0,+\infty)\times\Omega$, such that $u(t,\cdot) \in L^2(\Omega)$. Then we can write the $L^2(\Omega)$ decomposition of $u(t,\cdot)$ in eigenfunctions as

$$u(t,x) = \sum_{k=1}^{+\infty} u_k(t)\,\phi_k(x), \qquad \text{with} \qquad u_k(t) := \int_\Omega u(t,y)\,\phi_k(y)\,dy, \tag{1.2}$$

where the convergence of the series in (1.2) is meant with respect to the $L^2(\Omega)$-norm.

Definition 1.3. *Let $s \in (0,1)$. We define the Dirichlet spectral fractional Laplacian as*

$$\begin{aligned} (-\Delta)^s_{D,\Omega} : H^{2s}_D(\Omega) \subset & L^2(\Omega) \to L^2(\Omega) \\ & u \longmapsto (-\Delta)^s_{D,\Omega} u := \sum_{k=1}^{+\infty} u_k\,\lambda_k^s\,\phi_k \end{aligned} \tag{1.3}$$

where

$$H^{2s}_D(\Omega) := \left\{ u \in L^2(\Omega) \ \ s.t. \ \sum_{k=1}^{+\infty} u_k^2\,\lambda_k^{2s} < +\infty \right\}. \tag{1.4}$$

We stress that the "Dirichlet spectral fractional Laplacian" in (1.3) is different from the "integral fractional Laplacian" and from the "regional fractional Laplacian" that have been also widely studied in the recent literature: see e.g. [AV19] and the references therein for similarities and differences between these fractional operators.

From now on, we denote by $p^\Omega_D(t,x,y)$ the "fundamental solution" of the heat equation in Ω with homogeneous Dirichlet condition corresponding to an initial heat density concentrated at the point $y \in \Omega$, namely $p^\Omega_D(t,x,y)$ is the solution of

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) & \text{for all} \quad (t,x) \in (0,+\infty)\times\Omega, \\ u(t,x) = 0 & \text{for all} \quad (t,x) \in (0,+\infty)\times\partial\Omega, \\ u(0,x) = \delta_y(x) & \text{for all} \quad x \in \Omega. \end{cases} \tag{1.5}$$

The function p^Ω_D is often referred to with the name of *classical Dirichlet heat kernel in* Ω.

The following result is a consequence of Theorem 1 and Theorem 9 in [Itô57], together with Lemma A.2 in Appendix A and Lemma 2.12 in Section 2.2 (to be precise, we also mention that the uniqueness in the space $C^1((0,+\infty), H^1_0(\Omega))$ can be proved with the exact same approach used to show uniqueness for the system of equations (1.10) as stated in Theorem 1.8).

Theorem 1.4. *Let Ω be open, connected, bounded and smooth. Let us define the function*

$$\begin{aligned} &p_D^\Omega : (0,+\infty)\times\Omega\times\Omega \to \mathbb{R} \\ &(t,x,\,y) \mapsto p_D^\Omega(t,x,y) := \sum_{k=1}^{+\infty} \phi_k(x)\,\phi_k(y)\,\exp(-t\lambda_k). \end{aligned} \tag{1.6}$$

Then, for each $y \in \Omega$ the function $p_D^\Omega(\cdot,\cdot,y)$ is the unique solution to (1.5) *in $C^1((0,+\infty),H_0^1(\Omega))$. Moreover, $p_D^\Omega \in C^\infty([\varepsilon,+\infty)\times\overline{\Omega}\times\overline{\Omega})$ for each $\varepsilon > 0$.*

We also recall a Maximum Principle for the Dirichlet heat kernel:

Theorem 1.5 (See Theorem 10.3 in [Bre11])**.** *Let Ω be open, bounded, connected and with smooth boundary. For each $f \in L^2(\Omega)$ and $(t,y) \in (0,+\infty)\times\Omega$ we have that*

$$\min\Big\{0, \inf_\Omega f\Big\} \leqslant \int_\Omega p_D^\Omega(t,x,y)\,f(x)\,dx \leqslant \max\Big\{0, \sup_\Omega f\Big\}. \tag{1.7}$$

It is also useful to recall that, utilizing the classical Dirichlet heat kernel, one can equivalently express the spectral fractional Laplacian via the integration against a suitable kernel, plus a linear term of order zero. The precise details of this statement go as follows:

Proposition 1.6 (See [SV03] and [AD17])**.** *Let $u \in H_D^{2s}(\Omega)$. Then, for almost every $x \in \Omega$,*

$$(-\Delta)^s_{D,\Omega} u(x) = p.v.\int_\Omega [u(x)-u(y)]\,J_D(x,y)\,dy + K_D(x)\,u(x), \tag{1.8}$$

where

$$\begin{aligned} &J_D(x,y) := \frac{s}{\Gamma(1-s)}\int_0^{+\infty} p_D^\Omega(t,x,y)\,\frac{dt}{t^{1+s}} \\ \text{and}\quad &K_D(x) := \frac{s}{\Gamma(1-s)}\int_0^{+\infty}\left(1-\int_\Omega p_D^\Omega(t,x,y)\,dy\right)\frac{dt}{t^{1+s}}. \end{aligned} \tag{1.9}$$

The right-hand side of (1.8) *is defined for each $x \in \Omega$ if $u \in C^{2s+\varepsilon}(\Omega)\cap L^1(\Omega,\delta(x)\,dx)$ for some $\varepsilon > 0$, where $\delta(x) := dist(x,\partial\Omega)$.*

The functions J_D and K_D in (1.9) are called respectively the *jumping kernel* and *killing measure*. The killing measure has a probability interpretation that relates it to the lifetime of the stochastic process generated by the Dirichlet spectral fractional Laplacian (see Remark 2.9).

As customary, the notation "*p.v.*" in (1.8) stands for "the principal value sense" and it means that the integral is intended up to the removal of a ball around its singularity in the limit sense, that is

$$p.v.\int_\Omega [u(x)-u(y)]\,J_D(x,y)\,dy := \lim_{\varepsilon\to 0^+}\int_{\Omega\setminus B_\varepsilon(x)} [u(x)-u(y)]\,J_D(x,y)\,dy.$$

Equation (1.8) can be quite useful in practice: for instance, it leads to the following Maximum Principle for $(-\Delta)^s_{D,\Omega}$:

Lemma 1.7 (See Lemma 9 in [Aba16])**.** *Let $\varepsilon > 0$ and $u \in C^{2s+\varepsilon}_{loc}(\Omega)\cap L^1(\Omega,\delta(x)\,dx)$ be such that*

$$\begin{aligned} &(-\Delta)^s_{D,\Omega} u(x) \geqslant 0 \ \text{ for all } x \in \Omega \\ \text{and}\quad &\liminf_{\Omega\ni x\to p} u(x) \geqslant 0 \ \text{ for all } p \in \partial\Omega, \end{aligned}$$

where $(-\Delta)^s_{D,\Omega}$ is given as in (1.8)*. Then, $u(x) \geqslant 0$ for all $x \in \Omega$.*

For each $s \in (0,1)$ and $y \in \Omega$ the *Dirichlet spectral fractional heat equation* starting at y reads as

$$\begin{cases} \partial_t u(t,x) = -(-\Delta)^s_{D,\Omega}\, u(t,x) & \text{for all } (t,x) \in (0,+\infty) \times \Omega, \\ u(t,x) = 0 & \text{for all } (t,x) \in (0,+\infty) \times \partial\Omega, \\ u(0,x) = \delta_y(x) & \text{for all } x \in \Omega. \end{cases} \tag{1.10}$$

The last equation is meant in the sense of measure: more precisely, defining

$$C_0(\Omega) := \big\{ f \in C(\overline{\Omega}) \text{ s.t. } f(x) = 0 \text{ for all } x \in \partial\Omega \big\}, \tag{1.11}$$

the last line in (1.10) means that, for each $f \in C_0(\Omega)$,

$$\lim_{t \to 0^+} \int_\Omega u(t,x)\, f(x)\, dx = f(y). \tag{1.12}$$

No confusion should arise between the space $C_0(\Omega)$ introduced in (1.11) and the more standard space $C_c(\Omega)$, which is the space of continuous functions in $\overline{\Omega}$ whose support is contained in Ω: in particular, with this notation, we have that $C_c(\Omega) \subsetneq C_0(\Omega)$.

In this setting, one has the following basic structural results:

Theorem 1.8 (Existence, Uniqueness and Regularity of the solution to (1.10))**.** *Let $s \in (0,1)$ and Ω be open, bounded, connected and with smooth boundary. If we set*

$$\begin{aligned} & r^s_D : (0,+\infty) \times \Omega \times \Omega \to \mathbb{R} \\ & (t,x,\,y) \mapsto r^s_D(t,x,y) := \sum_{k=1}^{+\infty} \phi_k(x)\, \phi_k(y)\, \exp(-t\lambda_k^s), \end{aligned} \tag{1.13}$$

then for each $y \in \Omega$ the function $r^s_D(\cdot,\cdot,y)$ is the unique solution to (1.10) *in $C^1((0,+\infty), H^{2s}_D(\Omega))$. Moreover, $r^s_D \in C^\infty([\varepsilon,+\infty) \times \overline{\Omega} \times \overline{\Omega})$ for each $\varepsilon > 0$.*

We will refer to r^s_D as the *Dirichlet spectral fractional heat kernel.* Using the Maximum Principle for the classical heat kernel in equation (1.7) one can prove its fractional counterpart. More precisely we state that:

Theorem 1.9 (Maximum Principle)**.** *Let $s \in (0,1)$ and Ω be open, bounded, connected and with smooth boundary. For each $f \in L^2(\Omega)$ and $(t,y) \in (0,+\infty) \times \Omega$ we have that*

$$\min\Big\{0, \inf_\Omega f\Big\} \leqslant \int_\Omega r^s_D(t,x,y)\, f(x)\, dx \leqslant \max\Big\{0, \sup_\Omega f\Big\}. \tag{1.14}$$

Theorems 1.8 and 1.9 will be proven in Section 2.2. Additionally, we will establish in the forthcoming Theorem 2.6 and Proposition 2.8 that $r^s_D(\cdot,\cdot,y)$ is the *transition density* in Ω of the aforementioned subordinate killed Brownian motion, which will be denoted by $Y^\Omega_{D,y} = (\mathfrak{S}, \{\mathcal{P}_t\}_{t\geqslant 0}, \mathcal{P}, \{Y_{D,t}\}_{t \geqslant 0}, \mathbb{P}^s_{D,y})$. A detailed construction for this stochastic process with its main properties is contained in Section 2.1. This means that $r^s_D(t,x,y)$ represents the density of probability that the process starting at $y \in \Omega$ will be in the position $x \in \Omega$ at a time $t > 0$. Therefore, the probability $\mathbb{P}^s_{D,y}(Y^\Omega_{D,t} \in \Omega')$ of finding the process $Y^\Omega_{D,y}$ starting at $y \in \Omega$ in some subset Ω' at a time $t > 0$ is

$$\mathbb{P}^s_{D,y}(Y_{D,t} \in \Omega') = \int_{\Omega'} r^s_D(t,x,y)\, dx. \tag{1.15}$$

We now point out one effect of subordination on the killed Brownian motion and a direct link between the geometry of the domain and the notion of probability set forth in equation (1.16). In particular, one can define the conditional probability

$$\mathbb{P}^s_D(Y_{D,t} \in \Omega'' | Y_{D,0} \in \Omega') := \frac{1}{|\Omega'|} \int_{\Omega' \times \Omega''} r^s_D(t,x,y)\, dx\, dy, \tag{1.16}$$

for each $\Omega', \Omega'' \subset \Omega$, where Ω' has positive measure. The probability in equation (1.16) is the probability of being in a set Ω'' at some time $t > 0$ starting the process from a set Ω'.

The relation between the geometry of the domain and this conditional probability is stated in the next theorems, which are the main results of this paper. Their statements detect the monotonicity properties in the fractional parameter s in relation to the diffusive process induced by the Dirichlet fractional Laplacian. Interestingly, these monotonicity properties differ according to the size of the first Dirichlet eigenvalue of the Laplacian. The precise results that we have go as follows:

Theorem 1.10. *Let $0 < s_0 < s_1 \leqslant 1$ and $x \in \Omega$.*

(i) *If $\lambda_1 \geqslant 1$, then for each $t > 0$ it holds that*

$$r_D^{s_0}(t,x,x) > r_D^{s_1}(t,x,x). \tag{1.17}$$

(ii) *If $\lambda_1 < 1$, then there exists some $T > 0$, depending on x, s_0 and s_1, such that for each $t > T$*

$$r_D^{s_0}(t,x,x) < r_D^{s_1}(t,x,x). \tag{1.18}$$

Theorem 1.11. *Let $0 < s_0 < s_1 \leqslant 1$ and $\Omega' \subset \Omega$ measurable with positive measure.*

(i) *If $\lambda_1 \geqslant 1$, then for each $t > 0$ it holds that*

$$\mathbb{P}_D^{s_0}(Y_{D,t} \in \Omega' | Y_{D,0} \in \Omega') > \mathbb{P}_D^{s_1}(Y_{D,t} \in \Omega' | Y_{D,0} \in \Omega'). \tag{1.19}$$

(ii) *If $\lambda_1 < 1$, then there exists some $T > 0$, depending on Ω', s_0 and s_1, such that for each $t > T$*

$$\mathbb{P}_D^{s_0}(Y_{D,t} \in \Omega' | Y_{D,0} \in \Omega') < \mathbb{P}_D^{s_1}(Y_{D,t} \in \Omega' | Y_{D,0} \in \Omega'). \tag{1.20}$$

Theorems 1.10 and 1.11 will be proven in Section 2.3. The relation between the inequalities (1.17)-(1.20) and the geometry of the domain is encoded into the first eigenvalue λ_1. As a matter of fact it is well known that the first eigenvalue of the Dirichlet Laplacian is strictly related to the volume of the domain and (under convexity assumptions) to its perimeter (see [GN13] and references therein). Nevertheless, the volume of the domain is not the only geometrical feature of the domain that plays some role on the estimation of the first eigenvalue.

To make this observation clear, for a general bounded domain Ω, we have the following upper and lower bound (see formulas (4.9) and (4.12) in [GN13])

$$\left(\frac{v_n}{|\Omega|}\right)^{\frac{2}{n}} \beta^2_{\frac{n}{2}-1,1} \leqslant \lambda_1 \leqslant \left(\frac{\beta_{\frac{n}{2}-1,1}}{\rho}\right)^2, \tag{1.21}$$

where $\beta_{\frac{n}{2}-1,1}$ is the first positive zero of the Bessel function $\mathcal{J}_{\frac{n}{2}-1}$, v_n the Lebesgue measure of the unit ball in $\mathbb{R}^n$ and

$$\rho := \max_{x\in\Omega} \min_{y\in\partial\Omega} |x-y| \tag{1.22}$$

is the *inradius* of Ω, namely the radius of the largest ball inscribed in Ω.

As a consequence of (1.21), one sees that if the domain Ω has a small enough measure, then $\lambda_1 \geqslant 1$, and therefore the monotonicity properties in (1.17) and (1.19) hold true. In the case of equation (1.19) this means that, for a small domain, in the sense of measure, for lower values of $s \in (0,1)$ the probability of getting out of any starting set Ω' at any time t is lower.

This phenomenon is consistent with the heuristic idea that lower values of s correspond to a "less regular" solution of the fractional heat equation, therefore, for lower values of s, if we focus our attention only to the vicinity of the original source, a relatively large proportion of the mass of the solution tends to remain confined for quite a long time near its initial location.

While it is tempting to consider this situation as an occurrence of "slow diffusion", a pinch of salt is needed when trying to make these considerations consistent, also in view of the following observation.

If the largest ball B_ρ inscribed in Ω is large enough, then thanks to the right hand side inequality in (1.21) we infer that $\lambda_1 < 1$. In this case, we have that if we choose $0 < s_0 < s_1 < 1$ and $\Omega' \subset \Omega$, then there exists some $T > 0$ such that for all $t > T$ the probability $\mathbb{P}^{s_0}(Y_{D,t} \in \Omega' | Y_{D,0} \in \Omega')$ that $Y_{D,t}$ will be in the starting set Ω' is smaller than the probability $\mathbb{P}^{s_1}(Y_{D,t} \in \Omega' | Y_{D,0} \in \Omega')$.

This phenomenon is also consistent with the heuristic idea that lower values of s correspond to a "fat tail" probability of jumping far away from the original location: in this sense, if the inradius is sufficiently large, thus allowing these long jumps to occur, the long time behavior of the solution will exhibit this sort of "fast diffusion".

From these comments it should now be clear that the distinction between "slow" and "fast" diffusion is possibly more delicate than what it may seem at a first glance and it cannot be reduced merely to quantitative considerations about the fractional exponent of the equation (since, in light of the above comments, small fractional exponents can exhibit both "slow" and "fast" diffusion with respect to Dirichlet boundary conditions, and the size and shape of the domain may play an essential role in selecting the appropriate phenomenon in different situations).

Note that in the scenario where Ω has very large Lebesgue measure, but is narrow enough so that ρ is sufficiently small, inequality (1.21) does not give us a suitable upper or lower bound in order to apply Theorem 1.11. Nevertheless, there are other inequalities available relating the first eigenvalue of the Laplacian to the geometric properties of the domain Ω.

For instance, if the domain Ω is convex, then one can make the right-hand side in (1.21) more precise. In particular (see equation (4.14) in [GN13]) we have that

$$\lambda_1 \leqslant \beta^2_{\frac{n}{2}-1,1} \frac{H^{n-1}(\partial\Omega)}{n\,\rho\,|\Omega|}, \tag{1.23}$$

where H^{n-1} denotes, as usual, the $(n-1)$-dimensional Hausdorff measure.

Remark 1.12. We observe that, for convex sets, the upper bound for λ_1 in (1.23) is sharper than that in (1.21). This is due to the geometrically interesting fact that, for convex sets,

$$\rho\, H^{n-1}(\partial\Omega) \leqslant n\,|\Omega|. \tag{1.24}$$

To check this, up to a translation, we can assume that

$$B_\rho \subseteq \Omega. \tag{1.25}$$

Thus, we recall that, given $x \in \partial\Omega$, if $\nu(x)$ is the unit exterior normal of Ω at x, then

$$x \cdot \nu(x) \geqslant \rho. \tag{1.26}$$

This is possibly a classical inequality in convex geometry (see e.g. [BM20]), but we present its simple proof for the convenience of the reader. Let $y \in \overline{\Omega}$. By convexity, we know that $\overline{\Omega}$ is contained in its supporting halfspace at x, namely

$$\overline{\Omega} \subseteq \{z \in \mathbb{R}^n \text{ s.t. } (z-x)\cdot\nu(x) \leqslant 0\},$$

hence $(y-x)\cdot\nu(x) \leqslant 0$. In particular, recalling (1.25) and choosing $y := \rho\nu(x) \in \overline{B_\rho} \subseteq \overline{\Omega}$, we obtain that

$$\rho = \rho\nu(x)\cdot\nu(x) = y\cdot\nu(x) \leqslant x\cdot\nu(x)$$

and this proves (1.26).

From (1.26) we deduce that

$$n\,|\Omega| = \int_\Omega \operatorname{div}(x)\,dx = \int_{\partial\Omega} x\cdot\nu(x)\,dH^{n-1}(x) \geqslant \rho\,H^{n-1}(\partial\Omega),$$

and this proves (1.24), as desired.

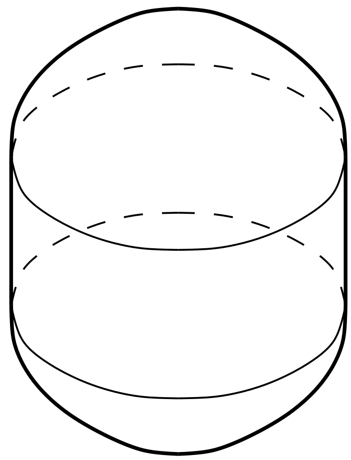


FIGURE 1. *The set described in Example 1.13.*

Example 1.13. To better appreciate the improvement provided by (1.23) for convex sets with respect to the general case in (1.21) and to recognize the effect of this improvement in the setting of Theorems 1.10 and 1.11, we provide here an elementary example of a convex set Ω such that

$$\beta^2_{\frac{n}{2}-1,1}\frac{H^{n-1}(\partial\Omega)}{n\,\rho\,|\Omega|}<1<\left(\frac{\beta_{\frac{n}{2}-1,1}}{\rho}\right)^2. \tag{1.27}$$

In particular, for this set, we infer from (1.23) that $\lambda_1<1$, therefore the claims in (ii) of Theorems 1.10 and 1.11 hold true: but we stress that (1.21) would have not guaranteed this, therefore this example showcases that inequality (1.21), only relying on the measure of a domain and on the inradius, does not provide optimal information on convex sets and instead other geometrical features play a decisive role in the estimation of λ_1, which is in turn crucial to detect the diffusivity properties of the spectral heat kernels according to Theorems 1.10 and 1.11.

Let us now construct an example of a convex set satisfying (1.27). The existence of sets with this property is somewhat a general fact (see e.g. the formula in display below (2.1) in [BM20]), but we present an elementary example to keep the discussion as simple as possible. The example that we construct is with boundary of class $C^{1,1}$, but one can also regularize the boundary and obtain a smooth set by exploiting this example.

The example that we present is the following. We pick $\varepsilon\in(0,1)$, to be conveniently chosen below. We let $h>0$, $\rho:=(1-\varepsilon)\beta_{\frac{n}{2}-1,1}$ and

$$\Omega:=\bigcup_{\tau\in(-h/2,h/2)}B_\rho(0,\dots,0,\tau).$$

See Figure 1 for a sketch of this set in $\mathbb{R}^3$. We point out that

$$\left(\frac{\beta_{\frac{n}{2}-1,1}}{\rho}\right)^2=\left(\frac{\beta_{\frac{n}{2}-1,1}}{(1-\varepsilon)\beta_{\frac{n}{2}-1,1}}\right)^2=\frac{1}{(1-\varepsilon)^2}>1,$$

therefore it only remains to check the first inequality in (1.27).

To this end, we denote by v_n the n-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^n$ and by a_{n-1} the $(n-1)$-dimensional surface area of the boundary of the unit ball in $\mathbb{R}^n$. It is well known (see e.g. [DV21b, equation (1.2.6)]) that

$$v_n=\frac{a_{n-1}}{n}.$$

We observe that Ω consists of two halfballs of radius ρ glued on the top and on the bottom of a cylinder of height h, therefore

$$|\Omega| = |B_\rho| + h\rho^{n-1}v_{n-1} = \rho^n v_n + h\rho^{n-1}v_{n-1}$$

and

$$H^{n-1}(\partial\Omega) = H^{n-1}(\partial B_\rho) + h\rho^{n-2}a_{n-2} = \rho^{n-1}a_{n-1} + h\rho^{n-2}a_{n-2} = n\rho^{n-1}v_n + (n-1)h\rho^{n-2}v_{n-1}.$$

As a consequence,

$$\begin{aligned} \beta^2_{\frac{n}{2}-1,1}\frac{H^{n-1}(\partial\Omega)}{n\,\rho\,|\Omega|} &= \left(\frac{\rho}{(1-\varepsilon)}\right)^2 \frac{n\rho^{n-1}v_n + (n-1)h\rho^{n-2}v_{n-1}}{n\,\rho\,(\rho^n v_n + h\rho^{n-1}v_{n-1})} \\ &= \frac{1}{(1-\varepsilon)^2}\,\frac{n\rho v_n + (n-1)hv_{n-1}}{n\rho v_n + nhv_{n-1}}. \end{aligned}$$

Hence, since

$$\frac{n\rho v_n + (n-1)hv_{n-1}}{n\rho v_n + nhv_{n-1}} < 1,$$

we deduce that, if ε is small enough, possibly in dependence of ρ and n, then also

$$\frac{1}{(1-\varepsilon)^2}\,\frac{n\rho v_n + (n-1)hv_{n-1}}{n\rho v_n + nhv_{n-1}} < 1.$$

The proof of (1.27) is thereby complete.

1.2. **The Neumann spectral fractional heat equation.** Now we deal with the counterpart of the spectral fractional heat equation presented in Section 1.1 when the boundary data are of Neumann, rather than Dirichlet, type. Most of the main features are in common between the Dirichlet and the Neumann case, since the functional analysis setting can be efficiently translated from one framework into another. However, for clarity and completeness, we provide explicitly the technical details in the Neumann case as well. For this, we proceed as follows.

We consider an orthonormal basis $\{\psi_k\}_k$ of $L^2(\Omega)$ satisfying

$$\begin{cases} -\Delta\psi_k = \mu_k\psi_k & \text{in } \Omega, \\ \dfrac{\partial\psi_k}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.28}$$

where the values $0 = \mu_0 < \mu_1 \leqslant \mu_2 \leqslant \mu_3 \ldots$ are the eigenvalues of the Laplace operator with a Neumann homogeneous boundary condition. In principle, the boundary condition in (1.28) should be intended in the weak sense (see e.g. page 296 in [Bre11]), but, since the boundary of Ω is assumed to be smooth, the eigenfunctions ψ_k belong to $C^\infty(\overline{\Omega})$ and (1.28) makes sense in the classical case.

Note also that the eigenfunction associated with the eigenvalue μ_0 is just the constant function normalized in $L^2(\Omega)$. If $u(t,x)$ with $(t,x) \in (0,+\infty)\times\Omega$ is such that $u(t,\cdot) \in L^2(\Omega)$, then

$$u(t,x) = \sum_{k=0}^{+\infty} u_k(t)\,\psi_k(x), \qquad \text{with} \qquad u_k(t) := \int_\Omega u(t,y)\,\psi_k(y)\,dy.$$

In line with the setting provided in Definition 1.3 for the Dirichlet case, we can thus introduce a fractional Neumann framework as follows:

Definition 1.14. *Let $s \in (0,1)$. We define the Neumann spectral fractional Laplacian as*

$$\begin{aligned} &(-\Delta)^s_{N,\Omega} : H^{2s}_N(\Omega) \subset L^2(\Omega) \to L^2(\Omega) \\ &u \longmapsto (-\Delta)^s_{N,\Omega}u := \sum_{k=0}^{+\infty} u_k\,\mu_k^s\,\psi_k, \end{aligned} \tag{1.29}$$

where

$$H^{2s}_N(\Omega) := \left\{ u \in L^2(\Omega) \ s.t. \ \sum_{k=0}^{+\infty} u_k^2 \, \mu_k^{2s} < +\infty \right\}. \tag{1.30}$$

We observe that the space $H^{2s}_N(\Omega)$ introduced in (1.30) coincides with the space $H^{2s}_D(\Omega)$ introduced in (1.4), except that the weights used here are the Neumann eigenvalues of the Laplacian, instead of the Dirichlet ones.

It is also useful to consider the classical "fundamental solution" of the heat equation with homogeneous Neumann condition, that is the solution of

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) & \text{for all } \ (t,x) \in (0,+\infty) \times \Omega, \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0 & \text{for all } \ (t,x) \in (0,+\infty) \times \partial\Omega, \\ u(0,x) = \delta_y(x) & \text{for all } x \in \Omega \end{cases} \tag{1.31}$$

where $\nu(x)$ is the outward pointing normal to $\partial\Omega$ at the point $x \in \partial\Omega$, and the last equation is meant in the sense of measure. The solution of (1.31) will be denoted by $p^\Omega_N(t,x,y)$ and it is often referred to with the name of *classical Neumann heat kernel in* Ω.

In what follows we denote by

$$W^{1,2}(\Omega) := \left\{ f \in W^{1,2}_N(\Omega) \quad \text{s.t} \ \frac{\partial f}{\partial \nu} = 0 \right\}.$$

Theorem 1.1 and equation (2.4) in [Hsu84] together with Lemma A.2 in Appendix A and Lemma 3.8 in Section 3.2 (with the clarification that the uniqueness in the space $C^1((0,+\infty), W^{1,2}_N(\Omega))$ can be proved with the exact same approach used to show uniqueness for the system of equations (1.35) as stated in Theorem 1.20) give us the following result:

Theorem 1.15. *Let Ω be open, connected, bounded and smooth. Let us define the function*

$$\begin{aligned} p^\Omega_N : (0,+\infty) \times & \Omega \times \Omega \to \mathbb{R} \\ (t,x,\,y) \mapsto p^\Omega_N(t,x,y) &:= \sum_{k=0}^{+\infty} \psi_k(x)\,\psi_k(y)\,\exp(-t\mu_k). \end{aligned} \tag{1.32}$$

Then, for each $y \in \Omega$ the function $p^\Omega_N(\cdot,\cdot,y)$ is the unique solution to (1.5) *in $C^1((0,+\infty), W^{1,2}_N(\Omega))$. Moreover, $p^\Omega_N \in C^\infty([\varepsilon,+\infty) \times \overline{\Omega} \times \overline{\Omega})$ for each $\varepsilon > 0$.*

The following Maximum Principle is a straightforward consequence of Theorem 1.2 in [Hsu84].

Theorem 1.16. *Let Ω be open, bounded, connected and with smooth boundary. For each $f \in L^2(\Omega)$ and $(t,y) \in (0,+\infty) \times \Omega$ we have that*

$$\min\left\{0, \inf_\Omega f\right\} \leqslant \int_\Omega p^\Omega_N(t,x,y)\, f(x)\,dx \leqslant \max\left\{0, \sup_\Omega f\right\}.$$

In agreement with Proposition 1.6, one can give an equivalent definition for the Neumann spectral fractional Laplacian by exploiting an integral operator built via a jumping kernel of Neumann type. The explicit details go as follows:

Proposition 1.17 (See [AV19])**.** *Let $u \in H^{2s}_N(\Omega)$. Then,*

$$(-\Delta)^s_{N,\Omega} u(x) = p.v. \int_\Omega [u(x) - u(y)]\, J_N(x,y)\,dy \tag{1.33}$$

for almost every $x \in \Omega$, where

$$J_N(x,y) := \frac{s}{\Gamma(1-s)} \int_0^{+\infty} p_N^\Omega(t,x,y)\,\frac{dt}{t^{1+s}}. \tag{1.34}$$

The right-hand side of (1.33) *is well defined for each $x \in \Omega$ if $u \in C^{2s+\varepsilon}_{loc}(\Omega) \cap L^\infty(\Omega)$ for some $\varepsilon > 0$.*

The function in (1.34) is called *Neumann jumping kernel.*

Remark 1.18. An interesting structural difference between (1.8) and (1.33) consists in the fact that the latter does not present any additional zero order term (equivalently, the Neumann type "killing measure" vanishes identically). Indeed, suppose that there exists some $k : \Omega \to \mathbb{R}$ such that for each $u \in H^{2s}_N(\Omega)$ one has

$$(-\Delta)^s_{N,\Omega} u(x) = \int_\Omega [u(x) - u(y)]\, J_N(x,y)\,dy + k(x)\,u(x)$$

for almost $x \in \Omega$. Then, since $1 \in H^{2s}_N(\Omega)$, we would have thanks to (1.29) that

$$0 = (-\Delta)^s_{N,\Omega} 1(x) = \int_\Omega (1-1)\, J_N(x,y)\,dy + k(x) = 0 + k(x).$$

Moreover, one obtains the following Maximum Principle for the Neumann spectral fractional Laplacian, which can be seen as the Neumann counterpart of Lemma 1.7:

Lemma 1.19. *Let $\varepsilon > 0$, $u \in C^{2s+\varepsilon}_{loc}(\Omega) \cap L^\infty(\Omega)$ be non-constant and $x^* \in \Omega$ such that $u(x^*) = \inf_\Omega u(x)$. Then it holds that*

$$(-\Delta)^s_{N,\Omega} u(x^*) < 0.$$

Now, for each $s \in (0,1)$ and $y \in \Omega$, we consider the *Neumann spectral fractional heat equation* starting at y, which reads as

$$\begin{cases} \partial_t u(t,x) = -(-\Delta)^s_{N,\Omega}\, u(t,x) & \text{for all } (t,x) \in (0,+\infty) \times \Omega, \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0 & \text{for all } (t,x) \in (0,+\infty) \times \partial\Omega, \\ u(0,x) = \delta_y(x) & \text{for all } x \in \Omega. \end{cases} \tag{1.35}$$

The last equation is meant in the sense of measure, see equation (1.12). In Section 3.2 we will establish the following two basic results:

Theorem 1.20 (Existence, Uniqueness and Regularity of the solution to (1.35))**.** *Let $s \in (0,1)$ and Ω be open, bounded, connected and with smooth boundary. If we set*

$$\begin{aligned} &r^s_N : (0,+\infty) \times \Omega \times \Omega \to \mathbb{R} \\ &\qquad (t,x,\,y) \mapsto r^s_N(t,x,y) := \sum_{k=0}^{+\infty} \psi_k(x)\,\psi_k(y)\,\exp(-t\mu_k^s). \end{aligned} \tag{1.36}$$

then for each $y \in \Omega$ the function $r^s_N(\cdot,\cdot,y)$ is the unique solution to (1.35) *in $C^1((0,+\infty), H^{2s}_N(\Omega))$. Moreover, $r^s_N \in C^\infty([\varepsilon,+\infty) \times \overline{\Omega} \times \overline{\Omega})$ for each $\varepsilon > 0$.*

We call r^s_N the *Neumann spectral fractional heat kernel.*

Theorem 1.21 (Maximum Principle)**.** *Let $s \in (0,1)$ and Ω be open, bounded, connected and with smooth boundary. For each $f \in L^2(\Omega)$ and $(t,y) \in (0,+\infty) \times \Omega$ we have that*

$$\min\left\{0, \inf_\Omega f\right\} \leqslant \int_\Omega r^s_N(t,x,y)\, f(x)\,dx \leqslant \max\left\{0, \sup_\Omega f\right\}. \tag{1.37}$$

Later on (see Theorem 3.3 and Proposition 3.5) we will also prove that $r^s_N(\cdot,\cdot,y)$ is the *transition density* of the aforementioned subordinate standard reflecting Brownian motion, which will be denoted by $Y^\Omega_{N,y} = (\mathfrak{S}, \{\mathcal{P}_t\}_{t\geqslant 0}, \mathcal{P}, \{Y_{N,t}\}_{t\geqslant 0}, \mathbb{P}^s_{N,y})$. A detailed presentation of this stochastic process is contained in Section 3.1. In this setting, $r^s_N(t,x,y)$ represents the density of probability that the process starting at $y \in \Omega$ will occupy the position $x \in \Omega$ at a time $t > 0$. Therefore, the probability $\mathbb{P}^s_{N,y}$ of finding the process Y^Ω_N starting at $y \in \Omega$ in some subset $\Omega' \subset \Omega$ at a time $t \in (0,+\infty)$ is

$$\mathbb{P}^s_{N,y}(Y_{N,t} \in \Omega') := \int_{\Omega'} r^s_N(t,x,y)\,dx. \tag{1.38}$$

Additionally (see Lemma 3.10) we will establish that the transition density $r^s_N(t,\cdot,y)$ has $L^1(\Omega)$-norm that is constantly equal to 1 for each t (and $y \in \Omega$). This property is in agreement with the fact that the process, once it reaches the boundary of Ω, gets reflected. Indeed, this reflection property encodes the main difference between the killed motion, corresponding to an equation with homogeneous Dirichlet boundary data, and the reflected motion, which produces a homogeneous Neumann condition.

We now showcase two results which can be seen as the counterpart of Theorems 1.10 and 1.11 for the Neumann case. To this end, assuming that a starting point $y \in \Omega$ is chosen uniformly in Ω, one defines the conditional probability

$$\mathbb{P}^s_N(Y_{N,t} \in \Omega'' | Y_{N,0} \in \Omega') = \frac{1}{|\Omega'|}\int_{\Omega'\times\Omega''} r^s_N(t,x,y)\,dx\,dy, \tag{1.39}$$

for each $\Omega', \Omega'' \subset \Omega$, where Ω' has positive measure. Note that $\mathbb{P}^s(Y_{N,t} \in \Omega''|Y_{N,0} \in \Omega')$ is the probability of being in the set Ω'' at a time $t > 0$ starting from a set Ω' at $t = 0$ and following the subordinate standard reflecting Brownian motion.

Given $\Omega' \subset \Omega$, we also define $\mu_{k(\Omega')}$ as the first non-zero eigenvalue associated with an eigenfunction satisfying

$$\int_{\Omega'} \psi_k(x)\,dx \neq 0. \tag{1.40}$$

Moreover, for each $x \in \Omega$ we denote with $\mu_{k(x)}$ the first non-zero eigenvalue associated with an eigenfunction satisfying

$$\psi_k(x) \neq 0. \tag{1.41}$$

The existence of $\mu_{k(\Omega')}$ and $\mu_k(x)$ is straightforward (see Proposition 3.14 for details). With this notation, we have the following two results, which will be proven in Section 3.3:

Theorem 1.22. *Let $0 < s_0 < s_1 \leqslant 1$ and $x \in \Omega$.*

(i) *If $\mu_{k(x)} \geqslant 1$, then for each $t > 0$ it hods*

$$r^{s_0}_N(t,x,x) > r^{s_1}_N(t,x,x). \tag{1.42}$$

(ii) *If $\mu_{k(x)} < 1$, then there exists some $T > 0$, depending on s_0, s_1 and x, such that for each $t > T$*

$$r^{s_0}_N(t,x,x) < r^{s_1}_N(t,x,x). \tag{1.43}$$

Theorem 1.23. *Let $0 < s_0 < s_1 \leqslant 1$, $\Omega' \subset \Omega$ measurable with positive measure.*

i) *If $\mu_{k(\Omega')} \geqslant 1$, then for each $t > 0$ we have*

$$\mathbb{P}^{s_0}_N(Y_{N,t} \in \Omega'|Y_{N,0} \in \Omega') > \mathbb{P}^{s_1}_N(Y_{N,t} \in \Omega'|Y_{N,0} \in \Omega'). \tag{1.44}$$

ii) *If $\mu_{k(\Omega')} < 1$, then there exists some $T > 0$, depending on s_0, s_1 and Ω', such that for each $t > T$*

$$\mathbb{P}^{s_0}_N(Y_{N,t} \in \Omega'|Y_{N,0} \in \Omega') < \mathbb{P}^{s_1}_N(Y_{N,t} \in \Omega'|Y_{N,0} \in \Omega'). \tag{1.45}$$

Similarly to Theorems 1.10 and 1.11, also Theorems 1.22 and 1.23 relate the local behavior of the subordinate standard reflecting Brownian motion to the geometry of the domain, through the Neumann eigenvalues. However, differently from the Dirichlet case, the first eigenfunction μ_0 does not play any role in Theorems 1.22 and 1.23 concerning the behavior of $\mathbb{P}^s(Y_{N,t} \in \Omega' | Y_{N,0} \in \Omega')$ in relation to s, since obviously $\mu_0 = 0$ and therefore a finer investigation is needed. Interestingly, as stated in Theorems 1.22 and 1.23, the essential ingredient in this setting becomes the first eigenvalue μ_k with $k \neq 0$ whose corresponding eigenfunction has either non-zero average in Ω' (as detailed in (1.40), corresponding to the statement in Theorem 1.22) or non-zero value at the point (as specified in (1.41), in correspondence with Theorem 1.23).

We stress that, by construction, in (1.40) and (1.41) we have that $k(\Omega') \geqslant 1$ and $k(x) \geqslant 1$, respectively. Therefore, $\mu_1 \leqslant \mu_{k(\Omega')}$ and $\mu_1 \leqslant \mu_{k(x)}$. Accordingly, if $\mu_1 \geqslant 1$ we get that also $\mu_{k(\Omega')} \geqslant 1$ and $\mu_{k(x)} \geqslant 1$. For this reason, a simple consequence of Theorems 1.22 and 1.23 is that if $\mu_1 \geqslant 1$ then the monotonicity claims with respect to the fractional parameter stated in (1.42) and (1.44) hold true.

It is therefore desirable to have explicit lower bounds on μ_1 in order to conveniently apply point (i) in Theorems 1.22 and 1.23. To this end, we recall that, if Ω is convex, one has that

$$\mu_1 \geqslant \frac{\pi^2}{\delta^2},$$

where we have denoted by

$$\delta := \max_{x,y \in \partial\Omega} |x-y|,$$

see [PW60, equation (1.9)]. Thus, if Ω is convex and its diameter δ is smaller than π, it follows that $\mu_{k(x)}$, $\mu_{k(\Omega')} \geqslant 1$, which in turns gives the validity of (1.42) and (1.44).

When $\mu_1 < 1$, it could still be possible that (1.42) and (1.44) hold true and therefore in this case a finer analysis is needed to detect the size of $\mu_{k(\Omega')}$ and $\mu_{k(x)}$ respectively. To this end, we recall (see [Krö92]) that, for all $k = 0, 1, 2 \ldots$,

$$\mu_k \leqslant \left(\frac{n+2}{2}\right)^{\frac{2}{n}} \frac{4\pi^2 k^{2/n}}{(v_n|\Omega|)^{\frac{2}{n}}},$$

where v_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. In particular, if the measure of Ω is large enough with respect to either $k(x)$ or $k(\Omega)$, one can deduce from the above inequality that either $\mu_{k(x)} < 1$ or $\mu_{k(\Omega)} < 1$, which permits to conclude that point (ii) in either Theorem 1.22 and 1.23 hold true in this case.

2. The spectral fractional heat equation with Dirichlet boundary conditions

In this section we study the Dirichlet spectral fractional heat equation (1.10) from a probabilistic and analytic point of view.

Section 2.1 introduces some background on a stochastic process called the subordinate killed Brownian motion. More precisely, we show that the transition density of this process coincides with the Dirichlet spectral fractional heat kernel r_D^s given in equation (1.13).

Section 2.2 is devoted to the analytical study of the solution to the system of equations (1.10). Here we prove Theorems 1.8 and 1.9. The first result states that the unique solution to (1.10) is the Dirichlet spectral fractional heat kernel r_D^s, while Theorem 1.9 is a Maximum Principle for the spectral fractional heat equation with Dirichlet boundary conditions.

In Section 2.3 we prove two new results, namely Theorems 1.10 and 1.11, which give us some interesting monotonicity properties (with respect to the parameter s) of the kernel r_D^s and the conditional probability given in equation (1.16).

2.1. **From the killed Brownian motion to the subordinate killed Brownian motion.** In this section we present a brief and self-contained construction of the subordinate killed Brownian motion, which is a stochastic process built from the killed Brownian motion by introducing a random choice of the time via a procedure called subordination (see [Boc49]). The detailed construction of such a process is given in Theorem 2.6. More precisely, in Theorem 2.6 and Proposition 2.8 we prove that the transition density associated with this stochastic process is the Dirichlet spectral fractional heat kernel r^s_D defined in equation (1.13). Further details on this subject can be found for instance in [SSV12].

Now we recall some basic facts related to the notion of Killed Brownian process. Roughly speaking, the gist is that when the particle leaves the reference set Ω, it is placed into a "cemetery" where it remains forever.

To formalize this heuristic idea, we let

$$X := \big(\mathfrak{S}, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \{X_t\}_{t\geqslant 0}, \{\mathbb{P}_x\}_{x\in\Omega}\big) \tag{2.1}$$

be a family of classical n-dimensional Brownian motions in $\mathbb{R}^n$ starting at $x \in \Omega$, namely

$$\mathbb{P}_x(X_0 = x) = 1.$$

For a definition of Brownian motion see for instance [Bal17]; in practice, in (2.1) one can consider $\mathfrak{S} = C([0, +\infty), \mathbb{R}^n)$ and $X_t(\omega) = \omega(t)$ for each $\omega \in \mathfrak{S}$ and $t \geqslant 0$.

Let us now recall the notion of *first exit time* random variable $\tau_\Omega := \inf\{t : X_t \notin \Omega\}$, and[1] a point $\partial \in \mathbb{R}^n \setminus \overline{\Omega}$.

Theorem 2.1 (See Theorems 2.2 and 2.4 in [CZ95])**.** *Let Ω be open, bounded, smooth and connected and $x \in \Omega$. Consider the process $X_{D,x} := (\mathfrak{S}, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \{X^\Omega_t\}_{t\geqslant 0}, \mathbb{P}_x)$, where the random variable $X^\Omega_t : \mathfrak{S} \to \mathbb{R}^n$ is defined by*

$$X^\Omega_t := \begin{cases} X_t & \text{if } t < \tau_\Omega, \\ \partial & \text{if } t \geqslant \tau_\Omega. \end{cases}$$

Then, the process $X_{D,x}$ is a Markov process and for each Lebesgue measurable set $\Omega' \subset \Omega$ and $t > 0$ it holds that

$$\mathbb{P}_x(X^\Omega_t \in \Omega') = \int_{\Omega'} p^\Omega_D(t, x, y)\, dy. \tag{2.2}$$

Definition 2.2. *Let $x \in \Omega$. The Markov process $X_{D,x} := (\mathfrak{S}, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \{X^\Omega_t\}_{t\geqslant 0}, \mathbb{P}_x)$ as defined in Theorem 2.1 is called killed Brownian motion.*

In view of Theorem 2.1, the semigroup $\{T_t\}_{t\geqslant 0}$ associated with $X_{D,x}$ is

$$T_t f(x) := \int_\Omega f(y)\, p^\Omega_D(t, x, y)\, dy = \mathbb{E}_x[f(X^\Omega_t)],$$

for each $f \in L^2(\Omega)$. Note that we are using the term "transition density" for $p^\Omega_D(t, x, y)$ even though p^Ω_D is just the part of the transition density concentrated in Ω, which is not the whole state space, that is instead $\Omega \cup \{\partial\}$. Furthermore, for each $x \in \Omega$,

$$\lim_{t\to+\infty} \mathbb{P}(X^\Omega_t \in \Omega) = \lim_{t\to+\infty} \int_\Omega p^\Omega_D(t, x, y)\, dy = 0, \tag{2.3}$$

[1]The point ∂ is often called the "cemetery" of the stochastic process. Using "∂" to denote the cemetery is a typical notation in the literature, hence we stick to it. No confusion should arise with the boundary of a set (as a matter of fact, the two notations somewhat coincide for a classical Brownian motion, in the sense that the cemetery in this case can be seen as the boundary of the domain, with all points identified into a single one).

see e.g. [CZ95, Chapter 2]. From a physical point of view, the result in (2.3) highlights that the process will eventually reach the boundary of Ω and will therefore be constrained in ∂, or, in other terms, it will be killed.

The *generator* of the killed Brownian motion coincides with the Dirichlet Laplacian, namely

$$\frac{d}{dt} T_t f|_{t=0} = \lim_{t\to 0^+} \frac{T_t f - f}{t} = \Delta_\Omega f$$

for each $f \in H^1_0(\Omega)$.

We now frame the notion of Brownian motion into a more general concept, according to the following classical framework:

Definition 2.3 (Lévy processes)**.** *Let $(\tilde{D}, \mathcal{B}, \mathbb{P})$ be a probability space. Then, we say that the stochastic process $X = (\tilde{D}, \mathcal{B}, \{\mathcal{B}_t\}_{t\geqslant 0}, \{X_t\}_{t\geqslant 0}, \mathbb{P})$ is a Lévy process if:*

1) $\mathbb{P}(X_0 = 0) = 1$*;*
2) *for any $n \in \mathbb{N}$ and $0 \leqslant t_1 < t_2 < \cdots < t_{n+1}$, the random variables $(X_{t_{j+1}} - X_{t_j}, 1 \leqslant j \leqslant n)$ are independent, and also $X_{t_{j+1}} - X_{t_j}$ and $X_{t_{j+1}-t_j} - X_0$ are identically distributed for each $1 \leqslant j \leqslant n$;*
3) *X is stochastically continuous, namely for each $a > 0$ and for all $s \geqslant 0$ it holds*
$$\lim_{t\to s} \mathbb{P}(|X_t - X_s| > a) = 0.$$

The Brownian motion is the prototype example of Lévy process, see for instance [App09].

We now recall the notion of s-stable subordinator. In a nutshell, a Lévy process is a subordinator if its realization is increasing as a function of time. Also, a subordinator is stable when its Laplace transform reduces to an exponential function of a power, as made precise through the following setting:

Definition 2.4 (s-stable subordinator)**.** *Let $s \in (0, 1)$ and $\mathfrak{S}_1 := C([0, +\infty), [0, +\infty))$. We say that a Lévy process*

$$S = (\mathfrak{S}_1, \mathcal{G}, \{S_t\}_{t\geqslant 0}, \widetilde{\mathbb{P}}^s) \tag{2.4}$$

is an s-stable subordinator if the following properties are satisfied:

1) *$S_t(\omega)$ is an increasing function of $t > 0$ for each $\omega \in \mathfrak{S}_1$;*
2) *for each $\lambda > 0$*
$$\widetilde{\mathbb{E}}^s[\exp(-\lambda S_t)] := \int_0^{+\infty} \exp(-x\lambda)\, \mu^s_t(dx) = \exp(-t\lambda^s), \tag{2.5}$$
where μ^s_t is the law associated with the random variable S_t, namely
$$\widetilde{\mathbb{P}}^s(S_t \in A) = \mu^s_t(A) \tag{2.6}$$
for each Borel subset $A \subset [0, +\infty)$.

Some comments are in order about the reason behind the name of s-stability in Definition 2.4. While the parameter s obviously refers to the exponent in (2.5), the notion of stability refers to the fact that

$$\begin{aligned} &\text{if } \overline{S}_t \text{ and } \underline{S}_t \text{ are independent and satisfy (2.5),}\\ &\text{then } \tilde{S}_t := \overline{S}_{t/2} + \underline{S}_{t/2} \text{ satisfies (2.5) as well.} \end{aligned} \tag{2.7}$$

To check this, we recall that the probability density $\tilde{\mu}^s_t$ of $\tilde{S}_t$ satisfies the condition

$$\int_0^{+\infty} f(z)\, \tilde{\mu}^s_t(dz) = \int_0^{+\infty} \int_0^{+\infty} f(x+y)\, \overline{\mu}^s_{t/2}(dx)\, \underline{\mu}^s_{t/2}(dy)$$

for each f which is Borel measurable, where $\overline{\mu}^s_{t/2}$ and $\underline{\mu}^s_{t/2}$ are the corresponding probability densities respectively for $\overline{S}_{t/2}$ and $\underline{S}_{t/2}$. Therefore, we obtain that for each $\lambda > 0$

$$\begin{aligned}
\int_0^{+\infty} e^{-\lambda x}\, \tilde{\mu}^s_t(dx) &= \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda(x+y)}\, \overline{\mu}^s_{t/2}(dx)\, \underline{\mu}^s_{t/2}(dy) \\
&= \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda x}\, e^{-\lambda y}\, \overline{\mu}^s_{t/2}(dx)\, \underline{\mu}^s_{t/2}(dy) \\
&= e^{-\lambda^s t/2} \int_0^{+\infty} e^{-\lambda x}\, \underline{\mu}^s_{t/2}(dy) \\
&= e^{-\lambda^s t},
\end{aligned} \tag{2.8}$$

which confirms (2.7).

Remark 2.5. We also remark that the density μ^s_t of a s-stable subordinator is absolutely continuous with respect to the Lebesgue measure for each $s \in (0,1)$. As a matter of fact, see for instance Proposition 3.1 in [KV18], if we call

$$\mu^s_t(l) := \int_0^{+\infty} e^{-lu - tu^s \cos(\pi s)} \sin(tu^s \sin(\pi s))\, du \quad \text{for all } l > 0,$$

then, $\mu^s_t \in L^1((0,+\infty))$ for each $t > 0$ and $s \in (0,1)$, and also for each Borel subset $A \subset [0,+\infty)$ it holds that

$$\mu^s_t(A) = \int_A \mu^s_t(l)\, dl.$$

Since the absolute continuity with respect to the Lebesgue measure is not needed in this paper, we will keep using the notation

$$\mu^s_t(A) = \int_A \mu^s_t(dl),$$

for each Borel subset $A \subset [0,+\infty)$.

Now we introduce the subordinate killed Brownian motion through the operation of subordination, which was first introduced by Bochner in [Boc49]. Roughly speaking, in this framework subordination determines a random choice of the time t through the s-stable subordinator S, providing in this way a new process (a formalization of this concept will be given in equation (2.9) here below). In the following result we recall a classical formula for the transition density of this subordinate process, which will be called *subordinate killed Brownian motion*. The formula is given in (2.10) and we provide here a detailed analytic proof for the convenience of the reader.

Theorem 2.6. *Let $s \in (0,1)$, Ω be open, bounded smooth and connected. Furthermore, let $S = (\mathfrak{S}_1, \mathcal{G}, \{S_t\}_{t\geqslant 0}, \widetilde{\mathbb{P}}^s)$ be an s-stable subordinator and $X_{D,x} := (\mathfrak{S}, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \{X^\Omega_t\}_{t\geqslant 0}, \mathbb{P}_x)$ be a killed Brownian motion as in Definition 2.2.*

Let us define the family of stochastic processes

$$\begin{aligned}
&Y^\Omega_D := (\mathfrak{S}\times\mathfrak{S}_1, \{\mathcal{F}_t\times\mathcal{G}\}_{t\geqslant 0}, \mathcal{F}\times\mathcal{G}, \{Y_{D,t}\}_{t\geqslant 0}, \{\mathbb{P}^s_{D,x}\}_{x\in\Omega}) \\
\textit{and}\qquad &Y_{D,t}(\omega,\omega_1) := X^\Omega_{S_t(\omega_1)}(\omega),
\end{aligned} \tag{2.9}$$

where $\mathbb{P}^s_{D,x} := \mathbb{P}_x \times \widetilde{\mathbb{P}}^s$.

Then, for each $x \in \Omega$ the stochastic process $Y^\Omega_{D,x}$ is a Markov process. Moreover, for each Lebesgue measurable subset $\Omega' \subset \Omega$ we have that

$$\mathbb{P}^s_{D,x}(Y_{D,t} \in \Omega') = \int_{\Omega'}\int_0^{+\infty} p^\Omega_D(l,x,y)\, \mu^s_t(dl)\, dy. \tag{2.10}$$

Proof. The process $Y^{\Omega}_{D,x}$ is a Markov process (see for instance [Bou84]). Thus, we only have to prove formula (2.10). Let $\Omega' \subset \Omega$ and $t > 0$, then we see that

$$\begin{aligned}\mathbb{P}^s_{D,x}(Y_{D,t} \in \Omega') &= \mathbb{P}^s_{D,x}(\{(\omega,\omega_1) \in \mathfrak{S} \times \mathfrak{S}_1 \quad \text{s.t. } X_{S_t(\omega_1)}(\omega) \in \Omega'\}) \\ &= \int_{\mathfrak{S}_1} \mathbb{P}_x(\{\omega \in \mathfrak{S} \quad \text{s.t. } X_{S_t(\omega_1)}(\omega) \in \Omega'\})\, d\widetilde{\mathbb{P}}^s(\omega_1).\end{aligned} \tag{2.11}$$

Now we consider $h > 0$ and define the following quantities

$$\begin{aligned}\mathcal{I}_k(h) := \int_{\{S_t(\omega_1)\in[kh,(k+1)h)\}} \Big(&\mathbb{P}_x(\{\omega \in \mathfrak{S} \quad \text{s.t } X_{S_t(\omega_1)}(\omega) \in \Omega'\}) \\ &- \mathbb{P}_x(\{\omega \in \mathfrak{S} \quad \text{s.t } X_{kh}(\omega) \in \Omega'\})\Big) d\widetilde{\mathbb{P}}^s(\omega_1) \\ \text{and} \quad \mathcal{J}_k(h) := \int_{kh}^{(k+1)h} \big(&\mathbb{P}_x(\{\omega \in \mathfrak{S} \quad \text{s.t } X_{kh}(\omega) \in \Omega'\}) - \mathbb{P}_x(\{\omega \in \mathfrak{S} \quad \text{s.t } X_l(\omega) \in \Omega'\})\big)\, \mu^s_t(l) dl.\end{aligned} \tag{2.12}$$

With the notation in (2.12), we observe that

$$\begin{aligned}&\int_{\mathfrak{S}_1} \mathbb{P}_x(\{\omega \in \mathfrak{S} \quad \text{s.t } X_{S_t(\omega_1)}(\omega) \in \Omega'\}) d\widetilde{\mathbb{P}}^s(\omega_1) \\ &= \sum_{k=0}^{+\infty} \int_{\{S_t(\omega_1)\in[kh,(k+1)h)\}} \mathbb{P}_x(\{\omega \in \mathfrak{S} \quad \text{s.t } X_{S_t(\omega_1)}(\omega) \in \Omega'\}) d\widetilde{\mathbb{P}}^s(\omega_1) \\ &= \sum_{k=0}^{+\infty} \int_{\{S_t(\omega_1)\in[kh,(k+1)h)\}} \mathbb{P}_x(\{\omega \in \mathfrak{S} \quad \text{s.t } X_{kh}(\omega) \in \Omega'\}) d\widetilde{\mathbb{P}}^s(\omega_1) + \sum_{k=0}^{+\infty} \mathcal{I}_k(h) \\ &= \sum_{k=0}^{+\infty} \mathbb{P}_x(\{\omega \in \mathfrak{S} \quad \text{s.t } X_{kh}(\omega) \in \Omega'\}) \int_{\{S_t(\omega_1)\in[kh,(k+1)h)\}} d\widetilde{\mathbb{P}}^s(\omega_1) + \sum_{k=0}^{+\infty} \mathcal{I}_k(h) \\ &= \sum_{k=0}^{+\infty} \mathbb{P}_x(\{\omega \in \mathfrak{S} \quad \text{s.t } X_{kh}(\omega) \in \Omega'\}) \widetilde{\mathbb{P}}^s(S_t(\omega_1) \in [kh,(k+1)h)) + \sum_{k=0}^{+\infty} \mathcal{I}_k(h) \\ &= \sum_{k=0}^{+\infty} \mathbb{P}_x(\{\omega \in \mathfrak{S} \quad \text{s.t } X_{kh}(\omega) \in \Omega'\}) \int_{kh}^{(k+1)h} \mu^s_t(dl) + \sum_{k=0}^{+\infty} \mathcal{I}_k(h) \\ &= \sum_{k=0}^{+\infty} \int_{kh}^{(k+1)h} \mathbb{P}_x(\{\omega \in \mathfrak{S} \quad \text{s.t } X_l(\omega) \in \Omega'\}) \mu^s_t(dl) + \sum_{k=0}^{+\infty} \mathcal{J}_k(h) + \sum_{k=0}^{+\infty} \mathcal{I}_k(h) \\ &= \int_0^{+\infty} \int_{\Omega'} p^{\Omega}_D(l,x,y)\, dy \mu^s_t(dl) + \sum_{k=0}^{+\infty} \mathcal{J}_k(h) + \sum_{k=0}^{+\infty} \mathcal{I}_k(h).\end{aligned} \tag{2.13}$$

Now we note that since $\mathbb{P}_x(A) \leqslant 1$ for each $A \in \mathcal{F}$, then

$$\begin{aligned}&\sum_{k=0}^{+\infty} \|\mathcal{I}_k(h)\|_{L^\infty((0,+\infty))} \leqslant \sum_{k=0}^{+\infty} \int_{\{S_t(\omega_1)\in[kh,(k+1)h)\}} d\widetilde{\mathbb{P}}^s(\omega_1) = 1 \\ \text{and} \quad &\sum_{k=0}^{+\infty} \|\mathcal{J}_k(h)\|_{L^\infty((0,+\infty))} \leqslant \sum_{k=0}^{+\infty} \int_{kh}^{(k+1)h} \mu^s_t(l)\, dl = 1.\end{aligned} \tag{2.14}$$

Furthermore, using (2.2) and the regularity of p^{Ω}_D stated in Theorem 1.4 one gets that for each $k \in \mathbb{N}$

$$\lim_{h\to 0} \mathcal{I}_k(h) = 0 \quad \text{and} \quad \lim_{h\to 0} \mathcal{J}_k(h) = 0. \tag{2.15}$$

By equations (2.14) and (2.15) we deduce that

$$\lim_{h\to 0}\sum_{k=0}^{+\infty}\mathcal{I}_k(h)=0 \quad\text{and}\quad \lim_{h\to 0}\sum_{k=0}^{+\infty}\mathcal{J}_k(h)=0.$$

Therefore, using the identity in (2.11) and taking the limit for $h\to 0$ in (2.13) we find that

$$\mathbb{P}^s_{D,x}(Y_{D,t}\in\Omega')=\int_0^{+\infty}\int_{\Omega'}p^{\Omega}_D(l,x,y)\,dy\,\mu^s_t(dl). \tag{2.16}$$

Hence we have proved formula (2.10), which concludes the proof of Theorem 2.6. □

According to Theorem 2.6, we can give the following:

Definition 2.7 (Subordinate killed Brownian motion, see [SV03])**.** *For each $x\in\Omega$ the process $Y^{\Omega}_{D,x}:=(\mathfrak{S}\times\mathfrak{S}_1,\{\mathcal{F}_t\times\mathcal{G}\}_{t\geqslant 0},\mathcal{F}\times\mathcal{G},\{Y_{D,t}\}_{t\geqslant 0},\mathbb{P}^s_{D,x})$ defined in Theorem 2.6 is called subordinate killed Brownian motion starting at x and it is given by*

$$Y_{D,t}(\omega,\omega_1)=\begin{cases}X^{\Omega}_{S_t(\omega_1)}(\omega) & \text{if } S_t(\omega_1)<\tau_\Omega(\omega),\\ \partial & \text{if } S_t(\omega_1)\geqslant\tau_\Omega(\omega).\end{cases} \tag{2.17}$$

Now we show that the transition density of the subordinate killed Brownian motion coincides with the Dirichlet spectral fractional heat kernel r^s_D defined in equation (1.13).

Proposition 2.8. *Let $r^s_D:(0,+\infty)\times\Omega\times\Omega\to\mathbb{R}$ be as in equation* (1.13)*. Then,*

$$r^s_D(t,x,y)=\int_0^{+\infty}p^{\Omega}_D(l,x,y)\,\mu^s_t(dl)\quad \text{for all } (t,x,y)\in(0,+\infty)\times\Omega\times\Omega. \tag{2.18}$$

Proof. Thanks to Theorem 1.4 we have that

$$p^{\Omega}_D(t,x,y)=\sum_{k=1}^{+\infty}\phi_k(x)\,\phi_k(y)\,\exp(-t\lambda_k).$$

Using this and equation (2.5) we obtain that

$$\begin{aligned}\int_0^{+\infty}p^{\Omega}_D(l,x,t)\,\mu^s_t(dl)&=\int_0^{+\infty}\sum_{k=1}^{+\infty}\phi_k(x)\,\phi_k(y)\,\exp(-l\lambda_k)\,\mu^s_t(dl)\\ &=\sum_{k=1}^{+\infty}\phi_k(x)\,\phi_k(y)\int_0^{+\infty}\exp(-l\lambda_k)\,\mu^s_t(dl)\\ &=\sum_{k=1}^{+\infty}\phi_k(x)\,\phi_k(y)\,\exp(-t\lambda^s_k)\\ &=r^s_D(t,x,y).\end{aligned} \tag{2.19}$$

The identity between the first and second line in (2.19) is guaranteed by the Dominated Convergence Theorem, which can be applied thanks to (A.1) in Proposition A.1 and Lemma A.2. □

As a straightforward consequence of Theorem 2.6 and Proposition 2.8, the transition density associated with the subordinate killed Brownian motion coincides with the Dirichlet spectral fractional heat kernel given in equation (1.13). Therefore,

$$\mathbb{P}^s_{D,x}(Y_{D,t}\in\Omega')=\int_{\Omega'}r^s_D(t,x,y)\,dy$$

and accordingly

$$\mathbb{P}^s_D(Y_{D,t} \in \Omega''|Y_{D,0} \in \Omega') = \frac{1}{|\Omega'|} \int_{\Omega' \times \Omega''} r^s_D(t,x,y)\, dx\, dy, \tag{2.20}$$

as anticipated in equation (1.16). The term $|\Omega'|^{-1}$ in equation (2.20) is a normalizing factor, and it is introduced so that $\mathbb{P}^s(\cdot|Y_{D,0} \in \Omega')$ is actually a probability in $\mathfrak{S} \times \mathfrak{S}_1$. Clearly, since the state space is not Ω (but $\Omega \cup \{\partial\}$), for $t > 0$ we have $\mathbb{P}^s(Y_{D,t} \in \Omega|Y_{D,0} \in \Omega') < 1$. By defining

$$\mathbb{P}^s_D(Y_{D,t} \in \partial|Y_{D,0} \in \Omega') := 1 - \mathbb{P}^s_D(Y_{D,t} \in \Omega|Y_{D,0} \in \Omega'),$$

we deduce that $\mathbb{P}^s(\cdot|Y_{D,0} \in \Omega')$ is a probability with state space $\Omega \cup \{\partial\}$.

In equation (1.8) of Proposition 1.6 we have stated that the Dirichlet spectral fractional Laplacian admits an integral representation. In that expression, a zero order term K_D, called the "killing measure", is present. In the following remark we point out its relation with the subordinate killed Brownian motion.

Remark 2.9. For each $x \in \Omega$ there is a connection between the killing measure $K_D(x)$ computed in x and the rapidity with which the process $Y^\Omega_{D,x}$ exits the domain Ω. More specifically, we have the identity (see Lemma 3.1 in [SV03])

$$K_D(x) = \frac{1}{\Gamma(1-s)} \mathbb{E}_x(\tau_\Omega^{-s}).$$

In particular this means that a subordinate killed process with low values for the exit time τ_Ω will have a greater "killing component". In this sense the killing measure keeps track of the killing rate.

From Proposition 2.8 we also obtain that:

Proposition 2.10. *Let $s \in (0,1)$ and Y^Ω_D as in Definition 2.7. The semigroup associated with Y^Ω_D*

$$\begin{aligned} R_t :&L^2(\Omega) \to L^2(\Omega) \\ &f \mapsto R_t f(x) := \int_\Omega f(y)\, r^s_D(t,x,y)\, dy, \end{aligned} \tag{2.21}$$

satisfies

$$\|R_t f\|_{L^2(\Omega)} < \|f\|_{L^2(\Omega)}.$$

Proof. We can rewrite (2.21) as

$$\begin{aligned} R_t f(x) &= \int_\Omega f(y)\, r^s_D(t,x,y)\, dy \\ &= \int_\Omega \sum_{k=1}^{+\infty} \phi_k(y)\, \phi_k(x)\, \exp(-t\lambda_k^s)\, f(y)\, dy \\ &= \sum_{k=1}^{+\infty} \phi_k(x)\, \exp(-t\lambda_k^s) \int_\Omega \phi_k(y)\, f(y)\, dy \\ &= \sum_{k=1}^{+\infty} f_k\, \phi_k(x)\, \exp(-t\lambda_k^s) \end{aligned} \tag{2.22}$$

for each $f \in L^2(\Omega)$. The identity between the second and third line is due to the bounds for the eigenfunctions given in Proposition A.1, to Lemma A.2 and the Dominated Convergence Theorem. Hence, we find that

$$\|R_t f\|_{L^2(\Omega)} = \left\| \sum_{k=1}^{+\infty} \phi_k(x) f_k \exp(-t\lambda_k^s) \right\|_{L^2(\Omega)} = \sum_{k=1}^{+\infty} f_k^2 \exp(-2\lambda_k^s t) < \sum_{k=1}^{+\infty} f_k^2 = \|f\|_{L^2(\Omega)}. \qquad \square$$

As well known, the generator associated with the killed Brownian motion is the Dirichlet Laplacian (see e.g. [CZ95, Theorem 2.13]). Now we show a similar result for the subordinate killed Brownian motion, namely that the Dirichlet spectral fractional Laplacian in (1.3) is the generator of Y_D^Ω.

Proposition 2.11. *The generator of the process Y_D^Ω is given by equation (1.3). Namely, for each $f \in H_D^{2s}(\Omega)$,*

$$\lim_{t\to 0^+} \frac{R_t f - f}{t} = -(-\Delta)^s_{D,\Omega} f,$$

where the convergence is meant in $L^2(\Omega)$.

Proof. We will prove the statement first for the eigenfunctions of the Laplacian, and then by density for all $f \in H_D^{2s}(\Omega)$. Let ϕ_k be as in equation (1.1). Then we have that for each $x \in \Omega$

$$\begin{aligned}
\frac{R_t\phi_k(x) - \phi_k(x)}{t} &= \frac{1}{t}\left(\int_\Omega \phi_k(y)\, r_D^s(t,x,y)\, dy - \phi_k(x)\right)\\
&= \frac{1}{t}\left(\int_\Omega \phi_k(y) \sum_{j=1}^{+\infty} \phi_j(x)\,\phi_j(y)\, \exp(-t\lambda_j^s)\, dy - \phi_k(x)\right) \qquad (2.23)\\
&= \phi_k(x)\left(\frac{\exp(-t\lambda_k^s) - 1}{t}\right).
\end{aligned}$$

Let now $f \in H_D^{2s}(\Omega)$, and consider its $L^2(\Omega)$ decomposition $f = \sum_{k=1}^{+\infty} f_k\,\phi_k$. Given $\varepsilon > 0$, we pick $N \in \mathbb{N}$ sufficiently large such that

$$\left\| f - \sum_{k=1}^{N} f_k\,\phi_k \right\|_{L^2(\Omega)} \leqslant \varepsilon.$$

Thus, owing to Proposition 2.10,

$$\left\| R_t f - \sum_{k=1}^{N} f_k\, R_t\phi_k \right\|_{L^2(\Omega)} = \left\| R_t\left(f - \sum_{k=1}^{N} f_k\,\phi_k\right)\right\|_{L^2(\Omega)} \leqslant \varepsilon.$$

This gives that

$$R_t f = \sum_{k=1}^{+\infty} f_k\, R_t\phi_k,$$

where the convergence is intended in $L^2(\Omega)$.

Since $f \in H_D^{2s}(\Omega)$, we have that $(-\Delta)^s_{D,\Omega} f \in L^2(\Omega)$, hence

$$(-\Delta)^s_{D,\Omega} f = \sum_{k=1}^{N} \lambda_k^s f_k\,\phi_k.$$

As a consequence, by (2.23),

$$\frac{R_t f - f}{t} + (-\Delta)^s_{D,\Omega} f = \sum_{k=1}^{+\infty} f_k\left(\frac{R_t\phi_k - \phi_k}{t} + \lambda_k^s\phi_k\right) = \sum_{k=1}^{+\infty} f_k\phi_k\left(\frac{\exp(-t\lambda_k^s) - 1}{t} + \lambda_k\right)$$

This gives that

$$\left\|\frac{R_t f - f}{t} + (-\Delta)^s_{D,\Omega} f\right\|^2_{L^2(\Omega)} = \sum_{k=1}^{+\infty} f_k^2\left(\frac{\exp(-t\lambda_k^s) - 1}{t} + \lambda_k^s\right)^2.$$

We observe that $-\lambda_k^s \leqslant \frac{\exp(-t\lambda_k^s)-1}{t} < 0$ for each $t>0$, which means that

$$\sum_{k=1}^{+\infty}\left\| f_k^2\left(\frac{\exp(-t\lambda_k^s)-1}{t}\right)^2\right\|_{L^\infty(0,+\infty)} \leqslant \sum_{k=1}^{+\infty} f_k^2\lambda_k^{2s} < +\infty,$$

where the last inequality is due to the fact that $f\in H_D^{2s}(\Omega)$. Therefore,

$$\lim_{t\to 0}\left\|\frac{R_tf-f}{t}+(-\Delta)^s_{D,\Omega}f\right\|^2_{L^2(\Omega)} = \sum_{k=1}^{+\infty}\lim_{t\to 0} f_k^2\left(\frac{\exp(-t\lambda_k^s)-1}{t}+\lambda_k^s\right)^2 = 0. \qquad \square$$

2.2. **Existence, Uniqueness and Maximum Principle for the Dirichlet spectral fractional heat equation.** We now focus on the existence, uniqueness and regularity of the solution to the Dirichlet spectral fractional heat equation (1.10) and on a Maximum Principle for its solution, addressing the proofs of Theorems 1.8 and 1.9.

To start with, the following result establishes the regularity of the kernel r_D^s, which will be later proved to be the solution to (1.10).

Lemma 2.12. *Let $s\in(0,1]$ and Ω be open, bounded, smooth and connected. Then, $r_D^s\in C^\infty([\varepsilon,+\infty)\times\overline{\Omega}\times\overline{\Omega})$ for each $\varepsilon>0$.*

Proof. Consider the truncated series

$$S_M(t,x,y) := \sum_{k=1}^{M}\phi_k(x)\,\phi_k(y)\,\exp(-t\lambda_k^s) \tag{2.24}$$

and $(t,y)\in(0,+\infty)\times\Omega$. Thanks to Proposition A.1 and Lemma A.2

$$\sum_{k=1}^{+\infty}|\phi_k(y)|^2\,\exp(-2t\lambda_k^s) \leqslant c^2_{m_0,\Omega,0}\sum_{k=1}^{+\infty}\lambda_k^{2\alpha(m_0)}\,\exp(-2t\lambda_k^s) < +\infty,$$

and therefore

$$S_M(t,\cdot,y) \text{ converges to } r^s_{D,y}(t,\cdot) := r^s_D(t,\cdot,y) \text{ in } L^2(\Omega) \text{ for each } (t,y)\in(0,+\infty)\times\Omega. \tag{2.25}$$

From the $L^2(\Omega)$ convergence of the series we obtain, possibly up to a subsequence, that $S_M(t,\cdot,y)$ is converging a.e. to $r^s_{D,y}(t,\cdot)$ in Ω. Using the estimates (A.1) and Lemma A.2 we observe that

$$\sum_{k=1}^{+\infty}\|\phi_k(\cdot)\,\phi_k(y)\,\exp(-t\lambda_k^s)\|_{C^r(\overline{\Omega})} \leqslant c_{m_r,\Omega,r}c_{m_0,\Omega,0}\sum_{k=1}^{+\infty}\lambda_k^{\alpha(m_0)+\alpha(m_r)}\,\exp(-t\lambda_k^s) < +\infty.$$

This gives that $S_M(t,\cdot,y)$ is converging in $C^r(\overline{\Omega})$ to $r_D^s(t,\cdot,y)$ for each $r\in\mathbb{N}$. From this, we conclude that $r_D^s(t,\cdot,y)\in C^\infty(\overline{\Omega})$, and since r_D^s is symmetric in the last two variables, we have that $r_D^s(t,\cdot,\cdot)\in C^\infty(\overline{\Omega}\times\overline{\Omega})$ for each $t\in(0,+\infty)$.

The next step is to show that $r_D^s\in C^\infty([\varepsilon,+\infty)\times\overline{\Omega}\times\overline{\Omega})$ for each $\varepsilon>0$. Given $(x,y)\in\overline{\Omega}\times\overline{\Omega}$ and $\varepsilon>0$, we note that

$$\begin{aligned}&\sum_{k=1}^{+\infty}\|D^\gamma\phi_k(x)\,D^\beta\phi_k(y)\,\frac{d^r}{dt^r}\exp(-t\lambda_k^s)\|_{C^0([\varepsilon,+\infty))}\\ \leqslant&\sum_{k=1}^{+\infty}\|\phi_k\|_{C^{|\gamma|}(\overline{\Omega})}\,\|\phi_k\|_{C^{|\beta|}(\overline{\Omega})}\,\|\exp(-t\lambda_k^s)\|_{C^r([\varepsilon,+\infty))}\,(\lambda_k^s)^r\\ \leqslant& c_{m_{|\gamma|},\Omega,|\gamma|}c_{m_{|\beta|},\Omega,|\beta|}\sum_{k=1}^{+\infty}\lambda_k^{\alpha(m_{|\gamma|})+\alpha(m_{|\beta|})+rs}\,\exp(-\varepsilon\lambda_k^s)\\ <&+\infty\end{aligned}$$

for each multindex $\beta, \gamma \in \mathbb{N}^n$ and $r \in \mathbb{N}$. In particular, this estimate gives that

$$\frac{d^r}{dt^r} D^\gamma D^\beta S_M \to \frac{d^r}{dt^r} D^\gamma D^\beta r_D^s \quad \text{for } M \to +\infty$$

in $C^0([\epsilon, +\infty))$, leading to $r_D^s \in C^\infty([\varepsilon, +\infty) \times \overline{\Omega} \times \overline{\Omega})$ for any $\varepsilon > 0$. □

Now we prove that r_D^s is a solution to the first two equations in (1.10). The proof of the fact that r_D^s solves also the third equation in (1.10) and that the solution is unique can be found at the end of this section (see the proof of Theorem 1.8).

Lemma 2.13. *Let $s \in (0,1)$ and Ω be open, bounded, smooth and connected. Then, for each $y \in \Omega$ the function $r_{D,y}^s(t,x) := r_D^s(t,x,y)$ is a solution to the first and second equation in* (1.10)*.*

Proof. First we show that $r_{D,y}^s$ solves the first equation in (1.10). For this, we observe that Proposition A.1 and Lemma A.2 yield that we can differentiate with respect to t inside the sum, and thus

$$\partial_t r_{D,y}^s(t,x) = -\sum_{k=1}^{+\infty} \phi_k(x)\,\phi_k(y)\,\lambda_k^s\,\exp(-t\,\lambda_k^s) = -(-\Delta)_{D,\Omega}^s r_{D,y}^s(t,x),$$

which proves that $r_{D,y}^s$ solves the first equation in (1.10).

Furthermore, we recall the uniform convergence of the series $S_M(t,\cdot,y)$ in Ω for each $(t,y) \in (0,+\infty) \times \Omega$ to $r_{D,y}^s(t,\cdot)$, as given by (2.25). Since $\phi_k(x) = 0$ for each $x \in \partial\Omega$ and $k \geqslant 1$, then we get that also $r_{D,y}^s(t,x) = 0$ for each $x \in \partial\Omega$. These observations show that $r_{D,y}^s$ solves also the second equation in (1.10), and this concludes the proof of Lemma 2.13. □

We now state and prove a technical result that will be useful later in order to prove the Maximum Principle stated in Theorem 1.9. More specifically, we will obtain a continuity with respect to the initial data (see the forthcoming equation (2.27)) which will be used to prove that r_D^s satisfies the third equation in (1.10) (as stated in Theorem 1.8 below).

Lemma 2.14. *Let $s \in (0,1)$ and Ω be open, bounded, smooth and connected. For each $f \in L^2(\Omega)$ the function $R_t f$ defined in* (2.21) *is the unique solution in $C([0,+\infty), L^2(\Omega)) \cap C^1((0,+\infty), H_D^{2s}(\Omega))$ to the system of equations*

$$\begin{cases} \partial_t u(t,y) = -(-\Delta)_{D,\Omega}^s u(t,y) & \text{for all } (t,y) \in (0,+\infty) \times \Omega, \\ u(t,y) = 0 & \text{for all } (t,y) \in (0,+\infty) \times \partial\Omega, \\ u(0,y) = f(y) & \text{for all } y \in \Omega. \end{cases} \tag{2.26}$$

The last equation in (2.26) *is meant as*

$$\lim_{t\to 0} \|u(t,y) - f(y)\|_{L^2(\Omega)} = 0.$$

Furthermore, for each $\varepsilon > 0$, it holds that $R_t f(y) \in C^\infty([\varepsilon,+\infty) \times \overline{\Omega})$. Moreover, if $f \in C_c^\infty(\Omega)$, then

$$\lim_{t\to 0} \|R_t f(y) - f(y)\|_{L^\infty(\Omega)} = 0. \tag{2.27}$$

Proof. Let $f \in L^2(\Omega)$. First we show that $R_t f$ satisfies the first equation in (2.26). By Lemma 2.12 we have that $r_D^s \in C^\infty([\varepsilon,+\infty) \times \overline{\Omega} \times \overline{\Omega})$ for each $\varepsilon > 0$. From this regularity result one easily deduces that $R_t f(y) \in C^\infty([\varepsilon,+\infty) \times \overline{\Omega})$ for each $\varepsilon > 0$. In view of (2.22), we know that

$$R_t f(y) = \sum_{k=1}^{+\infty} f_k\,\phi_k(y)\,\exp(-t\lambda_k^s), \tag{2.28}$$

and using Proposition A.1 and Lemma A.2 we obtain that

$$\partial_t R_t f(y) = \sum_{k=1}^{+\infty} f_k\, \phi_k(y)\, \frac{d}{dt} \exp(-t\lambda_k^s) = -\sum_{k=1}^{+\infty} \lambda_k^s\, f_k\, \phi_k(y)\, \exp(-t\lambda_k^s) = -(-\Delta)^s_{D,\Omega} R_t f(y).$$

This proves that $R_t f$ satisfies the first equation in (2.26).

Also the second equation in (2.26) is satisfied by $R_t f$. Indeed, thanks to Proposition A.1 and Lemma A.2 for each $t > 0$ the series on the right-hand side of (2.28) converges uniformly in $y \in \Omega$, and since the ϕ_k's are all vanishing on $\partial\Omega$, we can conclude that the second equation in (2.26) holds true.

Now we verify that $R_t f$ satisfies the last equation in (2.26). Using (2.28) and the fact that $f \in L^2(\Omega)$ we obtain that

$$\begin{aligned} \lim_{t\to 0^+} \int_\Omega |R_t f(y) - f(y)|^2\, dy &= \lim_{t\to 0^+} \sum_{k=1}^{+\infty} \left(\exp(-t\lambda_k^s) - 1\right)^2\, f_k^2 = 0 \\ &= \sum_{k=1}^{+\infty} \lim_{t\to 0^+} \left(\exp(-t\lambda_k^s) - 1\right)^2\, f_k^2 = 0. \end{aligned} \tag{2.29}$$

Therefore, we have proved that $R_t f$ satisfies also the third equation in (2.26). Now, from the fact that $R_t f(y) \in C^\infty([\varepsilon, +\infty) \times \overline{\Omega})$ for each $\varepsilon > 0$ together with equations (2.29) and (2.28) we deduce that $R_t f(y) \in C([0, +\infty), L^2(\Omega)) \cap C^1((0, +\infty), H^{2s}_D(\Omega))$. Therefore, we have proved that $R_t f$ is a solution to (2.26) in the space $C([0, +\infty), L^2(\Omega)) \cap C^1((0, +\infty), H^{2s}_D(\Omega))$.

Let us now discuss the uniqueness of the solution of (2.26). It is showed by using the monotonicity[2] of the operator $(-\Delta)^s_{D,\Omega}$. In particular suppose that $g(t, y) \in C([0, +\infty), L^2(\Omega)) \cap C^1((0, +\infty), H^{2s}_D(\Omega))$ is another solution to (2.26), then

$$\begin{aligned} \frac{d}{dt} \int_\Omega |R_t f(y) - g(t, y)|^2\, dy &= \frac{d}{dt} \int_\Omega \left(R_t f(y) - g(t, y)\right)\, \left(R_t f(y) - g(t, y)\right)\, dy \\ &= -\int_\Omega (-\Delta)^s_{D,\Omega} \left(R_t f(y) - g(t, y)\right)\, \left(R_t f(y) - g(t, y)\right)\, dy \\ &\leqslant 0 \end{aligned}$$

Therefore, the function $\kappa(t) := \|R_t f(y) - g(t, y)\|^2_{L^2(\Omega)}$ is positive, decreasing, continuous and by hypothesis $\kappa(0) = 0$. This implies that $\kappa(t) = 0$ for each $t > 0$, which means that $R_t f(y) = g(t, y)$ for each $t \in (0, +\infty)$ and for almost every $y \in \Omega$.

We now prove the limit in equation (2.27). Let $f \in C^\infty_c(\Omega)$. For this, we note that, by Proposition A.4,

$$\sum_{k=1}^{+\infty} \|f_k \phi_k\|_{L^\infty(\Omega)} \|(1 - \exp(-t\lambda_k^s))\|_{L^\infty([0,+\infty))} \leqslant \sum_{k=1}^{+\infty} \|f_k \phi_k\|_{L^\infty(\Omega)} < +\infty.$$

[2]An operator $A : \mathcal{D}(A) \subset H \to H$ is said *monotone* if $(Av, v) \geqslant 0$ for each $v \in \mathcal{D}(A)$, where $(H, (\cdot, \cdot))$ is an Hilbert space. For further details see [Bre11].

Therefore, using (2.28) we conclude that

$$\begin{aligned}
\lim_{t\to 0^+} \|R_t f - f\|_{L^\infty(\Omega)} &= \lim_{t\to 0^+} \left\| \sum_{k=1}^{+\infty} f_k\, \phi_k\, \exp(-t\lambda_k^s) - \sum_{k=1}^{\infty} f_k\phi_k \right\|_{L^\infty(\Omega)} \\
&\leqslant \lim_{t\to 0^+} \sum_{k=1}^{+\infty} \|f_k\phi_k\|_{L^\infty(\Omega)} (1-\exp(-t\lambda_k^s)) \\
&= \sum_{k=1}^{+\infty} \|f_k\phi_k\|_{L^\infty(\Omega)} \lim_{t\to 0^+} (1-\exp(-t\lambda_k^s)) \\
&= 0. \qquad \square
\end{aligned}$$

Now we are ready to prove the Maximum Principle for the Dirichlet spectral fractional heat equation (1.10) as stated in Theorem 1.9. With a rather standard method in parabolic PDEs, in the following proof we use an argument based on the Maximum Principle for the operator $(-\Delta)^s_{D,\Omega}$.

Alternatively, Theorem 1.9 can be proved by making use of the probabilistic framework developed in Section 2.1. More specifically, one can deduce the two-sided inequality in (1.14) from the Maximum Principle for the classical Dirichlet heat equation and formula (2.18). A proof using the probabilistic approach can be found in Appendix B. Instead, here we present a purely analytical proof which does not require any probability result.

Proof of Theorem 1.9. We focus on the proof of the right-hand side inequality in (1.14). Let us first assume that $f \in C^\infty_c(\Omega)$ (the general case will then be treated by using an approximation argument). By Lemma 2.14 we know that $R_t f(y) =: f(t,y) \in C([0,+\infty)\times\overline{\Omega}) \cap C^\infty([\delta,+\infty)\times\overline{\Omega})$ for each $\delta > 0$. Let us call $K := \max\{0, \sup_\Omega f\}$, and assume by contradiction that there exists $(t^*, y^*) \in (0,+\infty)\times\Omega$ such that

$$f(t^*, y^*) > K.$$

We define a new function $W(t,y) := f(t^*,y^*) - \varepsilon - f(t,y)$, where $\varepsilon > 0$ is a constant satisfying

$$\varepsilon < \inf_\Omega \left(f(t^*,y^*) - f(x)\right). \tag{2.30}$$

Note that $W(t,y) \in C([0,+\infty)\times\overline{\Omega}) \cap C^\infty([\delta,+\infty)\times\overline{\Omega})$ for each $\delta > 0$. Then, we observe that

$$W(0,y) > 0 \quad \text{and} \quad W(t^*,y^*) < 0,$$

and therefore there exists a point $(\tilde{t},\tilde{y}) \in (0,+\infty)\times\Omega$ such that

$$\begin{aligned}
&W(t,y) > 0 \quad \text{for all} \quad (t,y) \in [0,\tilde{t}) \times \Omega, \\
&W(\tilde{t},\tilde{y}) = 0 \quad \text{and} \quad W(\tilde{t},y) \geqslant 0 \quad \text{for all } y \in \Omega.
\end{aligned} \tag{2.31}$$

In particular this means that

$$\partial_t W(\tilde{t},\tilde{y}) \leqslant 0.$$

Moreover, using the Maximum Principle for $(-\Delta)^s_{D,\Omega}$ given in Lemma 1.7 and the fact that $\tilde{y}$ is a minimum for $W(\tilde{t},\cdot)$, we deduce that

$$(-\Delta)^s_{D,\Omega} W(\tilde{t},\tilde{y}) \leqslant 0.$$

Therefore, we get

$$\partial_t W(\tilde{t},\tilde{y}) + (-\Delta)^s_{D,\Omega} W(\tilde{t},\tilde{y}) \leqslant 0.$$

On the other hand, using the fact that $f(t,y)$ is a solution to the first equation in (2.26) (as established by Lemma 2.14) we obtain that

$$\begin{aligned}
\partial_t W(\tilde{t},\tilde{y}) + (-\Delta)^s_{D,\Omega} W(\tilde{t},\tilde{y}) &= -\partial_t f(\tilde{t},\tilde{y}) + (-\Delta)^s_{D,\Omega}(f(t^*,y^*)-\varepsilon) - (-\Delta)^s_{D,\Omega} f(\tilde{t},\tilde{y}) \\
&= (-\Delta)^s_{D,\Omega}(f(t^*,y^*)-\varepsilon) > 0,
\end{aligned}$$

providing a contradiction. This completes the proof of the right-hand side inequality in (1.14) when $f \in C_c^\infty(\Omega)$.

We now prove the right-hand side inequality in (1.14) when $f \in L^2(\Omega)$. To do so let us first observe that

$$r_D^s(t,x,y) \geqslant 0 \text{ for each } (t,x,y) \in (0,+\infty) \times \Omega \times \Omega. \tag{2.32}$$

Indeed, if by contradiction there exist (t_0,x_0,y_0) such that $r_D^s(t_0,x_0,y_0) < 0$, then by continuity there exists a neighbourhood V_{x_0} of x_0 such that $r_D^s(t_0,\cdot,y_0)|_{V_{x_0}} < 0$. Now, given a function $g \in C_c^\infty(V_{x_0})$ such that $g \geqslant 0$, we define

$$g(t,y) := \int_{V_{x_0}} r_D^s(t,x,y)\, g(x)\, dx$$

and we see that

$$g(t_0,y_0) = \int_{V_{x_0}} r_D^s(t_0,x,y_0)\, g(x)\, dx < 0.$$

On the other hand, $g(t,y)$ is the solution to (2.26) with initial condition $g(x)$, and therefore $g(t,x) \geqslant 0$ for each $(t,x) \in (0,+\infty) \times \Omega$, thus providing the desired contradiction and establishing (2.32).

Let us now consider a sequence of invading compact sets $K_n \subset \Omega$, namely $\cup_{k=1}^{+\infty} K_n = \Omega$, and $K_n \subset K_{n+1}$ for each n, and a sequence of smooth functions $f_n \in C_c^\infty(\Omega)$ such that $f_n|_{K_n} = 1$ and $0 \leqslant f_n \leqslant 1$. Then for each $(t,y) \in (0,+\infty) \times \Omega$ we have that

$$\int_\Omega r_D^s(t,x,y)\, dx = \lim_{n\to+\infty} \int_\Omega r_D^s(t,x,y)\, f_n(x)\, dx,$$

where the identity is guaranteed by the Dominated Convergence Theorem.

Furthermore, notice that, for each n,

$$\int_\Omega r_D^s(t,x,y)\, f_n(x)\, dx \leqslant \max\left\{0, \sup_\Omega f_n\right\} = 1.$$

These observations give that

$$\int_\Omega r_D^s(t,x,y)\, dx \leqslant 1. \tag{2.33}$$

Therefore, if $f \in L^2(\Omega)$ we have that

$$f(t,y) = \int_\Omega r_D^s(t,x,y)\, f(x)\, dx \leqslant \max\left\{0,\ \sup_\Omega f\right\} \int_\Omega r_D^s(t,x,y)\, dx \leqslant \max\left\{0,\ \sup_\Omega f\right\},$$

and this proves the right-hand side inequality of (1.14) when $f \in L^2(\Omega)$.

The left-hand side inequality of (1.14) can be proved analogously. □

For the sake of completeness, we now improve the statement in (2.32) by showing the strict positivity of the Dirichlet spectral fractional heat kernel r_D^s.

Corollary 2.15. *Let $s \in (0,1)$ and Ω be open, bounded, smooth and connected. Then, for each $(t,x,y) \in (0,+\infty) \times \Omega \times \Omega$ it holds that*

$$r_D^s(t,x,y) > 0.$$

Proof. We recall (2.32) and we argue by contradiction supposing that there exists some $(t_0,x_0,y_0) \in (0,+\infty) \times \Omega \times \Omega$ such that $r_D^s(t_0,x_0,y_0) = 0$. Then, using (1.8) and Lemma 2.13 we have that

$$\begin{aligned}\partial_t r_D^s(t_0,x_0,y_0) &= -(-\Delta)^s_{D,\Omega} r_D^s(t_0,x_0,y_0)\\ &= -\int_\Omega \left(0 - r_D^s(t_0,x,y_0)\right) J_D(x_0,x)\, dx > 0.\end{aligned}$$

Therefore, the function $t \mapsto r_D^s(t,x_0,y_0)$ is strictly increasing in t in a neighbourhood of t_0.

Since $r^s_D(t_0,x_0,y_0)=0$, we conclude that there exists some $t<t_0$ such that $r^s_D(t,x_0,y_0)<0$, which is in contradiction with (2.32). $\square$

As a consequence of Corollary 2.15 we establish that $R_t : C_0(\Omega)\to C_0(\Omega)$ is continuous with Lipschitz constant 1. This continuity result will be fundamental when proving that $r^s_D(t,x,y)$ solves also the third equation in (1.10).

Corollary 2.16. *Let $s\in(0,1)$ and $\Omega\subset\mathbb{R}^n$ be open, bounded, smooth and connected. Then,*

$$\begin{aligned} R_t : C_0(\Omega) &\to C_0(\Omega)\\ f &\longmapsto R_t f \end{aligned}$$

satisfies

$$\|R_t f\|_{L^\infty(\Omega)} \leqslant \|f\|_{L^\infty(\Omega)}. \tag{2.34}$$

Proof. From Corollary 2.15 and inequality (2.33) we obtain that

$$|R_t f(y)| = \left|\int_\Omega r^s_D(t,x,y)\,f(x)\,dx\right| \leqslant \|f\|_{L^\infty(\Omega)}\int_\Omega r^s_D(t,x,y)\,dx \leqslant \|f\|_{L^\infty(\Omega)},$$

for every $y\in\Omega$. Taking the maximum in $y\in\Omega$ of both sides we get (2.34), as desired. $\square$

Now we address the proof of Theorem 1.8.

Proof of Theorem 1.8. In Lemma 2.12 we already proved that $r^s_D\in C^\infty([\varepsilon,+\infty)\times\overline{\Omega}\times\overline{\Omega})$ for each $\varepsilon>0$. Moreover, thanks to Lemma 2.13 we know that $r^s_{D,y}$ is a solution to the first and second equation in (1.10). It is left to show that it satisfies the third equation, and that is unique in $C^1((0,+\infty),H^{2s}_D(\Omega))$. Let us first prove that $r^s_D(\cdot,\cdot,y)$ satisfies the third equation in (1.10).

Let $f\in C_0(\Omega)$ and consider a sequence of smooth functions $\{f_n\}_n\subset C^\infty_c(\Omega)$ such that f_n converges uniformly in Ω to f. Then we have that for each $y\in\Omega$,

$$\begin{aligned} |R_tf(y)-f(y)| &\leqslant |R_tf(y)-R_tf_n(y)|+|R_tf_n(y)-f_n(y)|+|f_n(y)-f(y)|\\ &\leqslant \|R_tf-R_tf_n\|_{L^\infty(\Omega)}+\|R_tf_n-f_n\|_{L^\infty(\Omega)}+\|f_n-f\|_{L^\infty(\Omega)}. \end{aligned}$$

Given $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ such that for all $n\geqslant N_\varepsilon$ with $n\in\mathbb{N}$ it holds $\|f_n-f\|_{L^\infty(\Omega)}<\varepsilon$. Thanks to Corollary 2.16 we know that $R_t : C_0(\Omega)\to C_0(\Omega)$ is continuous, and therefore if $n\geqslant N_\varepsilon$ we get that

$$\|R_tf-R_tf_n\|_{L^\infty(\Omega)}\leqslant\|f_n-f\|_{L^\infty(\Omega)}<\varepsilon.$$

Given $n\geqslant N_\varepsilon$, thanks to equation (2.27), we have that for each $t>0$ small enough it holds $\|R_tf_n-f_n\|_{L^\infty(\Omega)}<\varepsilon$. Therefore, we have just proved that for each $f\in C_0(\Omega)$ and $y\in\Omega$ it holds that

$$\lim_{t\to0}\int_\Omega r^s_D(t,x,y)\,f(x)\,dx=f(y),$$

which establishes that the function $r^s_{D,y}(t,x):=r^s_D(t,x)$ satisfies also the third equation in (1.10). Note that $r^s_{D,y}(t,x)\in C^1((0,+\infty);H^{2s}_D(\Omega))$. Indeed, in view of Proposition A.1, Lemma A.2 and equation (1.4) we have that for each $\delta>0$

$$\begin{aligned} &\sum_{k=1}^{+\infty}\lambda_k^{2s}\phi_k(y)^2\|\exp(-2t\lambda_k^s)\|_{L^\infty((\delta,+\infty))}<+\infty\\ \text{and}\quad &\sum_{k=1}^{+\infty}\lambda_k^{2s}\phi_k(y)^2\left\|\frac{d}{dt}\exp(-2t\lambda_k^s)\right\|_{L^\infty((\delta,+\infty))}<+\infty. \end{aligned}$$

Now that we have showed that $r^s_{D,y}\in C^1((0,+\infty);H^{2s}_D(\Omega))$ is a solution to (1.10), let us prove that it is unique in $C^1((0,+\infty);H^{2s}_D(\Omega))$. Suppose by contradiction that there exists, for some $y\in\Omega$, a map $v_y(t,x)$ that satisfies equations (1.10) and such that $v_y(t,x) \in C^1((0,+\infty), H^{2s}_D(\Omega))$. Then we can write

$$v_y(t,x) = \sum_{k=1}^{+\infty} c_k(t,y)\,\phi_k(x),$$

where $c_k(t,y) \in C^1((0,+\infty))$ as a consequence of the hypothesis $v_y(t,x) \in C^1((0,+\infty); H^{2s}_D(\Omega))$. Since v_y is a solution to the Dirichlet spectral fractional heat equation (1.10) we have that

$$\sum_{k=1}^{+\infty} \frac{d}{dt} c_k(t,y)\,\phi_k(x) = \partial_t v_y(t,x) = -(-\Delta)^s_{D,\Omega} v_y(t,x) = -\sum_{k=1}^{+\infty} \lambda_k^s\, c_k(t,y)\,\phi_k(x),$$

from which we deduce that $c_k(t,y) = m_{k,y}\,\exp(-t\lambda_k^s)$, for some constant $m_{k,y}$ to be specified. Since $v_y(t,x)$ satisfies the third equation in (1.10) we finally get that for each $k \geqslant 1$

$$\begin{aligned}
\phi_k(y) &= \lim_{t\to 0^+} \int_\Omega v_y(t,x)\,\phi_k(x)\,dx \\
&= \lim_{t\to 0^+} \int_\Omega \sum_{j=1}^{+\infty} m_{j,y}\,\exp(-t\lambda_j^s)\,\phi_j(x)\,\phi_k(x)\,dx \\
&= \lim_{t\to 0^+} m_{k,y}\,\exp(-t\lambda_k^s) \\
&= m_{k,y},
\end{aligned}$$

and therefore $v_y(t,x) = r_{D,y}(t,x)$, which proves that the solution to (1.10) in $C^1((0,+\infty); H^{2s}_D(\Omega))$ is unique. This last step concludes the proof of Theorem 1.8. □

2.3. **Proof of Theorems 1.10 and 1.11.** In this section we prove Theorems 1.10 and 1.11. We begin by pointing out that (2.18) means that $r^s_D(t,x,y)$ is equal to the probability that S_t has values in the infinitesimal interval dl, times the density of probability $p^\Omega_D(l,x,y)$ that a point starting at y, following a killed Brownian motion, will be in the position x at a time l, integrated over all possible values of l. Therefore, the action of subordination introduces a random s-dependent choice of the time in which we consider the classical Dirichlet heat kernel, so that $r^s_D(t,x,y)$ results in a time-weighted superposition of $p^\Omega_D(l,x,y)$.

In Theorems 1.10 and 1.11 we analyze the monotonicity with respect to s of the fractional Dirichlet heat kernel r^s_D on the diagonal, and the monotonicity of the conditional probability in equation (2.20) when $\Omega'' = \Omega'$.

Proof of Theorems 1.10 and 1.11. First we show point (i) of Theorems 1.10 and 1.11. We take the derivative of $r^s_D(t,x,y)$ with respect to s, which gives us

$$\begin{aligned}
\frac{d}{ds} r^s_D(t,x,y) &= \frac{d}{ds} \sum_{k=1}^{+\infty} \phi_k(x)\,\phi_k(y)\,\exp(-t\lambda_k^s) \\
&= -t \sum_{k=1}^{+\infty} \phi_k(x)\,\phi_k(y)\,\exp(-t\lambda_k^s)\,\lambda_k^s\,\ln(\lambda_k).
\end{aligned} \tag{2.35}$$

Note that we can differentiate inside the sum thanks to Proposition A.1 and Lemma A.2. Therefore, if we take the integral over $\Omega' \times \Omega'$ of both sides of (2.35) we get

$$\begin{aligned}
\frac{d}{ds} \int_{\Omega'\times\Omega'} r^s_D(t,x,y)\,dx\,dy &= \int_{\Omega'\times\Omega'} \frac{d}{ds} r^s_D(t,x,y)\,dx\,dy \\
&= -t \sum_{k=1}^{+\infty} c^2_{k,\Omega'}\,\exp(-t\lambda_k^s)\,\lambda_k^s\,\ln(\lambda_k),
\end{aligned} \tag{2.36}$$

where we have defined $c_{k,\Omega'} := \int_{\Omega'} \phi_k(x)\,dx$. The first identity in (2.36) is a consequence of Proposition A.1, Lemma A.2 and the Dominated Convergence Theorem. Therefore, if $\lambda_1 \geqslant 1$, the last term in (2.36) is strictly negative, and this implies (i) of Theorem 1.11.

Moreover, if $x = y \in \Omega$, then from (2.35) we also have that

$$r_D^{s_0}(t,x,x) > r_D^{s_1}(t,x,x),$$

for each $t \in (0,+\infty)$, and $0 < s_0 < s_1 \leqslant 1$, which proves point (i) of Theorem 1.10.

Let us now focus on the proof of point (ii) of Theorems 1.10 and 1.11. If $\lambda_1 < 1$, then from (2.36) we obtain that

$$\frac{\exp(t\lambda_1^s)}{t}\left(\frac{d}{ds}\int_{\Omega'\times\Omega'} r_D^s(t,x,y)\,dx\,dy\right) = c_{1,\Omega'}^2\lambda_1^s|\ln(\lambda_1)| - \sum_{k=2}^{+\infty} c_{k,\Omega'}^2 \exp(-t(\lambda_k^s-\lambda_1^s))\lambda_k^s\ln(\lambda_k).$$

Since the first eigenfunction is strictly positive, see for instance [Eva10], recalling also the classical estimates on the Dirichlet eigenvalues (see e.g. (A.7)), the above computation yields that $c_{1,\Omega'} > 0$.

We claim that for each $t > 0$ large enough (depending on s and Ω') we get

$$\frac{d}{ds}\int_{\Omega'\times\Omega'} r_D^s(t,x,y)\,dx\,dy > 0. \tag{2.37}$$

Indeed, using Hölder's inequality and the fact that for each $k \geqslant 1$ the ϕ_k's are normalized in $L^2(\Omega)$, we obtain

$$c_{k,\Omega'}^2 = \left(\int_{\Omega'} \phi_k(x)\,dx\right)^2 \leqslant \left(\left(\int_{\Omega'} \phi_k(x)^2\,dx\right)^{\frac{1}{2}} |\Omega'|^{\frac{1}{2}}\right)^2 \leqslant |\Omega'|.$$

Thus, we can employ this last inequality to obtain the bound

$$\begin{aligned}\frac{\exp(t\lambda_1^s)}{t}\left(\frac{d}{ds}\int_{\Omega'\times\Omega'} r_D^s(t,x,y)\,dx\,dy\right) &= c_{1,\Omega'}^2\lambda_1^s|\ln(\lambda_1)| - \sum_{k=2}^{+\infty} c_{k,\Omega'}^2 \exp(-t(\lambda_k^s-\lambda_1^s))\lambda_k^s\ln(\lambda_k)\\ &\geqslant c_{1,\Omega'}^2\lambda_1^s|\ln(\lambda_1)| - |\Omega'|\sum_{k=k''}^{+\infty} \exp(-t(\lambda_k^s-\lambda_1^s))\lambda_k^s\ln(\lambda_k),\end{aligned} \tag{2.38}$$

where $k'' = \min\{k \geqslant 1 \quad \text{s.t. } \lambda_k \geqslant 1\}$.

Also, thanks to equation (A.7), we can apply Lemma A.3 to the last series in (2.38) with $a_k = \lambda_k^s\ln(\lambda_k)$ and $b_k = (\lambda_k^s - \lambda_1^s)$, and deduce that

$$\lim_{t\to+\infty} |\Omega'|\sum_{k=k''}^{+\infty} \exp(-t(\lambda_k^s-\lambda_1^s))\lambda_k^s\ln(\lambda_k) = 0.$$

By combining this limit with the inequality in (2.38) we conclude the proof of the claim in (2.37).

Therefore, as a consequence of (2.37), if we fix $0 < s_0 < s_1 \leqslant 1$, there exists some $T > 0$ depending on s_0, s_1 and Ω' such that for each $t > T$

$$\int_{\Omega'\times\Omega'} r_D^{s_0}(t,x,y)\,dx\,dy < \int_{\Omega'\times\Omega'} r_D^{s_1}(t,x,y)\,dx\,dy.$$

This concludes the proof of point (ii) of Theorem 1.11.

Point (ii) of Theorem 1.10 is proved similarly. To show it, we note that from equation (2.35), we obtain that if $x = y \in \Omega$, then

$$\frac{\exp(t\lambda_1)}{t}\frac{d}{ds}r_D^s(t,x,x) = \phi_1(x)^2\lambda_1^s|\ln\lambda_1| - \sum_{k=2}^{+\infty}\phi_k(x)^2\exp(-t(\lambda_k^s-\lambda_1^s))\lambda_k^s\ln(\lambda_k).$$

Now, thanks to this last identity and Proposition A.1 we deduce the lower bound

$$\frac{\exp(t\lambda_1)}{t}\frac{d}{ds}r_D^s(t,x,x) \geqslant \phi_1(x)^2\lambda_1^s|\ln\lambda_1| - \sum_{k=k''}^{+\infty}\lambda_k^{2\alpha(m_0)}\exp(-t(\lambda_k^s-\lambda_1^s))\lambda_k^s\ln(\lambda_k), \tag{2.39}$$

where $k'' = \min\{k \geqslant 1 \quad \text{s.t. } \lambda_k \geqslant 1\}$.

Therefore, thanks to equation (A.7) we can apply Lemma A.3 to the last series in (2.39) with $a_k = \lambda_k^{2\alpha(m_0)+s}\ln(\lambda_k)$ and $b_k = (\lambda_k^s - \lambda_1^s)$, and obtain that

$$\lim_{t\to+\infty}\sum_{k=k''}^{+\infty}\lambda_k^{2\alpha(m_0)}\exp(-t(\lambda_k^s-\lambda_1^s))\lambda_k^s\ln(\lambda_k) = 0. \tag{2.40}$$

By this limit and (2.39) we prove that for each $t > 0$ large enough, depending on $s \in (0,1]$ and $x \in \Omega$ we get

$$\frac{d}{ds}r_D^s(t,x,x) > 0.$$

Hence, if we fix some $0 < s_0 < s_1 \leqslant 1$ and $x \in \Omega$ there exists some $T' > 0$ depending on s_0, s_1 and x such that for each $t > T'$

$$r_D^{s_0}(t,x,x) < r_D^{s_1}(t,x,x). \tag{2.41}$$

This proves point (ii) of Theorem 1.10. □

3. The spectral fractional heat equation with Neumann boundary conditions

In this section we provide an analytical and probabilistic exposition on the Neumann spectral fractional heat equation (1.35).

Section 3.1 is dedicated to the introduction of the stochastic process called the subordinate standard reflecting Brownian motion. We show that the transition density of this process is the Neumann spectral fractional heat kernel r_N^s given in equation (1.36).

In Section 3.2 we propose an analytical study of the solution to the system of equations (1.35). Specifically, we prove Theorems 1.20 and 1.21. The first result states that the unique solution to (1.35) is the Neumann spectral fractional heat kernel r_N^s, while Theorem 1.21 is a Maximum Principle for the spectral fractional heat equation with Neumann boundary conditions.

Section 3.3 consists in the proof of the original Theorems 1.22 and 1.23. These results provide some insight on the monotonicity properties (with respect to the parameter s) of the Neumann heat kernel r_N^s and the conditional probability introduced in equation (1.39).

3.1. **From the standard reflecting Brownian motion to the subordinate standard reflecting Brownian motion.** In this section, through a suitable modification of the setting introduced in Section 2.1, we will construct the subordinate standard reflecting Brownian motion. This stochastic process is obtained by starting from the standard reflecting Brownian motion via the introduction of a random choice of time. This type of procedure belongs to a class of operations between Markov processes which takes the name of subordination (see [Boc49]). We give a more detailed presentation on the subordinate standard reflecting Brownian motion in Theorem 3.3 below. More precisely, in Theorem 3.3 and Proposition 3.5 we will show that the transition density associated with this stochastic process is the Neumann spectral fractional heat kernel r_N^s given in equation (1.36).

We now briefly recall the main properties of the standard reflecting Brownian motion. There are different ways of defining a standard reflecting Brownian motion, and they all lead to the same Markov process. There exists a broad literature treating this problem, see e.g. [LS84, Fuk67, BH90, DI08]. In this paper we will follow the more analytic approach to define this stochastic process, namely by first constructing a transition density starting from a system of parabolic PDEs. In particular the main definitions and procedures here follow [Hsu84]. We now reconsider the Neumann heat kernel $p_N^\Omega(t,x,y)$ introduced in (1.32). As stated in Theorem 1.15, we can write the heat kernel as the $L^2(\Omega)$ expansion in eigenfunctions

$$p_N^\Omega(t,x,y)=\sum_{k=0}^{+\infty}\psi_k(x)\,\psi_k(y)\,\exp(-t\mu_k). \tag{3.1}$$

Let us denote by $\mathfrak{S}=C([0,+\infty),\Omega)$. Then, we have the following result (see [Hsu84]).

Theorem 3.1. *Let $y\in\Omega$. There exists a Markov process $X_N^\Omega=(\mathfrak{S},\mathcal{F},\mathcal{F}_t,\{X_t\}_{t\geqslant 0},\mathbb{P}_y)$ such that for each $t>0$ and $\Omega'\subset\Omega$ it holds*

$$\mathbb{P}_y(X_t\in\Omega')=\int_{\Omega'}p_N^\Omega(t,x,y)\,dx. \tag{3.2}$$

Definition 3.2. *Let $y\in\Omega$. A stochastic process $X_{N,y}^\Omega=(\mathfrak{S},\mathcal{F},\mathcal{F}_t,\{X_t^\Omega\}_{t\geqslant 0},\mathbb{P}_y)$ is defined as a standard reflecting Brownian motion starting at $y\in\Omega$ if its transition density satisfies the equations in* (1.31)*.*

In view of Theorem 3.1, the semigroup N_t associated with $X_{N,x}$ is

$$N_tf(x):=\int_\Omega p_N^\Omega(t,x,y)\,f(y)\,dy=\mathbb{E}_x[f(X_t^\Omega)],$$

for each $f\in L^2(\Omega)$. Therefore, the *generator* of the standard reflecting Brownian motion coincides with Neumann Laplacian, namely

$$\frac{d}{dt}N_tf|_{t=0}=\lim_{t\to 0^+}\frac{N_tf-f}{t}=\Delta_{N,\Omega}f,$$

for each $f\in W_N^{1,2}(\Omega)$.

We observe (see [Hsu84]) that for each $(t,x)\in(0,+\infty)\times\Omega$

$$\mathbb{P}_y(X_t\in\Omega)=\int_\Omega p_N^\Omega(t,x,y)dy=1. \tag{3.3}$$

We mention that this is a consequence of the fact that the first eigenvalue of the Neumann Laplacian is zero. In particular, the identity in (3.3) highlights the fact that the process is restricted to Ω and cannot escape the domain. Note that this is in contrast with the behavior of the killed Brownian motion, which was instead eventually doomed to be confined in the cemetery ∂ (see equation (2.3)).

In the following result we build the subordinate standard reflecting Brownian motion, by introducing in the previously discussed standard reflecting Brownian motion a random choice of the time. More specifically, we will assume that such a choice follows the motion of a s-stable subordinator (see Definition 2.4). The proof of the next result is analogous to the one of Theorem 2.6 and therefore will be omitted.

Theorem 3.3. *Let $s\in(0,1)$, $S=(\mathfrak{S}_1,\mathcal{G},\{S_t\}_{t\geqslant 0},\widetilde{\mathbb{P}}^s)$ be an s-stable subordinator and $X_{N,x}:=(\mathfrak{S},\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},\{X_t^\Omega\}_{t\geqslant 0},\mathbb{P}_x)$ be a standard reflecting Brownian motion as in Definition 3.2. Let us define the family of stochastic processes*

$$\begin{aligned} &Y_N^\Omega:=(\mathfrak{S}\times\mathfrak{S}_1,\{\mathcal{F}_t\times\mathcal{G}\}_{t\geqslant 0},\mathcal{F}\times\mathcal{G},\{Y_{N,t}\}_{t\geqslant 0},\{\mathbb{P}_{N,y}^s\}_{y\in\Omega})\\ \text{and}\quad &Y_{N,t}(\omega,\omega_1):=X_{S_t(\omega_1)}(\omega)\end{aligned} \tag{3.4}$$

where $\mathbb{P}_{N,y}^s:=\mathbb{P}_y\times\widetilde{\mathbb{P}}^s$. Then, for each $x\in\Omega$ the stochastic process $Y_{N,x}^\Omega$ is a Markov process. Moreover, for each Lebesgue measurable set $\Omega'\subset\Omega$,

$$\mathbb{P}_{N,y}^s(Y_{N,t}\in\Omega')=\int_{\Omega'}\int_0^{+\infty}p_N^\Omega(l,x,y)\,\mu_t^s(dl)\,dy. \tag{3.5}$$

Definition 3.4 (Subordinate standard reflecting Brownian motion). *For each $x \in \Omega$ the process $Y^{\Omega}_{N,x} := (\mathfrak{S} \times \mathfrak{S}_1, \{\mathcal{F}_t \times \mathcal{G}\}_{t\geqslant 0}, \mathcal{F} \times \mathcal{G}, \{Y_{N,t}\}_{t\geqslant 0}, \mathbb{P}^s_{N,x})$ defined in Theorem 3.3 is called subordinate standard reflecting Brownian motion starting at x.*

Now we show that the transition density of the subordinate standard reflecting Brownian motion coincides with the Neumann spectral fractional heat kernel r^s_N defined in equation (1.36).

Proposition 3.5. *Let $r^s_N : (0,+\infty) \times \Omega \times \Omega \to \mathbb{R}$ be as in equation (1.36). Then,*

$$r^s_N(t,x,y) = \int_0^{+\infty} p^{\Omega}_N(l,x,y)\,\mu^s_t(dl) \quad \text{for all } (t,x,y) \in (0,+\infty) \times \Omega \times \Omega. \tag{3.6}$$

Proof. Thanks to Theorem 1.15 we have that

$$p^{\Omega}_N(t,x,y) = \sum_{k=0}^{+\infty} \psi_k(x)\,\psi_k(y)\,\exp(-t\mu_k).$$

Using this and equation (2.5) we conclude that

$$\begin{aligned}
\int_0^{+\infty} p^{\Omega}_N(l,x,t)\mu^s_t(dl) &= \int_0^{+\infty} \sum_{k=0}^{+\infty} \psi_k(x)\psi_k(y)\exp(-l\mu_k)\mu^s_t(dl)\\
&= \sum_{k=0}^{+\infty} \psi_k(x)\psi_k(y) \int_0^{+\infty} \exp(-l\mu_k)\mu^s_t(dl)\\
&= \sum_{k=0}^{+\infty} \psi_k(x)\psi_k(y)\exp(-t\mu^s_k)\\
&= r^s_N(t,x,y).
\end{aligned} \tag{3.7}$$

The identity between the first and second line in (3.7) is guaranteed by the Dominated Convergence Theorem, which can be applied thanks to Proposition A.1 and Lemma A.2. □

Thanks to Theorem 3.3 and Proposition 3.5, the transition density associated with the subordinate standard reflecting Brownian motion is the Neumann spectral fractional heat kernel r^s_N given in equation (1.36). Hence, we can consider the probability function

$$\mathbb{P}^s_{N,x}(Y_{N,t} \in \Omega') = \int_{\Omega'} r^s_N(t,x,y)\,dy.$$

Similarly to what we have done in Section 2.1, we can also take into account the conditional probability

$$\mathbb{P}^s_N(Y_{N,t} \in \Omega' | Y_{N,0} \in \Omega'') := \frac{1}{|\Omega''|} \int_{\Omega' \times \Omega''} r^s_N(t,x,y)\,dx\,dy. \tag{3.8}$$

Note that in this case the state space coincide with Ω, and thus $\mathbb{P}^s(Y_{N,t} \in \Omega | Y_{N,0} \in \Omega'') = 1$ for each $t > 0$. This last observation follows from the fact that for each $(t,x) \in (0,+\infty) \times \Omega$

$$\mathbb{P}^s_{N,x}(Y_{N,t} \in \Omega) = \int_{\Omega} r^s_N(t,x,y)\,dy = 1,$$

thanks to (3.3) and (3.6). An alternative analytical proof of this fact will be given in Lemma 3.10.

Another consequence of Theorem 3.3 and Proposition 3.5 is the following result on the continuity of the semigroup associated to the subordinate standard reflecting Brownian motion Y^{Ω}_N.

Proposition 3.6. *Let $s \in (0,1)$ and Y_N^Ω as in Definition 3.4. For each $t > 0$ the semigroup associated with Y_N^Ω*

$$
\begin{aligned}
Q_t :& L^2(\Omega) \to L^2(\Omega) \\
& f \mapsto Q_t f(x) := \int_\Omega f(y)\, r_N^s(t,x,y)\, dy,
\end{aligned}
\tag{3.9}
$$

satisfies

$$\|Q_t f\|_{L^2(\Omega)} \leqslant \|f\|_{L^2(\Omega)}.$$

In Proposition 2.11 we established that the generator associated with the subordinate killed Brownian motion is the Dirichlet spectral fractional Laplacian. The next result provides the corresponding property for the subordinate standard reflecting Brownian motion, namely that the Neumann spectral fractional Laplacian $(-\Delta)^s_{N,\Omega}$ in (1.29) is the generator of Y_N^Ω. Since the proof of this fact is similar to the one of Proposition 2.11, we omit it.

Proposition 3.7. *The generator of the process Y_N^Ω is given by the spectral fractional Laplacian with Neumann boundary conditions. Namely for each $f \in H_N^{2s}(\Omega)$ the following limit holds*

$$\lim_{t\to 0} \frac{Q_t f - f}{t} = -(-\Delta)^s_{N,\Omega} f$$

where the convergence is meant in $L^2(\Omega)$.

3.2. **Existence, Uniqueness and Maximum Principle for the Neumann spectral fractional heat equation.** Now we state the main auxiliary results that lead to Theorems 1.20 and 1.21, on the existence, uniqueness and regularity of the solution to the Neumann spectral fractional heat equation (1.35) and on the Maximum Principle for its solution. The procedure to prove these results is the same that we used in Section 2.2 for the Dirichlet case. For this reason most of the proof will be omitted.

Thus, following the same steps of Section 2.2 we begin this discussion by establishing the following regularity result for the Neumann spectral fractional heat kernel r_N^s given in equation (1.36).

Lemma 3.8. *Let $s \in (0,1]$ and Ω be open, bounded, smooth and connected. Then, $r_N^s \in C^\infty([\varepsilon,+\infty)\times \overline{\Omega} \times \overline{\Omega})$ for each $\varepsilon > 0$.*

In the following result we state that the Neumann kernel r_N^s is a solution to the first and second equation in (1.35).

Lemma 3.9. *Let $s \in (0,1)$ and Ω be open, bounded, smooth and connected. Then, for each $y \in \Omega$ the function $r_{N,y}^s(t,x) := r_N^s(t,x,y)$ is a solution to the first and second equation in* (1.35).

We now prove that the Neumann Kernel r_N^s has unit mass.

Lemma 3.10. *Let $s \in (0,1]$ and Ω be open, bounded, smooth and connected. Then for each $(t,x) \in (0,+\infty)\times\Omega$ it holds*

$$\int_\Omega r_N^s(t,x,y)\, dy = 1. \tag{3.10}$$

Proof. Using Lemma 3.8, Proposition A.1 and Lemma A.2, together with the integration by parts formula, we deduce that for each $(t,x) \in (0,+\infty) \times \Omega$ it holds that

$$\begin{aligned}\frac{d}{dt}\int_\Omega r_N^s(t,x,y)\,dy &= \frac{d}{dt}\int_\Omega \sum_{k=0}^{+\infty} \psi_k(x)\,\psi_k(y)\,\exp(-t\mu_k^s)\,dy \\ &= -\int_\Omega \sum_{k=1}^{+\infty} \mu_k^s\,\psi_k(x)\,\psi_k(y)\,\exp(-t\mu_k^s)\,dy \\ &= \sum_{k=1}^{+\infty} \psi_k(x)\,\exp(-t\mu_k^s)\int_\Omega -\mu_k^s\,\psi_k(y)\,dy \\ &= \sum_{k=1}^{+\infty} \psi_k(x)\,\exp(-t\mu_k^s)\int_\Omega \Delta\psi_k(y)\,dy \\ &= \sum_{k=1}^{+\infty} \psi_k(x)\,\exp(-t\mu_k^s)\int_{\partial\Omega} \frac{\partial\psi_k}{\partial\nu}(y)\,dH_y^{n-1}(y) \\ &= 0.\end{aligned}$$

Furthermore, one easily observes that

$$\lim_{t\to+\infty}\int_\Omega r_N^s(t,x,y)\,dy = \lim_{t\to+\infty}\int_\Omega \frac{1}{|\Omega|} + \sum_{k=1}^{+\infty}\psi_k(x)\,\psi_k(y)\,\exp(-t\mu_k^s)\,dy = 1,$$

as desired. $\square$

For completeness, we mention that an alternative approach to establish (3.10) relies on the probabilistic framework previously developed, using equation (3.3) and equation (3.6), which allow one to write r_N^s as a time weighted superposition of the classical Neumann heat kernel p_N^Ω. The computation provided above instead does not make use of the probabilistic setting, but relies on the fact that the first eigenvalue of $(-\Delta)^s_{N,\Omega}$ is vanishing.

Now we remark that a result in the spirit of Lemma 2.14 holds true in the Neumann case as well, thus providing the existence, uniqueness and regularity of the solution to the Neumann spectral fractional heat equation with initial datum $f \in L^2(\Omega)$. This result, which is showcased here below, will be employed in this paper to prove a Maximum Principle in the Neumann framework. Moreover, the uniform continuity with respect to the initial datum will be used to prove that r_N^s satisfies also third equation in (1.35), in analogy with what we already did for the Dirichlet case.

Lemma 3.11. *Let $s \in (0,1)$ and Ω be open, bounded smooth and connected. For each $f \in L^2(\Omega)$ the function $Q_t f$ defined in* (3.9) *is the unique solution in $C([0,+\infty), L^2(\Omega)) \cap C^1((0,+\infty), H_N^{2s}(\Omega))$ to the system of equations*

$$\begin{cases} \partial_t u(t,y) = -(-\Delta)^s_{N,\Omega} u(t,y) & \text{for all } (t,y) \in (0,+\infty)\times\Omega, \\ \dfrac{\partial u}{\partial\nu}(t,y) = 0 & \text{for all } (t,y) \in (0,+\infty)\times\partial\Omega, \\ u(0,y) = f(y) & \text{for all } y \in \Omega. \end{cases} \tag{3.11}$$

The last equation in (3.11) *is meant as*

$$\lim_{t\to0}\|u(t,y)-f(y)\|_{L^2(\Omega)} = 0.$$

Furthermore, for each $\varepsilon > 0$, it holds that $Q_t f(y) \in C^\infty([\varepsilon,+\infty)\times\overline{\Omega})$. Moreover, if $f \in C_c^\infty(\Omega)$, then

$$\lim_{t\to0}\|Q_t f(y)-f(y)\|_{L^\infty(\Omega)} = 0. \tag{3.12}$$

Below we give a proof of the Maximum Principle in the Neumann setting that was stated in Theorem 1.21. The argument that we present here can be seen as a natural modification of the analytic proof that we provided in the case of Dirichlet boundary conditions.

Alternatively, one can also obtain this Maximum Principle by using the probabilistic framework previously developed, relying on the Maximum Principle for the classical Neumann heat equation (this approach is developed in Appendix B for the Dirichlet case, but the Neumann case can be treated similarly).

Proof of Theorem 1.21. We prove the right-hand side inequality in (1.37). First we assume that $f \in C_c^\infty(\Omega)$, since the general case of $f \in L^2(\Omega)$ will be addressed below using an approximation argument. Thanks to Lemma 3.11 we know that $Q_t f(y) =: f(t,y) \in C([0,+\infty) \times \overline{\Omega}) \cap C^\infty((0,+\infty) \times \Omega)$. Let $K := \max\{0, \sup_\Omega f\}$ and assume, by contradiction, that there exists $(t^*, y^*) \in (0,+\infty) \times \Omega$ such that

$$f(t^*, y^*) > K.$$

We define the function $W(t,y) := f(t^*, y^*) - \varepsilon - f(t,y)$, where ε is as in (2.30). Applying the same reasoning as in proof of Theorem 1.9 we get the existence of $(\tilde{t}, \tilde{y}) \in (0,+\infty) \times \Omega$ as in equation (2.31). In particular, this means that

$$\partial_t W(\tilde{t}, \tilde{y}) \leqslant 0.$$

We note that $W(\tilde{t}, \cdot)$ cannot identically vanish in Ω. Indeed, $W(t,y) = f(t^*, y^*) - \varepsilon - f(t,y)$, and $f(t,y)$ cannot be constant unless its initial data f is, as one can easily infer from the $L^2(\Omega)$ decomposition of $f(t,y)$ in eigenfunctions. Therefore, since $\tilde{y}$ is a minimum for $W(\tilde{t}, \cdot)$, Lemma 1.19 gives us that

$$(-\Delta)^s_{N,\Omega} W(\tilde{t}, \tilde{y}) < 0.$$

Therefore

$$\partial_t W(\tilde{t}, \tilde{y}) + (-\Delta)^s_{N,\Omega} W(\tilde{t}, \tilde{y}) < 0.$$

On the other hand,

$$\begin{aligned}\partial_t W(\tilde{t}, \tilde{y}) + (-\Delta)^s_{N,\Omega} W(\tilde{t}, \tilde{y}) &= -\partial_t f(\tilde{t}, \tilde{y}) + (-\Delta)^s_{N,\Omega}(f(t^*, y^*) - \varepsilon) - (-\Delta)^s_{N,\Omega} f(\tilde{t}, \tilde{y}) \\ &= (-\Delta)^s_{N,\Omega}(f(t^*, y^*) - \varepsilon) = 0,\end{aligned}$$

providing a contradiction. Therefore we have proved the right-hand side inequality in (1.37) for each solution with initial condition smooth with compact support. The left-hand side inequality is proved similarly.

With the same procedure followed in the proof of Theorem 1.9 one can show that also $r^s_N(t,x,y) \geqslant 0$ for each $(t,x,y) \in (0,+\infty) \times \Omega \times \Omega$.

If $f \in L^2(\Omega)$, according to equation (3.10) we have

$$f(t,y) = \int_\Omega r^s_N(t,x,y)\, f(x) dx \leqslant \max\left\{0, \sup_\Omega f\right\} \int_\Omega r^s_N(t,x,y)\, dx = \max\left\{0, \sup_\Omega f\right\},$$

which proves the right-hand side of (1.37) when $f \in L^2(\Omega)$. Similarly we also obtain the left-hand side inequality in (1.37). ☐

The following result is the analogous of Corollary 2.15 for the Neumann kernel r^s_N.

Corollary 3.12. *Let $s \in (0,1)$ and Ω be bounded, smooth and connected. Then, for each $(t,x,y) \in (0,+\infty) \times \Omega \times \Omega$ it holds that*

$$r^s_N(t,x,y) > 0.$$

While a modification of the proof of Corollary 2.15 would produce a purely analytical argument to establish Corollary 3.12, for completeness we also note that the proof of this result can be alternatively achieved by using equation (3.6) and the strict positivity of the classical Neumann kernel p^Ω_N (see [Hsu84]).

As a consequence of Corollary 3.12 we have the following result:

Corollary 3.13. *Let $s \in (0,1)$ and $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. Then,*

$$\begin{aligned} Q_t : C_0(\Omega) &\to C(\overline{\Omega}) \\ f &\longmapsto Q_t f \end{aligned}$$

satisfies

$$\|Q_t f\|_{C(\overline{\Omega})} \leqslant \|f\|_{C_0(\Omega)}.$$

Now we prove Theorem 1.20.

Proof of Theorem 1.20. In Lemma 3.8 we have already stated that $r_N^s \in C^\infty([\varepsilon,+\infty)\times\Omega\times\Omega)$ for each $\varepsilon > 0$, while in Lemma 3.9 we showed that $r_{N,y}^s(t,x) := r_N^s(t,x,y)$ satisfies the first and second equation in (1.35) for each $y \in \Omega$.

It remains to prove that $r_{N,y}^s$ satisfies the third equation and that such a solution is unique in the space $C^1((0,+\infty);H_N^{2s}(\Omega))$. One can obtain that $r_N^s \in C^1((0,+\infty);H_N^{2s}(\Omega))$ proceeding analogously to the Dirichlet case.

Using Proposition 3.6 and Corollary 3.13, one can show that, if $f \in C_0(\Omega)$, then

$$\lim_{t\to 0^+} Q_t f(y) = f(y)$$

for each $y \in \Omega$. Therefore, we have showed that $r_{N,y}^s$ is a solution to (1.35) for each $y \in \Omega$. The proof of the uniqueness statement is analogous to the one in the Dirichlet case. □

3.3. **Proof of Theorems 1.22 and 1.23.** Here we give the proof of Theorems 1.22 and 1.23 related to the monotonicity properties, with respect to the fractional parameter of subordination s, of the Neumann heat kernel r_N^s and of the conditional probability in equation (3.8). To prove these results, we account for the effects of subordination on the standard reflecting Brownian motion from a probabilistic point of view, in dependence of the geometric properties of the domain Ω encoded by the first non-trivial eigenvalue of the Neumann Laplacian.

To prove Theorems 1.22 and 1.23 we also rely on the following ancillary result:

Proposition 3.14. *Let $\Omega' \subset \Omega$ with positive measure, and $x \in \Omega$. Then, the eigenvalues $\mu_{k(\Omega')}$ and $\mu_{k(x)}$ defined respectively in* (1.40) *and* (1.41) *exist.*

Proof. We argue by contradiction. Suppose that for each $k \in \mathbb{N}$ and $k \geqslant 1$ we have that

$$\int_{\Omega'} \psi_k(x)\,dx = 0,$$

and let $f \in C_c^\infty(\Omega)$ such that $f \geqslant 0$ and $f = 0$ in Ω'. In view of Proposition A.4 the Fourier series in eigenfunctions of a smooth and compactly supported function converges uniformly. Then we see that

$$\begin{aligned} 0 = \int_{\Omega'} f(y)\,dy &= \int_{\Omega'} \sum_{k=0}^{\infty} f_k\,\psi_k(y)\,dy = \sum_{k=0}^{\infty} f_k \int_{\Omega'} \psi_k(y)\,dy \\ &= f_0\,\frac{|\Omega'|}{|\Omega|^{\frac12}} = \left(\frac{1}{|\Omega'|^{\frac12}}\int_\Omega f(y)\,dy\right)\frac{|\Omega'|}{|\Omega|^{\frac12}} \neq 0, \end{aligned}$$

which provides a contradiction.

Similarly, suppose that for each $k \geqslant 1$ one has that

$$\psi_k(x) = 0,$$

and take $f \in C_c^\infty(\Omega)$ such that $f(x) = 0$ and $f \geqslant 0$. Then,

$$0 = f(x) = \sum_{k=0}^{\infty} f_k\, \psi_k(x) = f_0\, \psi_0 = \frac{1}{|\Omega|} \int_\Omega f(y)\, dy \neq 0,$$

which provides a contradiction. □

Proof of Theorems 1.22 and 1.23. Let us first prove point (i) of Theorems 1.22 and 1.10. We take the derivative of $r_N^s(t,x,y)$ with respect to s and obtain

$$\begin{aligned}\frac{d}{ds} r_N^s(t,x,y) &= \frac{d}{ds} \sum_{k=0}^{+\infty} \psi_k(x)\, \psi_k(y)\, \exp(-t\mu_k^s) \\ &= -t \sum_{k=1}^{+\infty} \psi_k(x)\, \psi_k(y)\, \exp(-t\mu_k^s)\, \mu_k^s\, \ln(\mu_k).\end{aligned} \tag{3.13}$$

Note that we can differentiate with respect to s inside the sum thanks to Proposition A.1 and Lemma A.2. Let us assume $x = y \in \Omega$, and let $\mu_{k(x)}$ be defined as in equation (1.41). Then, from (3.13) we deduce that if $\mu_{k(x)} \geqslant 1$, then all the terms in the sum are positive. Therefore,

$$\begin{aligned}\frac{d}{ds} r_N^s(t,x,y) &= \frac{d}{ds} \sum_{k=k(x)}^{+\infty} \psi_k(x)^2\, \exp(-t\mu_k^s) \\ &= -t \sum_{k=k(x)}^{+\infty} \psi_k(x)^2\, \exp(-t\mu_k^s)\, \mu_k^s\, \ln(\mu_k) < 0,\end{aligned}$$

which proves point (i) in Theorem 1.22. Similarly, if we take the integral over $\Omega' \times \Omega'$ of both sides of (3.13), with Ω' as in the statement of Theorem 1.23, we get

$$\begin{aligned}\frac{d}{ds} \int_{\Omega'\times\Omega'} r_N^s(t,x,y)\, dx\, dy &= \int_{\Omega'\times\Omega'} \frac{d}{ds} r_N^s(t,x,y)\, dx\, dy \\ &= -t \sum_{k=k(\Omega')}^{+\infty} c_{k,\Omega'}^2\, \exp(-t\mu_k^s)\, \mu_k^s\, \ln(\mu_k),\end{aligned} \tag{3.14}$$

where we have defined $c_{k,\Omega'} := \int_{\Omega'} \psi_k(x)\, dx$, and $k(\Omega')$ is the index of the eigenvalue $\mu_{k(\Omega')}$ defined in (1.40). Therefore, if $\mu_{k(\Omega')} \geqslant 1$, the right-hand side of (3.14) is strictly negative, which proves point (i) of Theorem 1.23.

We prove now point (ii) of Theorems 1.22 and 1.23. If $\mu_{k(\Omega')} < 1$, from (3.14) we obtain

$$\begin{aligned}&\frac{\exp\left(t\mu_{k(\Omega')}^s\right)}{t} \left(\frac{d}{ds} \int_{\Omega'\times\Omega'} r_N^s(t,x,y)\, dx\, dy\right) \\ &= c_{k(\Omega'),\Omega'}^2\, \mu_{k(\Omega')}^s\, |\ln(\mu_{k(\Omega')})| - \sum_{k=k(\Omega')+1}^{+\infty} c_{k,\Omega'}^2\, \exp(-t(\mu_k^s - \mu_{k(\Omega')}^s))\, \mu_k^s\, \ln(\mu_k).\end{aligned} \tag{3.15}$$

We claim that for each $t \in (0, +\infty)$ large enough (depending on s and Ω') one has

$$\frac{d}{ds} \int_{\Omega'\times\Omega'} r_N^s(t,x,y)\, dx\, dy > 0. \tag{3.16}$$

Indeed, using Hölder's inequality and the fact that ψ_k's are normalized in $L^2(\Omega)$ we observe that

$$c_{k,\Omega'}^2 = \left(\int_{\Omega'} \psi_k(x)\, dx\right)^2 \leqslant \left(\left(\int_{\Omega'} \psi_k(x)^2\, dx\right)^{\frac{1}{2}} |\Omega'|^{\frac{1}{2}}\right)^2 \leqslant |\Omega'|. \tag{3.17}$$

Thus from this latter inequality and (3.15) we deduce that

$$\begin{aligned}&\frac{\exp\left(t\mu^s_{k(\Omega')}\right)}{t}\left(\frac{d}{ds}\int_{\Omega'\times\Omega'} r^s_N(t,x,y)\,dx\,dy\right)\\ &\qquad\geqslant c^2_{k(\Omega'),\Omega'}\,\mu^s_{k(\Omega')}\,|\ln(\mu_{k(\Omega')})| - |\Omega'|\sum_{k=k(\Omega')+1}^{+\infty}\exp(-t(\mu^s_k-\mu^s_{k(\Omega')}))\,\mu^s_k\,\ln(\mu_k)\\ &\qquad\geqslant c^2_{k(\Omega'),\Omega'}\,\mu^s_{k(\Omega')}\,|\ln(\mu_{k(\Omega')})| - |\Omega'|\sum_{k=k''}^{\infty}\exp(-t(\mu^s_k-\mu^s_{k(\Omega')}))\,\mu^s_k\,\ln(\mu_k),\end{aligned} \tag{3.18}$$

where $k''=\min\{k\in\mathbb{N}\quad\text{s.t. }\mu_k\geqslant 1\}$.

Also, thanks to the classical estimates on the Neumann eigenvalues (see e.g. (A.8)), we can apply Lemma A.3 to the last series in (3.18) with $a_k=\mu^s_k\,\ln(\mu_k)$ and $b_k=(\mu^s_k-\mu^s_{k(\Omega')})$. In this way, we obtain that

$$\lim_{t\to+\infty}|\Omega'|\sum_{k=k''}^{+\infty}\exp(-t(\mu^s_k-\mu^s_{k(\Omega')}))\,\mu^s_k\,\ln(\mu_k)=0.$$

By combining this limit with the inequality in (3.18) we conclude the proof of the claim in (3.16).

Now, given $0<s_0<s_1\leqslant 1$, we deduce from inequality (3.16) that there exists some $T>0$ depending on s_0,s_1 and Ω' such that for each $t>T$

$$\int_{\Omega'\times\Omega'} r^{s_0}_N(t,x,y)\,dx\,dy<\int_{\Omega'\times\Omega'} r^{s_1}_N(t,x,y)\,dx\,dy.$$

This concludes the proof of point (ii) of Theorem 1.23.

Analogously, from (3.13) one obtains that, if $x=y\in\Omega$, then

$$\begin{aligned}&\frac{\exp(t\mu^s_{k(x)})}{t}\left(\frac{d}{ds}r^s_N(t,x,y)\,dx\,dy\right)\\ &\qquad=\psi_{k(x)}(x)^2\mu^s_{k(x)}|\ln(\mu_{k(x)})|-\sum_{k=k(x)+1}^{+\infty}\psi_k(x)^2\exp(-t(\mu^s_k-\mu^s_{k(x)}))\mu^s_k\ln(\mu_k).\end{aligned}$$

We claim that for each $t>0$ large enough (depending on x and s) it holds that

$$\frac{d}{ds}r^s_N(t,x,x)>0. \tag{3.19}$$

Indeed, thanks to Proposition A.1 and this latter inequality we obtain that

$$\begin{aligned}&\frac{\exp(t\mu^s_{k(x)})}{t}\left(\frac{d}{ds}r^s_N(t,x,y)\,dx\,dy\right)\\ &\qquad\geqslant\psi_{k(x)}(x)^2\mu^s_{k(x)}|\ln(\mu_{k(x)})|-\sum_{k=k''}^{+\infty}\mu_k^{2\alpha(m_0)}\exp(-t(\mu^s_k-\mu^s_{k(x)}))\mu^s_k\ln(\mu_k),\end{aligned} \tag{3.20}$$

where $k''=\min\{k\in\mathbb{N}\quad\text{s.t. }\mu_k\geqslant 1\}$. Using inequality (A.8), we are in the position of applying Lemma A.3 to the last series in (3.20) with $a_k=\mu_k^{2\alpha(m_0)+s}\,\ln(\mu_k)$ and $b_k=(\mu^s_k-\mu^s_{k(x)}))$, proving thus that

$$\lim_{t\to+\infty}\sum_{k=k''}^{+\infty}\mu_k^{2\alpha(m_0)}\exp(-t(\mu^s_k-\mu^s_{k(x)}))\mu^s_k\ln(\mu_k)=0. \tag{3.21}$$

This limit and the lower bound in (3.20) yield to the claim in (3.19). Hence if $0 < s_0 < s_1 \leqslant 1$, there exists some $T' > 0$ depending on s_0, s_1 and x such that

$$r_N^{s_0}(t,x,x) < r_N^{s_1}(t,x,x)$$

for each $t > T'$. This concludes the proof of point (ii) of Theorem 1.22. $\square$

Appendix A. Auxiliary and technical results

Proposition A.1. *Let $\{\phi_k\}_k$ be defined as in equation (1.1), and $\{\psi_k\}_k$ as in (1.28) with $k \geqslant 1$. Then for each $r \in \mathbb{N}$, we have the following estimates*

$$\begin{aligned} &\|\phi_k\|_{C^r(\overline{\Omega})} \leqslant c_{m_r,\Omega,r}\lambda_k^{\alpha(m_r)} \\ \textit{and} \qquad &\|\psi_k\|_{C^r(\overline{\Omega})} \leqslant \tilde{c}_{m_r,\Omega,r}\mu_k^{\alpha(m_r)} \end{aligned} \tag{A.1}$$

where $c_{m_r,\Omega,r}$, $\tilde{c}_{m_r,\Omega,r}$ are positive constants depending on m_r, r and Ω, and

$$\alpha(m) := \begin{cases} \dfrac{m}{2} + 1 & \textit{if } m \textit{ is even,} \\ \left[\dfrac{m}{2}\right] + \dfrac{3}{2} & \textit{if } m \textit{ is odd} \end{cases}$$

and

$$m_r := \begin{cases} \left[\dfrac{n+2r-4}{2}\right] + 1 & \textit{if } n+2r-4 \geqslant 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We apply the elliptic estimate (77) at page 323 of [Eva10] to the eigenfunctions ϕ_k's and get

$$\|\phi_k\|_{H^{2+m}(\Omega)} \leqslant c_{m,\Omega}\, \lambda_k \|\phi_k\|_{H^m(\Omega)} \quad \forall m \in \mathbb{N}, \tag{A.2}$$

where $c_{m,\Omega} > 0$ is a constant depending on m and on the domain Ω. We also observe that

$$\|\phi_k\|^2_{H^1_0(\Omega)} := \int_\Omega |\nabla\phi_k|^2 dx = -\int_\Omega \Delta\phi_k\, \phi_k dx = \lambda_k \int_\Omega |\phi_k|^2 dx = \lambda_k, \tag{A.3}$$

where we have used the fact that the ϕ_k's are eigenfunctions of the Laplacian, and that they are normalized in $L^2(\Omega)$. Applying iteratively inequality (A.2), we see that if $m = 2n_0$ for some $n_0 \in \mathbb{N}$, then

$$\|\phi_k\|_{H^{2+m}(\Omega)} \leqslant C_{m,\Omega}\lambda_k^{1+n_0}\|\phi_k\|_{L^2(\Omega)} = C_{m,\Omega}\lambda_k^{1+n_0}, \tag{A.4}$$

while if $m = 2n_0 + 1$ for some $n_0 \in \mathbb{N}$, using now also identity (A.3), we get

$$\|\phi_k\|_{H^{2+m}(\Omega)} \leqslant C_{m-1,\Omega}\lambda_k^{n_0+\frac{3}{2}}, \tag{A.5}$$

where $C_{m,\Omega}$ is a constant depending on $m \in \mathbb{N}$ and Ω. Let us take $r \in \mathbb{N}$ and call m_r the smallest integer with respect to which the following Sobolev embedding is satisfied

$$H^{2+m_r}(\Omega) \hookrightarrow C^r(\overline{\Omega}), \tag{A.6}$$

namely $m_r := [\frac{n+2r-4}{2}] + 1$, if $n + 2r - 4 \geqslant 0$, otherwise $m_r = 0$. Therefore, if m_r is even, using (A.4) and (A.6) we obtain

$$\|\phi_k\|_{C^r(\overline{\Omega})} \leqslant c_{r,\Omega}\|\phi_k\|_{H^{2+m_r}(\Omega)} \leqslant c_{m_r,\Omega,r}\lambda_k^{1+\frac{m_r}{2}},$$

where $c_{r,\Omega}$ is the constant of the Sobolev embedding, and if instead m_r is odd, then using the Sobolev embedding (A.6) and the estimate (A.5) we get

$$\|\phi_k\|_{C^r(\overline{\Omega})} \leqslant c_{r,\Omega}\|\phi_k\|_{H^{2+m_r}(\Omega)} \leqslant c_{m_r,\Omega,r}\lambda_k^{\frac{3}{2}+[\frac{m_r}{2}]}.$$

The constants $c_{m_r,\Omega,r}$ depends on Ω, r and m_r. Since the elliptic estimate in (A.2) and the identities in (A.3), with obvious adjustments, hold true also for the eigenfunctions ψ_k's, we can repeat the above reasoning to obtain the bound (A.1) for the family $\{\psi_k\}_k$. ☐

Lemma A.2. *Let λ_k's and μ_k's be defined respectively as in equations* (1.1) *and* (1.28)*. Then, for each $m \in \mathbb{N}$, $s \in (0, 1]$ and $t \in (0, +\infty)$, we have*

$$\sum_{k=1}^{+\infty} \lambda_k^m \exp(-t\,\lambda_k^s) < +\infty$$

$$\sum_{k=1}^{+\infty} \mu_k^m \exp(-t\,\mu_k^s) < +\infty$$

Proof. We prove the first inequality only, since the proof of the second one is the same. Indeed, thanks to Weyl's law (see [Pro87]), we know that there exists some $k_0 \in \mathbb{N}$ such that if $k \geqslant k_0$ then the following double sided inequality holds for both λ_k's and μ_k's

$$c\,k^{\frac{2}{n}} \leqslant \lambda_k \leqslant C\,k^{\frac{2}{n}}, \tag{A.7}$$

$$c\,k^{\frac{2}{n}} \leqslant \mu_k \leqslant C\,k^{\frac{2}{n}} \tag{A.8}$$

for some constants $C, c > 0$. Therefore, if $k \geqslant k_0$ we have that that

$$\lambda_k^m \exp(-t\,\lambda_k^s) \leqslant C^m k^{\frac{2m}{n}} \exp(-c^s t\, k^{\frac{2s}{n}}).$$

Since it holds that

$$\lim_{k\to+\infty} k^{\frac{2m}{n}} \exp\left(-\frac{c^s t\, k^{\frac{2s}{n}}}{2}\right) = 0,$$

then there exists $k_1 \in \mathbb{N}$ such that for all $k \geqslant k_1$

$$k^{\frac{2m}{n}} \exp(-c^s t\, k^{\frac{2s}{n}}) \leqslant \exp\left(-\frac{c^s t\, k^{\frac{2s}{n}}}{2}\right).$$

Therefore, if we call $k_2 := \max\{k_0, k_1\}$ we have that

$$\begin{aligned}\sum_{k=1}^{+\infty} \lambda_k^m \exp(-t\,\lambda_k^s) &\leqslant \sum_{k=1}^{k_2-1} \lambda_k^m \exp(-t\,\lambda_k^s) + C^m \sum_{k=k_2}^{+\infty} k^{\frac{2m}{n}} \exp(-c^s t\, k^{\frac{2s}{n}}) \\ &\leqslant \sum_{k=1}^{k_2-1} \lambda_k^m \exp(-t\,\lambda_k^s) + \sum_{k=k_2}^{+\infty} \exp\left(-\frac{c^s t\, k^{\frac{2s}{n}}}{2}\right) \\ &< +\infty.\end{aligned}$$

This concludes the proof of Lemma A.2. ☐

Lemma A.3. *Let $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in N}$ be sequences of real numbers such that*

$$\lim_{t\to+\infty} \frac{a_k}{k^M} = C \tag{A.9}$$

$$\lim_{t\to+\infty} \frac{b_k}{k^m} = c, \tag{A.10}$$

for some $M, m \in (0, +\infty)$ and $C, c > 0$. Then,

$$\lim_{t\to+\infty} \sum_{k=1}^{+\infty} a_k \exp(-tb_k) = 0.$$

Proof. We assume $d > 0$ and $t \in [d, +\infty)$. In view of (A.9) and (A.10), we obtain that there exists some $k_0 \in \mathbb{N}$ such that for each $k \geqslant k_0$ one has

$$a_k \exp(-tb_k) \leqslant C_1 k^M \exp(-tc_1 k^m),$$

for some $c_1, C_1 > 0$ and for all $t \in [d, +\infty)$. Moreover, since

$$\lim_{k\to+\infty} C_1 k^M \exp\left(-\frac{tc_1 k^m}{2}\right) = 0, \tag{A.11}$$

we deduce the existence of some $k_1 \in \mathbb{N}$ such that for all $k \geqslant k_1$ it holds

$$C_1 k^M \exp(-tc_1 k^m) \leqslant \exp\left(-\frac{c_1 t k^m}{2}\right).$$

Therefore, taking $k_2 = \max\{k_0, k_1\}$ we obtain that

$$\begin{aligned}\sum_{k=1}^{+\infty} \|a_k \exp(-tb_k)\|_{L^\infty([d,+\infty))} &= \sum_{k=1}^{k_2-1} \|a_k \exp(-tb_k)\|_{L^\infty([d,+\infty))} + \sum_{k=k_2}^{+\infty} \|a_k \exp(-tb_k)\|_{L^\infty([d,+\infty))} \\ &\leqslant \sum_{k=1}^{k_2-1} \|a_k \exp(-tb_k)\|_{L^\infty([d,+\infty))} + \sum_{k=k_2}^{+\infty} \exp\left(-\frac{c_1 d k^m}{2}\right) < +\infty.\end{aligned}$$

Thus, we can take the limit inside the sum as follows

$$\lim_{t\to+\infty} \sum_{k=1}^{+\infty} a_k \exp(-tb_k) = \sum_{k=1}^{+\infty} \lim_{t\to+\infty} a_k \exp(-tb_k) = 0. \qquad \square$$

Proposition A.4. *Let $f \in C_c^\infty(\Omega)$, and consider the $L^2(\Omega)$ expansion $f = \sum_{k=1}^{+\infty} f_k\, \phi_k$, where $f_k := (f, \phi_k)_{L^2(\Omega)}$ and the ϕ_k's are given in equation (1.1). Then the following estimate for the L^2 projection f_k holds*

$$|f_k| \leqslant \lambda_k^{-m} \|f\|_{H_0^{2m}(\Omega)} \qquad \forall m \in \mathbb{N}. \tag{A.12}$$

In particular, for each $r \in \mathbb{N}$, the sequence $f_M(x) := \sum_{k=1}^{M} f_k \phi_k(x)$ converges in $C^r(\overline{\Omega})$ to f as $M \to +\infty$.

Similarly, if we consider as $L^2(\Omega)$ Hilbert basis $\{\psi_k\}_k$, then by calling now $f_k := (f, \psi_k)_{L^2(\Omega)}$, for each $k \geqslant 1$ it holds

$$|f_k| \leqslant \mu_k^{-m} \|f\|_{H_0^{2m}(\Omega)} \qquad \forall m \in \mathbb{N}. \tag{A.13}$$

and, as $M \to +\infty$, the sequence $f_M(x) := \sum_{k=0}^{M} f_k \psi_k(x)$ converges to f in $C^r(\overline{\Omega})$ for each $r \in \mathbb{N}$.

Proof. As usual, we use the notation $\Delta^m f := \underbrace{\Delta \circ \Delta \cdots \circ \Delta}_{\text{m-times}} f$ and $\Delta^0 f := f$.

We prove by induction on $m \in \mathbb{N}$ the following identity

$$f_k = \frac{(-1)^m}{\lambda_k^m} \int_\Omega \Delta^m f(x)\, \phi_k(x)\, dx \tag{A.14}$$

Let $m = 1$, using first equation (1.1) and then the integration by parts formula we obtain

$$f_k := \int_\Omega f(x)\phi_k(x)\, dx = -\frac{1}{\lambda_k} \int_\Omega f(x)\Delta\phi_k(x)\, dx = -\frac{1}{\lambda_k} \int_\Omega \Delta f(x)\phi_k(x)\, dx.$$

Next we show that if the identity (A.14) holds true for $m-1$, then it is valid also for m. Indeed, using the induction hypothesis and repeating the latter computations we get

$$\begin{aligned} f_k &= \frac{(-1)^{m-1}}{\lambda_k^{m-1}} \int_\Omega \Delta^{m-1} f(x)\phi_k(x)\,dx \\ &= \frac{(-1)^m}{\lambda_k^m} \int_\Omega \Delta^{m-1} f(x)\Delta\phi_k(x)\,dx \\ &= \frac{(-1)^m}{\lambda_k^m} \int_\Omega \Delta^m f(x)\phi_k(x)\,dx. \end{aligned}$$

From equation (A.14), applying Hölder's inequality we obtain

$$|f_k| \leqslant \lambda_k^{-m} \|\phi_k\|_{L^2(\Omega)} \|f\|_{H_0^{2m}(\Omega)} = \lambda_k^{-m} \|f\|_{H_0^{2m}(\Omega)} \tag{A.15}$$

which is the desired estimate. The series $f_N(x)$ is converging a.e. in Ω to f, as consequence of the $L^2(\Omega)$ convergence. The estimates proved in Proposition A.1 lead us to the $C^k(\overline{\Omega})$ convergence of the series. Indeed,

$$\sum_{k=1}^{+\infty} \|f_k\phi_k\|_{C^r(\overline{\Omega})} \leqslant \sum_{k=1}^{+\infty} |f_k|\,\lambda_k^{\alpha(m_r)} \leqslant \|f\|_{H_0^{2m}(\Omega)} \sum_{k=1}^{+\infty} \lambda_k^{-m+\alpha(m_r)} =: \mathcal{I}_m$$

and choosing m such that it satisfies $m - \alpha(m_r) > 1$, we see that $\mathcal{I}_m$ is a finite quantity, proving our latter claim.

The identity (A.14), with obvious adjustments, holds true also for the ψ_k's when $k \geqslant 1$. By repeating the above reasoning we obtain the desired result. ☐

Appendix B. Alternative proofs of Theorems 1.9 and 1.21

Here we give an alternative proof of the Maximum Principle for the Dirichlet spectral fractional heat equation (see Theorem 1.9), using the probabilistic framework developed in Section 2.1. To be more precise, in Proposition 2.8 we showed that the Dirichlet spectral fractional heat kernel r_D^s can be rewritten as

$$r_D^s(t,x,y) = \int_0^{+\infty} p_D^\Omega(l,x,y)\mu_t^s(dl),$$

where $p_D^\Omega(t,x,y)$ is the classical Dirichlet heat kernel, and μ_t^s is the probability density in $(0,+\infty)$ given in Definition 2.4. From the above equation it is clear that starting from the Maximum Principle for the classical Dirichlet heat equation one can obtain the corresponding result for the fractional case.

The proof for the Neumann case is analogous, and therefore here we will stick to the one for the Dirichlet equation.

Proof of Theorem 1.9. Let us prove the right-hand side inequality in (1.14). Let $f \in L^2(\Omega)$ and consider the map

$$f(t,y) = \int_\Omega r_D^s(t,x,y)\,f(x)\,dx.$$

Then, using equation (2.18) one gets that

$$f(t,y) = \int_\Omega \int_0^{+\infty} p_D^\Omega(l,x,y)\,\mu_t^s(dl)\,f(x)\,dx = \int_0^{+\infty} \int_\Omega p_D^\Omega(l,x,y)\,f(x)\,dx\,\mu_t^s(dl).$$

Accordingly, by the Maximum Principle for the classical Dirichlet heat equation (see equation (1.7)) one gets that

$$f(t,y) \leqslant \int_0^{+\infty} \max\left\{0,\, \sup_\Omega f\right\}\,\mu_t^s(dl) = \max\left\{0,\, \sup_\Omega f\right\}.$$

This prove the right-hand side inequality in (1.14). The left-hand side inequality of (1.14) is obtained analogously using (2.18) and the left-hand side inequality in (1.7). □

References

[Aba16] Nicola Abatangelo, *On Dirichlet data for the spectral Laplacian*, Rend. Semin. Mat. Univ. Politec. Torino **74** (2016), no. 2, 31–41. MR3772597 ↑5

[AD17] Nicola Abatangelo and Louis Dupaigne, *Nonhomogeneous boundary conditions for the spectral fractional Laplacian*, Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017), no. 2, 439–467, DOI 10.1016/j.anihpc.2016.02.001. MR3610940 ↑5

[AV19] Nicola Abatangelo and Enrico Valdinoci, *Getting acquainted with the fractional Laplacian*, Contemporary research in elliptic PDEs and related topics, Springer INdAM Ser., vol. 33, Springer, Cham, 2019, pp. 1–105. MR3967804 ↑4, 11

[App09] David Applebaum, *Lévy processes and stochastic calculus*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 116, Cambridge University Press, Cambridge, 2009. MR2512800 ↑16

[Bal17] Paolo Baldi, *Stochastic calculus*, Universitext, Springer, Cham, 2017. An introduction through theory and exercises. MR3726894 ↑15

[BH90] Richard F. Bass and Pei Hsu, *The semimartingale structure of reflecting Brownian motion*, Proc. Amer. Math. Soc. **108** (1990), no. 4, 1007–1010, DOI 10.2307/2047960. MR1007487 ↑30

[Boc49] S. Bochner, *Diffusion equation and stochastic processes*, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 368–370, DOI 10.1073/pnas.35.7.368. MR30151 ↑15, 17, 30

[Bou84] N. Bouleau, *Quelques résultats probabilistes sur la subordination au sens de Bochner*, Seminar on potential theory, Paris, No. 7, Lecture Notes in Math., vol. 1061, Springer, Berlin, 1984, pp. 54–81, DOI 10.1007/BFb0099017 (French). MR779295 ↑18

[BM20] Lorenzo Brasco and Dario Mazzoleni, *On principal frequencies, volume and inradius in convex sets*, NoDEA Nonlinear Differential Equations Appl. **27** (2020), no. 2, Paper No. 12, 26, DOI 10.1007/s00030-019-0614-2. MR4056837 ↑8, 9

[Bre11] Haim Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. MR2759829 ↑5, 10, 24

[CZ95] Kai Lai Chung and Zhong Xin Zhao, *From Brownian motion to Schrödinger's equation*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 312, Springer-Verlag, Berlin, 1995. MR1329992 ↑15, 16, 21

[DGV] Serena Dipierro, Giovanni Giacomin, and Enrico Valdinoci, *The Lévy flight foraging hypothesis in light of subordinate Brownian motions*, Forthcoming. ↑2

[DROV17] Serena Dipierro, Xavier Ros-Oton, and Enrico Valdinoci, *Nonlocal problems with Neumann boundary conditions*, Rev. Mat. Iberoam. **33** (2017), no. 2, 377–416, DOI 10.4171/RMI/942. MR3651008 ↑2

[DV21a] Serena Dipierro and Enrico Valdinoci, *Description of an ecological niche for a mixed local/nonlocal dispersal: an evolution equation and a new Neumann condition arising from the superposition of Brownian and Lévy processes*, Phys. A **575** (2021), Paper No. 126052, 20, DOI 10.1016/j.physa.2021.126052. MR4249816 ↑3

[DV21b] ———, *Elliptic partial differential equations from an elementary viewpoint* (2021), available at 2101.07941. ↑9

[DI08] Paul Dupuis and Hitoshi Ishii, *Correction: "SDEs with oblique reflection on nonsmooth domains" [Ann. Probab.* **21** *(1993), no. 1, 554–580; MR1207237]*, Ann. Probab. **36** (2008), no. 5, 1992–1997, DOI 10.1214/07-AOP374. MR2440929 ↑30

[Eva10] Lawrence C. Evans, *Partial differential equations*, 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010. MR2597943 ↑4, 29, 39

[FJ15] Mouhamed Moustapha Fall and Sven Jarohs, *Overdetermined problems with fractional Laplacian*, ESAIM Control Optim. Calc. Var. **21** (2015), no. 4, 924–938, DOI 10.1051/cocv/2014048. MR3395749 ↑2

[Fuk67] Masatoshi Fukushima, *A construction of reflecting barrier Brownian motions for bounded domains*, Osaka Math. J. **4** (1967), 183–215. MR231444 ↑30

[GN13] D. S. Grebenkov and B.-T. Nguyen, *Geometrical structure of Laplacian eigenfunctions*, SIAM Rev. **55** (2013), no. 4, 601–667, DOI 10.1137/120880173. MR3124880 ↑7, 8

[Hsu84] Pei Hsu, *Reflecting Brownian motion, boundary local time and the Neumann problem*, ProQuest LLC, Ann Arbor, MI, 1984. Thesis (Ph.D.)–Stanford University. MR2633560 ↑11, 30, 31, 35

[Itô57] Seizô Itô, *Fundamental solutions of parabolic differential equations and boundary value problems*, Jpn. J. Math. **27** (1957), 55–102, DOI 10.4099/jjm1924.27.0_55. MR98240 ↑4

[KV18] K. K. Kataria and P. Vellaisamy, *On densities of the product, quotient and power of independent subordinators*, J. Math. Anal. Appl. **462** (2018), no. 2, 1627–1643, DOI 10.1016/j.jmaa.2018.02.059. MR3774308 ↑17

[Krö92] P. Kröger, *Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space*, J. Funct. Anal. **106** (1992), no. 2, 353–357, DOI 10.1016/0022-1236(92)90052-K. MR1165859 ↑14

[LS84] P.-L. Lions and A.-S. Sznitman, *Stochastic differential equations with reflecting boundary conditions*, Comm. Pure Appl. Math. **37** (1984), no. 4, 511–537, DOI 10.1002/cpa.3160370408. MR745330 ↑30

[PW60] L. E. Payne and H. F. Weinberger, *An optimal Poincaré inequality for convex domains*, Arch. Rational Mech. Anal. **5** (1960), 286–292 (1960), DOI 10.1007/BF00252910. MR117419 ↑14

[Pro87] M. H. Protter, *Can one hear the shape of a drum? revisited*, SIAM Rev. **29** (1987), no. 2, 185–197, DOI 10.1137/1029041. MR889243 ↑40

[SSV12] René L. Schilling, Renming Song, and Zoran Vondraček, *Bernstein functions*, 2nd ed., De Gruyter Studies in Mathematics, vol. 37, Walter de Gruyter & Co., Berlin, 2012. Theory and applications. MR2978140 ↑15

[SV03] Renming Song and Zoran Vondraček, *Potential theory of subordinate killed Brownian motion in a domain*, Probab. Theory Related Fields **125** (2003), no. 4, 578–592, DOI 10.1007/s00440-002-0251-1. MR1974415 ↑5, 19, 20

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