Mathematical Physics Preprint Archive, University of Texas,, *June 2021*

Non-deterministic Quantum Mechanics, the Two-Slit Experiment, Measurement, Wave-Particle Duality, Spin, Mass and Energy[1]

Damon Wai Kwan So[2]

Oxford Centre for Mission Studies, Woodstock Road, Oxford OX2 6HR.

Abstract

A previous paper [1] proposed an alternative interpretation of quantum mechanics which is distinct from the Copenhagen Interpretation and the pilot wave theory. This alternative interpretation and its associated non-deterministic surfing velocity on the *S* (phase) surface are applied here to model point-like entities (particles and photons) with non-deterministic trajectories in the one-slit experiment and the two-slit experiment. The diffraction integral for these point-like entities, as a solution to the Helmholtz equation, is in the same form as the diffraction integral found in optical studies so that the mathematical results in optical studies regarding diffraction and interference patterns can be readily applied to the point-like entities modelled here. The non-relativistic model for slowly moving particles and the relativistic model for fast moving particles and photons employed in this paper are therefore consistent with the the observed particle nature and the observed wavy behaviour. A crucial factor accounting for the common diffraction behaviour of the point-like entities (particles and photons) in the two models is that these models share the same Helmholtz equation (where the mass parameter does not appear) which gives rise to their identical diffraction integral. Therefore, these models resolve the problem of wave-particle duality for slowly moving particles, fast moving photons and fast moving particles.

The disappearance of interference pattern when observation is made of the point-like entities in the two-slit experiment can be explained by an exchange of momentum between such an entity and the measuring agent (e.g., in scattering) such that the existing wave function goes through a sudden wholesale transformation, not collapse, in order to maintain consistency with the new momentum. The probability density distribution of the new wave function corresponding to the new momentum is much more localised; hence the disappearance of the interference pattern. Measurement is thus defined as an event where an exchange of momentum between the measured point-like entity and the measuring agent takes place. Hence, the two proposed models are consistent with the observed impact of measurement on the interference pattern.

The two models also reproduce the observed de Broglie's formula for momentum and wavelength, and the observed Planck-Einstein relation for energy and frequency. In addition, a non-deterministic component of the surfing velocity on the *S* (phase) surface in the models is set to be responsible for satisfying Born's rule. As Born's rule is satisfied, the angular momentum generated by the deterministic component of the surfing velocity is integrated over the relevant domain, yielding the expected spin value of the particle in consideration. Hence, the models reproduce the two aforementioned relations and the particle's spin value. The probability density in Born's rule is also interpreted as the non-dimensional time density or the time-averaged probability density such that the rule is satisfied over a given period of time, not at an instant. This implies that the quantum mechanical equations of the two models describe the time-averaged statistical behaviour of the particle, not its instantaneous motion; hence the room for non-determinacy in its instantaneous motion set within the bound of the time-averaged statistical constraint (Born's rule).

The notions of rest mass, rest energy and massless particles are investigated with reference to the particle's surfing momentum on the *S* surface. A new definition of energy is proposed with partial support by data. If validated by experiments, this definition of energy could shed some light on the puzzle of dark matter. The crucial non-deterministic root in quantum mechanics is identified and Heisenberg's Uncertainty Principle is critiqued. Finally, a possible relationship between a quantum-potential-like term and dark energy is posed.

---

[1] This paper is dedicated to the community at the Oxford Centre for Mission Studies which has supported my research leading to this paper. An almost identical paper can be found at *Mathematical Physics Preprint Archive*, University of Texas, Oct. 2020, **20**-94: https://web.ma.utexas.edu/mp_arc/c/20/20-94.pdf . This version ties the theory closer to observation in the Abstract and in the section on Discussion and Conclusion.

[2] Email address: dso@ocms.ac.uk. The author did his PhD in physics at Imperial College, London, before writing another doctoral thesis in theology in Oxford, hence the present academic affiliation.

1.0 Introduction

According to Richard Feynman, the two-slit experiment 'has in it the heart of quantum mechanics. In reality, it contains the only mystery'.[3] He also said,

> No one can "explain" any more than we have just "explained." No one will give you any deeper representation of the situation. We have no ideas about a more basic mechanism from which these results can be deduced.[4]

Since the two-slit experiment is such a fundamental experiment for Quantum Mechanics and it is an experiment that can be performed (compared to cosmological phenomena where experiments are not possible), any interpretation of Quantum Mechanics, if it is to be deemed credible, will do well to be tested with respect to this experiment. A previous paper [1] proposed an alternative interpretation of quantum mechanics which is distinct from the Copenhagen Interpretation and the pilot wave theory. It is the aim of this paper to apply this interpretation to model the two-slit experiment.

Paper [1] identifies a deterministic component and two non-deterministic components of the velocity of a particle which renders the total velocity non-determinate. The three velocity components neatly form an orthogonal set. The two non-deterministic velocity components are tangential to the S (phase) surface and the particle can be visualised as 'surfing' non-deterministically on the S surface as it is carried forward by the deterministic velocity component which is in the direction of ∇S and is called the translational velocity. In the second paper [2], by prescribing or determining one of the two components of the surfing velocity on the S surface (called the spin component or spin velocity), the degree of non-determinacy is effectively reduced from two to one. The overall system is still non-deterministic because there is still a non-deterministic velocity component on the S surface. This configuration yields a promising mechanism for the generation of spin for a particle travelling in free space with a constant translational velocity in a system of cylindrical co-ordinates. This is called the cylindrical mode of motion and the probability density distribution on a S surface (in this case a z-plane which is perpendicular to the translational velocity), obtained from the solution of the Helmholtz equation, exhibits a certain pattern of concentric circles with the probability density as a function of radius. The pattern of

---

[3] R. Feynman, *The Feynman Lectures on Physics*, Vol. 3, Section 1-1; https://www.feynmanlectures.caltech.edu/III_01.html; accessed on 27th Oct., 2020.

[4] *ibid*., Section1-7.

probability density distribution could be tested with respect to real experiments, by comparing it with the density distribution of particles on a detecting screen after these particles have been released from the same source, one at a time, with the same velocity direction at some distance from the screen. It is suggested that such an experiment will be difficult to perform since it is not easy to guarantee that these particles will have the same direction in their translational velocity as they are released because invariably the particles in the experiment will need to be released through a narrow slit which will cause the particles to be diffracted into different directions, rendering the motion no longer in the cylindrical mode.[5] It seems that since such experiments will invariably involve particles passing through a slit with the diffraction therein, for the suggested interpretation to be verified by experiments, the interpretation needs to be applied to model particles diffracted through slits. Paper [2] deals with free particles in the cylindrical mode of motion using cylindrical co-ordinates but this is not able to deal with diffraction. However, it is possible to model the mechanics of a diffracted particle involving spherical co-ordinates.

In Section 5, this paper will model particle diffraction through a single slit using the spherical solution to the Helmholtz Equation which is derived from the Schrödinger equation. The spherical solution corresponds to what is called the spherical mode of motion which is appropriate for a point source only. This spherical mode can be interpreted as representing the motion of a particle through a slit of infinitesimal width. To represent the dynamics with respect to a slit of finite width, the spherical solution will need to be integrated across the width of the slit with respect to some suitable weighting across the slit. It will be seen that the required integral is no different in form from the Fresnel-Kirchhoff diffraction integral used in the study of optics if the boundary condition used at the slit is the same as Kirchhoff's boundary condition; and equally the required integral is no different in form from the Rayleigh-Sommerfield diffraction integral (also used in the study of optics) if the boundary conditions used at the slit are the same as the Rayleigh-Sommerfield boundary conditions. The very close resemblance between the diffraction integral for particles and the diffraction integral in optics explains the wave-like behaviour of particles. This solution for diffraction through a slit of finite width corresponds to what is called the intermediate mode,

---

[5] If it turns out that the particles can be released with the same or nearly the same direction in their velocities in some sophisticated experiment, then this experiment can be used to verify if indeed the the probability density distribution given by the suggested mechanism matches the density distribution of particles on the detecting screen.

intermediate between the cylindrical mode and the spherical mode. The intermediate mode tends to the cylindrical mode as the slit width tends to infinity and tends to the spherical mode as the slit width tends to zero. The solution to the diffraction problem with two slits can be simply obtained by evaluating the integral over the appropriate boundary covering the two slits, thus allowing the two intermediate modes corresponding to the two slits to interact with one another to produce the interference pattern (see Section 6).

The question of measurement is dealt with in Section 7. The measurement of a particle will be defined as an event where an exchange of momentum between the measured particle and the measuring agent takes place. Such an event causes the existing wave function over space to change discontinuously to a new wave function over space which is consistent with the new momentum of the particle. It is because of this kind of change in the wave function when the particle in the two-slit experiment is 'watched' or observed through a measuring agent that the interference pattern in the original wave function disappears, giving way to a localised pattern corresponding to the cylindrical mode. Also, it will be shown that the very insertion of a measuring instrument into a localised vicinity of the particle will also effect a change in the wave function.

To further explore the close similarity between diffraction of particles and photons, they will be modelled in a relativistic framework in Section 8. The paper will justify rigorously that, contrary to popular understanding, the Gordon-Klein equation can incorporate a positive definite probability density in a conservative system and is therefore suitable for studying particles in a special relativistic framework. Furthermore, the components of the electric field and the magnetic field of the Maxwell Equations also satisfy the Gordon-Klein equation under certain simple conditions. It will be shown that because both particles and photons satisfy the Gordon-Klein equation with different values for the mass parameter, this sharing of the same equation lies at the heart of the close similarity between diffraction of particles and photons. And photons and particles are discrete point-like entities whose dynamic behaviour is influenced but not determined by the wave function which is the solution to the Gordon-Klein equation. Also, the square of the amplitude of the wave function can properly be interpreted according to Born's rule, i.e., as the probability density of the point-like entity in question. The relationship between momentum and wavelength, as given by de Broglie's formula, will become evident from the analysis of the equations both for the particle case and the photon case.

The derivation of particle spin in a relativistic framework will be treated in Section 8.3. The question of the meaning of rest mass will be raised and it will be suggested that the 'rest mass' is intimately related to the surfing momentum on the *S* surface and the particle's 'inherent mass'. An implication of this will be that the Lorentz factor is underestimated in calculating the energy according to Einstein's formula, leading to an underestimation of the energy of the particle. A revised definition of energy, partially supported by data, will be suggested to remove the underestimation. If validated by further experiments, it may help to partially account for the missing energy called dark matter. Also, it is possible for a particle with non-zero inherent mass to have zero rest mass when its surfing momentum on the *S* surface is zero. This raises questions about the nature of so-called massless particles.

The crucial factor for the non-determinacy of the motion of a particle will be identified in the non-deterministic direction of the non-deterministic velocity component on the *S* surface. This non-deterministic velocity has to satisfy a certain bulk constraint, i.e., in a budgetary statistical manner over time its evolution should lead to the satisfaction of Born's rule which will in turn result in the generation of the expected spin of the particle. But there are innumerable ways in which this can be achieved, hence the non-determinacy of the motion of a particle. This raises questions about the source of the information which is necessary for the satisfaction of Born's rule and the generation of the particle's spin since this information is definitely not deterministic. This leads to the question of how much we can know about non-deterministic processes and the causes behind them. Finally, the finding of this paper is discussed in relation to the Heisenberg Uncertainty Principle.

The following is the section headings of the paper to help the reader to have map of the paper.

Before this paper attempts to find the solution of the diffraction problem using spherical co-ordinates, it will be helpful for the reader to have a summary of the previous two papers outlining the interpretation suggested there and its application to model particle spins in a cylindrical system of co-ordinates. This summary will only briefly refer to or engage with the papers by other authors referenced in the previous two papers.

2.0 Non-Deterministic Quantum Mechanics

The Schrödinger equation is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi \qquad (\S1)$$

where $\hbar$ is the reduced Planck constant, *m* is the mass of the particle and *U* is the potential 'experienced' by the particle. Writing $\psi = Re^{iS}$, the Schrödinger equation can be exactly re-written as the following two equations by considering its real and imaginary parts:

$$\hbar\frac{\partial S}{\partial t} + \frac{\hbar^2}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + U = 0 \qquad (\S2)$$

$$\frac{\partial R^2}{\partial t} + Div\left(\frac{\hbar}{m}R^2\nabla S\right) = 0\ . \qquad (\S3)$$

Equation (2) is called the energy equation. It is in a modified form of the Hamilton-Jacobi equation. The term, $-\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$, is called the quantum potential.[6] Equation (3) is called the pseudo continuity equation for reasons which will become clear. Apart from these two

[6] The quantum potential is like the pressure term in the Navier-Stokes equation for fluid dynamics where the pressure can also be considered as a kind of energy potential.

equations derived from the Schrödinger equation, there is also the following equation which according to paper [1] (and other papers implicitly, e.g., [3, 4, 5]) should also be satisfied,

$$\frac{\partial \rho}{\partial t} + Div\left(\rho \underset{\sim}{v}\right) = 0 \tag{§4}$$

where $\rho \equiv R^2$ is the probability density, $\underset{\sim}{v}$ is the particle's total velocity in three dimensions and $\rho \underset{\sim}{v}$ is the flux of probability. This equation is called the generic continuity equation or the general continuity equation. It is called the generic continuity equation in paper [1] because its counterparts can be found in other branches of physics, e.g., fluid mechanics. Here, it is also called the general continuity equation in contrast with the pseudo-continuity equation (3). This general continuity equation, involving the explicit velocity of the particle, expresses the conservation of probability in a point-wise manner rather than as a bulk or budgetary constraint as in the normalisation constraint. This paper will later raise the question whether this general continuity equation should also be satisfied in addition to the pseudo continuity equation, even though in paper [1] and [2] it was assumed that the general continuity equation ought to be satisfied. The general continuity equation (§4) is often conflated with the pseudo-continuity equation (§3) but they must be seen as distinct. This is so because the general continuity equation explicitly includes the total velocity of the particle, $\underset{\sim}{v}$, as it should (since the second term is the divergence of the probability flux or probability current) while the pseudo continuity equation does not explicitly includes the velocity (hence the word 'pseudo'). The conflation of the general continuity with the pseudo-continuity equation, as seen in many papers on the pilot wave theory, is not hard to appreciate but it must be resisted. In the pilot wave theory, an explicit velocity is prescribed,

$$\underset{\sim}{v} = \underset{\sim}{v}_1 \equiv \frac{\hbar}{m} \nabla S \ .$$

It can be easily seen that, with this definition of the particle's velocity, the pseudo-continuity equation (§3) is equivalent to the general continuity equation (§4); hence the conflation. And since (§3) must be satisfied as it is derived from the Schrödinger equation, prescribing the velocity in the above manner guarantees that the general continuity equation is also satisfied. However, this prescription of the particle velocity makes the particle velocity deterministic in time since the time evolution of S according to (§2) and (§3), and hence the time evolution of ∇S , are deterministic. Nevertheless, paper [1] shows rigorously that two other components of the particle velocity can be added to $\underset{\sim}{v}_1$ (which is called the translational velocity) such that the general continuity equation is still satisfied:

$$\underset{\sim}{v}_2 \equiv \lambda_2(t)\frac{\hbar}{2m}\frac{\nabla\rho}{\rho}\wedge\vec{s}$$

$$\underset{\sim}{v}_3 \equiv \lambda_3(t)\frac{\hbar}{m}\frac{\nabla S}{\rho}\wedge\vec{a}$$

where $\vec{s}$ is a unit vector constant over space (see paper [1]), defined at the point where the particle is (which is called the reference point in that paper), $\vec{a}$ is the unit vector constant over space in the direction of $\underset{\sim}{v}_2$ defined at the point where the particle is (see paper [1]), $\lambda_2(t)$ and $\lambda_3(t)$ are arbitrary functions of time but they are not functions of space (see paper [1] for further details; in papers [1] and [2], $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ are initially defined without $\lambda_2(t)$ and $\lambda_3(t)$ which are later inserted as multiplicative factors for $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$). $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ neatly form an orthogonal set of velocity components at the point where the particle is; their sum represents the total velocity of the particle and this sum satisfies the general continuity equation (§4):

$$\frac{\partial\rho}{\partial t} + Div\big(\rho(\underset{\sim}{v}_1 + \underset{\sim}{v}_2 + \underset{\sim}{v}_3)\big) = 0 \ .$$

With this definition of the total velocity of the particle, the general continuity equation (§4) containing the full version of the particle's velocity is different from pseudo continuity equation (§3) containing only one component of the particle's velocity. Without explicitly including the general continuity equation as a governing equation in addition to the Schrödinger equation, one is tempted to use the pseudo continuity equation (§3) derived from the Schrödinger equation as the general continuity equation and limits the velocity to only one component, $\underset{\sim}{v}_1$. This is the usual approach taken in the study of pilot wave theory (see, e.g., Bohm's paper [6]) which renders the velocity of the particle deterministic. In paper

[1], by explicitly including the general continuity equation (§4) as a governing equation, two surfing velocity components can be added with the arbitrary functions, $\lambda_2(t)$ and $\lambda_3(t)$, which afford two degrees of freedom for the velocity of the particle which is therefore non-deterministic. In this paper, the forms of $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ will be retained even though the reason for retaining them here is for the *consistency with observations* (see the whole paper for such consistency), rather than satisfying the general continuity equation (see Section 8).

It should be noted that $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$, evaluated at the position where the particle is, are perpendicular to ∇S (and hence perpendicular to $\underset{\sim}{v}_1$); therefore $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ lie on a plane tangential to the *S* surface at the position where the particle is. This means that the sum of $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ represents a velocity on that tangential plane so that the particle can be said to be surfing on the *S* surface while moving forward with velocity $\underset{\sim}{v}_1$. Since the sum of $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$, which is called the surfing velocity $\underset{\sim}{v}_s$, is quite arbitrary in its magnitude (at least at this stage) and since it can be in any direction on that plane tangential to the *S* surface, the particle can be visualised as surfing *freely* and therefore *non-deterministically* on the *S* surface while being carried forward by its deterministic translational velocity component, $\underset{\sim}{v}_1$. Because (i) the particle's position in space is non-deterministic due to its non-deterministic surfing velocity and (ii) the deterministic translational velocity can vary over space (i.e., it is a function of space), generally all three components of the particle's velocity are non-deterministic in time and therefore cannot be predicted in advance.

The interpretation of Quantum Mechanics presented in paper [1] takes the position that while the particle has an unambiguous position and an unambiguous momentum at any instant in time, the total velocity of a particle at any instant in time is non-determinate. This interpretation therefore echoes the non-deterministic or contingent nature of our universe in consonance with the Copenhagen interpretation while it maintains the existence of the objective position and momentum of a particle before measurement which is held by those in the pilot wave theory camp. In this sense, this interpretation has similarities to both of these interpretations but it is also fundamentally distinct from them.

In a disorderly and violent universe, the non-deterministic free motion of a particle on the *S* surface has little constraint; the particles in such a chaotic universe will not give rise to any stable chemical property to sustain life as we know it in this our orderly universe.

However, in a reasonable and orderly universe like ours, even though the free surfing velocity on the *S* surface is not determinate, it may still be subject to some overall budgetary constraints. Paper [2] investigates the possible constraints on a particle's surfing velocity on the *S* surface and how the statistically constrained surfing may generate the particle spin.

3.0 Conservation of Energy and Its Implications

If we define

$$E \equiv \frac{\hbar^2}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + U \qquad (\S5)$$

where *E* is considered as the energy of the particle, then in quantum mechanics the energy of the particle consists of the kinetic energy (first term), the potential energy (third term) and the quantum potential (second term). This energy differs from the classical energy of a particle by the quantum potential and can be reduced to the classical energy if we put $\hbar$ to zero. In the absence of any event in which the particle gains or loses energy, this energy is conserved regardless where the particle is. In such a conservative regime, $\frac{\partial E}{\partial t} = 0$, $\nabla E = 0$. Mathematically, it is possible not to insist that these conservative conditions are satisfied but it will not correspond to the physical phenomena as we know them in our universe. We therefore treat *E* as constant in time and in space in the absence of any event where the particle's energy changes. In the case of scattering, the particle's energy will not be conserved and this case will be considered in Section 7. For the conservative case, from (§2)

$$\hbar\frac{\partial S}{\partial t} = -E \ , \ S = -\frac{E}{\hbar}t + S_p$$

where S_p is a spatial function independent of time. From this, we see that

$$\nabla\left(\frac{\partial S}{\partial t}\right) = \frac{\partial \nabla S}{\partial t} = 0 \ .$$

Taking the gradient of the energy equation (§2), we have

$$\hbar\frac{\partial \nabla S}{\partial t}+\nabla\left[\frac{\hbar^2}{2m}(\nabla S)^2-\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}+U\right]=0$$

which consistently gives us $\nabla E=0$. Furthermore, if U is independent of time, since $\frac{\partial E}{\partial t}=0$ and $\frac{\partial \nabla S}{\partial t}=0$, the quantum potential is also independent of time. Now, this raises the question whether R is also independent of time. It is conceivable that R is a function of time, $R=g(t)I(\underset{\sim}{r})$, $\frac{\nabla^2 R}{R}=\frac{\nabla^2 I}{I}$ will make the quantum potential independent of time. However, the normalisation constraint requires that $\int_V R^2 dv=g^2\int_V I^2 dv=1$ where V is the relevant entire physical space. Since I is independent of time, g has to be independent of time. Hence, we have R independent of time.

The following three conditions follow from the starting point that for a conservative system E is independent of space and time:

$$S=-\frac{E}{\hbar}t+S_p\ ,\ \frac{\partial \nabla S}{\partial t}=0\ ,\ \frac{\partial R}{\partial t}=0\ . \qquad (\S 6)$$

We call the state of such a conservative system Steady Motion State (*SMS*). The only steady part of the motion field is the deterministic velocity component, $\underset{\sim}{v}_1$, which is proportional to ∇S and is a function of space only while the non-deterministic surfing component can vary in time and space and hence can be unsteady. As explained above, if one follows a particle, one will see that the total velocity of the particle is non-deterministic. This is invariably true for Steady Motion State despite the word ‘Steady’ in the term.

4.0 Particle in Free Space in Cylindrical Co-ordinates

In the particular case of the particle travelling in free space, U is constant in space and time. For conservative Steady Motion State which applies in free space, we have already established that $\underset{\sim}{v}_1$ and therefore ∇S are constant in time. In free space, in the absence of any exchange of energy or momentum with any other entity, we can safely propose that $\underset{\sim}{v}_1$ is also independent of space. We call this conservative Steady Motion State in free space the cylindrical mode of motion and the reason for such a naming will shortly become clear. We introduce the constant *a* by defining $a^2 \equiv \frac{2m}{\hbar^2}\left(E - U - \frac{\hbar^2}{2m}(\nabla S)^2\right)$. Thus (§5) can be written as

$$\nabla^2 R + a^2 R = 0 \tag{§7}$$

which is in the form of a Helmholtz equation. In a system of cylindrical co-ordinates, (*r, θ, z*), we adopt the convention that $\underset{\sim}{v}_1$ is in the *z* direction. It can be easily shown that a *S* surface is then identical to the (*r, θ*) plane (the *z* plane) on which the particle surfs non-deterministically. Also, we can appeal to isotropy with respect to *θ* and take $\frac{\partial}{\partial\theta} = 0$ so that the Helmholtz equation becomes

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + a^2 R = 0$$

which is the radial equation with one independent variable, *r*. This equation is in a form of the Bessel equation and has the solution

$$R = AJ_0(ra) + BY_0(ra)$$

where J_0, Y_0 are Bessel functions of the first and second kind respectively. It turns out that the only physical solution is the one with real value of *a* and involving the Bessel function of the first kind (see [2]). Physically, the *R* surfaces are circular tube extending along the direction of *z*, and the *S* surfaces, being perpendicular to the *z* direction, are also perpendicular to the circular tubes of *R*.

$\underset{\sim}{v}_2$, as defined above, in this case of a particle in free space is the velocity which is tangential to the *R* contour and hence the *ρ* contour on a *S* surface; it generates angular momentum with respect to the origin and is therefore called the spin velocity. And $\underset{\sim}{v}_3$ is the velocity perpendicular to $\underset{\sim}{v}_2$ on the *S* surface; it goes through the origin and is therefore called the radial velocity. This radial velocity generates zero angular momentum with respect to the origin. $\underset{\sim}{v}_1$ is perpendicular to the *S* surface such that $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ form an orthogonal set of velocities as expected.

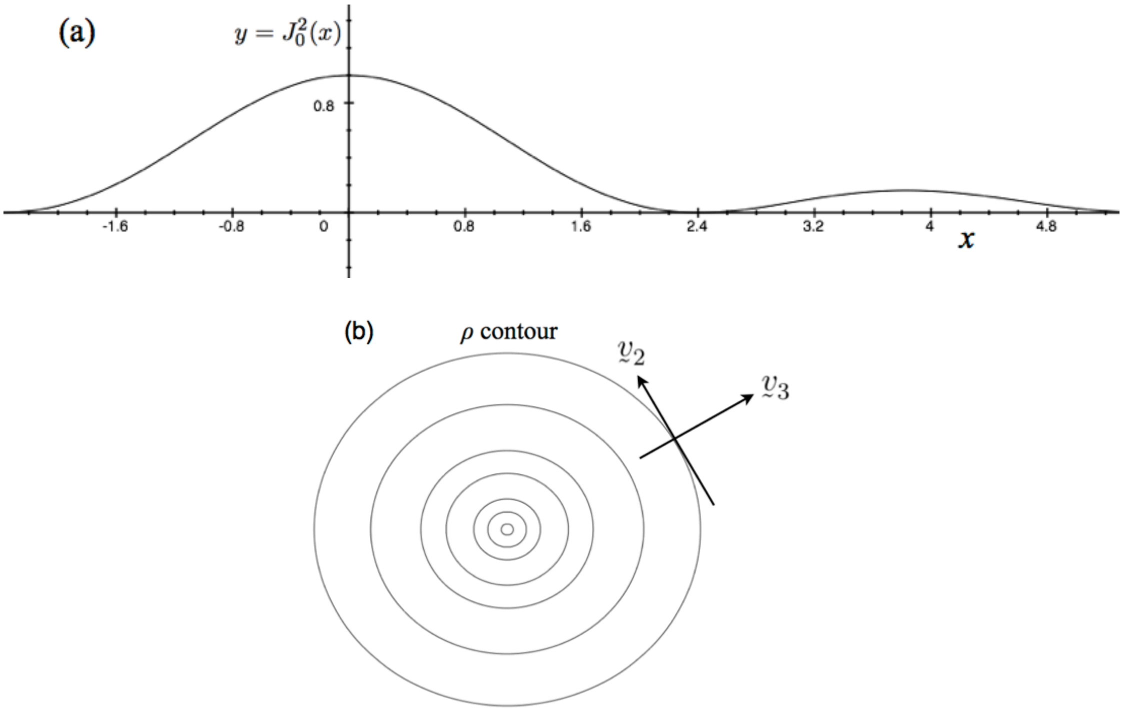

Figure 1: (a) $y = J_0^2(x)$; (b) contours of constant ρ , $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ on the (r, θ) plane (or *S* surface); the *z* direction and hence the direction of $\underset{\sim}{v}_1$ is perpendicular to the page

Since the angular momentum of the particle depends on $\underset{\sim}{v}_2$ which in turn depends on $\lambda_2(t)$ which is arbitrary, for consistency with observed constant angular momentum $\lambda_2(t)$ cannot be an arbitrary function of time. We therefore make it a constant for the particular particle in question, constant in terms of space and time, which is to be validated by observation. This means that we have made the spin velocity, $\underset{\sim}{v}_2$, deterministic. This is the approach taken by Salesi and Esposito [3, 4, 5] but in a deterministic framework because they did not include the non-deterministic $\underset{\sim}{v}_3$. Now, the specification of the spin velocity does not come from the Schrödinger equation. The Schrödinger equation in itself does not give sufficient information to determine the spin angular momentum of the particle. This is well known. Additional constraints, such as the ones given here, are necessary to determine the angular momentum. At this point, one is reminded of Einstein's idea to find more governing equation(s) to close a quantum system, i.e., to make the system totally deterministic (see, e.g., [7]). However, it is not the aim of paper [1], paper [2] and this paper to close a quantum system, i.e., the aim is not to make it deterministic by adding a sufficient number of governing equations to reduce the degree of freedom to zero. Rather, the aim of paper [2] was to retain the non-deterministic nature of a quantum system but giving due allowance for the consistency of the particle's spin velocity. This can be achieved by setting λ_2 to be a constant but leaving $\lambda_3(t)$ to be non-deterministic over time. This means that while the spin velocity, $\underset{\sim}{v}_2$, is deterministic, the radial velocity, $\underset{\sim}{v}_3$, is non-deterministic, in contrast with Salesi and Esposito ([3, 4, 5]) who did not include the radial velocity in their models.

One can picture the particle to be on one hand circulating around a circular ρ contour with the deterministic spin velocity appropriate to the position where the particle is, while on the other hand moving towards or away from the centre with the non-deterministic radial velocity. However, in order for Born's rule to be satisfied in some manner, there is a certain budgetary statistical constraint (or bulk constraint) to be satisfied by the non-deterministic radial velocity. That is, over a suitable time *period* the radial velocity has to be such that it places the particle at any given narrow finite circular strip bounded by two circular ρ contours for a length of time which is a certain fraction of the *period*, and this fraction is equal to the area of the circular strip multiplied by ρ for that circular strip (and this will be seen below as the integrated probability for that circular strip and is therefore consonant with Born's rule; see [2] for details). It is important to note that this budgetary/bulk statistical constraint on the non-deterministic radial velocity does not make this velocity deterministic. There are innumerable ways in which this constraint can be satisfied: at any moment in time the particle still has its own freedom to move in a particular direction along the radius, i.e., inward or outward, with a certain speed.

It is not yet clear how this way of interpreting ρ relates to Born's rule for interpreting ρ which sees ρ as the probability density. Further clarification is necessary and can be found in Section 6 of paper [2]. A concise clarification for the reader here may be helpful. ρ is a function of r as can be seen in the solution of the radial equation. Let us say the particle spends $\triangle t$ length of time in a certain narrow circular strip with area $2\pi r \triangle r$ within the period T. $\triangle t \,/\, T$ is a non-dimensional measure of the time spent in the strip with respect to the period T. The 'non-dimensional time density', defined for the circular strip, is the *expected* non-dimensional time, $\triangle t \,/\, T$, to be spent by the particle within that strip over the period of T

divided by the area of that strip. We interpret ρ to be this 'non-dimensional time density' for the circular strip:

$$\rho = \frac{\triangle t/T}{2\pi r \triangle r} \;.$$

For a given r, ρ tends to a limit as $\triangle r$ and $\triangle t$ tend to zero. Note that ρ is defined with respect to expected events *over the period T* and is thus not yet defined with respect to an *instant* in time as in the usual interpretation of ρ according to Born's rule. However, since the particle spends $\triangle t$ length of time in the strip over the period T, at any random instant within the period T the expected probability of finding the particle at that particular circular strip is $\triangle t \,/\, T$; and this probability, as seen from the above expression, is $2\pi r \triangle r \rho$, which is the area of the circular strip multiplied by ρ. In that sense, ρ which has just been interpreted as the 'non-dimensional time density' is also the 'probability density' for that circular strip as normally interpreted according to Born's rule. Hence, we have two equivalent interpretations of ρ, as the 'non-dimensional time density' and as the 'probability density'. This equivalence is highly reasonable since the more time a particle spends in a circular strip within the period T (high non-dimensional time density), the more likely it is to be found in that strip at any random point of time within T (high probability density).

It should be pointed out that there are innumerable ways in which the particle can spent a total time of $\triangle t$ in the circular strip. The governing equations do not put a definite constraint on the specific timings of the particle visiting that strip. The only constraint is not a specific one but a budgetary statistical one, i.e., according to Born's rule the particle is expected to spend a total time of $\triangle t = 2\pi r \triangle r \rho T$ in a circular strip during the period T. There is nothing to determine in what specific way or order this $\triangle t$ is accumulated within this period. For example, the radial velocity of the particle can be such that it visits that strip n number of times during the period and the accumulated time is $\triangle t$ as expected; however, the same amount of time can be accumulated over $2n$ number of visits or over other numbers of visits in many different sequences or orders, just as there are many different ways or sequences to closely reproduce the following statistical property of throwing a dice over many throws – i.e., the average probability of each of the six outcomes is close to one sixth. In this sense, Born's rule serves as an overall budgetary statistical rule for the length of time to be spent in a strip by the particle so that this rule exerts a bulk constraint on its radial velocity. But this rule is not a deterministic rule dictating the radial velocity at any point in time. In this way, the inherent non-deterministic nature of the quantum system under consideration is maintained even after Born's rule has been imposed. Section 6 of paper [2] gives yet another equivalent interpretation of ρ, which is the 'time averaged probability density' for the period T. This interpretation is also consonant with Born's rule. For this paper, it will be sufficient to concentrate on interpreting ρ as the non-dimensional time density and as the probability density.

Paper [2] calculates the time averaged angular momentum of the particle over the period T which turns out to be $\lambda_2 \hbar$ which is independent of the parameter, a, in the Helmholtz equation. λ_2 can take on opposite signs corresponding to opposite spins and its magnitude corresponds to the particular kind of particle in question. In Section 8.3, the calculation of particle spin will be extended to include special relativity; critical comparison with Dirac's theory will be made in a forthcoming paper. Section 10.1 on Discussion will give a graphical illustration of ρ involving cat and deal with the question of superposition.

5.0 Diffraction of a Single Particle Through a Single Slit in Free Space

This section attempts to model the mechanics of a single particle going through a single slit and being diffracted by the slit. The attempt will concentrate only on one particle. In physical reality the slit wall consists of many other particles which will influence the mechanics of the particle going through the slit. One could make a theoretical attempt to represent this physical reality by treating it as a many-body problem. However, such an attempt is beyond the scope of this paper.

5.1 The Spherical Mode (the details in this section, apart from the result at the end, may be skipped)

We now attempt to solve the governing equations – the energy equation and the pseudo continuity equation – in spherical co-ordinates. We again consider the Steady Motion State (*SMS*) where $\frac{\partial \nabla S}{\partial t}$ is zero. As seen above and in paper [2], this implies that the deterministic velocity component, $\underset{\sim}{v}_1$, does not vary with time (though it can vary with space) and the energy of the particle

$$E \equiv \frac{\hbar^2}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + U$$

is constant over space. The second term on the r.h.s. is the quantum potential. If we define

$$D \equiv \frac{2m}{\hbar^2}\left(E - U - \frac{\hbar^2}{2m}(\nabla S)^2\right) ,$$

then

$$\nabla^2 R + DR = 0 \qquad (\S8)$$

and $\frac{\hbar^2}{2m}D$ is the quantum potential. Equation (§8) is in a similar form to equation (§7) and both equations are in the form of the Helmholtz equation. However, the parameter *a* in equation (§7) is a constant while the parameter, *D*, here cannot be assumed to be constant. Since ∇S is independent of time and *E* is conserved in time, if *U* is also independent of time, then *D* and the quantum potential are independent of time. According to equation (§8) and the reasoning for (§6), *R* has to be independent of time so that $\frac{\partial R^2}{\partial t}$ in the pseudo continuity equation is zero. The pseudo continuity equation becomes

$$\nabla^2 S + \nabla S \cdot \frac{\nabla \rho}{\rho} = 0 \ . \qquad (\S9)$$

We now proceed to solve for *R* in equation (§8) in spherical co-ordinates,(r, θ, ϕ). The solution for *R* is then input into equation (8) which is solved for *S* in spherical co-ordinates. If *D* is a function of *r* only and appealing to symmetry, $\frac{\partial}{\partial \phi} = 0$, we then separate the variables, $R = F(r)f(\theta)$, equation (§8) can be written as two separate equations:

$$\frac{d}{dr}(r^2\frac{dF}{dr}) + (Dr^2 - k)F = 0 \qquad (\S10)$$

$$\frac{1}{sin\theta}\frac{d}{d\theta}(sin\theta\frac{df}{d\theta}) + kf = 0 \qquad (\S11)$$

where *k* is the constant of separation. If $D = k/r^2$, a possible solution for the radial equation (§10) is $F = \chi/r$, where χ is a constant. The solution for equation (§11) is the Legendre polynomial in $cos\ \theta$ for *k=l(l+1)* where *l* is an integer.

For equation (§9), we posit *S* to be a function of *r* only and $\dfrac{dS}{dr} = v_T$ is constant over space. This implies that the radial (translational) velocity has constant speed, $\dfrac{\hbar}{m}v_T$, regardless of its direction. Now, $\nabla^2 S = \dfrac{2}{r}v_T$. Since ∇S is in the radial direction and has magnitude v_T, and $R = \dfrac{\chi}{r}f(\theta)$,

$$\nabla S \cdot \frac{\nabla\rho}{\rho} = v_T\frac{\partial\rho}{\partial r}/\rho = -\frac{2}{r}v_T$$

where the solution for *F* of equation (§10) has been used for *R* and *ρ*. Equation (§9) is exactly satisfied by the proposed solution for *S* and *R,* and this solution will hold for various constant values of v_T and χ. It can be seen that χ will be fixed by the normalisation constraint while v_T is a free parameter. Crucially, the value of *k=l(l+1)*, where *l* is an integer, is also a parameter to be determined. At this point, a physical interpretation of this solution will help us to visualise the dynamics of the diffracted particle.

Firstly, the *S* surfaces are concentric spherical surfaces and it is evident that the particle is 'radiating' from the centre with a constant radial translational speed even if the direction of the radial translational velocity ($\underset{\sim}{v}_1$) varies over space. However, since the particle is to go forward through the slit in the diffraction, in order to model the diffraction one needs to use only one half of a spherical *S* surface, i.e., use only a hemispheric *S* surface. Here is a cross section of some hemispheric *S* surfaces which can be rotated by *π* radians with respect to the 'vertical' axis to give the hemispheric picture.

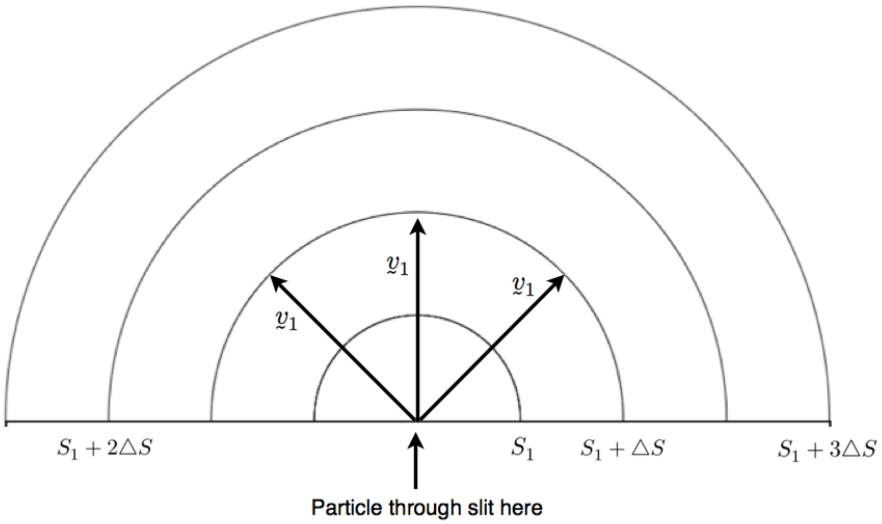

Figure 3: Cross Section of Some Hemispheric *S* Surfaces with Constant $|\underset{\sim}{v}_1|$

$\underset{\sim}{v}_1$ can have its θ between 0 and $\pi/2$ radian, and its ϕ between 0 and 2π radian, while its magnitude is constant.

Secondly, since this theoretical model seeks to represent the three-dimensional dynamics of a diffracted particle in a real experiment, the slit in the experiment is envisaged to be a circular slit instead of a rectangular slit (which is effectively one-dimensional), i.e., the slit is a small circular hole through which the particle goes through.

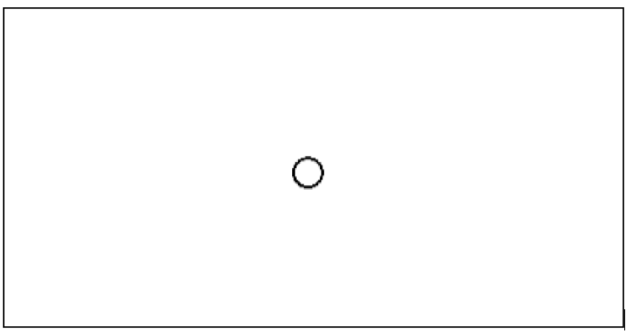

Figure 4: A Wall with a Small Circular Slit in an Experiment

Thirdly, regarding the ρ distribution on the hemisphere, if we imagine the particle is going through the slit vertically 'upward' through the radial centre of the 'northern' hemisphere, i.e., with θ=0, ρ is independent of ϕ by symmetry as assumed above in the solution for equation (§8), then a ρ contour on an latitudinal plane is a latitudinal circle. And since there will be zeros of the Legendre polynomial which describes the co-latitudinal distribution of ρ as a function of the co-latitude, θ, there will be bands of non-zero ρ in the form of circular strips on the hemispheric S surface which are bounded by zero latitudinal contours of ρ (see Figure 5 and compare with Figure 1 for the cylindrical case). $\underset{\sim}{v}_2$ as defined above can be shown to be (i) lying on the same plane as the ρ latitudinal contour and (ii) tangential to the ρ contour which is the latitudinal circle. $\underset{\sim}{v}_3$ as defined above can be shown to be (i) lying on the longitudinal plane and (ii) tangential to the longitudinal circle. Since $\underset{\sim}{v}_1$ is in the radial direction, $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ form an orthogonal set of velocity components, as expected. $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ together form a surfing velocity on the S hemispheric surface, i.e., they form a velocity which is on the plane tangent to the hemispheric surface at the point where the particle is. While this surfing velocity is perpendicular to the radial direction, $\underset{\sim}{v}_1$ is along the radial direction. One can picture the particle as being carried away from the radial centre by the translational velocity $\underset{\sim}{v}_1$ while surfing on the hemispheric S surface. And while $\underset{\sim}{v}_2$ is determined according to the expression given above with constant λ_2, $\underset{\sim}{v}_3$ with non-constant $\lambda_3(t)$ is still non-determinate rendering the surfing velocity on the S surface non-determinate. Since the radial translational speed, $|\underset{\sim}{v}_1| = \dfrac{\hbar}{m}\dfrac{dS}{dr}$, is constant, at a time after leaving the centre (slit) the particle will have the same radial distance from the centre regardless of the surfing motion on the S surfaces so that it will be on the same S surface regardless of the surfing motion (see Figure 3). This means that the particle's position on the S hemispheric surface is non-determinate. However, overall during a period T, there is a budgetary constraint on the probability of the particle in various neighbourhoods on a S surface. To look into this, we need a closer look at the probability density distribution on different S surfaces.

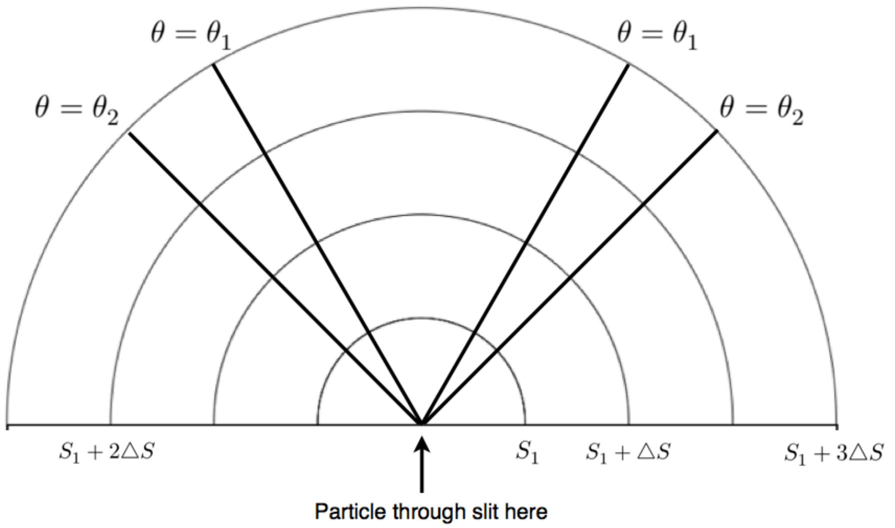

Figure 5: Integrated Probability on Circular Strips of Hemispheric *S* Surfaces (cross section is to be rotated by *π* radian around the $\theta = 0$ axis)

Here we have two cones, one with co-latitude angle $\theta = \theta_1$ and the other with angle $\theta = \theta_2$. The area on a hemispheric *S* surface between the two latitude circles forms a circular strip. The probability density (*ρ*) integrated over this circular strip is

$$\int_{\theta_1}^{\theta_2} 2\pi r^2 sin\theta\rho d\theta = \int_{\theta_1}^{\theta_2} 2\pi sin\theta\chi^2 f^2(\theta)d\theta$$

which is independent of *r*. That is, the integrated probability on any circular strip bounded by the two latitude circles $\theta = \theta_1$ and $\theta = \theta_2$ on any hemispheric *S* surface has the same value. The integrated probability is preserved between the two latitudes, regardless of the hemispheric *S* surfaces, i.e., regardless of their radii. During a period *T*, the particle will spend the following length of time within these two latitudes while it moves through different *S* surfaces: *T* multiplied by the integrated probability between the two latitudes, remembering that *ρ* has two equivalent meanings – probability density and non-dimensional time density. This overall budgetary time requirement needs to be satisfied by $\underset{\sim}{v}_3$ tangential to the longitudinal circles with its suitable non-determinate $\lambda_3(t)$.

The function $f(\theta)$ is a Legendre polynomial in $cos\ \theta$ and has its maximum amplitude at $\theta = 0$, that is, the probability density will be highest there so that the particle is most likely to be found in the neighbourhoods around the $\theta = 0$ axis. If it happens that $f(\theta_1) = f(\theta_2) = 0$, then the probability density, ρ, is zero on the two cones with these two angles, and there will be non-zero ρ between these two cones with a maximum ρ somewhere in between the two latitudes.

$\frac{\hbar^2}{2m}D$ is the quantum potential; $D = k/r^2$; $k=l(l+1)$ is derived from the parameter of the Legendre polynomial, l. Since k is positive (even for negative l), the positive quantum potential initially decreases rapidly with distance from the centre and later approaches the asymptotic value of zero. Also,

$$U = \left(E - \frac{\hbar^2}{2m}(\nabla S)^2\right) - \frac{\hbar^2}{2m}D\ ,$$

where the first two terms on the r.h.s are constant, U initially increases rapidly with distance from the centre and later approaches an asymptotic value which is the sum of the first two terms, as the quantum potential (the third term) approaches its asymptotic value of zero. With large distance from the centre, both the quantum potential and U tend to their asymptotic values. However, with small distance from the centre, both take on very large magnitude so that at r=0 their magnitudes reach infinity except for the case of k=0. When k>0, the Legendre polynomial is zero at some values of θ; when k=0, the Legendre polynomial has the constant value of 1. Even though the solution for k>0 is interesting as it varies as a function of θ and may be useful in some physical scenarios, for the slit experiment under consideration we cannot really account for the infinite value of the potential U and the infinite value of the quantum potential at the slit. Therefore, the only solution we can accept is the case of k=0. This special case of k = 0 is called the *spherical mode*. In this case, using the normalisation constraint to fix χ, the wave function is

$$\psi = \frac{1}{\sqrt{2\pi r}} e^{i(v_T r - \frac{E}{\hbar}t)}\ .$$

We have already pointed out that the radial translational speed, $|\underset{\sim}{v}_1| = \dfrac{\hbar}{m}\dfrac{dS}{dr}$, is constant, at any time after leaving the centre the particle will have the same radial distance from the centre so that it will be on the same *S* hemispheric surface regardless of the surfing motion on such surfaces in the past. With the probability density already independent of ϕ, the case of *k*=0 implies that the probability density is also independent of *θ*, meaning that the particle is equally likely to be found at any point of the hemispheric *S* surface. That is, the surfing motion on the *S* surface has to be such that statistically it will satisfy (or approximate in finite time) the constant probability density requirement for *k*=0. This particular scenario corresponds to a particle emerging from a point source and radiating out along the radial direction with constant radial speed while surfing non-deterministically on the *S* hemispheric surfaces to produce the uniform probability density across all angles of *θ* and ϕ. Since the particle emerges from a point source, this scenario is equivalent to a particle coming through a slit of infinitesimal width. In the usual theoretical treatment of one-slit experiment where the detecting screen is flat rather than a hemisphere as considered here, the equivalent scenario for that treatment is this: as the slit width tends to zero, the distance of the first minimum tends to infinity on either side of the flat detecting screen and the intensity or probability density is uniform across the flat detecting screen.

5.2 The Diffraction Integral for Single Slit in Free Space

In a one-slit experiment in the real world, the slit area has finite width, as in Figure 4. How can one construct the solution to the governing equations when the slit width or slit area is no longer infinitesimal? An intuitive idea is to (i) assume that the above spherical solution is representative of a small neighbourhood of the slit which has a certain representative probability density and (ii) integrate the spherical solution, weighted by the probability density, over the whole slit area. The probability density distribution over the slit area depends on the kind of motion the particle is in prior to entering the slit. If it has been in a cylindrical mode of motion in free space, then the pattern of its probability density distribution before entering the slit will have concentric circles of *ρ* contours given by the Bessel function of the first kind (as summarised above) and this can be taken as the approximation of the probability density at the slit area. If the particle has already been in a near spherical mode of motion in free space due to the fact that it has been released from a source with an extremely small opening, then the probability and phase of the wave function from this first mode of motion can be evaluated at the slit area and taken as the approximation of the probability density at the slit area. This intuitive idea is akin to (but not identical to) Huygens' principle usually applied to wave diffraction but this principle can be rigorously clarified and made more precise by the following proper mathematical treatment of the governing partial differential equation and its boundary conditions for a slit with finite width or finite area. According to (§6), the wave function can be written as

$$\psi = Re^{iS_p}e^{iS_t}$$

where $S = S_p + S_t$, i.e., the phase is split into the spatial part denoted by *p* and the temporal part denoted by *t*. A S_p surface at a given point in time comprises positions of a certain constant S_p value. On that surface, $S = S_p + S_t$ is also constant since S_t is a constant over space at any given point in time. Hence, a S_p surface at any given point in time is also a *S* surface and the term '*S* surface' will be used consistently throughout the paper. The above expression for *S* means that ∇S is independent of time; hence, we are considering Steady Motion State and we will justify below rigorously why our attention should be focussed on such states. According to (§6), for Steady Motion State, an accompanying condition is *R* is also independent of time. Substituting this ψ into the Schrödinger equation, noting from (§2) that $\frac{\partial S_t}{\partial t} = -E/\hbar$, writing $\psi_p = Re^{iS_p}$ and $b^2 = 2m(E-U)/\hbar^2 = (\nabla S)^2 - \frac{\nabla^2 R}{R}$ yield

$$\nabla^2\psi_p + b^2\psi_p = 0 \; . \qquad (§12)$$

Note that since *E*, the energy of the particle, is conserved (hence constant) and *U* is constant in free space, *b* is a constant. In the cylindrical mode of motion, the translational kinetic energy is constant and the quantum potential can be a non-zero constant. In the spherical mode discussed above, the translational kinetic energy is also constant but the quantum potential is zero. In either mode, the sum of the translational kinetic energy and the quantum potential, $\frac{\hbar^2}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$, is equal to *E–U* and is therefore constant over space and time. This is a consequence of the total energy of a particle being conserved; therefore, it is also true in other modes of motion in free space, such as the intermediate mode which will be discussed next.

At this point of analysis, the parameter, *b*, has yet to be determined but it will be determined. This Helmholtz equation with complex variable can be solved with the help of an appropriate Green function for some appropriate boundary conditions at the slit and the slit wall. For this solution, we can borrow from the well established *mathematical procedure* used in the study of optical diffraction where the Helmholtz equation of the same form is to be solved, e.g., see *Introduction to Fourier Optics* by Joseph W. Goodman, chapters 3 and 4. Note at this point it is the *mathematical procedure* we are borrowing, not the physics from optical studies.

The free space Green function is defined as

$$G(\underset{\sim}{r}, \underset{\sim}{r}_0) \equiv \frac{e^{ib\,|\underset{\sim}{r} - \underset{\sim}{r}_0|}}{|\underset{\sim}{r} - \underset{\sim}{r}_0|}$$

with the parameter *b* given by the *b* in (§12), $\underset{\sim}{r}_0$ is the reference point on the diffracted side of the slit and $\underset{\sim}{r}$ is the position vector. This Green function, apart from a multiplicative constant, in effect is virtually equivalent to the spatial part of the wave function of the spherical mode obtained above with $b = v_T$ and $\underset{\sim}{r}_0$ treated as the origin. This function also satisfies (§12) but on its own it cannot satisfy the boundary condition of a slit with finite width. Using the Green's Theorem relating an integral over a volume and an integral over the enclosing surface and after some derivation,

$$\psi_p(\underset{\sim}{r}_0) = \frac{1}{4\pi}\int\int_{S_1}\left(\frac{\partial \psi_p}{\partial n}G - \psi_p\frac{\partial G}{\partial n}\right)ds \qquad (\S 13)$$

where the surface S_1 is the surface comprising the slit wall and the slit, the direction of the normal derivative is pointing to the outside of S_1 in relation to $\underset{\sim}{r}_0$ and the integration is carried out by varying $\underset{\sim}{r}$ and the corresponding element on the S_1 surface.

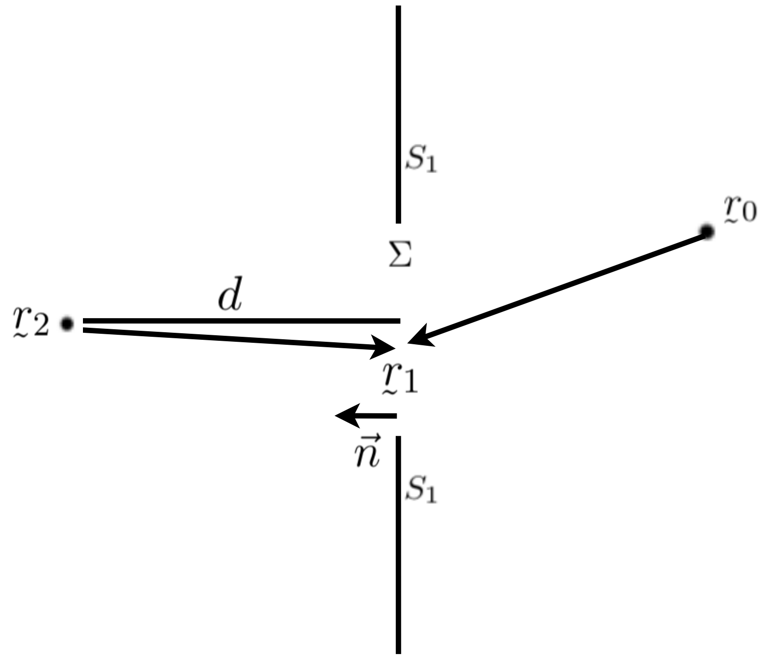

Figure 6: Circular slit, Σ, with point source at $\underset{\sim}{r}_2$ which is at distance *d* from the slit and situated midway of the slit; the wave function in the ambient environment on the left of the slit is diffracted on the right of the slit

Let the point source at $\underset{\sim}{r}_2$ give rise to this wave function to the left of the slit wall:

$$\psi_p = \frac{e^{ib\left|\underset{\sim}{r}-\underset{\sim}{r}_2\right|}}{\sqrt{2\pi}\left|\underset{\sim}{r}-\underset{\sim}{r}_2\right|}$$

which is again of the same form as the spherical mode above, and the particle from $\underset{\sim}{r}_2$ can only visit the half of any spherical *S* surface towards the slit, hence the 2π (not 4π) in the expression of the wave function. This expression for ψ_p models the scenario where a particle is released through an extremely small circular hole at $\underset{\sim}{r}_2$ towards the slit, Σ . The particle can freely surf on the hemispheric *S* surfaces on the left of the slit wall and on the distorted hemispheric *S* surfaces on the right in the diffraction region. This ψ_p also satisfies (§12) because the same parameter, *b,* is used in the expression for ψ_p here. In fact, the parameter *b* in (§12) is set by this parameter in this expression for ψ_p for the region to the left of the slit wall. What we have is that (§12) should be satisfied throughout the whole domain which has been considered as a free space with constant *U* throughout. This requirement for (§12) to be satisfied throughout the whole domain (left and right of the slit wall) is a consequence of the conservation of energy as discussed above. Therefore, ψ_p for the region to the left of the slit with its *b* represents the ambient environment the particle is in before entering the diffraction region to the right of the slit where the same equation, (§12) with the same *b*, also applies and is to be satisfied, subject to the appropriate boundary conditions. (There can be other ambient environments before the slit which set the parameter *b* as will be seen below.)

If the position of $\underset{\sim}{r}_2$ is far enough from the slit, then the distance between any point $\underset{\sim}{r}_1$ of the slit and $\underset{\sim}{r}_2$ is very close to *d*. Then, the solution can be well approximated by

$$\psi_p(\underset{\sim}{r}_0) = \frac{-ib}{(2\pi)^{3/2}} \int\int_{\Sigma} \frac{e^{ibd}}{d} \frac{e^{ib|\underset{\sim}{r}_1 - \underset{\sim}{r}_0|}}{|\underset{\sim}{r}_1 - \underset{\sim}{r}_0|} \left(\frac{1 + cos(\vec{n}, \underset{\sim}{r}_1 - \underset{\sim}{r}_0)}{2} \right) ds \qquad (§14)$$

where $(\vec{n}, \underset{\sim}{r}_1 - \underset{\sim}{r}_0)$ is the angle between the normal to the slit surface and $\underset{\sim}{r}_1 - \underset{\sim}{r}_0$ (see Figure 6), and the integration is carried out on the slit surface, Σ, by varying $\underset{\sim}{r}_1$ and the corresponding element on it. This is the same as the Fresnel-Kirchhoff diffraction formula except that (i) the distance between $\underset{\sim}{r}_1$ on the slit and $\underset{\sim}{r}_2$ is approximated by the constant *d* and (ii) $\underset{\sim}{r}_2$ is situated midway of the slit. This approximation and simplification have made the mathematical presentation cleaner and easier to read. The following Kirchhoff boundary conditions have been assumed to yield the above analytic integral:

1. on the slit wall away from the slit surface, $\psi_p(r_1)$ and $\frac{\partial \psi_p(r_1)}{\partial n}$ are assumed to vanish; hence, no integration needs to be carried out there;
2. on the slit surface, $\psi_p(r_1)$ and $\frac{\partial \psi_p(r_1)}{\partial n}$ are completely determined by the source at $\underset{\sim}{r}_2$ (as if the slit wall is not there).

(1) can be alleviated by either setting the first Rayleigh-Sommerfield boundary condition or the second Rayleigh-Sommerfield boundary condition by using one of the two modified Green functions respectively. The two resulting integrals will be of slightly different forms to (§14). However, interestingly the average of the two integrals is identical to the integral obtained by applying the Kirchhoff's boundary condition as has been done above. The boundary condition (2) can be alleviated by numerical method where the effect of the slit wall's presence on the the wave function being generated by the point source at $\underset{\sim}{r}_2$ can be

taken into account. But this effect is expected to be very small especially if the distance between $\underset{\sim}{r}_2$ and the slit, *d*, is large.

One can see from (§14) that the spherical mode, given by the second factor in the integral, makes contribution to the overall solution while being weighted by (i) a factor which is the wave function due to the particle source at $\underset{\sim}{r}_2$ (see first term of the integral) and (ii) a second factor to do with the geometry of $\underset{\sim}{r}_0$ and $\underset{\sim}{r}_1$(see third term of the integral). The former intuition that the weighting factor is the probability density at the point of the slit, $\underset{\sim}{r}_1$, needs to be adjusted here in light of the more rigorous mathematical derivation above which gives the weighting factor in (i) instead of the the probability density. However, it is still true that the square of the modulus of $\psi_p(\underset{\sim}{r}_0)$ will reproduce the intuited probability density.

The integration in (§14) allow the various weighted spherical modes 'radiating' from the points on the slit surface to interact with one another in terms of phase and magnitude such that the solution will not be the same as the pure solitary spherical mode and the resulting *S* surfaces will not be hemispheric. This interaction between the spherical modes gives rise to maximums and minimums (or zeros) in the modulus of the diffracted wave function, hence generating the interference pattern of the one-slit experiment. The magnitude of ∇S most certainly will no longer be uniform but as shown above $\frac{\hbar^2}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$, which is the sum of the translational kinetic energy of the particle and the quantum potential, will still be constant. This means that the quantum potential and the translational kinetic energy will adjust to one another to maintain that constant.

In terms of non-determinacy, a particle is free to surf on the *S* surfaces. This is true both in the region before diffraction happens and in the region where diffraction happens. If we set the translational velocity $\underset{\sim}{v}_1$ and the velocity $\underset{\sim}{v}_2$ as before, i.e., as determined by the given formulae, then the third velocity component $\underset{\sim}{v}_3$ which is non-deterministic will have to be such that it will move the particle to satisfy the budgetary statistical constraint given by the probability density distribution. Again, there are an infinite number of ways this budgetary statistical constraint can be satisfied; hence the inherent non-determinacy in particle motion in quantum mechanics also applies here.

The uniform probability density on a hemispheric *S* surface centred at $\underset{\sim}{r}_2$ (the point source) is a function of distance from the point source only and this means that the particle can surf to any neighbourhood of equal size on a hemispheric *S* surface with equal probability. This in turn means that for large *d* the probability of reaching any small neighbourhood of equal size on the slit surface is virtually the same (since the distance between $\underset{\sim}{r}_1$and $\underset{\sim}{r}_2$ is virtually constant for any $\underset{\sim}{r}_1$ on the slit surface). And the phase will be virtually constant on the slit surface. These good approximations have been assumed in (§14). The possible path from $\underset{\sim}{r}_1$ to $\underset{\sim}{r}_0$ is also subject to non-determinacy as the particle is also free to surf on the diffracted *S* surfaces on the right side of the slit. The likelihood of the particle reaching $\underset{\sim}{r}_0$ from $\underset{\sim}{r}_1$ or from other points on the slit surface depends on the probability density at $\underset{\sim}{r}_0$, which is the square of the modulus of $\psi_p(\underset{\sim}{r}_0)$. Hence, this modulus squared determines the probability density distribution or the interference pattern on a detecting screen. The best detecting screen for a circular slit should be an hemispheric screen. (Similar surfing on *S* surfaces and the resultant non-determinacy will also operate in the two-slit experiment; see below.)

For the cases where the perpendicular distance between $\underset{\sim}{r}_0$ and the slit plane is also large, the angle $(\vec{n}, \underset{\sim}{r}_1 - \underset{\sim}{r}_0)$is almost zero radian and the third factor in the integral of (§14) is virtually 1. Further approximations can be made to simplify the integral of (§14), e.g., the Fraunhofer approximation, but since these are well known they will not be treated here. The more important issue to be studied here is how the spherical modes at the slit surface contribute together towards the diffracted wave function. Since the Fresnel-Kirchhoff diffraction formula and its approximations are known to produce interference pattern even for the one-slit experiment in optical studies, the same kind of mathematical result applies here so that we can borrow the mathematical results of those formulae to affirm the interference pattern of the one-slit experiment of particles considered here.

5.3 The Ambient Environment, The Slit Size and De Broglie's Formulae

Firstly, the ambient environment to the left of the slit where the particle is before going through the slit can be different from the point source (the spherical mode at $\underset{\sim}{r}_2$) as described above and the resulting diffraction will be different. For example, if the particle released at $\underset{\sim}{r}_2$ has to first go through a slit of small but finite width at the source before reaching the slit Σ later, then diffraction also happens at this first slit and this will affect the ambient environment so that the boundary condition at the Σ slit involving the wave function there will need to be adjusted according to the ambient environment in evaluating the more general integral of (§13) which will be different from (§14). Alternatively, if the particle happens to come towards the slit Σ in a cylindrical mode of motion as described in Section 4.0, and in a direction perpendicular to the slit, then the radially varying circular contours of *R* on a constant *S* surface will apply in the wave function in the ambient environment and, as a good approximation, this can be used for the boundary condition at the slit surface in the integral of (§13) which again will be different from (§14). In this case, the second Rayleigh-Sommerfield solution is better than the one in (§14) which is based on the Fresnel-Kirchhoff diffraction formula. The second Rayleigh-Sommerfield solution for diffraction from an ambient cylindrical mode is

$$\psi_p(\underset{\sim}{r}_0) = \frac{1}{2\pi}\int\int_{\Sigma} \frac{\partial \psi_p}{\partial n} G\, ds = \frac{i|\nabla S|}{2\pi}\int\int_{\Sigma} \psi_p(\underset{\sim}{r}_1)\frac{e^{ib\,|\underset{\sim}{r}_1-\underset{\sim}{r}_0|}}{|\underset{\sim}{r}_1-\underset{\sim}{r}_0|}\, ds\ . \qquad (\S 15)$$

Again, the integration is carried out on the slit surface, Σ , by varying $\underset{\sim}{r}_1$ and the corresponding element on it.

In both of these examples, (§12) applies on both sides of the slit and the constant value of $\frac{\hbar^2}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$ in the ambient environment determines the constant value of $b^2 = 2m(E-U)/\hbar^2 = (\nabla S)^2 - \dfrac{\nabla^2 R}{R}$ in (§12). For the cylindrical mode of motion in the ambient environment, unlike the spherical mode the quantum potential will be non-zero and will need to be included in calculating the value of b^2 for (§12).

Secondly, apart from the ambient environment, the size of the slit area also influences the diffraction to the right of the slit. If the slit area is infinite, then a cylindrical mode of motion will not be perturbed by any diffraction and continues in that mode – the particle merely continues with that mode of motion with constant translational velocity. If the slit area is infinitesimal, the diffracted solution is the spherical mode with zero quantum potential. This can be seen from (§15): since $\psi_p(\underset{\sim}{r}_1)$ and indeed the whole integrand over a very small slit surface will be almost constant, the expression for $\psi_p(\underset{\sim}{r}_0)$ is in the form for the spherical mode of motion. It can be shown that the same is true if the wave function in the ambient environment to the left of the slit is given by a point source instead of a cylindrical mode of motion.

The cylindrical mode (corresponding to infinite slit width) and the spherical mode (corresponding to infinitesimal slit width) are two modes at the ends of a spectrum of possible modes which are called the intermediate modes. These intermediate modes correspond to diffraction by slit of finite width or area and are given by (§14).

Thirdly, the two de Broglie formulae relating (i) wavelength and momentum and (ii) energy and frequency can be explored. The translational velocity of a particle, $\underset{\sim}{v}_1$, is given by $\frac{\hbar}{m}\nabla S$. The translational momentum, $\underset{\sim}{p}$, of the particle is given by $\hbar\nabla S$. If λ is the wavelength of the particle and since $|\nabla S|$ is the rate of change of phase with distance, $\lambda|\nabla S| = 2\pi$. Then

$$\left|\underset{\sim}{p}\right| = \hbar|\nabla S| = \hbar\frac{2\pi}{\lambda} = \frac{h}{\lambda}$$

which is de Broglie's formula relating the momentum of a particle to the 'wavelength' of a particle. It has to be added here that this momentum is the translational momentum of the particle, which is distinct from the surfing momentum on the *S* surface. It is seen here that the 'wavelength' of a particle comes through the rate of change of phase, *S*, with distance. This 'wavelength' is the wavelength of the wave field, $\psi = Re^{iS}$, which influences the motion of the particle, rather than being an inherent property of the particle. As the translational momentum of the particle changes as $|\nabla S|$ changes through diffraction, the wavelength in the wave field influencing the particle will also change accordingly. It will be seen later that this formula and this idea will carry over to the relativistic case. However, here it needs to be mentioned that in the diffracted wave function solution of (§12), $|\nabla S|$ cannot be guaranteed to be constant even though it can be constant in the ambient environment before the slit. This raises the meaning of the wavelength since it can have different values for the same particle as $|\nabla S|$ varies in the region on the right side of the slit through diffraction. Perhaps the wavelength should be defined with respect to the constant $|\nabla S|$ in the ambient environment. However, in pre-university physics textbook and even in some university elementary lecture notes, in the illustration of how interference pattern can occur in the presence of one slit or two slits, in working out the path difference the wavelength and thus $|\nabla S|$ is assumed to be constant even in the region where diffraction occurs. This can only be an approximation and in an average sense it could be a reasonable approximation when $|\nabla S|$ is averaged over a sufficiently long distance. Such an approximation can be assessed either by experiment or by looking at the actual solution of the diffracted wave function on the right side of the slit. Or indeed experiments can be used to verify the actual solution of the diffracted wave function as presented here which should be more accurate than those based on constant $|\nabla S|$.

From the energy equation (§2),

$$E = -\hbar\frac{\partial S}{\partial t} = -h\frac{\partial(S/(2\pi))}{\partial t} = hf \ ,$$

where $f \equiv -\dfrac{\partial(S/(2\pi))}{\partial t}$ is the angular frequency. Hence, de Broglie's two formulae can be derived from the spatial derivative of *S* and the time derivative of *S* respectively in the analytic framework adopted in this paper, the key elements of which are (§2) and (§3).

6.0 Diffraction of a Single Particle Through Two Slits in Free Space

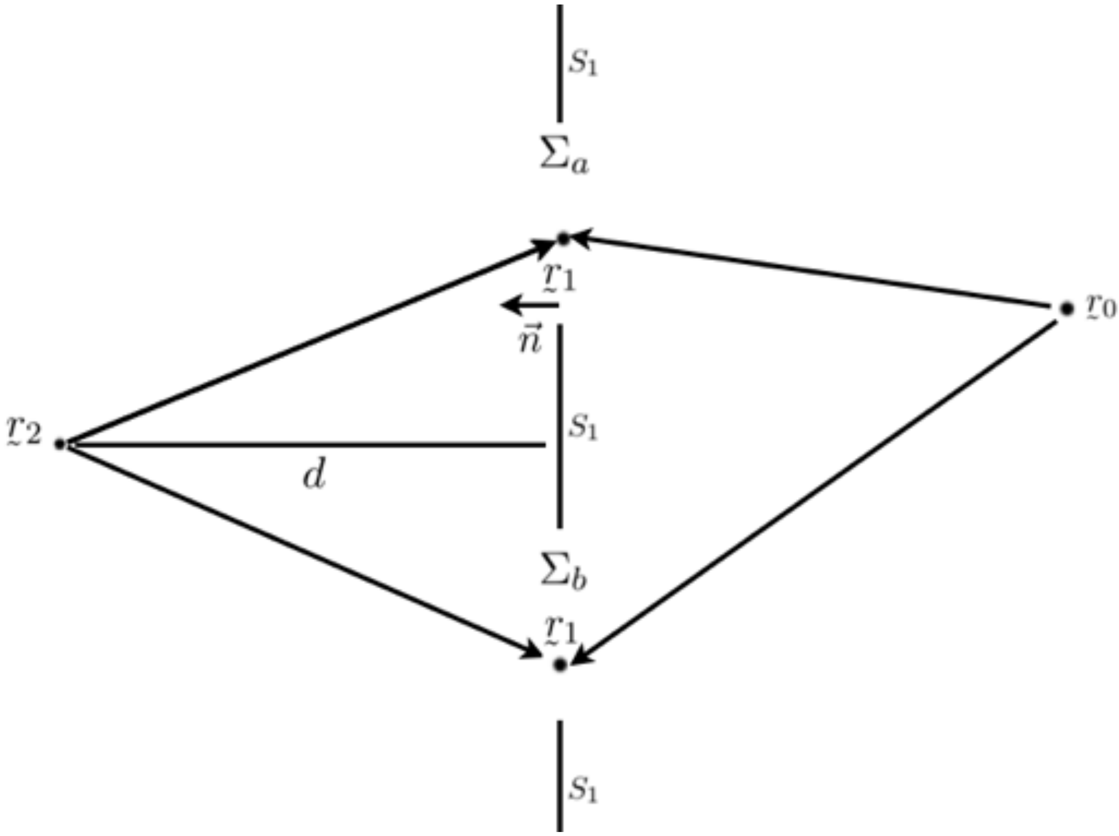

Figure 7: Two circular slits, $\Sigma_a$and Σ_b, with point source at $\underset{\sim}{r}_2$ which is at distance *d* from the slit and situated midway of the two slits; the wave function in the ambient environment on the left of the slit is diffracted on the right of the slit

The mathematical expression for the diffracted wave function in this two-slit experiment is almost the same as for the one-slit case except that the surface integral covers two slits rather than one. For large *d*, the diffracted wave function is well approximated by

$$\psi_p(\underset{\sim}{r}_0) = \frac{-ib}{(2\pi)^{3/2}} \int \int_{\Sigma_a,\Sigma_b} \frac{e^{ibd}}{d} \frac{e^{ib\,|\underset{\sim}{r}_1 - \underset{\sim}{r}_0|}}{|\underset{\sim}{r}_1 - \underset{\sim}{r}_0|} \left(\frac{1 + cos(\vec{n}, \underset{\sim}{r}_1 - \underset{\sim}{r}_0)}{2} \right) ds \qquad (\S 16)$$

where the Kirchhoff boundary conditions have been applied, the free space Green function, *G*, has been used and the integration is carried out on the two slit surfaces by varying $\underset{\sim}{r}_1$ and the corresponding element on it. The wave function at $\underset{\sim}{r}_0$ is the sum of the two diffracted wave functions corresponding to the two slits. Again, $|\nabla S|$ cannot be assumed to be constant in the diffraction region even though it is constant on the left side of the slits. As in the single slit case, the particle is free to surf on the *S* surfaces on the left or right side of the slits. Hence, it can either reach slit Σ_a or slit Σ_b (at any point of the two slits); and after going through one of the two slits, the particle surfs on the distorted hemispheric S surfaces (distorted by the diffraction) so that non-determinacy continues throughout the particle's trajectory. If the particle is detected by a screen on the right side of the slits, there is no way of telling which slit the particle has gone through, as Feynman pointed out in his lecture. Again, if we set the translational velocity $\underset{\sim}{v}_1$ and the velocity $\underset{\sim}{v}_2$ as before, then the third velocity component $\underset{\sim}{v}_3$ which is non-deterministic will have to be such that it will move the particle to satisfy or approximate the budgetary statistical constraint given by the probability density distribution. Obviously, one particle cannot fulfil this budgetary statistical constraint which can only be approximated by a large number of particles going through the two-slit system.

Regarding the interference pattern, since the Fresnel-Kirchhoff diffraction formula and its approximations are again known to produce interference pattern for the two-slit experiment in optical studies, the same kind of mathematical result applies here so that we can borrow the mathematical results of those formulae to affirm the interference pattern of the two-slit experiment of particles considered here.

Lastly, a brief discussion on the interference pattern is in order. (§16) can be written as

$$\psi_p(\underset{\sim}{r}_0) = R_a e^{iS_a} + R_b e^{iS_b}$$

where the first term on the right hand side corresponds to the contribution to the integral from slit Σ_a and the second term corresponds to the contribution from slit Σ_b. The usual approach in elementary textbooks to find the interference pattern on a detecting screen of a two-slit experiment is to assume that (i) $|\nabla S|$ and hence the diffracted wavelength are constant and (ii) R_a and R_b are equal; but these two assumptions in most cases are not absolutely valid. To find a point of zero intensity (or zero probability density), this approach looks for two paths to the same $\underset{\sim}{r}_0$ such that S_a and S_b differs by an odd number of π radians, i.e., they are completely out of phase. However, because of the above two assumed approximations or simplifications, the point of zero intensity identified by this approach is only an approximation and the actual intensity or probability density at such a point may well not be zero even though it may be small. For the probability density to be truly zero at r_0, (i) R_a and R_b have to be exactly equal *and* (ii) S_a and S_b are exactly and completely out of phase. (i) can be satisfied by appealing to symmetry, i.e., symmetry about the line from left to right midway between the two slits so that R_a and R_b are exactly equal. However, such symmetry also implies that S_a and S_b are exactly in phase, instead of exactly out of phase which is required for the wave function and probability density to vanish. The author suggests that it is highly improbable that these two conditions can be met exactly *and* simultaneously. Hence, it is highly improbable that there can be a point of absolutely zero intensity or zero probability density even though there can be points of minimal intensity whose value can be slightly above zero. This has important implications for the surfing motion of the particle on the *S* surfaces and the generation of the observed interference pattern. If there are some points or contours of truly zero probability density, then these nodal points or nodal contours are in theory out of bound for the particle's surfing motion on the *S* surfaces. In that way, the particle is not fully free to surf on the *S* surfaces to produce the full range of the interference pattern observed in experiments. However, our theoretical analysis suggests strongly that such restriction imposed by alleged nodal points or contours is highly improbable such that the particle is indeed fully free to surf on the *S* surfaces to produce the full range of the observed interference pattern. Numerical evaluation of (§16) can be carried out to confirm the theoretical analysis here but such a numerical endeavour is beyond the scope of this paper which is theoretical in nature.

What about the nulling effect on the interference pattern which comes from making observation on the particle to see which slit it goes through? Feynman discussed this strange phenomenon of observation.[7] A light source is placed on the right side of the slit wall to detect or observe which slit a particular electron goes through. If the electron passes through slit Σ_a, for example, then a photon scattered by the electron reaching our eye will allow us to see that the flash of light came from the vicinity of Σ_a instead of Σ_b, and vice versa. Hence, we know which slit the electron has passed through. And if we repeat this many

[7] R. Feynman, *The Feynman Lectures on Physics*, Vol. 3, Section 1-6.

times to build up the pattern in the detecting screen, we will see that the interference pattern which was there without observation disappears. And if we plot the distribution of electrons detected on the screen according to which slit they went through, the distribution pattern for each slit will be a localised distribution. It seems that the interference pattern is shy of such kind of observation and withdraws itself when the process of an electron passing through which slit is spied on. Feynman wrote,

> Anyway, the light exerts a big influence on the electrons. By trying to "watch" the electrons we have changed their motions. That is, the jolt given to the electron when the photon is scattered by it is such as to change the electron's motion ... [T]he effect of the photons being scattered is enough to smear out any interference effect.[8]

No doubt the motion of the electron is changed by its scattering of the photon and somehow this scattering smears out the interference pattern. Feynman indicated the change of momentum of the electron when he referred to 'the jolt given to the electron'. He went no further than this but we need to go further. More specifically, the electron's momentum has changed in the process of scattering the photon since there is an exchange of translational momentum between the electron and the photon in the scattering. Now, since the momentum of the electron has changed, the parameter in the Helmholtz Equation (§12), b which depends on ∇S and thus on the translational momentum, has also changed. This means that the diffracted wave function which applied before the scattering and corresponded to the original momentum and original b is no longer valid. *A new wave function satisfying the new Helmholtz Equation (§12) with the new value of b corresponding to the new momentum is necessary*. At this point, we recall that in free space there can be three modes of motion for a particle (at least those considered in this paper) – the cylindrical mode, the spherical mode and the intermediate mode. The spherical mode corresponds to diffraction by a slit of infinitesimal width, the intermediate mode corresponds to diffraction by a slit of finite width, while the cylindrical mode corresponds to no diffraction. When the electron scatters the photon, there is momentum exchange between them. The resultant momentum of the electron will be in a certain direction. If the resultant momentum is such that the electron is approaching a region with no slit 'in sight', i.e., it is not approaching any slit, the only possible mode of motion for the electron will be the cylindrical mode. And this will be the case if the electron is moving away from the two slits after the scattering. In the unlikely case where the electron has received such a strong jolt in scattering the photon that instead of

[8] *Ibid*.

moving away from the slits it is now moving back towards the slits, it will approach the slits in the intermediate mode of motion. Whether the particle will move in the cylindrical mode of motion or in the intermediate mode of motion, a new $|\nabla S|$ corresponding to the new momentum will apply so that the electron will have a new 'wavelength' after the scattering.

With the new momentum or new ∇S , the new parameter *b* for the Helmholtz equation (§12) is to be set by this relation: $b^2 = 2m(E-U)/\hbar^2 = (\nabla S)^2 - \dfrac{\nabla^2 R}{R}$. This means the new ∇S on its own is not sufficient to fix the parameter *b*, the quantum potential, $\dfrac{\hbar^2}{2m}\dfrac{\nabla^2 R}{R}$, is also required. Since immediately after the scattering the particle will not go into the spherical mode of motion (where the direction of ∇S is ambiguous), we should not use the zero value for the quantum potential which is valid for the spherical mode. This raises the question of the value of this quantum potential. It is proposed that *immediately after the scattering the electron goes into the cylindrical mode of motion and will continue in this mode if there is no slit in sight* but will later continue in the intermediate mode of motion if it is approaching one or more slits. For the cylindrical mode of motion immediately after the scattering, the value of *b* is set by the new ∇S and the quantum potential energy for cylindrical mode of motion in free space. The author suggested in paper [2] that the quantum potential energy in the cylindrical mode of motion in free space is a constant free parameter which needs to be input into the system in order to solve the Helmholtz equation in *R* (§7). Once *b* is set by this parameter and the new ∇S in the cylindrical mode, the Helmholtz Equation (§12) is solved solve for ψ_p; the solution of this can be found in Section 4.

One needs to justify further the input of the quantum potential energy as a parameter into the system. Evidently in the cylindrical mode of motion in free space where the particle's energy (*E*), ∇S , the kinetic energy and the potential energy (*U*) are constant; the quantum potential energy must also be a constant in free space since the particle's constant energy is the sum of the constant kinetic energy, constant potential energy and the quantum potential energy. In fact, the quantum potential energy can be considered as a universal constant in free space regardless of the magnitude of ∇S ; see its possible connection with dark energy in the Conclusion. Whether this connection is correct or not, the value of the quantum potential in free space as a constant has to be supplied as a parameter to yield the necessary parameter, *b*. *Once b is set, equation (§12) can be solved with the appropriate spatial boundary condition, whether the boundary condition involves slits or no slits*. If there is no slit, then the solution to (§12) corresponds to the cylindrical mode of motion. If there are one or more slits, then the intermediate mode of motion will ensue with the same parameter *b*. In whatever case it may be, the new wave function appears as soon as ∇S (or the translational momentum) changes due to the scattering with the measuring agent.

For the case of the electron moving away from the two slits after the scattering, no slit is in sight and the electron will move in the cylindrical mode of motion in free space. And since the probability density distribution on the flat *S* surfaces of the cylindrical mode of motion has the highest density at the centre (or central axis) of motion and generally decreases radially, when a large number of such electrons, after scattering, are detected on the detecting screen, one expects the particle intensity pattern on the detecting screen will be localised. And this localised pattern is consistent with the disappearing of the interference pattern when the electrons are observed by the use of the light source. Hence, the disappearance of the interference pattern is not due to it being shy of being spied upon; it has to do with the sudden change of the wave function into the cylindrical mode due to the sudden change of momentum of the electron when scattering occurs. In terms of timing, a new regime of motion is *immediately* necessary to satisfy the Helmholtz Equation (§12) with new parameter *b*. This understanding of scattering leads to the following understanding of measurement or observation.

7.0 The Meaning of Measurement

The above analysis prompts one to define measurement of a particle as an event where an exchange of momentum between the measured particle and the measuring agent takes place. In the above analysis, the example of the measured particle is the electron and the example of the measuring agent is the photon. Or one can reverse their roles, seeing the electron as the measuring agent and the photon as the measured particle. The precise naming of roles for the two particles is immaterial but what is important is the exchange of momentum between the two. Such exchange invariably entails a sudden change of the wave field, or equivalently the wave function, of the particle. This sudden change in the wave function in the process of measurement is not a collapse of the original wave function to a point, which is usually invoked in the Copenhagen Interpretation for describing measurement, but it is a wholesale and sudden change of the wave function over the span of the relevant physical space necessitated by the exchange of momentum which takes place in measurement. The former evolution of the wave function in the Schrödinger equation is suddenly jolted by the sudden (or discontinuous) insertion of a new ∇S , a new momentum and therefore a new *b* into the system – the new evolution of the wave function according to the Schrödinger equation takes place according to the new *b*. The parameter *b* influences the spatial part of the wave function through (§12) while *E* determines the temporal evolution of the wave function ($\frac{\partial S_t}{\partial t} = -E/\hbar$). If there is also an exchange of energy in addition to the exchange of momentum in the process of measurement, then the particle's energy will also change. In that case, one may say that instead of the wave function collapsing discontinuously to a point, at the moment of measurement the wave function changes discontinuously over the physical space and its phase evolves in time at a different rate according to the new energy. And as long as there is no further energy and momentum exchange (i.e., no measurement) in the meantime, since the time derivative of ∇S is zero according to (§6), ∇S continues in time without change. Hence, the conservative Steady Motion State, where the time derivative of ∇S is zero, always applies when the system is not disturbed by any measurement. ∇S can only be changed by a sudden exchange of momentum, or by a change of *U* in time which is non-uniform over space (see later). Therefore, the conservative Steady Motion State, which has been the focus of this and the previous paper, is widely applicable, and the event between two distinctive and successive Steady Motion States is either the event of measurement or a non-uniform change in *U* as will be seen later.

This above way of understanding measurement is entirely self-consistent and logical – it does not involve the mysterious sudden collapse of the possible states into a single state as in the Copenhagen Interpretation and therefore it does not have the challenge of the paradox of the Schrödinger's cat. It is also consistent with what we know of a measuring process where the measured particle is disturbed by a measuring agent. Such a measuring process may give us some information about the measured particle, e.g., its position regarding which of the two slits the particle went through, but there is much information of the measured particle which cannot be recovered by the measuring process, e.g., its original momentum and original energy.

An interesting question which follows from the above analysis arises. If some measuring equipment or agent is inserted into the vicinity of a particle but there is no exchange of momentum and energy between the particle and the measuring equipment (because there is no direct contact or scattering between them), will the particle 'feel' the presence of the measuring equipment which has been newly inserted into its vicinity? If we think about the insertion of the measuring equipment into the vicinity of a particle, which is otherwise free, as a change in the the classical potential, *U*, in a limited area in the particle's vicinity, then we expect this change of *U* in a limited area to have effect on the wave function of the particle even when there is no exchange of momentum and energy between the particle and the measuring equipment. This can be confirmed rigorously as follows.

In a conservative system *E* is constant and $\frac{\partial \nabla S}{\partial t} = 0$, i.e., the energy and the kinetic energy field do not change in time. If *U* changes in time which is permissible in a conservative system, it seems that the only way to accommodate this change in the classical potential is to have the quantum potential and hence *R* change in time (see (§5)). But in Section 3 by assuming $R = g(t)I(\underset{\sim}{r})$, $\frac{\nabla^2 R}{R} = \frac{\nabla^2 I}{I}$, it was shown that *R* cannot change in time in a conservative Steady Motion State, i.e., g has to be a constant. This is reasonable when *U* is not changing in time. For *U* changing in time, let us *suppose* that *R* is more complicated and more exotic than the above assumed separable form so that it can be a function of time and satisfy the energy balance as *U* changes in time. Let $R = R_1$ before *U* changes; R_1 is independent of time. In that case, the pseudo continuity equation (§3) becomes

$$Div\left(R_1^2 \nabla S\right) = 0 \ .$$

Let $R = R_2$ after *U* has finished changing; R_2 is independent of time. In that case, the pseudo continuity equation (§3) becomes

$$Div\left(R_2^2 \nabla S\right) = 0 \ .$$

Since *R* changes as *U* changes, $R_1 \neq R_2$. Now, since $\frac{\partial \nabla S}{\partial t} = 0$, ∇S has not changed throughout the time when *U* was changing for a certain period. This implies that the same ∇S is compatible with R_1 and R_2 in satisfying the above two divergence equations. However, since the change in *U* can be quite arbitrary, R_2 can take a great variety of form so that in general the second divergence equation involving R_2 will not be satisfied, even though we can be sure that the first divergence equation involving R_1 is satisfied. But the two divergence equations, corresponding to the time before *U* changes and the time after *U* has changed, have to be satisfied. In view of this, we have to conclude that $R_1 = R_2$, i.e., *R* has not changed in time despite *U* has changed in time. Hence, we have a contradiction against our presupposition that there can be a *R* which can change in time and has changed in time as *U* has gone through a period of change in time. Therefore, *R* has not changed in time.

The above conclusion about the constancy of *R* and ∇S implies that the quantum potential remains unchanged while the kinetic energy also remains unchanged in a conservative regime even when the classical potential *U* changes in time. But this means that

$$E \equiv \frac{\hbar^2}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + U$$

is not conserved in a conservative system! We have a formidable contradiction. However, what we have learnt about measurement above will help us to resolve this contradiction. One of the assumptions above is that ∇S has not changed throughout the time when *U* was changing. However, this assumption can be relaxed to avoid the contradiction. Since *E* must remain constant in a conservative system, when *U* changes in time, ∇S and *R* instantaneously adjust to the new *U* to maintain constant *E*. This adjustment cannot be made via the time integration of the Schrödinger equation because, as we have seen, this equation does not allow ∇S and *R* to change in a conservative system. Therefore, the changes in ∇S and *R* are like the corresponding changes in ∇S and *R* when the particle exchanges momentum with another entity in scattering which is a form of measurement – i.e., these changes happen outside the remit of the Schrödinger equation. We are thus led to conclude that the remit of the Schrödinger equation only covers those conservative systems where the momentum and the classical potential do not change in time. When the momentum or the classical potential changes within a period of time, however short or long, the Schrödinger equation has nothing to say during this period, even though what happens after the period of change is within its remit since the momentum and the classical potential remains settled and constant after the period of change.

We can summarise the above discussion as follows. When a particle is measured through an exchange of momentum with the measuring agent, the wave function with its ∇S and *R* is jolted into a new regime with a new momentum. But before the measurement and the exchange of momentum take place, the very insertion of any measuring equipment into the vicinity of the particle generates a change in the classical potential in the localised area occupied by the measuring equipment so that this insertion in itself already moves the original regime (or wave function) to a new regime before any event of measurement. The presence of the measuring equipment in the vicinity of the particle is sufficient to influence the behaviour and regime (or wave function) of the particle. If this presence of the measuring equipment is further followed by an actual event of measurement, the behaviour and regime of the particle will be changed again. Hence, it is virtually impossible to know precisely the behaviour and regime of the particle before the measuring equipment is introduced into the particle's vicinity.

It has to be noted that the change in the classical potential has to be a localised one for the behaviour and regime of the particle of a conservative system to change. We first look at the opposite situation, i.e., the classical potential changes from U to $U + \triangle U$ where $\triangle U$ is constant in space, i.e, the change is not localised. This wholesale uniform change of classical potential over all space amounts to setting a different reference point as the level of zero potential. And since the reference point of such a potential is arbitrary, one should not allow the change of reference point to affect the dynamics of the system. The dynamics of the system can be safeguarded by changing the particle's energy from E (with respect to the original potential reference point which gives the potential as *U*) to $E + \triangle U$ (with respect to the new potential reference point which gives the potential as $U + \triangle U$). $E + \triangle U$ is constant in space and therefore satisfies the requirement of the constancy of energy in space for a conservative particle. The energy relations with respect to the original reference point (with subscript 1) and the new reference point (with subscript 2) are:

$$\frac{\hbar^2}{2m}(\nabla S_1)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R_1}{R_1} = E - U$$

$$\frac{\hbar^2}{2m}(\nabla S_2)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R_2}{R_2} = (E + \triangle U) - (U + \triangle U) = E - U \ .$$

These two equations are exactly the same except the subscripts. Their accompanying pseudo continuity equations involving *R* and *S* are also exactly the same except the subscripts. The solution for *R* and *S* in one set of two equations (energy and pseudo continuity) is identical to the solution of the other set. Hence, the dynamics is safeguarded and there is no change to the conservative system despite the wholesale uniform change in the potential. However, the same cannot be said of the case where the change in potential is not uniform across the whole physical space. In that case, one cannot change the energy of the particle from *E* to $E + \triangle U$ (as in the case of uniform potential change) since $E + \triangle U$ is not constant in space and the energy of a conservative particle must be constant in space. In such a case of non-uniform (or localised) change in potential, the energy of the particle should remain as *E* and such a non-uniform localised change in potential should effect a change in the wave function (with its *R* and *S*) as described above.

In practice, the only possible change to the classical potential is the non-uniform localised one since it is pointless and impossible to impose a blanket uniform change of the classical potential over the whole physical space. Any insertion of a measuring equipment into the vicinity of the particle to be measured will therefore introduce a non-uniform localised change in classical potential which will inevitably change the behaviour and regime of the particle, even before any measurement has taken place.

Finally, if the localised change in classical potential happens discontinuously, e.g., through an abrupt or impulsive introduction of a measuring equipment, the wave function field will also change accordingly, instantaneously and thus discontinuously. That is, there will be no time lag in the wave function's response to the sudden change in the classical potential. Often when a dynamical system is disturbed from its equilibrium by an additional forcing, there will be a time lag before the system's response settles into a new equilibrium state. However, in a conservative quantum system which goes through a local change in the classical potential such as discussed above, and in a non-conservative quantum system which is disturbed by measurement, the responses in the wave function in both cases are immediate with no time lag. This brings to mind the perplexing problem of entangled particles. When one of the two particles is measured, there is a momentum change which triggers the wave function to jump into a new regime with no time lag. Further research is necessary to explore the relationships between measurement, instantaneous transformation of the wave function and entanglement but this is beyond the scope of this paper.

8.0 Photons and Particles in Special Relativistic Framework and the Definition of Energy

It is known that the three components of the electric fields and the three components of the magnetic field in the Maxwell equations all satisfy the equation

$$\nabla^2\psi - \frac{n^2}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

if the propagation medium is linear, isotropic, homogeneous and non-dispersive, where n is the refractive index of the medium and n=1 if the medium is vacuum. For this paper, we are only interested in the case of vacuum medium; hence the following wave equation will be used:

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0\ . \tag{§17}$$

This is the same as the following Gordon-Klein equation if the parameter *m,* the rest mass, in that equation is set to zero:

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \frac{m^2c^2}{\hbar^2}\psi \tag{§18}$$

The Gordon-Klein equation is the quantised version of Einstein's famous equation,

$$E^2 = m^2c^4 + p^2c^2 \tag{§19}$$

where *p* is treated as $-ih\nabla$ and *E* is treated as $ih\frac{\partial}{\partial t}$. The Gordon-Klein equation with rest mass set to zero is identical to the wave equation describing the six components of the Maxwell equations. In this sense, *the Gordon-Klein equation is valid for describing the behaviour of photons*. That is, with *m* aptly set to zero for massless photon it is saying no more and no less than the wave equation (§17) which describes the behaviour of photons. But according to conventional wisdom, the Gordon-Klein equation cannot incorporate a positive definite probability density; hence, it appears that it is not suitable for describing the particle-like behaviour of photons. The same can be said of the capacity of the Gordon-Klein equation to describe particles in a special relativistic framework.. But we have just arrived at the clear conclusion that the Gordon-Klein equation is valid for describing the behaviour of photons. This means that conventional wisdom has been challenged by the above logical conclusion and this calls for a questioning of conventional wisdom. Section 8.1 will indeed rigorously show that there is a proper positive definite probability density for the Gordon-Klein equation. Hence, it will be shown that this particular hurdle for using the Gordon-Klein equation for describing photons and particles can be overcome. However, there is a second hurdle – according to conventional wisdom the Gordon-Klein equation can only describe spinless particles while photons have spin 1 and other particles can have non-zero spins. Neither the Gordon-Klein equation in itself nor the Schrödinger equation in itself contains sufficient information to determine the spin angular momentum of a photon or particle. However, as demonstrated in Paper [2] and mentioned above, additional constraints are necessary to determine the angular momentum. In Section 8.3, this demonstration will be extended to the relativistic case where critical questions will be raised about Dirac's theory.

If the two hurdles mentioned above can be overcome with confidence, then one can conclude that the Gordon-Klein equation with the additional constraints can indeed describe both particles and photons including their spins, albeit with different values for the parameter *m*, the rest mass (zero for the photon case). From this fundamental sharing of the same governing equation by particles and photons, we will be able to appreciate better the similarities between the behaviour of particles and the behaviour of photons in a relativistic quantum mechanical framework, including their similarities in diffraction which will be demonstrated in Section 8.2. Indeed, because of the fundamental sharing of the same governing equation, we should expect such similarities between particles and photons.

8.1 Probability Density, the Translational Velocity, and Other Relations for the Gordon-Klein Equation

We will use the Gordon-Klein equation with the rest mass parameter, *m*, and will set *m* to zero for the case of photons. Again writing $\psi = Re^{iS}$ and substituting this into (§18), we have two equations corresponding to the real and imaginary parts of ψ. The equation corresponding to the real part concerns the energy of the particle or photon and the equation corresponding to the imaginary part is the pseudo continuity equation. Both equations have their counterpart in the non-relativistic case and will be considered here. Firstly, for the real part,

$$\hbar^2\left[\left(\frac{\partial S}{\partial t}\right)^2 - \frac{1}{R}\frac{\partial^2 R}{\partial t^2}\right] = c^2\hbar^2(\nabla S)^2 + m^2c^4 - c^2\hbar^2\frac{\nabla^2 R}{R}\,. \qquad (\S 20)$$

Using our insights gained from the non-relativistic case, we are looking for an energy-like expression which will be conserved by a particle or photon in the absence of exchange of energy with other entities. The r.h.s. of (§19) has dimension energy squared which is not surprising since the Gordon-Klein equation is the quantised version of (§19) which has dimension of energy squared. For particles, the last term on the r.h.s. of (§20) is the quantum potential multiplied by $2mc^2$, i.e., twice the particle's rest energy. Taking the gradient of (§20), we have

$$\hbar^2\nabla\left(\left(\frac{\partial S}{\partial t}\right)^2 - \frac{1}{R}\frac{\partial^2 R}{\partial t^2}\right) = \nabla\left(c^2\hbar^2(\nabla S)^2 + m^2c^4 - c^2\hbar^2\frac{\nabla^2 R}{R}\right)\,. \qquad (\S 21)$$

Again using our insights gained from the non-relativistic case,

$$K^2 \equiv c^2\hbar^2(\nabla S)^2 + m^2c^4 - c^2\hbar^2\frac{\nabla^2 R}{R} \qquad (\S 22)$$

has dimension of energy squared and could be a conserved quantity of the particle or photon if the l.h.s. and the r.h.s. of (§21) are zero. If we take the time derivative of *R* to be zero as in the non-relativistic case, and $S = -\frac{K}{\hbar}t + S_p$ similar to the non-relativistic case (with the constant *K* replacing *E* and with S_p a function of space only), then the l.h.s. of (§21) is indeed zero. Furthermore, again $\dfrac{\partial \nabla S}{\partial t} = 0$ and (§20) reproduces the expression for K^2 in (§22). For the quantity in (§22) to be a conserved quantity for the particle or photon, it is necessary that its time derivative is also zero. Since $\dfrac{\partial \nabla S}{\partial t} = 0$, this requirement is ensured if $\nabla^2 R / R$ is also independent of time. This independence of time has been argued for in the conservative non-relativistic case (see Section 3 above) and the same argument can be made for the relativistic case. Indeed, this has already been assumed above when we stated that the time derivative of *R* is zero as in the non-relativistic case. In summary,

$$S = -\frac{K}{\hbar}t + S_p \, , \ \frac{\partial \nabla S}{\partial t} = 0 \, , \ \frac{\partial R}{\partial t} = 0 \qquad (§23)$$

is a set of consistent conditions which yield the conserved quantity of K^2 (of dimension energy squared) for a particle or photon. (§23) is the relativistic equivalence of (§6). Again, note that *K* has replaced *E* of (§6). Conditions other than (§23) are mathematically possible for the Gordon-Klein equation; however, they are highly unlikely to correspond to any real physical phenomenon in our universe as they will be non-conservative in terms of K^2.

If we follow the lead from the non-relativistic case and identify $\hbar \nabla S$ as the translational momentum, then the translational momentum of a particle for the relativistic case is

$$\underset{\sim}{p} \equiv \gamma m \underset{\sim}{v}_1 = \hbar \nabla S$$

where γ is the Lorentz factor. For the translational momentum of a photon necessarily in the relativistic case, we have no need to reference the rest mass and write $\underset{\sim}{p} \equiv \hbar \nabla S$. From this definition of translational momentum for particle *and* photon as $\hbar \nabla S$, we see that

$$K^2 = p^2c^2 + m^2c^4 - c^2\hbar^2\frac{\nabla^2 R}{R} = E^2 - c^2\hbar^2\frac{\nabla^2 R}{R} \, . \qquad (§24)$$

Here, we see that this quantity *K* (with *E* as defined by Einstein in (§19)) is conserved by the particle or photon. In addition, the form of the Gordon-Klein equation being used assumes that the classical potential is uniformly zero. And this is reflected in the definition of K^2 where no classical potential is involved. This is not a problem if we are interested in particles or photons in free space. (It is instructive to compare the conserved quantity of K^2 for the relativistic case with the conserved quantity in (§5) for the non-relativistic case. The similarity between the two is evident.)

For the imaginary part of the Gordon-Klein equation, we have

$$\frac{\partial}{\partial t}\left(R^2\frac{\partial S}{\partial t}\right) - c^2 Div(R^2\nabla S) = 0 \ . \qquad (§25)$$

This pseudo continuity equation for the relativistic case can be written as

$$\frac{\partial(R^2)}{\partial t} + \frac{c^2\hbar}{K}Div(R^2\nabla S) = 0 \ . \qquad (§26)$$

The general continuity equation is satisfied if we set

$$\rho = R^2 \ , \ \underset{\sim}{v} = \frac{c^2\hbar}{K}\nabla S \ .$$

Note that the probability density as defined here is positive definite. (de Broglie derived a similar probability density and continuity equation for the Gordon-Klein equation but he did not exploit the constant value of $\frac{\partial S}{\partial t}$.)[9] The usual approach to finding an expression for the probability density from the Gordon-Klein equation is to subtract the conjugate equation (times ψ) from the original equation (times conjugate of ψ) so that something like a continuity equation emerges and a possible probability density is identified. It is often assumed that the identified probability density cannot be positive definite. However, if we write $\psi = Re^{iS}$ and $\psi^* = Re^{-iS}$, subtracting the conjugate equation multiplied by the wave function from the original Gordon-Klein equation multiplied by the conjugate wave function, and *taking the forms of S and R in (§23) for a conservative system*, will produce (§26) so that there can be a positive definite probability density derived from the Gordon-Klein equation.

[9] Louis de Broglie, *Non-linear Wave Mechanics : a causal interpretation,* p.38.

Since *R* does not depend on time, from (§26) $Div(\rho\nabla S) = 0$, the velocity can be set as ∇S multiplied by any constant and the general continuity equation, $\frac{\partial\rho}{\partial t} + Div(\rho \underset{\sim}{v}) = 0$, will be satisfied. The velocity and momentum of a particle can be written as

$$\underset{\sim}{v} = \beta\nabla S \ , \ \underset{\sim}{p} = m\gamma\underset{\sim}{v} = m\gamma\beta\nabla S$$

where β is a constant and γ is the Lorentz factor which is dependent on the magnitude of $\underset{\sim}{v}$ and therefore not a constant. If we take seriously de Broglie's relation for momentum and wavelength,

$$\underset{\sim}{p} = \frac{h}{2\pi}\nabla S = \hbar\nabla S$$

where $\lambda|\nabla S| = 2\pi$ and λ is the wavelength, we see that this contradicts the previous expression for momentum since $m\gamma\beta \neq \hbar$ because γ in general is not a constant while $\hbar$ is. This means that if de Broglie's relation holds, the velocity cannot simply be ∇S multiplied by a constant, except in the case where γ is one, i.e., the case of non-relativity, or the very special relativistic case where *the magnitude of* ∇S*, and hence the magnitude of* $\underset{\sim}{v}$ *and* γ *, are constant*. This very special relativistic case only happens in the cylindrical or the spherical mode (see below), and is not expected to hold for the intermediate mode for diffraction.

Because of the above difficulty in using the expression for the velocity derived from the pseudo continuity equation to satisfy the general continuity equation, there is now a question whether the general continuity equation can be or needs to be satisfied. In paper [1] which deals with the non-relativistic case, it was indeed suggested that this equation needs to be satisfied and can be satisfied. However, the situation for the relativistic case is not as simple as that paper suggested and it will be shown below that in the relativistic case, the satisfaction of this equation will involve some unphysical constraint on the system.

The expression for the velocity above contradicts de Broglie's relation for momentum and wavelength of particles. A natural second attempt is to begin with this relation for particles and see if this will lead to the satisfaction of the general continuity equation. Throughout this section we are interested in the translational velocity, i.e., not the surfing velocity, so we set $\underset{\sim}{v} = \underset{\sim}{v}_1$. In this case, $\underset{\sim}{p} = \frac{h}{2\pi}\nabla S = \hbar\nabla S$ and

$$\underset{\sim}{v}_1 = \frac{\hbar}{m\gamma}\nabla S$$

where γ is again the Lorentz factor which is dependent on the magnitude of $\underset{\sim}{v}_1$, the translational velocity (γ is reduced to 1 for the non-relativistic case and the expression for $\underset{\sim}{v}_1$ in that case is recovered). Since *R* does not depend on time, the general continuity equation becomes

$$Div(\rho\nabla S/\gamma) = \rho\nabla S \cdot \nabla(1/\gamma) = 0$$

where the pseudo continuity equation, $Div(\rho\nabla S) = 0$, has been used. This means that γ is constant along ∇S. Since γ is a function of the magnitude of $\underset{\sim}{v}_1$, this in turn implies successively that

1. $m\gamma|\underset{\sim}{v}_1|$ is constant along ∇S
2. $\hbar|\nabla S|$ is constant along ∇S
3. $|\nabla S|$ is constant along ∇S
4. the magnitude of $\underset{\sim}{v}_1$ is constant along ∇S.

Even though this time we begin with de Broglie's relation for momentum and wavelength, we still end up with a similar constraint, i.e., the constancy of the magnitude of the velocity along ∇S , for satisfying the general continuity equation. Of course, in the non-relativistic case, γ is one and such a constraint is not necessary to satisfy the general continuity equation.

For photons, the translational velocity is written as

$$\underset{\sim}{v}_1 = \frac{c}{|\nabla S|}\nabla S \ , \ \alpha_1 \equiv \frac{c}{|\nabla S|}$$

to maintain the speed of the photon to be *c* and the direction to be given by ∇S. Since *R* and *ρ* are independent of time, the general continuity equation is $Div\left(\rho \underset{\sim}{v}\right) = 0$ while the pseudo continuity equation is $Div(\rho\nabla S) = 0$; if we set $\underset{\sim}{v} = \underset{\sim}{v}_1$, the general continuity equation becomes

$$Div(\rho\alpha_1\nabla S) = \rho\nabla S\cdot\nabla\alpha_1 + \alpha_1 Div(\rho\nabla S) = \rho\nabla S\cdot\nabla\alpha_1 = 0$$

which means that $|\nabla S|$ in the direction of ∇S does not change.

We have now attempted to use three velocity expressions to satisfy the general continuity equation. The resulting constraint is either the magnitude of ∇S has to be constant throughout the whole domain or $|\nabla S|$ is constant along ∇S. These constraints can only be relieved in the non-relativistic case for particles where γ is one. In the relativistic case, these constraints can be met only in the cylindrical or the spherical mode of motion, but they are not expected to be met for the intermediate mode for diffraction. This means that in the relativistic framework, in general the constraints will not be met and the general continuity equation may not need to be satisfied. We have the following three reasons to further support this.

Firstly, the general continuity equation is about the conservation of probability but the conservation of probability has already been expressed through the following normalisation constraint: the integrated probability density over the whole space domain must be one. We therefore have reason to believe that we do not need another equation for the conservation of probability. Secondly, the general continuity equation

$$\frac{\partial\rho}{\partial t} + Div\left(\rho\underset{\sim}{v}\right) = 0$$

when applied to the conservation of probability involves a flux term $\rho\underset{\sim}{v}$ – the probability flux or the probability current. The physical meaning of this term is far from clear. Can probability be carried around by the velocity field like mass, heat or electric charge? While mass, heat and electric charge can be carried around in physical space, what does it mean for some probability of a particle to be carried around in physical space? The notion of the probability of a particle being carried around in physical space according to the general continuity equation is akin to the particle being 'dissolved' in physical space with a certain distribution of concentration in space (corresponding to the probability density distribution) so that at each point in space there is no net source or sink of any part of the 'dissolved' particle. This kind of advection model normally applies to some dissolved solute advected by the velocity of the solvent, or the heat carried by some medium and so on. But this advection model does not apply to a discrete particle. This and the previous papers see a particle to be at one definite place at any one time. It is not present in all of the physical space in the form of a dissolved solute to be carried around (or advected around) as suggested by the general continuity equation. To summarise the second point, the probability of a particle is not a material entity like mass, heat or charge which can be moved around; hence the application of the general continuity equation in quantum mechanics is highly questionable. Thirdly, the general continuity equation is a point-wise constraint, i.e., it has to be satisfied at every physical point in the whole space domain, while the normalisation constraint is a bulk constraint involving integrating the probability density over the whole space domain. It seems that a point-wise constraint will be over-constraining and will unphysically limit the system, as seen in the above very restrictive constraints. All these three reasons question the status, and therefore the applicability, of the general continuity equation in quantum mechanics. Nevertheless, the pseudo continuity equation still applies because it is derived from the Gordon-Klein equation which is the governing equation.

In the non-relativistic case, it was shown in paper [1] and above that the over-constraining by the general continuity equation does not happen as the pseudo continuity equation and a suitable choice of the translational velocity easily ensures that the general continuity equation is satisfied. And this gave the author (and perhaps other authors) the false impression that the general continuity equation ought to be satisfied (see paper [1]). However, as has been shown above, in the relativistic case the general continuity equation overstrains the system. For these reasons, we have good ground to relieve a quantum system of the unnecessary constraint of the general continuity equation but we ensure the probability conservation by using the usual normalisation constraint involving the volume integral which therefore is a bulk or budgetary constraint, not a stringent point-wise constraint like the general continuity equation. Furthermore, since the possible scenarios we are considering here are those where *R* and therefore the probability density does not vary with time, once the normalisation constraint is applied to set the magnitude of *R* at any one time, the conservation of the summed probability density of the particle is ensured at all times. And if the *R* distribution is transformed by a measurement event or by the intrusion of a localised potential, the normalisation constraint will also apply to the new *R* distribution and the conservation of the summed probability density of the particle will continue to be ensured.

This is a suitable point to re-think how the velocity components in the relativistic case relate to the governing equations, (§20) and (§25), and the normalisation constraint (the consideration for the non-relativistic will be virtually identical, as will be seen). (§20) and (§25) are equations in *R* and *S* and they do not in any way provide direct prescription of any velocity component. In different possible universes where these equations hold, there can be different prescriptions of the velocity components which are based on some possible functions of *R* and *S,* and the behaviour of particles and photons will be very different in these universes. *R and S are the basic fields from which many different kinds of velocities can be constructed and the governing equations can have no objection to these constructions.* Some universes will be extremely chaotic because of the way the velocities are prescribed from *R* and *S*. In our orderly universe, we have *observed* that for both particles and photons,

$$p = \frac{h}{\lambda} = \frac{h|\nabla S|}{2\pi} = \hbar|\nabla S| \ .$$

Since we are dealing with a momentum derived with reference to ∇S , the velocity component involved should be in the direction of ∇S, so that we are not dealing with the surfing velocity on the *S* surface. Hence, we have called this velocity component the translational velocity, $\underset{\sim}{v}_1$. For particles, the above expression for the momentum corresponds to $\underset{\sim}{v}_1 = \dfrac{\hbar}{m\gamma}\nabla S$. For the non-relativistic case, γ the Lorentz factor is one and $\underset{\sim}{v}_1$ is in the same form as that derived by using the pseudo continuity equation to satisfy the general continuity equation. However, this does not make the derivation using the two continuity equations valid for the following reason. The two continuity equations will always yield the same form for $\underset{\sim}{v}_1$. In another universe where $\underset{\sim}{v}_1$ has a different form, e.g., the factor multiplying ∇S is not a constant, the derivation using the two continuity equations will be seen as misguided. In any case, in the relativistic universe like ours, such derivation using the two continuity equations has been shown to be inappropriate. Here, instead of deriving the form of $\underset{\sim}{v}_1$ from the two continuity equations, we have derived the form for $\underset{\sim}{v}_1$ *from observation in our universe* so that we have confidence that this form is appropriate for this universe. And for photons, we have suggested that the translational velocity is given by $\underset{\sim}{v}_1 = \frac{c}{|\nabla S|}\nabla S$ based on our observation that photons travel at the constant speed of *c*. Again, the two governing equations, (§20) and (§25), and the normalisation constraint, can have no objection to it. What this paragraph seeks to show is that *the two governing equations and the normalisation constraint are not sufficient to close or determine a quantum system*. The two governing equations and the normalisation constraint constitute a very open system. For this open system to become less open in order to describe our universe, more constraints are required. Evidently, the necessary constraints are the constraints on the velocity components since the open system only prescribes *R* and *S* but does not prescribe any velocity component. Hence, we have chosen the above form for the translational velocity, $\underset{\sim}{v}_1$, based on our *observation* in this universe. Yet, there are still two velocity components together constituting the surfing velocity on the *S* surface which is perpendicular to the translational velocity and thus cannot be captured by this translational velocity. These two velocity components again are not provided by the governing equations in *R* and *S* so that in different universes where the same equations apply, these velocity components can be very different and the particle or photon behaviour will be very different to those in our universe. In Section 8.3, based on the constant particle spin in the non-relativistic framework, we will suggest the forms of the two surfing velocity components which will accord with the *observed* particle spins in this universe while maintaining a sufficient degree of non-determinacy. In sum, *we are seeking to find prescriptions for the three components of the velocity according to the observations we have been able to make in our universe.* Surely, this is a robust approach for prescribing the velocity components.

It is instructive to look at some relations involving the energy of a particle and the energy of a photon (the relation between momentum and wavelength has already been given).

From (§19), for photon, *E=pc*; hence,

$$E = c\hbar|\nabla S| = \frac{2\pi}{\lambda}c\hbar = \frac{ch}{\lambda} \tag{§27}$$

where c/λ is the frequency. This is the Planck-Einstein relation. In this form, it is readily seen that the energy of the photon is inversely proportional to its wavelength but proportional to its frequency. The same cannot be said of particles since in that case the mass in (§19) is non-zero. For particles in free space, according to (§23),

$$K = -\hbar\frac{\partial S}{\partial t} = -h\frac{\partial(S/(2\pi))}{\partial t} = hf \tag{§28}$$

where $f \equiv -\dfrac{\partial(S/(2\pi))}{\partial t}$ is the frequency and is constant since *K* is constant. Hence, this relation for particles is not in the same form as the Planck-Einstein relation for photons as de Broglie suggested since *h* times frequency is not equal to *E* but *K*. From (§24), in the relativistic framework one can see that *E* is not a constant while *K* is so that *E* does not fit (§28) which is an equation involving constants only. However, *K* and *E* are very close to one another according to (§24):

$$K = E\left(1 - \frac{c^2\hbar^2}{E^2}\frac{\nabla^2 R}{R}\right)^{\frac{1}{2}} = E\left(1 + \frac{QP}{\gamma E} + \cdots\right) = E + \frac{QP}{\gamma} + \cdots$$

where γ is the Lorentz factor, *QP* stands for the quantum potential and is expected to be small compared to *E*. In (§28), *K* can only be replaced by *E* if the quantum potential is zero but this is only true in rare cases, e.g., in the spherical mode.

In this section, based on the two equations derived from the real and imaginary part of the wave function in the Gordon-Klein equation, we have set the probability density to be R^2 which is positive definite. We have set an appropriate form for the translational velocity by invoking the observed relationship between momentum and wavelength, rather than by using the pseudo continuity equation to satisfy the general continuity equation whose status has been put to doubt. The relation between energy, wavelength and frequency for photons has been derived. Contrary to de Broglie, a slightly different relationship between energy and frequency has been derived for particles.

8.2 Diffraction in Special Relativistic Framework

For the special relativistic framework, the Gordon-Klein equation has been shown above as valid for describing both the behaviour of photons and the behaviour of particles, at least as far as the translational velocity is concerned. For the photon case, *m* as a parameter of the Gordon-Klein equation is appropriately set to zero. To consider diffraction of particles and photons in the framework of special relativity, we need to work out the deterministic translational velocity of the particle or photon in the slit experiments (with one slit and two slits) and explore the possible surfing velocity on the *S* surfaces which crucially render the system non-deterministic. This requires solving the Gordon-Klein equation with the appropriate boundary conditions. The wave function can again be written as

$$\psi = Re^{iS_p}e^{iS_t}$$

where $S = S_p + S_t$, $S_t = -\frac{K}{\hbar}t$, i.e., the phase is split into the spatial part denoted by *p* and the temporal part denoted by *t*. Writing $\psi_p = Re^{iS_p}$ and substituting this into the Gordon-Klein equation, we have the Helmholtz equation:

$$\nabla^2\psi_p + b^2\psi_p = 0 \tag{§29}$$

where $b^2 = \frac{K^2 - m^2c^4}{c^2\hbar^2} = (\nabla S)^2 - \nabla^2 R/R$ according to (§22) is constant. This Helmholtz equation is exactly of the same form as (§12) which is for the non-relativistic case and b^2 has the same expression in *S* and *R* in both the non-relativistic and the relativistic cases. This is very convenient and means that the ψ_p part (spatial part) of the solution for the relativistic case is the same as that of the non-relativistic case. This applies to the cylindrical mode, the spherical mode and the intermediate mode for diffraction where the boundary conditions at the slit and slit wall have to be taken into account. (§29) is valid for both particles and photons and it is rather convenient that the mass parameter, *m*, does not feature in the equation at all since b^2 can be written purely in terms of *S* and *R* which can be considered as the dynamic variables of the system while *m* can be considered as a static parameter which does not feature in the Helmholtz equation, as in the non-relativistic case.

In both the relativistic and non-relativistic case, the translational momentum has the same expression, $\underset{\sim}{p} = \frac{h}{2\pi}\nabla S = \hbar\nabla S$. In this section, we concentrate on the relativistic case for particles and photons. The translational momentum of particles and photons have this same form of $\hbar\nabla S$ but their velocities have different expressions. The expression for the translational velocity of photon has already been given as $\underset{\sim}{v}_1 = \frac{c}{|\nabla S|}\nabla S$. For particles, the translational velocity is $\underset{\sim}{v}_1 = \frac{\hbar}{m\gamma}\nabla S$ (where the non-relativistic form can be recovered by setting γ to one).

With reference to Figure 7 for the two-slit experiment, for large *d* the mathematical expression for the diffracted wave function satisfying (§29) is well approximated by

$$\psi_p(\underset{\sim}{r}_0) = \frac{-ib}{(2\pi)^{3/2}} \int\int_{\Sigma_a,\Sigma_b} \frac{e^{ibd}}{d} \frac{e^{ib\left|\underset{\sim}{r}_1 - \underset{\sim}{r}_0\right|}}{\left|\underset{\sim}{r}_1 - \underset{\sim}{r}_0\right|} \left(\frac{1 + cos(\vec{n}, \underset{\sim}{r}_1 - \underset{\sim}{r}_0)}{2}\right) ds \quad (\S 30)$$

which is of exactly the same form as (§16) of the non-relativistic case since (§29) is the same as (§12) of the non-relativistic case. (For the one-slit case, the surface integral covers one slit rather than two.) Note that *this solution applies to both particles and photons and this shows that both particles and photons are subjected to the same kind of diffraction effect due to slits*. Furthermore, for slowly moving particles, the same expression applies to the diffracted wave function so that these slowly moving particles also exhibit the wave like behaviour of fast moving particles and photons.

The translational velocity is a deterministic velocity derived from the deterministic phase of the deterministic ψ_p field. For the system to be non-deterministic, as before we invoke the non-deterministic surfing velocity on the *S* surface which is perpendicular to the translational velocity. This surfing velocity, being perpendicular to the translational velocity, is not related to the translational velocity and is therefore independent of it. We recall that the solution of the Gordon-Klein equation, expressed for example in *R* and *S*, does not prescribe, or lead to any necessary expression for, any velocity component. In the previous section, we have chosen the above forms for the translational velocity, $\underset{\sim}{v}_1$, for particles and photons based on our observation in this universe. Likewise, the solution of the Gordon-Klein equation does not lead to any necessary expression for the surfing velocity. Again, we will rely on some observations in our universe to give us some idea of what the surfing velocity looks like. Because we do observe the interference pattern in a two-slit experiment (and to a lesser extent in the single slit experiment), the non-deterministic surfing velocity on the *S* surfaces has to be consistent with the interference pattern, i.e., the surfing velocity is free and non-deterministic at any instant of time but overall it is constrained in a bulk budgetary sense to yield the interference pattern which is a statistical budgetary pattern and can be produced in innumerable different ways – hence non-determinism. *In this statistical budgetary manner, non-determinism is compatible with the overall known or 'determined' interference pattern*. (An analogy is that one is free to spend one's salary in many different ways but the overall spending is limited by one's determined salary.) In the next section on particle spin, we will suggest the forms of the two surfing velocity components which will satisfy the observed constant spin criterion and is consistent with the interference pattern resulted from diffraction, i.e., it conforms to these two kinds of observation. In another universe where the Gordon-Klein equation still applies, the surfing velocity can be very different to that in this universe with the result that the behaviour of a quantum system there will be very different from those in our universe, e.g., non-constant spin and very different interference patterns.

Again, it is virtually certain that the solution in the form of (§30), when evaluated numerically or otherwise, will not have constant $|\nabla S|$ in some regions with the diffracted wave (e.g., near the slits) even though these regions may be small. This means that the wavelength is constant only in an approximate or average sense, i.e., when averaged over a distance.

In summary, the components of the electric field and the magnetic field of the Maxwell Equations also satisfy the Gordon-Klein equation under certain simple conditions (e.g., in free space) so that both particles and photons satisfy the Gordon-Klein equation, albeit with different *m* parameters. This sharing of the same equation underlies the close similarity between diffraction of particles and diffraction of photons. In this paper, photons and particles are treated as discrete point-like entities whose non-deterministic behaviour is influenced but not determined by the wave function of the Gordon-Klein equation. For both particles and photons, the square of the amplitude of the wave field can properly be interpreted according to Born's rule, i.e., as the probability density of the point-like entity in question. Finally, because the Helmholtz equation, derived from the Gordon-Klein equation and involving the spatial part of the wave function, is exactly of the same form as the Helmholtz equation for the non-relativistic case, the diffraction integral for the non-relativistic case (involving particles only) can be conveniently applied to the relativistic case involving both particles and photons. Hence, this paper has affirmed the *ontological* nature of both particles and photons – i.e., they are both particle entity – while maintaining the influence (not determination) of the wave function on the *functional* behaviour of particles and photons. In this sense, the question of wave-particle duality for particles and photons has been dealt with.

8.3 Particle Spins in Special Relativistic Framework

According to conventional wisdom the Gordon-Klein equation can only describe spinless particles. But photons have spin 1 and other particles can have non-zero spins. It is true that the Gordon-Klein equation in itself does not contain sufficient information to determine the spin angular momentum of a photon or particle. (The same can be said for the Schrödinger equation regarding particles.) However, as has been suggested above when dealing with the form of the translational momentum, the surfing momentum on the *S* surface is available for prescription. In the case of the translational momentum, its form was prescribed in such a way as to match observation. For the surfing momentum, we will also seek to prescribe its form which will match particles spins as observed in our universe. This amounts to adding another constraint to the quantum system but it will be seen that this will not make the system deterministic. In fact, it will be shown in Section 8.7 that if an energy conservation criterion is applied, the surfing momentum cannot be deterministic while yielding the appropriate constant spin of a particle. This approach of describing the mechanism for generating particle spin raises some critical questions about Dirac's equation which will be discussed later.

It is proposed to prescribe these three orthogonal components of the momentum of a particle:

$$\underset{\sim}{p}_1 \equiv \hbar \nabla S \ , \ \underset{\sim}{p}_2 \equiv \lambda_2 \frac{\hbar}{2} \frac{\nabla \rho}{\rho} \wedge \vec{s} \ , \ \underset{\sim}{p}_3 \equiv \lambda_3 \hbar \frac{\nabla S}{\rho} \wedge \vec{a}$$

where $\vec{s}$ at the particle's position is a unit vector perpendicular to $\nabla \rho$ and lies on the plane formed by ∇S and $\nabla \rho$, $\vec{a}$ is the unit vector in the direction of $\underset{\sim}{p}_2$ at the point where the particle is, λ_2 is a constant corresponding to the kind of particle (or photon) in question, λ_3 is constant over space but varies in time. This three orthogonal momentum components mimic the forms of the three orthogonal velocity components in the non-relativistic case in paper [2] and paper [1]. $\underset{\sim}{p}_2$ and $\underset{\sim}{p}_3$ are on a *S* surface. We now see if these two additional prescribed momentum on a *S* surface will generated the expected spin of a particle or photon.

8.3.1 Particle Spin in the Cylindrical Mode

For simplicity of illustration, it is best to consider the particle or photon spin in the cylindrical mode where the translational momentum, $\underset{\sim}{p}_1$, is constant. Since (§29) of the relativistic case is of exactly the same form as (§12) of the non-relativistic case, the solution for ψ_p should be the same in both cases. We can therefore make use of or adapt the solution of the non-relativistic case for the relativistic case in question here. Hence, the *S* surfaces are uniformly flat and perpendicular to the translational momentum. As shown in Figure 1, the contours of constant *ρ* on a *S* surface are concentric circles and we can calculate the angular momentum with respect to the centre of these circles. $\underset{\sim}{p}_2$ is tangential to the *ρ* contour. Since $\underset{\sim}{p}_3$ goes through this centre, it does not generate any angular momentum. The solution of the radial equation for (§29) is the Bessel function of the first kind (see paper [2]) and has an infinite number of zeros as *r* tends to infinity. Let us say the first zero is at *r=L*. We shall concentrate our attention on the region with *r* less than *L*. (The case of *r > L* will be considered in a future paper.) The integrated angular momentum for the region with *r* less than *L* is

$$\int_0^L r \left| \underset{\sim}{p}_2 \right| (2\pi r \rho) \, dr = - \int_0^L r \frac{\lambda_2 \hbar}{2\rho} \frac{d\rho}{dr} (2\pi r \rho) \, dr = - \int_0^L \lambda_2 \hbar \pi r^2 \frac{d\rho}{dr} \, dr \ .$$

Now, the last integral can be evaluated by integration by parts so that it becomes

$$-\lambda_2 \hbar \pi \left[\left[\rho r^2 \right]_0^L - 2 \int_0^L r\rho \, dr \right] .$$

Since ρ=0 at r=L, the first term is zero and the integrated angular momentum of the particle is

$$\lambda_2 \hbar \int_0^L 2\pi r\rho \, dr = \lambda_2 \hbar$$

where the integral on the l.h.s. after $\lambda_2 \hbar$ is the normalisation integral and is therefore 1. One can readily see the physical significance of λ_2, i.e., it corresponds to the kind of particle (or photon) being considered and is the spin number of the particle or the photon. Also the integrated spin of a particle or a photon does not involve the mass parameter or depend on the b parameter in (§29). Furthermore, the precise form of the Bessel function of the first kind is not necessary in the above integration; all that is required is that it has a zero at r=L.

We have obtained this form of integrated angular momentum by assuming the above forms of $\underset{\sim}{p}_2$ and $\underset{\sim}{p}_3$. Since this form of integrated angular momentum is consistent with the observed spins of particles and photons, we have good reason to believe that the the above forms of $\underset{\sim}{p}_2$ and $\underset{\sim}{p}_3$ are theoretically possible or probable.

In Section 4 and in paper [2] for the non-relativistic case, we considered the question of interpreting ρ according to Born's rule while maintaining that the particle is at one specific position at any one time. Similarly for the relativistic case, ρ can also be equivalently interpreted (i) as the 'time averaged probability density' for a suitable period T or (ii) as the non-dimensional time density for the same period. A neighbourhood of high ρ means that the particle or photon will spend more time in that neighbourhood and therefore is more likely to be in that neighbourhood. Born's rule is therefore satisfied over the period T. That is, when the presence of the particle over the region of $r < L$ (presence in terms of the time spent in the small neighbourhoods in the region) is averaged over the period T, such presence satisfies Born's rule. Hence, the spin derived above is the average spin over the period T. (See paper [2] for a detailed explanation for this.) It needs to be noted that Born's rule is satisfied in the above manner only if $\underset{\sim}{p}_3$ does take the particle or photon to visit different circular strips for the appropriate lengths of time as expected from Born's rule. At this point we cannot see any deterministic mechanism on $\underset{\sim}{p}_3$ which will ensure that Born's rule is satisfied. All we can say is that $\underset{\sim}{p}_3$ must play the role required by Born's rule. Later in Section 8.7 we shall look at the crucial role of $\underset{\sim}{p}_3$ in quantum mechanics.

How does this understanding of spin relate to the Dirac equation? Allegedly, the Dirac equation describes particles of half spin. This will be scrutinised closely in another paper but here briefly its potential problem is highlighted. The Gordon-Klein equation is a second order equation with four second order derivatives of the wave function corresponding to the time dimension and the three spacial dimensions. In order to reduce this equation to a system or set of first order equations without loss, four new variables need to be introduced and they correspond to the four first order derivatives of the wave function with respect to time and the three spatial dimensions. These four definitions of the four new variables constitute four first order equations. When the four new variables are substituted into the Gordon-Klein equation, that equation itself also becomes first order. Hence, in total there is a system of five first order equations which is entirely equivalent to the original Gordon-Klein equation. However, the Dirac equation is a system of four first order equations, allegedly derived from the Gordon-Klein equation through 'factorising' that equation. This is made possible by suitably choosing the gamma matrices in the Dirac equation. But this has the consequence that every component of the Dirac wave function (or Dirac spinor) satisfies the Gordon-Klein equation while only some combinations of the solutions to the Gordon-Klein equation can satisfy the Dirac equation. As a result, it imposes a further constraint on the solutions to the Gordon-Klein equation and filter out some of these solutions. In any case, mathematically the system of five first order equations corresponding to the original Gordon-Klein equation with no loss is different from the system of four first order equations in the Dirac equation. One has to decide which system of equations is more fundamental. Since the Dirac equation is derived from the Gordon-Klein equation but with some loss, this paper holds the position that the Gordon-Klein equation is more fundamental. This position is also vindicated by its satisfactory application in this paper for explaining the wave-particle duality of both particles and photons, and by the ability to explain the different values of particle or photon spins. Because the Dirac equation filters out some of the admissible solutions to the Gordon-Klein equation, it may also filter out some or many of the admissible solutions to the Gordon-Klein equation which have spins other than that admitted by the

Dirac equation. By using the Gordon-Klein equation and the additional prescriptions of the surfing momenta on a *S* surface, $\underset{\sim}{p}_2$ and $\underset{\sim}{p}_3$, this paper recovers the solutions with many different spin values.

Now we turn to the question of the magnitude of the surfing momentum on a *S* surface. $\underset{\sim}{p}_2$ depends on λ_2 which is constant in space and time; $\underset{\sim}{p}_2$ also depends on *ρ* and its gradient which are influenced by the parameter $-\nabla^2 R/R = b^2 - (\nabla S)^2$. Let us call $\underset{\sim}{p}_2$ the spin momentum since it is this term that is responsible for generating the spin. This spin momentum at a point in space is deterministic since (i) λ_2 is constant in space and time, and (ii) *ρ* is deterministic over space and constant in time. We have so far left the parameter λ_3 in $\underset{\sim}{p}_3$ undetermined save that it varies with time. Any non-determinacy of a quantum system therefore has to come from the non-deterministic nature of λ_3 in $\underset{\sim}{p}_3$. However, it seems highly reasonable to conserve the total momentum on a *S* surface which is quite independent of the translational momentum. Hence, we suggest that the magnitude of the surfing momentum

$$\underset{\sim}{p}_s \equiv \underset{\sim}{p}_2 + \underset{\sim}{p}_3$$

is constant. This does not make $\underset{\sim}{p}_3$ deterministic since its direction can be non-deterministic. Section 8.7 will discuss this crucial point of non-determinacy later.

Paper [2] for the non-relativistic case suggested that the magnitude of the surfing velocity $\underset{\sim}{v}_s$ and therefore the magnitude of $\underset{\sim}{p}_s$ are not constant on the *S* surfaces. There, the quantum potential (also called Total Quantum Energy) is split into quantum potential energy and quantum kinetic energy; and the quantum kinetic energy representing the surfing energy is allowed to vary on the *S* surfaces. However, it is now suggested that this is not realistic as it is not possible to account for the change of the surfing momentum and the consequent change of surfing energy. It makes more sense to conserve the surfing momentum on the *S* surfaces and let the magnitude of $\underset{\sim}{p}_3$ adjust to the varying magnitude of $\underset{\sim}{p}_2$. The next section will show that a constant magnitude of $\underset{\sim}{p}_s$ has the an important implication concerning the rest mass of a particle.

8.4 'Rest Mass', 'Rest Energy' and Einstein's Energy Formula

Hestenes [8] suggested 'The *Zitterbewegung* Interpretation of Quantum Mechanics' which encouraged other papers ([4,6]) along that line. Even though he was working with the Dirac equation, not the Gordon-Klein equation as in this paper, he suggested in Section 4 of his paper, 'The so-called "rest mass" of the electron is therefore a kinetic energy of the magnetic self-interaction. It is this that gives the electron its inertial properties.' He also referenced the 'flywheel-like nature of this inertia' but he did not investigate the dynamics of such inertia. In this section, we suggest that a particle has a genuine rest mass, called the 'inherent mass', which is not the same as the usual 'rest mass'. The surfing motion or the surfing momentum (with its surfing energy) on the *S* surfaces with this inherent mass generates what we usually call the 'rest mass'. (Photons will be considered in the next section.) And this has serious implication for the way we calculate the energy of a particle in the relativistic case, i.e., the usual way of calculating the energy of a particle will underestimate the value of the energy and this may well need to be corrected. Einstein's energy formula can be written as

$$E^2 \equiv p_{1E}^2 c^2 + m^2 c^4 \qquad (\S 31)$$

where p_{1E} is the momentum as Einstein envisaged it, and *m* is the usual 'rest mass'. Since his momentum plays the role of the translational momentum considered in this paper (see later), it is denoted by the subscript 1 while the subscript '*E*' in p_{1E} denotes Einstein's notion of momentum. This translational momentum is defined as

$$p_{1E} \equiv \frac{mv_1}{\sqrt{1 - \frac{v_1^2}{c^2}}}$$

where v_1 is the translational speed. Note that Einstein's energy formula does not explicitly take into account the surfing energy on the *S* surface which is an essential energy in the quantum world as suggested in this paper. When the translational momentum is zero, the energy is given solely by the second term of (§31) involving the 'rest mass' and it will be seen below that it is this 'rest mass' which *partially* incorporates the the contribution of the surfing motion to the energy.

Now, we define a new energy with subscript *N* (for new) to distinguish it from *E* given by (§31), and a new translational momentum:

$$E_N \equiv pc \tag{§32}$$

$$p_{1N} \equiv \frac{m_i v_1}{\sqrt{1-\frac{v_1^2+v_s^2}{c^2}}} \tag{§33}$$

where $p = \left|\underset{\sim}{p}\right|$, $\underset{\sim}{p} = \underset{\sim}{p}_{1N} + \underset{\sim}{p}_s$, $\underset{\sim}{p}_s = \underset{\sim}{p}_2 + \underset{\sim}{p}_3$, m_i is the 'inherent mass' of the particle. The term 'inherent mass' means the mass inherent to the particle which does not arise from any motion of the particle, not even its surfing motion on the *S* surface. $\underset{\sim}{p}$ is the total momentum of the particle taking into account both the translational momentum *and* the surfing momentum on the *S* surface. Similar to the translational momentum in (§33), the total momentum and the surfing momentum are consistently defined as

$$p \equiv \frac{m_i v}{\sqrt{1-\frac{v^2}{c^2}}} \ , \ p_s \equiv \frac{m_i v_s}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_i v_{so}}{\sqrt{1-\frac{v_{so}^2}{c^2}}} \tag{§34}$$

where v is the total speed, v_s is the surfing speed in general and v_{so} is the surfing speed when the translational speed is zero. Taking the square of (§32), we have

$$E_N^2 = p^2c^2 = p_{1N}^2c^2 + p_s^2c^2 \ . \tag{§35}$$

If we compare (§35) with (§31), we see that m^2c^4 could be equivalent to and therefore accounted for by $p_s^2c^2$ such that

$$p_s = mc \ .$$

The 'rest mass', *m*, is given by p_s divided by *c*. A constant *m* requires a constant p_s which has already been suggested above. Note that since p_s is constant, as the total speed v varies, v_s will need to adjust to maintain a constant p_s. In the particular case of zero translational speed, v_{so} is the surfing speed and the total speed, and it can be used to set the constant p_s value and then calculate the 'rest mass' which is p_s/c. At this point it will be good to begin to use a term different to the term 'rest mass' which can be misleading since the particle is not at rest due to the surfing motion on the *S* surface even though its translational speed may be zero. From now on we replace the term 'rest mass' with 'effective mass' which in general is not the same as the 'inherent mass', m_i.

If m^2c^4 is equivalent to $p_s^2c^2$ as suggested above, then p_{1E} in (§31) does not include the surfing momentum which has been taken care of by m^2c^4 . Hence, p_{1E} is the translational momentum as suggested above.

Note that, for zero p_s , the 'effective mass', *m*, will be zero even though its inherent mass, m_i , is not zero. This brings to mind the photon and other massless particles whose 'effective mass' is zero. But zero effective mass does not exclude the possibility of the photon or other 'massless' particles having a non-zero inherent mass. In the extreme case of zero p_s and zero p_{1N} (zero total momentum), while the inherent mass is non-zero, the effective mass is zero and the energy will be absolutely zero. In that case, the particle with zero energy may be 'invisible' or undetectable as far as their energy or mass is concerned and may become 'visible' only when it receives energy.

If v_1 is zero, then both p_{1E} and p_{1N} are zero so that they have the same value. However, if v_1 is greater than zero,

$$\frac{p_{1E}}{p_{1N}} = \frac{m}{m_i}\frac{\sqrt{1-\frac{v_1^2+v_s^2}{c^2}}}{\sqrt{1-\frac{v_1^2}{c^2}}} = \frac{m}{m_i}\frac{\sqrt{c^2-(v_1^2+v_s^2)}}{\sqrt{c^2-v_1^2}}$$

where it can be seen that p_{1N} will tend to be larger than p_{1E} when v_s is greater than zero. However, the ratio between the p_{1E} and p_{1N} also depends on the ratio between the 'effective mass' and the 'inherent mass'. Now,

$$\frac{m}{m_i} = \frac{\underset{\sim}{p_s}}{m_i c} = \frac{m_i v_s}{m_i \sqrt{c^2 - v^2}} = \frac{v_{so}}{\sqrt{c^2 - v_{so}^2}} \ .$$

For $v_{so} = \dfrac{c}{\sqrt{2}}$, $m = m_i$.

For $v_{so} > \dfrac{c}{\sqrt{2}}$, $m > m_i$.

For $v_{so} < \dfrac{c}{\sqrt{2}}$, $m < m_i$.

Without experimental observation, we cannot know which of the above three cases is correct. However, one can with some reason suggest that the third case is more realistic, i.e., v_{so} is not near to *c* or is not a significant fraction of *c*. In that case, p_{1N} is definitely larger than p_{1E}. Even if the second case is true, i.e., $m > m_i$ which is unlikely, the other factor in the ratio of p_{1E} to p_{1N} can still make p_{1N} greater than p_{1E} – this will be shown to be the case in a forthcoming paper.

It will be interesting to compare *E* with E_N. Since m^2c^4 of (§31) is identical to $p_s^2c^2$ of (§35), the difference between E^2 and E_N^2 lies in the magnitudes of p_{1E} and p_{1N}. *If we consider the most probable case of* p_{1N} *greater than* p_{1E}, *then* E_N *will be greater than E* (as will be shown in the next paper). If v_1 is zero, then the two energies are equal to the same 'rest energy' since both p_{1N} and p_{1E} are zero, i.e., $E_N = p_s c = E = mc^2$. In this case, the particle is not genuinely at rest since its surfing momentum is non-zero but is only 'at rest' in the sense that its translational momentum is zero. Hence, instead of the term 'rest energy', the term 'base energy' corresponding to the energy for zero translational momentum is used from now on. This base energy is the base on which further energy due to translational motion (kinetic energy) can be added.

In (§22), we have the following quantity with the unit of energy squared:

$$K^2 \equiv c^2\hbar^2(\nabla S)^2 + m^2c^4 - c^2\hbar^2\frac{\nabla^2 R}{R}$$

where $\hbar\nabla S$ is the translational momentum. The sum of the first two terms of K^2 is equivalent either to E_N^2 or E^2, depending on whether one equates $\hbar\nabla S$ with $\underset{\sim}{p}_{1N}$ or $\underset{\sim}{p}_{1E}$. Since $\underset{\sim}{p}_{1N}$ and E_N^2 take into account the surfing motion on the *S* surface while E^2 and $\underset{\sim}{p}_{1E}$ do not, this paper suggests that the sum of the first two terms of K^2 is equivalent to E_N^2, not E^2. E_N is therefore considered to be a better measure of the energy of a particle than *E* which underestimates E_N. However, E_N is not yet the total energy since there is the third term in K^2 which is related to the quantum potential.

The reason why *E* underestimates E_N is because p_{1E} underestimates p_{1N} and this in turn is due to the crucial fact that p_{1N} takes into account the effect of the surfing velocity on the *S* surface while p_{1E} does not. The Lorentz factor in p_{1N} is naturally greater than the Lorentz factor in p_{1E} which lacks the surfing speed in its definition. Also, m_i tends to be greater than *m* which accentuates the difference between p_{1N} and p_{1E}.

One can visualise the effect of the surfing velocity on the energy of a particle by imagining a bee buzzing rapidly in a small scale matchbox while the matchbox is being carried in the large scale motion by a train. The large scale velocity of the train is like the translational velocity of the particle while the rapid buzzing of the bee in the small matchbox is like the surfing motion of the particle on the *S* surface in the quantum scale. Let us imagine that the bee is the inherent mass. When the train is at rest, the bee’s buzzing motion with its energy in the matchbox gives rise to the ‘effective mass’ which will be less than its inherent mass unless it buzzes around with speed greater than $\frac{c}{\sqrt{2}}$. At the same time, the bee’s buzzing motion gives rise to its base energy (conventionally called ‘rest energy’) when the train is at rest. When the train moves from rest to a higher speed, the bee’s energy will increase beyond the base energy. As the bee moves forward with the train, its buzzing speed increases the total speed of the bee beyond the speed of the train so that the Lorentz factor for the bee's translational momentum is greater than it would have been without the buzzing motion. Compared to the conventional way of calculating the energy contribution due to the translational momentum which uses the smaller Lorentz factor and the effective mass (which is normally less than the inherent mass), calculating the energy contribution due to the translational momentum using the larger Lorentz factor and the larger inherent mass, as suggested in this section, will produce a larger energy value. That is, the bee has more energy than conventionally expected.

Physicists have been looking for the dark matter necessary to hold the galaxies together. If the energy distribution of a galaxy is estimated without including all the contributions from the surfing (or buzzing) motion of its constituent particles in the quantum scale, the energy will inevitably be underestimated. It is probable that the existing energy of the existing matter in the galaxy has not been fully taken into account because some effects of the the surfing motion of the particles have been unintentionally excluded. Conversely, including all the effects of the surfing motion of the particles in the galaxy could partially account for the elusive 'dark matter'.

As pointed out above, in general because p_{1E} underestimates p_{1N}, *E* underestimates E_N and is therefore not equal to E_N, except when v_1 and therefore p_{1E} and p_{1N} are zero. For non-zero v_1, *E* can be written as

$$E = m_r c^2$$

where m_r is the relativistic mass, $m_r = m/\sqrt{1 - v_1^2/c^2}$. However, if the new definition of energy

$$E_N \equiv pc$$

is a better definition of energy than the traditional *E* as justified in this section, then the new definition of energy should be used and its implications should be explored.

8.5 The Case of Photons

It is puzzling how a massless object can have momentum and energy. Some physicists have entertained the possibility of photons having mass by appealing to experimental observation ([9, 10, 11, 12]). The above analysis shows the possibility of zero effective mass while the inherent mass is non-zero: since $p_s = mc$, zero p_s implies zero effective mass. However, if we set p_s to be zero for photon, then it cannot have the surfing velocity to generate its angular momentum, i.e., its spin value will be zero. If the photon spin is $\hbar$, we need to relax this constraint on p_s and allow it to be non-zero. But this will mean that its effective mass is non-zero. A non-zero effective mass for photon will go some way towards unravelling the mystery or paradox of a massless object having angular momentum, translational momentum and energy. Hence, it is worth investigating this possibility of non-zero effective mass of photon while maintaining the other characteristics of a photon in good shape.

It is noteworthy that the Gordon-Klein can be written in this form:

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \frac{p_s^2}{\hbar^2}\psi \tag{§36}$$

which is arguably more original than the form in (§18) since the more fundamental surfing momentum squared is used instead of m^2c^2 which involves the derived effective mass. If p_s is small, then treating it as zero, i.e., treating the effective mass as zero, is a good approximation. Under what condition can p_s be very small? According to the definition $p_s \equiv \frac{m_i v_s}{\sqrt{1-\frac{v^2}{c^2}}}$, p_s of a photon could be small if the inherent mass of a photon, $m_i = m_{ip}$, is small or its surfing speed, v_s, is small. However, the Lorentz factor will be very large since we expect the total speed, v, to be near to *c* (it cannot be *c* for the Lorentz factor to be finite). But if both m_{ip} and v_s are small, their product will be very small indeed such that p_s could still be very small. In that case, the last term of (§36) can be ignored as a good approximation.

Looking at the definition $p_{1N} \equiv \frac{m_i v_1}{\sqrt{1-\frac{v_1^2+v_s^2}{c^2}}}$, the total speed, v, has to be less than *c* and the translational speed, v_1, has to be close to *c* and therefore very large. The Lorentz factor as mentioned above will also be very large such that even though m_{ip} is small, p_{1N} will be of a significant size. Looking at the definition $E_N^2 = p^2c^2 = p_{1N}^2c^2 + p_s^2c^2$, one can see that the energy is dominated by p_{1N} at the expense of p_s such that $E_N = p_{1N}c$ is a good approximation. It is worthy to note that in the case of particles the energy could be dominated by the contribution from p_s in the form of rest energy or base energy, whereas in the case of photons the opposite is true, i.e., the contribution from p_{1N} dominates.

Photons of different translational momenta correspond to different energies and wavelengths. What is the most appropriate way for the translational momentum to vary? Since v_1 is close to *c*, small variation of v_s can yield great variation in the translational momentum and energy. Also, non-zero v_s means that the angular momentum of photon can be non-zero, which is consistent with observation. This and other possibilities will be explored in another paper.

Finally, strictly speaking in this modelling framework, *c* is no longer the speed of light. as v_1 is the speed of light, which has the value c' say, which is very close to some universal constant *c*. This modelling framework will inevitably throw up many questions in relation to other theories but it is beyond the scope of this paper to investigate these in great details (but see next section). What this framework has sought to show is that one can conceive of a photon with small inherent mass and small effective mass, while maintaining that the translational speed is close to *c* with the possibility of varying translational momentum, varying wavelength and varying energy. Perhaps, this idea of photon is not unreasonable compared to the idea of a massless photon with non-zero energy and momentum; this idea will be dealt with in another paper.

8.6 Particle Mass

It was suggested above that for zero p_s, the 'effective mass' of a particle, *m*, will be zero even though its inherent mass, m_i, is not zero. This point of zero effective mass of a particle reminds us of the massless particles invoked in the Standard Model. Regarding such massless particles, Chang [13] wrote,

> [I]n the Higgs model, the rest mass of a particle is just a parameter associated with the strength of coupling between the particle field and the Higgs field. If this is true, we may have a philosophical problem. That is, the physical meaning of the rest mass *m* would be intrinsically different from energy *E* or momentum *p*. This does not seem to be very satisfactory in view of our traditional understanding of the physics concept. (p. 23)

He also listed some challenging problems for the Standard Model (pp. 21, 24). He thus attempted to 'explore a different approach to explain the origin of particle mass based on a less complicated physical picture' (p. 24). He proposed a *V*-medium which is 'a continuous medium, which can be excited by an energetic stimulation' and '[d]ifferent excitation waves of the *V*-medium make up different types of particles observed in the physical world' (p. 24). He deduced that the 4-dimensional wave equation (three-dimensions in space and one dimension in time) is suitable for modelling these particles in free space (equation 19). However, the mass of the particle is not a parameter of the equation but it can be deduced from the wave equation. What is required as a parameter of the wave equation is the quantity, $-\frac{\nabla^2\psi_T}{\psi_T} = l^2$, which effectively is the quantum potential (apart from some constants and their units) as identified in this paper. ψ_T is the component of the wave function on the plane perpendicular to the translational (or called longitudinal in his paper) momentum; this 'transverse' plane is effectively equivalent to the *S* surface in this paper. He suggested that the particle has motion on this plane which is controlled by l^2 but gave insufficient details about this motion. (This paper and previous papers also affirm the particle's motion on the *S* surface but give much more details about such motion which is crucial in generating the rest or effective mass.) The three dimensions in the spatial Laplacian in the wave equation are reduced to one dimension in the direction of the translational (or longitudinal) momentum as the Laplacian in the other two dimensions on the transverse plane yield what is effectively the quantum potential (multiplied by some constants), l^2, which is a constant. Then, the mass is set as $m = \frac{\hbar l}{c}$ and the square of this mass is incorporated into the wave equation (via l^2) which has become a reduced Gordon-Klein equation but this reduced Gordon-Klein equation only operates in the translational or longitudinal direction since it lacks the Laplacian in the other two dimensions on the transverse plane to operate on the wave function (as they have been replaced by l^2).

It is laudable that the paper seeks an alternative explanation of a particle's mass and identify some necessary motion on the transverse plane (*S* surface in this paper); it also sees the wave equation or the Gordon-Klein equation as useful for modelling particles (as this paper affirms). However, it identifies the quantum potential, or l^2, as the crucial parameter in the wave equation to generate the particle's mass. There are three disadvantages or difficulties in this approach. Firstly, it does not take into account the motion of the particle on the transverse plane (*S* surface in this paper) with sufficient details. This motion on the transverse plane certainly adds the surfing momentum and surfing energy to the particle which yield the effective/rest mass and base/rest energy of the particle respectively when the the longitudinal (or translational) momentum is zero; hence the transverse motion should be explicitly included with sufficient details in the model to generate the particle's effective/rest mass, as suggested in this paper. Furthermore, the Lorentz factor for the total momentum is also dependent on the magnitude of the transverse motion (surfing motion in this paper); hence sufficient details of the transverse motion should be incorporated explicitly, as suggested in this paper. Secondly, since the effective or rest mass of the particle, $m = \frac{p_s}{c}$, can be expressed in terms of m_i (inherent mass) and v_{so} (the surfing velocity when the translational momentum is zero), this effective mass as a parameter can appear explicitly in the full Gordon-Klein equation, while the quantum potential, or l^2, is available to be used to correspond to another elusive energy in the universe, dark energy (see Conclusion). But by using the quantum potential term to account for the particle's mass, this quantum potential term cannot be used to account for another energy such as the dark energy. Thirdly, relying on the quantum potential to account for the particle's mass reduces the Gordon-Klein equation to effectively have only one spatial dimension instead of three spatial dimensions in its Laplacian. However, maintaining all three spatial dimensions in the Laplacian in that equation gives explicit room to express the quantum potential alongside the effective mass. It allows one to see how the various energy terms fit together as a whole, as illustrated in the following.

This paper, in its relativistic section, began with the full Gordon-Klein equation with the effective/rest mass parameter. Eventually, it expresses the effective/rest mass in terms of the surfing momentum and writes the full Gordon-Klein equation as

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \frac{p_s^2}{\hbar^2}\psi$$

where we can see heuristically from (§21) and (§22) that the translational dimension of the Laplacian on the l.h.s. yields the translational momentum squared while the other two dimensions of the Laplacian operating on the *S* surfing surface yield the quantum potential multiplied by a suitable constant, the second order time derivative on the r.h.s. yields the square of *K* which can be called the super-energy (the signs and exact units are not included in this heuristic description of the terms). It is highly reasonable that the last term of this equation is representing the square of the surfing momentum of the particle since this surfing momentum is an important aspect of the particle's motion which has not been represented by any other term in the equation. It is this term representing the surfing motion on the *S* surface which accounts for the effective/rest mass of the particle.

It is possible for particles to have zero effective mass, or zero rest mass, through having zero surfing momentum, but they can still have non-zero inherent mass. How these inherently massive particles with zero effective mass relate to massless particles (with zero effective/rest mass) in the Standard Model remains to be investigated. For example, it may still be possible for the massless particles in the Standard Model (with implied zero surfing momentum) to have non-zero inherent mass and non-zero translational momentum.

Finally, in terms of angular momentum, how does the model presented in this paper, based on the Gordon-Klein equation with two added velocity components on the *S* surface generating a particle's spin, relates to the spins of elementary particles in the Standard Model?

8.7 The Nature of The Surfing Velocity and Non-Determinacy

We now come to the final consideration of the surfing velocity component on the *S* surface, $\underset{\sim}{v}_3$. In the cylindrical mode, we see that the surfing velocity component on the *S* surface, $\underset{\sim}{v}_2$, is responsible for generating the spin angular momentum of the particle while $\underset{\sim}{v}_3$ is perpendicular to it. In paper [2], it was indicated that it might be possible to derive a deterministic prescription for $\underset{\sim}{v}_3$. If that is the case, then since $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ are deterministic, a deterministic prescription for $\underset{\sim}{v}_3$ will make the total velocity deterministic. However, paper [2] suggests some objection to this deterministic prescription for $\underset{\sim}{v}_3$. Here, a stronger objection will be presented.

Figure 8 shows the plot of $r\rho$ against r on a *S* surface for the cylindrical mode.

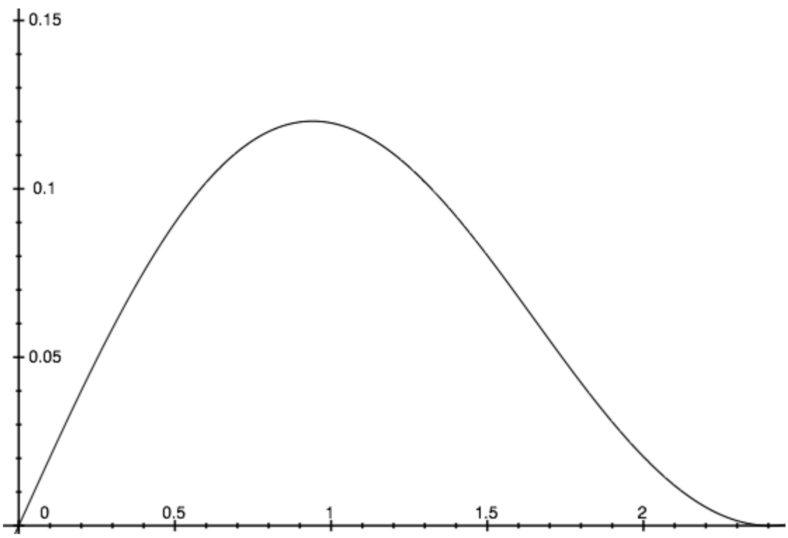

Figure 8: $r\rho$ as a function of r for illustrative purpose; L=2.4

Remember the interpretation of ρ as the non-dimensional time density, for a circular strip around the centre with $\triangle r$ as the width of the strip, the time to be spent in that circular strip should be $\triangle t = 2\pi r \triangle r \rho T$ where T is a suitable period (for more details, see sections 6 and 7 of paper [2]). The magnitude of the radial velocity, $\underset{\sim}{v}_3$, could be obtained by taking the limit of

$$\frac{\Delta r}{\Delta t} = \frac{1}{2\pi T r \rho}\,. \qquad (\S 37)$$

This seems to have fixed the radial speed. Since $\underset{\sim}{v}_3$ is perpendicular to $\underset{\sim}{v}_2$, it seems that the direction and magnitude of the radial velocity is fixed, $\underset{\sim}{v}_3$ is determined and the total velocity of the particle is determined. However, apart from the challenge of infinite radial velocity at the centre and r=L, there is also the following unsurmountable difficulty. According to the above expression for the radial speed, this speed should be at a minimum when $r\rho$ is at the maximum. Since p_s is conserved, $\underset{\sim}{v}_2$ should be at the maximum when the radial speed v_3 is at a minimum. However, the maximum $\underset{\sim}{v}_2$ does not occurs at the point when $r\rho$ is at the maximum; it occurs somewhere else near or at L as seen in the expression for $\underset{\sim}{v}_2$.[10] Hence, this contradicts the suggestion that the magnitude of $\underset{\sim}{v}_3$ is determined by (§37).

Even though the magnitude of $\underset{\sim}{v}_3$ cannot be determined by (§37), its magnitude is nevertheless determined in a different way while its direction/orientation cannot be determined, i.e., $\underset{\sim}{v}_3$ is still non-deterministic. Since (i) p_{1N} is deterministic, (ii) the surfing momentum p_s is conserved and thus deterministic, according to (§33) and (§34) $\underset{\sim}{v}_1$ and v_s are deterministic. And since $\underset{\sim}{v}_2$ is deterministic, the magnitude of $\underset{\sim}{v}_3$, v_3 , is also deterministic. However, even though $\underset{\sim}{v}_3$ is perpendicular to $\underset{\sim}{v}_2$, *the direction of* $\underset{\sim}{v}_3$ *can be in two opposite directions* and hence is not deterministic so that $\underset{\sim}{v}_3$ is non-deterministic. We now look at how $\underset{\sim}{v}_3$ with its non-deterministic direction can satisfy the requirement given above, $\triangle t = 2\pi r \triangle r \rho T$, despite the magnitude of $\underset{\sim}{v}_3$ being deterministic.

It is possible for the particle to visit the same circular strip many times within the period T and the number of times it needs to visit a strip depends on v_3 . For example, a large v_3 means that the particle has to visit the same strip more times before accumulating the required amount of time for that strip. If N is the number of required visit, then

---

[10] The problem of the nodal circular contour at $r = L$ will be dealt with in another paper.

$$\frac{N\triangle r}{v_3} = \triangle t = 2\pi r\triangle r\rho T \ , \ N = 2\pi T r\rho v_3 \ .$$

An initial scale analysis suggests that v_3 is probably very large. If that is the case, *N* could be very large. *N* will not be an integer but because of its large size the fractional part can be neglected. The factor of *2πT* in *N* is constant. *N* is thus proportional to $r\rho v_3$ and it is instructive to plot $r\rho v_3$ as a function of *r*. Figure 9 is such a plot for illustrative purpose; hence the unit in the vertical axis is not given.

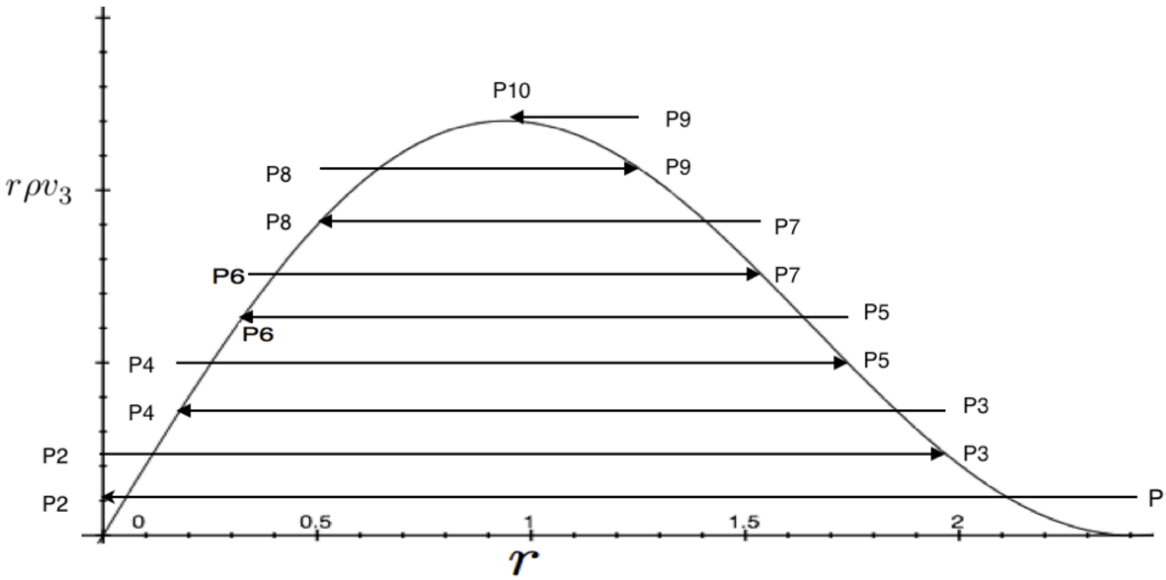

Figure 9: $r\rho v_3$ as a function of *r* for illustrative purpose; particle moves from P1 to P10

Figure 9 is like a frequency distribution as a function of *r* since *N* is proportional to $r\rho v_3$. The particle has to visit the circular strip at maximum $r\rho v_3$ for the largest number of times, and visit the strips with minimum $r\rho v_3$ the least number of times. The figure shows an illustrative example of a particle oscillating about the position of maximum $r\rho v_3$ in order to approximate the required frequency distribution. In reality, many more oscillations will be necessary to satisfy closely the required frequency distribution.[11] The figure may apparently suggest that the particle has to stop at many points as it changes its radial direction of

---

[11] If we let *T* tend to infinity, then *N* will tend to infinity so that the curve in Figure 9 will be satisfied very closely.

motion. But this is not the case. v_3 is in general non-zero; an initial scale analysis suggests that v_3 is probably very large and much larger than v_2 . How then can the frequency distribution be satisfied or closely approximated by taking multiple sweeps across the *r* domain without stopping? The only conceivable way is for the particle to reverse its radial direction instantaneously without stopping, while maintaining the speed of v_3 at the point of reversing the radial direction. v_3 as a function of *r* is determined (see above) and therefore has to be maintained even though the direction of $\underset{\sim}{v}_3$ (inward or outward) is not determined and is therefore free to change.

There are innumerable ways of satisfying the frequency distribution illustrated by Figure 9 and there are infinite number of positions where the particle's $\underset{\sim}{v}_3$ can suddenly and instantaneously reverse its direction. Hence, there is a high degree of freedom and non-determinacy. One could conceivably still prescribe a deterministic way of satisfying the frequency distribution, e.g., by fixing the positions of turning and other necessary determinations, but there are an infinite number of equally valid ways of doing so. There is no reason to pick any one way. Therefore, there is no justifiable reason to prescribe any specific deterministic way to satisfy the frequency distribution. It is best to see the quantum system as truly non-deterministic. *The key of non-determinism lies in the non-deterministic positions where $\underset{\sim}{v}_3$ suddenly reverses its direction. It is the ultimate source of non-determinism in quantum mechanics*. As long as the bulk statistics of the frequency distribution is satisfied, the system does not require that satisfaction to be accomplished in any specific way. This is the inherent non-determinism of a quantum system.

Note that the surfing speed of the particle on the *S* surface, v_s , will be constant and therefore continuous. But the velocity, $\underset{\sim}{v}_3$, is discontinuous in terms of direction at the points of sudden and unpredictable turning even though its magnitude is continuous. Hence, the magnitude of the surfing velocity is constant and continuous while the direction of the surfing velocity is discontinuous. This is not conceivable in classical mechanics, but in the quantum world our expectation from the study of classical mechanics can be set aside.

9.0 The Heisenberg Uncertainty Principle

According to the Heisenberg Uncertainty Principle, the product of the standard deviations of position and momentum should be greater than or equal to half of $\hbar$. However, if the energy of a particle is defined by $E_N \equiv pc$, in the cylindrical mode of motion the energy of the particle is conserved which implies that the magnitude of the total momentum is conserved even though the direction of the total momentum is unpredictable due to the unpredictable nature of the direction of $\underset{\sim}{v}_3$ (despite the translational momentum is predictable). This means that there is *certainty* regarding the magnitude of the momentum while there is *uncertainty* regarding the direction of the momentum and *uncertainty* regarding the position of the particle on the *S* surface. These uncertainties are inherent in quantum mechanics and are therefore not due to the effect of any observation. If these uncertainties are indeed correct, then the Heisenberg Uncertainty Principle does not really stand since there can be *certainty* in the magnitude of the momentum. A useful example for illustration is a particle with zero translational momentum. In this example, the magnitude of the momentum of the particle is purely given by the magnitude of the surfing momentum, p_s which is conserved and therefore deterministic with *no uncertainty*, even though (i) the direction of the surfing momentum is unpredictable due to the unpredictable nature of the direction of $\underset{\sim}{v}_3$ and (ii) the position of the particle on the surfing *S* surface is unpredictable. In the extreme case of zero translational momentum and zero surfing momentum, the momentum of the particle is zero, so is its energy. In that case, its momentum and its position will be determinate simultaneously but it will be invisible or undetectable as far as their energy or mass is concerned until it receives energy.

10.0 Discussion and Conclusion

10.1 Discussion

Hestenes [8] proposed a *Zitterbewegung* interpretation of quantum mechanics. He suggested that the spinning motion of a particle on the spin plane is crucial in understanding quantum mechanics. He worked with the Dirac equation but this equation is physically too restrictive and something in the Gordon-Klein equation is lost because of the questionable mathematical procedure used to derive it from the Gordon-Klein equation (see the forthcoming paper for more details). Hestenes' work encouraged others such as Salesi, Esposito and Recami to study the spinning motion of particles ([3, 4, 5]). However, their studies invariably adopted a deterministic framework but they did obtain one deterministic surfing velocity component which is in addition to the usual deterministic translational velocity identified in pilot wave theory. This paper and the previous two papers ([1,2]) adopt the non-deterministic framework and investigate two, rather than one, possible surfing velocity components on the S surface, on top of the translational velocity component. Einstein proposed that an open quantum system as described by the Schrödinger equation ought to be closed by adding more equations or constraints. His idea of additional constraints or equations is echoed in this paper by the two theoretical surfing velocity components on top of the translational velocity but this paper does not agree with him in that the three velocity components (translational and surfing) do not yield a deterministic closed system as Einstein hoped, since one of the surfing velocity components, $\underset{\sim}{v}_3$, is still non-deterministic even though it has to satisfy the overall statistical constraint imposed by Born's rule. This overall statistical or budgetary constraint allows non-determinacy in $\underset{\sim}{v}_3$ since there are many different ways of satisfying this constraint, just as there are many different ways of spending the money in one's limited budget coming from one's restricted salary. Because of the non-determinacy in $\underset{\sim}{v}_3$, the theoretical models adopted in this paper for the non-relativistic case and the relativistic case are fundamentally different from Einstein's picture of a closed deterministic quantum system. The choices of the forms of the two theoretical surfing velocity components and the translational velocity component ought to be consistent with observations in our universe if they are to have any validity, even though hypothetically these three velocity components can take on vastly different forms in other hypothetical universes whose properties and phenomena will be vastly different from those of our universe, with many of these universes being chaotic and not life sustaining. The theoretical choices of the particular forms of the three velocity components, as adopted in this paper for modelling our universe, will have to be vindicated by their consistencies with phenomena observed in our universe for them to be treated as credible and reliable. This paper therefore investigates whether such consistencies with observations indeed exist and does so with respect to a number of observed phenomena.

Firstly, the chosen form of the translational momentum in our modelling, $\underset{\sim}{p}_1 \equiv \hbar\nabla S$ involving the translational velocity, readily gives rise to the Louis de Broglie's formula for momentum and wavelength which is confirmed in multiple observations. Secondly, the observed relationship between energy and frequency (Planck-Einstein relation) for particles appears readily from the energy equation (§2) for the non-relativistic case. For the relativistic case, the same relationship between energy and frequency holds for photons but for particles a slightly different constant, *K* involving the quantum potential, replaces the constant *E*.[12] Our modelling is therefore consistent with these two observed relations, thus giving some credibility to the modelling.

Thirdly, the Schrödinger equation (for the non-relativistic case) and the Gordon-Klein equation (for the relativistic case) in themselves cannot incorporate spin. However, these equations with the two prescribed surfing velocity components on the *S* surface can be used to model the angular momentum of a particle. In order to match the observed constancy of the spin of a particle in our universe, λ_2 in $\underset{\sim}{v}_2$ is made constant; hence $\underset{\sim}{v}_2$ is deterministic, $\underset{\sim}{v}_1$ the translational velocity is deterministic while $\underset{\sim}{v}_3$ remains non-deterministic. Hence, the need for our model to match with the observed constancy of particle spin in our universe helps us to give a suitable specific form for $\underset{\sim}{v}_2$. The ability of our modelling to match the observed constancy of particle spin (and for that matter for different spin values) further enhance the credibility of the modelling.

Fourthly, in Section 6 the Schrödinger equation and the non-deterministic surfing motion of a non-relativistic particle on the *S* surface – together forming the non-relativistic model – is invoked to explain the particle's observed non-determinacy in the two-slit experiment (and the one-slit experiment) and the observed associated interference pattern. The integral yielding the wave function for this experiment satisfies the Helmholtz equation (§12) derived from Schrödinger equation. For the relativistic case (Section 8), in a conservative system the Gordon-Klein equation is shown to be capable of having a positive definite probability density which is the square of the modulus of its wave function. Thus it

[12] But later in the paper it is suggested that photons like particles can possibly have effective mass, albeit extremely small.

is the relativistic equivalent of the Schrödinger equation. While the Schrödinger equation can be applied to particles only, the Gordon-Klein equation can be applied to both particles and photons. This equation and the two surfing velocity components on the S surfaces constitute the modelling framework in Section 8 for studying relativistic particles and photons. This modelling framework is capable of explaining the observed non-determinacy and the observed interference pattern of the two-slit experiment (and the one-slit experiment) for *both particles and photons*. It is thus seen that particles and photons are point-like entities whose behaviour is influenced but not determined by the wave function. The similarities in the *ontological* nature and the *functional* behaviour between relativistic particles and photons, that is, the similarities in their particle nature and their wavy behaviour, can therefore be traced to their sharing of the same Gordon-Klein equation and the same consequent Helmholtz equation (§29), albeit with different values for the mass parameter in the former (but incidentally the mass parameter does not appear in the Helmholtz equation involving the spatial part of the wave function). Hence, the same Helmholtz equation applies in the relativistic cases for photons and particles (§29) and the non-relativistic case for slowing moving particles (§12). This explains the similar wave-like behaviour between particles and photons in the two-slit experiment while maintaining their point-like ontological nature. In this way, the question of wave-particle duality for both particles and photons is dealt with. Hence, the non-relativistic model and the relativistic model used in this paper gain more credibility through their consistencies with the observed phenomena of the two-slit experiments for particles and photons.

Fifthly, when observation is made of the particles or photons of the two-slit experiment, the interference pattern disappears. This can be explained in the two models by an exchange of momentum between a particle or photon and the measuring element such that the two interacting intermediate modes before the momentum exchange reduce to the single cylindrical mode whose probability pattern is much more localised. This prompts the definition of measurement as an event where an exchange of momentum between the measured particle or photon and the measuring agent takes place, leading to a wholesale, sudden and discontinuous change of of the wave function (Section 7). But this sudden change is not a collapse of the wave function, as understood in the Copenhagen interpretation of quantum mechanics. Also, similar change can take place if the classical potential changes in a localised manner, even in the absence of a measurement event. Hence, the introduction of a measurement instrument can already effect a transformed wave field. Again, the models' capacity to be consistent with the observed disappearance of the interference pattern at measurement increases the credibility of the models.

Sixthly, the two models with the suitable forms of the two surfing velocity components on the on the *S* surfaces is consistent with Born's rule which is observed in experiments. The probability density on the *S* surface can be interpreted equivalently (i) as the non-dimensional time density and (ii) as the time averaged probability density with respect to a certain period. For Born's rule to be satisfied, it is essential that the non-deterministic surfing velocity component v_3 evolves in time (within the period) in such a way as to satisfy that rule. The improved pilot wave models by Salesi, Esposito and Recami ([3, 4, 5]) do not have this non-deterministic surfing velocity component and hence cannot be consistent with Born's rule. The idea of satisfying Born's rule within a period of time alleviates us of the paradox of a particle (or an entity such as a cat) being in multiple states simultaneously at one instant – superposition. In our models, the particle can traverse different states at different instants during the period but in such a way that the overall times spent in different states satisfy Born's rule. One can take a time averaged picture of all the different pictures within the period and that average picture will satisfy Born's rule if v_3 behaves accordingly. This process of averaging is like compressing the history of the particle within the period into a single instant; hence, if one interprets the average picture as a single instant, one can conclude erroneously that the particle is in multiple states in that single instant which then leads to the paradox of the Schrödinger's cat, i.e., it can be dead *and* alive at the same instant. However, in reality in a series of the same Schrödinger's cat experiments, the cat in each experiment will be found to be dead or alive and the *average* probability of a cat being found dead or alive over all the experiments will approach the expected probability if the number of experiments tends towards infinity. Again, the averaging process covering the whole series of cat experiments will yield an instantaneous picture of the cat being dead *and* alive at the same time. The above understanding suggests that *the wave function and its governing equation (Schrödinger or Gordon-Klein) do not describe instantaneous events or states but only the time average of a series of instantaneous events or states*. Hence, the Schrödinger equation or the Gordon-Klein equation is not deterministic of individual events or states but deterministic of their time average; hence the room for non-determinacy. The following illustration, also using a cat, will further illustrate the discussion in this paragraph.

The cat sat on the left chair for 3 seconds.

The cat sat on the right chair for 7 seconds.

Figure 10: A Cat in Two Different States (On Two Different Chairs) Over 10 Seconds

The cat in Figure 10 spent 3 seconds and 7 seconds on the left and right chair respectively, but never on both chairs simultaneously. However, the time averaged picture over 10 seconds compresses the history of 10 seconds into an instant. If this average and compressed picture is mistaken to represent a real picture at an instant in time, then it gives the erroneous impression that the cat was on both chairs simultaneously (superposition), with probability density of 0.3 (left) and 0.7 (right) per chair respectively. Furthermore, the cat could spend the 3 seconds and 7 seconds on the respective chairs in many different way, i.e., over multiple visits, hence non-determinacy, while the overall budgetary constraint of 3 seconds and 7 seconds for the respective chairs (cf. Born's rule) is satisfied.[13] In this case, the probability densities of 0.3 (left) and 0.7 (right) can be interpreted as the non-dimensional time densities, or equivalently as the time-averaged probability densities, for the two chairs. The illustration here can be used to understand the particle in free space in cylindrical mode where the particle is free to surf on the *S* surfaces. Over time, the particle will visit all the circular strips within the circle of radius *L* (at that circle the probability density is zero) because of the suitable evolution of the radial velocity $\underset{\sim}{v}_3$, and more and more points in each circular strip will be visited by the particle as time increases because of its spinning velocity, $\underset{\sim}{v}_2$. The particle is at one position at any one time; hence no superposition. But the time averaged picture yields the probability density over space in accordance with Born's rule. Again, the probability density is a time averaged statistics; the Schrödinger equation or the Gordon-Klein equation with the wave function only describes the time averaged statistical behaviour of the particle, not its instantaneous motion; hence the room for non-determinacy

[13] We can say that the time spent by the cat on jumping between the two chairs is negligible in doing the time average. If we wish to include the times and trajectories of the jumping in the averaging process, this will make the process slightly more complicated by introducing more states. But the simple scenario in Figure 10 is sufficient to illustrate the point here.

within that time average. In this respect, one needs to note that the angular momentum of the particle in the free cylindrical mode depends on its radial distance and since its radial distance varies in time, its angular momentum also varies in time. The result of constant angular momentum can be obtained only by averaging the varying angular momentum in time and assuming that Born's rule is satisfied. And this has been taken as the implicit assumption in the integral for the particle's angular momentum in Section 8.3.1. In light of the observed 'constant' angular momentum of a particle, this implies that the radial velocity $\underset{\sim}{v}_3$ is very fast so that all the relevant circular strips can be visited by the particle in a very short time which should be smaller than a typical timescale of the experiment used to measure the particle's angular momentum. To summarise, the two models in this papers are consistent with Born's rule when the probability density in that rule is interpreted as the non-dimensional time density or the time-averaged probability density (for more details, see [2]). Since Born's rule has been verified by experimental observations, the two models here are consistent with these observations which again adds credibility to these models.

The above list of the models' consistencies with observed phenomena together should increase their credibility. However, the following suggests another experiment to further verify or falsify the models.

In Section 8.4, a new definition of energy is proposed: $E_N \equiv pc$ which can equivalently be written as $E_N^2 = p^2c^2 = p_{1N}^2c^2 + p_s^2c^2$. This energy definition prompts the identification between m^2c^4 and $p_s^2c^2$ which suggests that *m*, the effective mass, is generated by the surfing momentum on the *S* surface while the particle itself has an inherent mass which is different from, probably of a greater magnitude than, the rest or effective mass. Since $p_s = mc$, the effective mass, p_s/c , is identical to the rest mass which is the term used conventionally. For the case of zero translational momentum, the base or rest energy according the new definition of energy is also identical to the rest energy according to Einstein's formula. But for non-zero translational momentum, it is virtually certain here (and it will shown to be the case in a forthcoming paper) that the translational momentum p_{1N} is greater than the conventional translational momentum p_{1E} ; and the energy according the new definition will be higher than that given by Einstein's formula – an experiment can be designed to verify the new formula (see below). This revised understanding of energy may give us a better grasp of the nature of rest energy which is called base energy in this paper since the particle is not at rest even when the translational momentum is zero. This base energy is the particle's surfing energy on the S surface and it can be of very large magnitude. An initial scale analysis (not presented here) suggests that most of this energy is carried by the non-deterministic component of the surfing velocity.

A relatively simple experiment can be conducted to see if the new definition of energy is more accurate than Einstein's conventional formula. Since the energy of a particle according to the new formula is mostly likely to be higher than that according to Einstein's formula, and the rest energy is the same in both formulae, for a given non-zero translational velocity the kinetic energy (due to the translational motion, not the surfing motion) of a particle according to the new formula will be higher than that according to Einstein's formula. An experiment correlating the translational speed of a particle with its kinetic energy can therefore decide which formula is closer to observation. In fact, such experiments have already been performed. Bertozzi [14] carried out such an experiment at MIT with electrons in the 1960s. Figure 11 contains the plot of his result.[14]

[14] The last data point from Bertozzi is not used in the plot. For that data point, the kinetic energy is so large that the translational speed is very close to c such that he set it to be equal to c. But that requires infinite kinetic energy, not a finite large kinetic energy. This means that accurately measuring the translational speed so close to c is beyond the capacity of his experiment. Hence, that data point is not used in the plot in Figure 11.

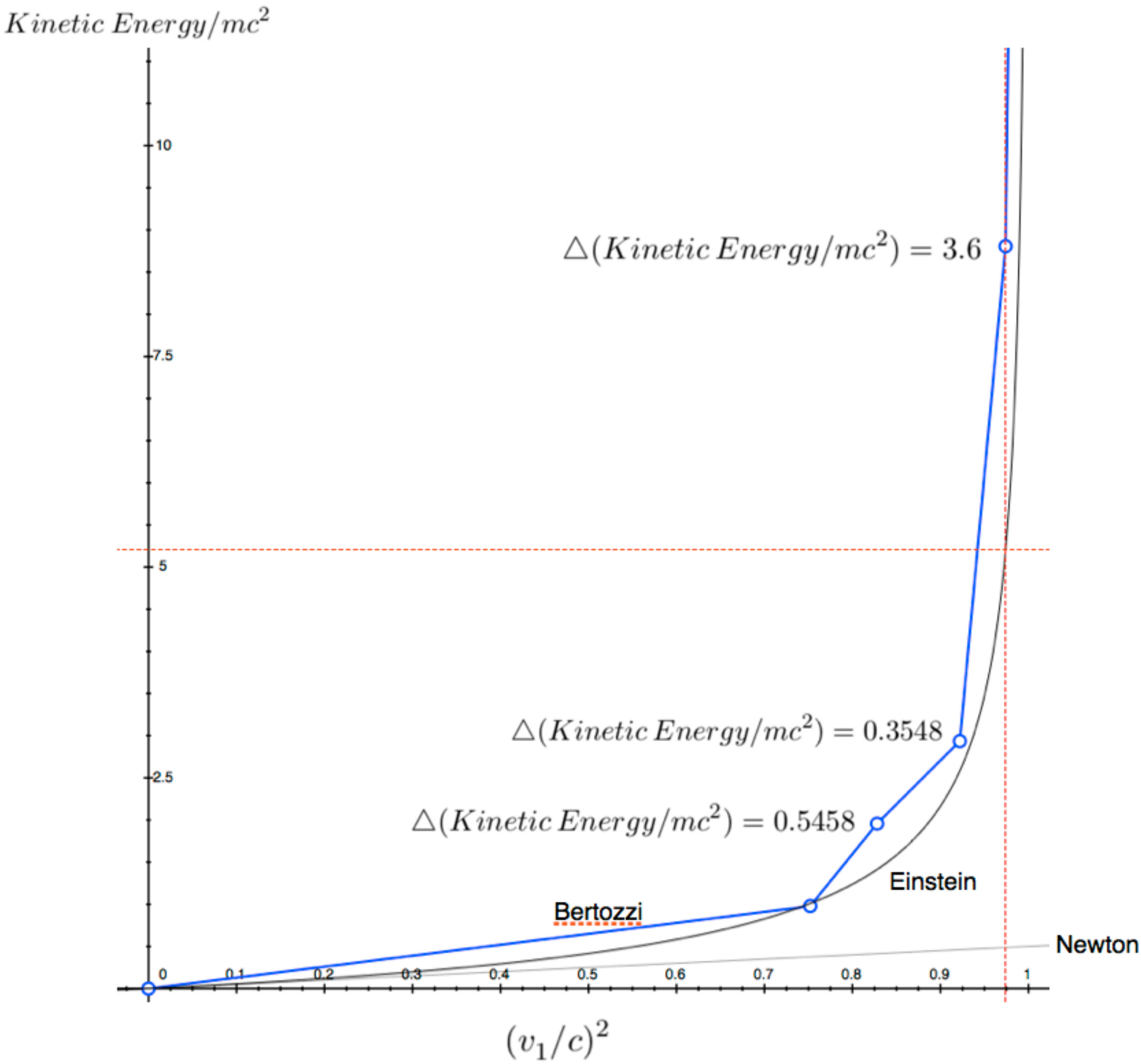

Figure 11: $Kinetic\ Energy/mc^2$ vs $(v_1/c)^2$ for Einstein, Bertozzi and Newton

Bertozzi's experimental data follow the general trend of the curve for Einstein's formula but we can see from his data that there is an anomaly, i.e., apart from the data point with the lowest speed, Einstein's formula consistently underestimates the kinetic energy of the electron at high speed, which is predicted by the new energy formula proposed in this paper; see the three values of $\triangle(Kinetic\ Energy/mc^2)$ which give the differences between Bertozzi's data and the values given by Einstein's formula. The greatest difference marked by the red dotted lines is 3.6 times of the electron's rest energy (or base energy) which is a very large amount of energy for the electron. The other two differences are 0.5458 and

0.3548 times of the electron's rest energy (or base energy) which are considerable amounts for an electron. Because Bertozzi was looking to see whether the data follow the trend of the curve corresponding to Einstein's relativistic formula or the curve corresponding to Newton's non-relativistic formula, he had his conclusive answer to this question, i.e., Einstein's formula is much better than Newton's formula. However, he failed to see that there are consistent and significant energy deficits in Einstein's formula compared to his observed data. Perhaps, he might have put it down to some experimental errors in measuring the electron's speed. However, the observed data for high speed electrons consistently fall on only one side of Einstein's curve, as the new formula predicts. Bertozzi did not discuss the energy anomaly probably because he was interested in his original question concerning Einstein's formula and Newton's formula, and nothing more.

Lund and Uggerhøj [15], two physicists from Denmark, also performed a similar experiment with electrons in 2009 as Bertozzi did in 1964 but with smaller kinetic energies.

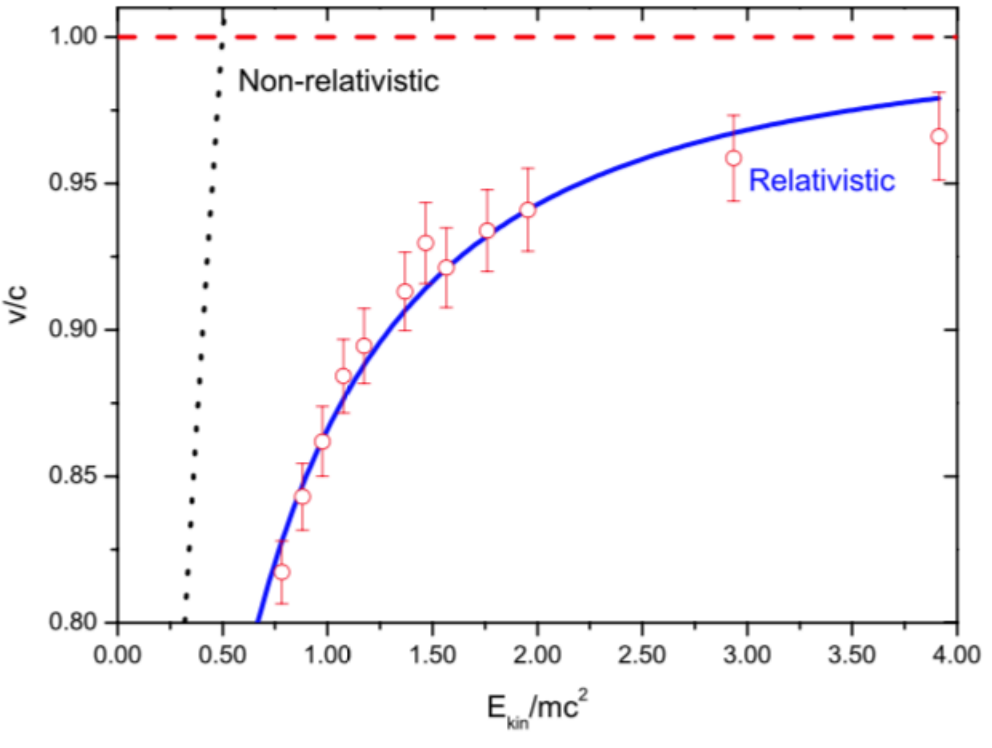

Figure 12: v_1/c (normalised translational speed, *v/c* in graph) vs $Kinetic\ Energy/mc^2$ for Einstein (blue curve), and Newton (black dotted line); with data by Lund and Uggerhøj; graph borrowed from [15]

Note that in Figure 12, the plot is v_1/c versus $Kinetic\ Energy/mc^2$ whereas the plot in Figure 11 is $Kinetic\ Energy/mc^2$ versus $(v_1/c)^2$. At high speeds with v_1/c greater than 0.95, the measured kinetic energies are again significantly larger than those provided by Einstein's formula, as found in Bertozzi's data. For the two data points with the two highest energies, Einstein's formula underestimates the measured kinetic energies by 0.424 and 1.039 times of the electron's rest energy.[15] The trend indicated in the data by Lund and Uggerhøj and the data by Bertozzi is that *the magnitude of the underestimation by Einstein's formula increases with speed*. This trend will be echoed in the next paper where it will be shown that the kinetic energy of a particle according to the new formula proposed in this paper exceeds the kinetic energy according to Einstein's formula and the difference will tends to infinity as the speed increases towards *c*.[16] The next paper will also analyse to what extent the new formula will fare better than Einstein's formula in comparing with the observed data. Because of the weightiness of the matter, even though there are the five data points (three from Bertozzi and two from Lund and Uggerhøj) confirming the deficit in Einstein's energy formula, more experimental data need to be collected in order to further confirm the underestimation in Einstein's formula as predicted by the new formula.

Because the new formula for energy is based on the notion of surfing momentum on the *S* surface and this notion has been used to construct the non-relativistic model and the relativistic model used in this paper, a confirmation of the new formula by experimental data will enhance the credibility of the notion of the surfing momentum and will add further credibility to the two models for quantum mechanics proposed in this paper.

Physicists have been looking for the mysterious dark matter required to hold galaxies together. If one takes the proposed revised definition of energy seriously, one can investigate further whether the higher value of the energy given by this definition, compared to that given by the conventional formula, can at least partially account for the missing energy. If this turns out to be possible, then we can see the connection between particle motion on the small quantum scale and the motion of celestial bodies of the large scale, i.e., small scale

---

[15] The data for the measured kinetic energy and speed of the electrons are kindly provided by Professor Uggerhøj. These data are used to calculate the underestimation of the kinetic energy by Einstein's formula.

[16] Because of the length of the present paper which is already long, the detailed comparison between the new formula for energy and Einstein's formula ought to be dealt with in another paper.

particle motion can at least partially contribute to the enhancement of energy necessary to account for the coherence of celestial galaxies.

If the new definition of energy, $E_N \equiv pc$, turns out to be a better definition of energy than the traditional formula, $E = m_r c^2$, then the new definition of energy should be used when higher accuracy is needed and its implications should be explored in physics.

The Gordon-Klein equation in (§36) with the surfing momentum replacing the rest/effective mass is arguably more original than the form in (§18) featuring the rest/effective mass since the rest/effective mass is derived from the surfing momentum. The form in (§36) gives us a heuristic appreciation of the various energy terms involved in the conservation of energy. Since this equation also incorporate the quantum-potential-like term, it embodies more energy terms than the definition of energy squared: $E_N^2 \equiv p_{1N}^2 c^2 + p_s^2 c^2$. Instead of seeing the Gordon-Klein equation in (§36) as an equation derived from quantising this energy definition, one should see the Gordon-Klein equation as being the more fundamental equation as E_N^2 only accounts for a subset of the energy terms in the equation. This is clearly seen in the new quantity derived in this paper:

$$K^2 \equiv c^2\hbar^2(\nabla S)^2 + m^2c^4 - c^2\hbar^2\frac{\nabla^2 R}{R} = E_N^2 - c^2\hbar^2\frac{\nabla^2 R}{R}$$

where the last term is like the quantum potential term. *K* has the unit of energy and is the conserved in the absence of exchange of energy. Its expression involves the new energy and the quantum-potential-like term. Let us call *K* the super-energy since it incorporates one more term than the revised definition of energy. One wonders if the quantum-potential-like term is related to the dark energy necessary to account for the increasing inflation rate of the universe (see attempts to relate the quantum potential to dark energy in [16, 17, 18]). It is certain that the total energy of a particle or a photon will have to include this term and it is ubiquitous even though its value may vary from the value in free space when particles interact with one another as in tunnelling. In tunnelling, it is possible that a particle enters a region of lower $c\hbar\sqrt{-\frac{\nabla^2 R}{R}}$ so that its E_N increases sufficiently to overcome the potential barrier which would have been impossible without this quantum-potential-like term. Paper

[2] shows that for finite surfing energy, $-c^2\hbar^2\frac{\nabla^2 R}{R}$ has to be positive. It therefore has the right sign for becoming a candidate to account for the dark energy. Even though the spin value does not depend on this quantum-potential-like term, the magnitude of the spin velocity, $\underset{\sim}{v}_2$, does depend on it since it influences the distribution of the probability density on the *S* surface; hence photons may have a low value for this term which corresponds to a small $\underset{\sim}{v}_2$. It seems that this term has different values for different particles and photons. It is plausible that for free space there is a universal density of $ch\sqrt{-\frac{\nabla^2 R}{R}}$ and its value for a particle or photon could be given by the product of this universal density and the volume of the particle or photon. If that is the case, one can possibly integrate the density of $ch\sqrt{-\frac{\nabla^2 R}{R}}$ over the whole universe to give us an estimate of this additional energy in the universe. Whether this can account for the dark energy in the universe remains to be seen.

Even though the motion of a particle on the *S* surface is non-determinate due to the non-deterministic direction of $\underset{\sim}{v}_3$, it does not mean that the motion is random. One must be careful to make the distinction between randomness and non-determinacy. The non-determinate motion on the *S* surfaces, e.g., in the cylindrical mode, has to be well co-ordinated to match the ρ distribution according to Born's rule; otherwise, that rule will be violated and the particle's characteristic spin will be lost leading to a chaotic and non-life-sustaining universe which could be the case in an alternative universe having the same governing Gordon-Klein equation. One may ask: how does the particle know how to behave in order to match the ρ distribution, satisfy Born's rule and in so doing produce the characteristic spin? It requires some kind of knowledge or information of the recent history of the trajectory of the particle, at least within the period of *T*. Yet, even with this information available, this information along with all other information and laws operating in the universe since its primordial beginning cannot prescribe a deterministic velocity or trajectory at any succeeding moment in time for the particle. All one can say is that over a suitable period of *T*, *somehow* the particle traces out a trajectory with suitable non-deterministic and non-random velocity to match the ρ distribution on the *S* surface according to Born's rule and thus produces the characteristic spin. This requires new information on top of the information of the particle's history. The baffling question is: what causes such delicately balanced non-deterministic trajectories and velocities without which our universe will have little order and will not sustain life as we know it? And what is the source of the new information which is required moment by moment to move the particles and photons of the universe forward in time? The generation of this information might be simulated by some stochastic process but such simulating processes do not inform us any more about the causes of the particle's highly organised non-determinate motion on the *S* surface since (i) these stochastic processes are simulations in the first place and therefore they can only simulate the effect but not the cause of such motion, and (ii) there could be multiple stochastic processes 'successfully' simulating the generation of the required information so that we have no way of knowing which simulation is nearer to reality. Here, we may be reaching the limit of what we can know. Deterministic processes are more amenable to our knowledge; the causes of non-deterministic but highly organised processes cannot be so easily pinned down, or cannot be pinned down.

10.2 Conclusion

Neither the Schrödinger equation nor the Gordon-Klein equation with their wave function gives any specific information of the three velocity components of a particle. This may not be an disadvantage because it allows us to prescribe the three velocity components according to *observation* in our universe (in another hypothetical universe, these velocities could be different). In this paper, firstly as in the usual pilot wave theory we have prescribed the translational velocity according to the observed de Broglie relation between momentum and wavelength (which is related to ∇S). Secondly, we have also prescribed the form of $\underset{\sim}{v}_2$ to match the different observed particle spins. Thirdly, we have prescribed the non-deterministic $\underset{\sim}{v}_3$ in such a way that Born's rule is satisfied which is also observed. Hence, all three velocities have been prescribed to match our observations in this universe. These correspondences yield the advantage that the non-relativistic model and the relativistic model proposed in this paper matches the above three observations in this universe.

Apart from the consistency with these three observations, the proposed models yield the observed relationship between energy and frequency. They also have the capacity to account for the non-determinacy and the interference pattern observed in the two-slit experiment. The models account for the observed ontological particle nature of *both* particles and photons. They also account for their observed wavy functional behaviour through their sharing of the same Helmholtz equation which is derived from the Schrödinger equation or the Gordon-Klein equation. The models give us a concrete notion of measurement and how measurement can destroy the interference pattern of the two-slit experiment as has been observed. Furthermore, it gives a reasonable account of tunnelling which is also observed.

The relativistic model gives us some physical understanding of the meaning of rest energy (or base energy) which is otherwise hard to imagine. Regarding the observations on cosmological scale, it has the potential capacity to at least partially account for the elusive dark matter which was postulated as a result of our observation of galaxies. Despite its potential, we do not yet know if the model can contribute positively to the understanding of the dark energy of the universe which has been inferred from observation. Nevertheless, the above has listed *eleven observations* with which the proposed models are consistent with:

1. non-determinacy
2. de Broglie relation between momentum and wavelength
3. Planck-Einstein relation between energy and frequency
4. Born's rule
5. spin
6. ontological particle nature of *both* particles and photons
7. the observed wavy functional behaviour of *both* particles and photons explained through their sharing of the same Gordon-Klein equation and the consequent Helmholtz equation
8. the interference pattern observed in the two-slit experiment for slowing moving particles and the interference pattern of the same experiment for relativistic particles and photons (explained by the sharing of the same Helmholtz equation in these three cases)
9. measurement which destroys the interference pattern of the two-slit experiment (explained by the sudden wholesale transformation of the existing wave function into a new wave function for a cylindrical mode of motion with localised probability density distribution)
10. tunnelling
11. rest energy.

Regarding the new formula for energy, some initial data from Bertozzi, Lund and Uggerhøj suggest that the new formula can possibly account for the kinetic energy of a particle better than Einstein's formula. If this is confirmed by further experiments, it may have some implication for our understanding of dark matter. Lastly, the relativistic model could possibly contribute to our understanding of dark energy.

The philosophy of science teaches us that no theory can be conclusively proved to be correct since in principle every theory is falsifiable and could be improved by a better theory. One can only speak of the credibility of a theory or model and its proximity to reality. The credibility of a theory or model should be gauged according to its consistency with observation, no matter how elegant a theory or model is. The proposed models can have some claim to elegance, e.g., the prescription of a set of orthogonal velocities which include the two crucial surfing velocity components on the *S* surface, the simple expression for the new definition of energy, $E_N \equiv pc$, and the more comprehensive energy, *K,* the super-energy. Concerning the claim to consistency, the two proposed models are consistent with at least eleven features observed in our universe (the issues of dark matter and dark energy need to be investigated further). Hence, the proposed models with their claim to consistency with observation, and with some claim to elegance, have a good claim to credibility. Their credibility may be further advanced if they prove to have the capacity to explain other observed experimental phenomena not listed above, e.g., the experiment suggested at the end of Section 11 of paper [2], and some experiments questioning the Heisenberg Uncertainty Principle [19]. It is for the experimentalists to explore further the models' explanatory capacity in assessing its credibility and its proximity to reality.

References

Books

Louis de Broglie, *Non-linear Wave Mechanics : a causal interpretation*; Amsterdam; New York : Elsevier Pub. Co. (1960).

Richard Feynman, *The Feynman Lectures on Physics*, Vol. 3; https://www.feynmanlectures.caltech.edu/III_01.html, accessed on 27th Oct., 2020.

Joseph W. Goodman, *Introduction to Fourier Optics*, 3rd edition, Englewood: Roberts and Company (2005).

Articles

[1] Damon W. K. So. 'Non-deterministic Quantum Mechanics', *Mathematical Physics Preprint Archive* at https://web.ma.utexas.edu/mp_arc/, University of Texas, Archive Number **18**-119 (2018).

[2] Damon W. K. So. 'Non-deterministic Quantum Mechanics and Particle Spin', *Mathematical Physics Preprint Archive* at https://web.ma.utexas.edu/mp_arc/, University of Texas, Archive Number **18**-120 (2018).

[3] S. Esposito. 'On the role of Spin in Quantum Mechanics', *Foundations of Physics Letters*, **12(2)**, 165-177 (1999).

[4] G. Salesi. 'Spin and Madelung Fluid', *Modern Physics Letters*, **A11**, 1815 (1996).

[5] E. Recami, G. Salesi. 'Hydrodynamics of Spinning Particles', *Physical Review A*, **57(1)**, 98-105 (1998).

[6] D. Bohm. 'A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. II', *Physical Review*, **85**(2), 166–179 (1952).

[7] A. Einstein, B. Podolsky, N. Rosen. 'Can Quantum-Mechanical Description of Physical Reality Be Considered Complete' ?', *Physical Review*, **47**, 777-780 (1935).

[8] David Hestenes. 'The *Zitterbewegung* Interpretation of Quantum Mechanics', *Foundations of Physics*, **20(10)**, 1213-1232 (1990).

[9] M.A. Grado-Caffaro , M. Grado-Caffaro. 'Photon Velocity from the Klein–Gordon Equation', *Optik*, **121**, 2094–2095 (2010).

[10] Bert Pariser. 'Photon Mass', presented to American Society for Engineering Education, Internet link to paper (2012).

[11] Jun-Jie Wei, Xue-Feng Wu. 'Robust Limits on Photon Mass from Statistical Samples of Extragalactic Radio Pulsars', arXiv:1803.07298 [astro-ph.HE] (2018).

[12] Roderic Lakes. 'Experimental Limits on the Photon Mass and Cosmic Magnetic Vector Potential', *Physical Review Letters,* **80**(9), 1826-1829 (1998).

[13] Donald C. Chang. 'A Classical Approach to the Modeling of Quantum Mass', *Journal of Modern Physics*, **4**, 21-30 (2013).

[14] William Bertozzi, 'Speed and Kinetic Energy of Relativistic Electrons', *American Journal of Physics*, **32** (7), 551-555 (1964). See also his video taken at MIT at https://www.youtube.com/watch?v=B0BOpiMQXQA, accessed 28th May, 2021.

[15] M. Lund, U. I. Uggerhøj, 'Experimental special relativity with a meter stick and a clock', *American Journal of Physics*, **77** (8), 757–761 (2009).

[16] Pedro F. Gonzalez-Diaz. 'Sub-Quantum Dark Energy', *Physical Review D*, **69,** 103512 (2004). Arxiv ID: astro-ph/0311244.

[17] Ahmed Farag Ali, Saurya Das. 'Cosmology from quantum potential', *Physics Letters B*, **741(4)**, 276-279 (2015).

[18] Dongshan He, Dongfeng Gao, Qing-yu Cai. 'Spontaneous creation of the universe from nothing', *Physical Review D*, **89(8) (**4/2014).

[19] Lee A. Rozema, Ardavan Darabi, Dylan H. Mahler, Alex Hayat, Yasaman Soudagar, and Aephraim M. Steinberg. 'Violation of Heisenberg's Measurement-Disturbance Relationship by Weak Measurements', *Physical Review Letters,* **109**, 100404 (2012)