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The D'Alembert–Lagrange Principle: a Geometrical Aspect

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Abstract—The d'Alembert–Lagrange Principle and the theory of ideal connections are considered from the viewpoint of modern differential geometry and tensor analysis.

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The D'Alembert–Lagrange Principle and the theory of ideal constraints play the central role in theoretical mechanics. In this article, we address this theory in terms of modern differential geometry and tensor analysis. This article is purely methodological.

The traditional presentation of these questions are contained, for example, in [1]. We use the apparatus of differential geometry [2].

1. FUNCTIONS ON A TANGENT BUNDLE

Let Y be a smooth s-dimensional manifold with local coordinates $y = (y^1, \dots, y^s)$ and let TY be its tangent bundle with local coordinates $(y, \dot{y}) = (y^1, \dots, y^s, \dot{y}^1, \dots, \dot{y}^s)$. In what follows, all objects (manifolds, mappings, and tensor fields) are assumed C^∞-smooth.

Introduce the function $\mathcal{L} : (t_1, t_2) \times TY \to \mathbb{R}$, $\mathcal{L} = \mathcal{L}(t, y, \dot{y})$.

Definition 1. The set of functions

$$[\mathcal{L}]_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}^i} - \frac{\partial \mathcal{L}}{\partial y^i}$$

is called the *Lagrangian derivative* of $\mathcal{L}$.

The Lagrangian derivative is linear and possesses the following property: For every function $f : (t_1, t_2) \times Y \to \mathbb{R}$ the identity holds:

$$\left[\frac{df}{dt}\right]_i = 0, \qquad \dot{f} = \frac{df}{dt} := \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y^l}\dot{y}^l.$$

Let X be a smooth r-dimensional manifold with local coordinates $x = (x^1, \dots, x^r)$, $r \le s$. Denote by $\varphi : (t_1, t_2) \times X \to Y$, $y^i = \varphi^i(t, x)$ a mapping that is an *embedding* for every fixed t. An *embedding* is a mapping such that for every t the image $M(t) = \varphi(t, X) \subset Y$ is a smooth manifold; and, moreover, in a neighborhood of every point $y \in M(t)$ of Y there exist some local coordinates $z^1, \dots, z^s$ (they smoothly depend on t) such that $M(t)$ is given by the system of equations

$$z^l = 0, \qquad l = 1, \dots, s - r; \qquad \operatorname{rang} \frac{\partial \varphi}{\partial x} = r.$$

Theorem 1. *The next formulas hold:*

$$\frac{\partial \varphi^i}{\partial x^j}[\mathcal{L}]_i = [L]_j,$$

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where $L : (t_1, t_2) \times TX \to \mathbb{R}$ *is determined by the formula*

$$L(t,x,\dot{x}) = \mathcal{L}\left(t, \varphi(t,x), \frac{\partial \varphi}{\partial x^l}\dot{x}^l + \frac{\partial \varphi}{\partial t}\right). \tag{1}$$

In particular, the Lagrangian derivative behaves like a covector field regarding the coordinate changes on Y.

Proof. Calculate the derivatives:

$$\frac{\partial L}{\partial x^i} = \frac{\partial \mathcal{L}}{\partial y^m}\frac{\partial \varphi^m}{\partial x^i} + \frac{\partial \mathcal{L}}{\partial \dot{y}^m}\left(\frac{\partial^2 \varphi^m}{\partial x^l \partial x^i}\dot{x}^l + \frac{\partial^2 \varphi^m}{\partial t \partial x^i}\right);$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}^m}\frac{\partial \varphi^m}{\partial x^i}\right) = \frac{\partial \varphi^m}{\partial x^i}\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}^m}\right) + \frac{\partial \mathcal{L}}{\partial \dot{y}^m}\left(\frac{\partial^2 \varphi^m}{\partial x^l \partial x^i}\dot{x}^l + \frac{\partial^2 \varphi^m}{\partial t \partial x^i}\right).$$

Theorem 1 is proved. □

Theorem 2. *If the quadratic form with the matrix*

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{y}^2}(t,y,\dot{y}) \tag{2}$$

is positive-definite for all $(t,y,\dot{y}) \in (t_1,t_2) \times TY$ *then the quadratic form with the matrix* $\dfrac{\partial^2 L}{\partial \dot{x}^2}(t,x,\dot{x})$ *is also positive-definite for all* $(t,x,\dot{x}) \in (t_1,t_2) \times TX$.

Proof. Indeed,

$$\frac{\partial^2 L}{\partial \dot{x}^k \partial \dot{x}^j} = \frac{\partial^2 \mathcal{L}}{\partial \dot{y}^i \partial \dot{y}^p}\frac{\partial \varphi^i}{\partial x^k}\frac{\partial \varphi^p}{\partial x^j}.$$

The matrix

$$a_{kj} = \frac{\partial^2 \mathcal{L}}{\partial \dot{y}^i \partial \dot{y}^p}\frac{\partial \varphi^i}{\partial x^k}\frac{\partial \varphi^p}{\partial x^j}$$

is positive-definite since it is the Gramian matrix of the vectors

$$u_k = \left(\frac{\partial \varphi^1}{\partial x^k}, \ldots, \frac{\partial \varphi^s}{\partial x^k}\right)$$

with respect to the inner product given by (2).

Theorem 2 is proved. □

2. THE COVARIANT VERSION OF THE D'ALEMBERT–LAGRANGE PRINCIPLE

Hereinafter, we consider that the quadratic form with matrix (2) is positive-definite.

Consider the following system of differential equations:

$$[\mathcal{L}]_i = F_i(t,y,\dot{y}), \qquad i = 1, \ldots, s. \tag{3}$$

The function $\mathcal{L}$ is called the *Lagrangian function* of (3).

The functions on the right side are called *active forces*, and they are supposed given. The functions F_i and $\mathcal{L}$ as well are defined on the extended phase space $(t_1,t_2) \times TY$ and are transformed according to the covector law.

System (3) is the system of differential equations of order $2s$ for $y(t)$. On the assumption made about (2), this system is solvable for $\ddot{y}$. The phase space of system (3) is TY. The manifold Y is called the *configurational space* of (3).

Introduce the 1-forms $\omega^q = \omega^q_j dy^j$, $q = 1, \dots, \nu < s$, in Y. It means that $\{\omega^q_j(t, y)\}$ behave as the components of a covector field under changes of the coordinates on the manifold Y. Wherein the coordinate changes either do not depend on time or time is considered as a parameter.

We assume that rang $(\omega^q_j(t, y)) = \nu$, $(t, y) \in (t_1, t_2) \times Y$. This condition means that the 1-forms ω^q are linearly independent on each tangent space T_yY for every t.

Introduce more functions:

$$\psi^q(t, y, \dot{y}) = \omega^q_j(t, y)\dot{y}^j + \beta^q(t, y). \tag{4}$$

The equations

$$\psi^q(t, y, \dot{y}) = 0, \qquad q = 1, \dots, \nu, \tag{5}$$

are called the *equations of differential constraints*, and they define in the extended phase space of the system (3) some smooth submanifold

$$W \subset (t_1, t_2) \times TY, \qquad \dim W = 2s + 1 - \nu.$$

The connection of W with the system itself is explained by the following definitions:

Definition 2. The subspace

$$\Delta(t, y) = \{\delta y = (\delta y^1, \dots, \delta y^s) \in T_yY \mid \omega^q_j(t, y)\delta y^j = 0,\ q = 1, \dots, \nu\} \tag{6}$$

is called the *space of virtual displacements*. The elements of this space are called the *virtual displacements*.

Then $\dim \Delta(t, y) = s - \nu$ is called the *number of degrees of freedom* of the system.

Definition 3. Assume that we can choose some R_i forces that are defined on $(t_1, t_2) \times TY$ so that

(1) W turns out to be the invariant manifold of the system

$$[\mathcal{L}]_i = F_i(t, y, \dot{y}) + R_i(t, y, \dot{y}), \qquad i = 1, \dots, s; \tag{7}$$

(2) the equality

$$R_i(t, y, \dot{y})\delta y^i = 0 \tag{8}$$

holds for every vector $\delta y \in \Delta(t, y)$.

In this case we say that the *ideal constraints* (5) *are imposed* on (3) or the *system with ideal constraints* is given.

The forces R_i are called the *reactions* of ideal constraints. If there is no constraints then

$$\Delta(t, y) = T_yY, \qquad W = (t_1, t_2) \times TY, \qquad R_i = 0.$$

Remark. In a system with ideal constraints we are interested in the dynamics of (7) only on W. Therefore, the behavior of the forces F_k and R_k outside W does not matter. However, in problems of mechanics, these forces turn out to be naturally determined all over the whole extended phase space $(t_1, t_2) \times TY$. If these forces are initially given on W then they can be extended to smooth functions on $(t_1, t_2) \times TY$. Of course, it is nonunique continuation but this does not influence the dynamics of the system on W.

Often constraint equations are of the form $f^q(t, y) = 0$, $q = 1, \dots, \nu$. These constraints are called *geometric*. They are reduced to the form (5) by time differentiation:

$$\frac{\partial f^q}{\partial t} + \frac{\partial f^q}{\partial y^i}\dot{y}^i = 0, \qquad q = 1, \dots, \nu. \tag{9}$$

Definition 4. If there is a set of functions $f^1, \dots, f^\nu$ such that W is defined by (9) then the constraints (5) are called *holonomic*. Otherwise, the constraints are *nonholonomic*.

The next is a direct corollary of (7) and (8):

Theorem 3. *Suppose that* (3) *is the system with ideal constraints* (5) *and* $y(t)$ *is a solution of* (7). *Then* $y(t)$ *satisfies the equation*

$$([\mathcal{L}]_i - F_i)\delta y^i = 0 \tag{10}$$

for all $\delta y \in \Delta(t, y(t))$.

Equation (10) is called the *general dynamic equation*.

Theorem 4. *For every sets of the forces* F_k *and the initial conditions*

$$(t_0, y_0, \dot{y}_0) \in W \tag{11}$$

there exists a unique $y(t)$ *such that*

(1) $y(t_0) = y_0$ *and* $\dot{y}(t_0) = \dot{y}_0$;

(2) $y(t)$ satisfies (5);

(3) $y(t)$ satisfies (10) for all $\delta y \in \Delta(t, y(t))$.

Proof. Suppose first that for each initial condition (11) the specified function $y(t)$ exists, and prove its uniqueness.

Lemma 1 [3]. *Let* E *be a vector space, and let* $u, u_1, \ldots, u_l : E \to \mathbb{R}$ *be some linear functionals. If*

$$\bigcap_{k=1}^{l} \ker u_k \subseteq \ker u$$

then there are reals $\lambda_1, \ldots, \lambda_l$ *such that*

$$u = \sum_{k=1}^{l} \lambda_k u_k.$$

From this lemma together with (6) and (10) for $t = t_0$ we obtain

$$[\mathcal{L}]_i\big(t_0, y_0, \dot{y}_0, \ddot{y}(t_0)\big) - F_i(t_0, y_0, \dot{y}_0) = \lambda_j \omega_i^j(t_0, y_0). \tag{12}$$

Let us show that as λ_j you can take functions from $C^\infty\big((t_1, t_2) \times TY\big)$; moreover, the restrictions $\lambda_j|_W$ are uniquely defined. We denote the inverse matrix to (2) by $g^{ij}(t, y, \dot{y})$. Then (12) takes the form

$$\ddot{y}^j(t_0) = g^{ij}(t_0, y_0, \dot{y}_0)\omega_i^k(t_0, y_0)\lambda_k + u_1^j(t_0, y_0, \dot{y}_0). \tag{13}$$

Here and below, u_p^j are some given smooth functions.

Let us differentiate (5): $\omega_l^q(t_0, y_0)\ddot{y}^l(t_0) = u_2^q(t_0, y_0, \dot{y}_0)$. Inserting here $\ddot{y}$ from (13), we find that

$$\omega_j^q g^{ij} \omega_i^k \lambda_k = u_3^q(t_0, y_0, \dot{y}_0). \tag{14}$$

We obtained a system of linear algebraic equations for λ_k. The matrix of this system $\omega_j^q g^{ij} \omega_i^k$ is nondegenerate since this is the Gramian matrix of vectors $\xi^q = \big(\omega_1^q, \ldots, \omega_s^q\big)$ with respect to the inner product g^{ij}.

Hence, $\lambda_k = \lambda_k(t_0, y_0, \dot{y}_0)$ is uniquely found from (14).

Since the functions in (14), are defined on the whole extended phase space $(t_1, t_2) \times TY$, we can consider the functions λ_k also defined on the whole extended phase space. However, since (14) was derived only for initial conditions (11), λ_k are uniquely determined only on W. Thus, $y(t)$ satisfies the equation

$$[\mathcal{L}]_i - F_i = \lambda_j(t, y, \dot{y})\omega_i^j. \tag{15}$$

Note that (15) is called the *Lagrange equation with multipliers*. The uniqueness of $y(t)$ follows from the Cauchy Theorem of the existence and uniqueness of solution to (15).

By the construction, ψ^q is the first integral of (15); therefore, the existence also follows from the Cauchy Theorem.

The proof of Theorem 4 is complete. □

As the consequence we obtain the following

Theorem 5 (release from principal bundles). *For every set of forces F_i there is a set of reactions R_i such that* (8) *is completed for all virtual displacements and W is an invariant manifold for* (7). *Moreover, narrowing $R_i|_W$ is defined unambiguously. The reactions can be taken as*

$$R_i(t, y, \dot{y}) = \lambda_j(t, y, \dot{y})\omega_i^j(t, y).$$

Theorems 3–5 present the d'Alembert–Lagrange Principle.

3. REDUCTION OF A SYSTEM WITH GEOMETRIC CONSTRAINTS

Suppose that in addition to the differential constraints (5) the geometric constraints are put on the system:

$$f^i(t, y) = 0, \qquad i = 1, \dots, \mu, \qquad \mu + \nu < s; \tag{16}$$

and the differential forms

$$\omega^i, \qquad \delta f^j := \frac{\partial f^j}{\partial y^l} dy^l, \qquad i = 1, \dots, \nu, \qquad j = 1, \dots, \mu, \tag{17}$$

are linearly independent everywhere. From (16) we have the differential constraints

$$\frac{\partial f^i}{\partial t} + \frac{\partial f^i}{\partial y^j}\dot{y}^j = 0, \quad i = 1, \dots, \mu. \tag{18}$$

We will consider (3) with ideal constraints (5) and (18). Now, W is given by (5) and (18). Let us remind that, by the definition of a system with ideal constraints, W is the invariant manifold of (7).

If we narrow (7) on W then f^i become the first integrals of the so-obtained the narrowing system.

The space of virtual displacements has the form

$$\Delta(t, y) = \left\{\delta y \in T_y Y \ \mid \ \omega_k^q(t, y)\delta y^k = 0,\ \frac{\partial f^j}{\partial y^l}\delta y^l = 0,\ j = 1, \dots, \mu,\ q = 1, \dots, \nu\right\}.$$

For every fixed t system of equations (16) defines in Y a submanifold of dimension $r = s - \mu$. Denote this manifold by $M(t)$. We will assume that this manifold is the embedding image $\varphi(t, \cdot)$ of some r-dimensional manifold X in Y:

$$\varphi(t, X) = M(t), \qquad f^i(t, \varphi(t, x)) = 0.$$

In what follows, we will use the theory and notations of Section 1.

In mechanics, the local coordinates x on the manifold X are called the *generalized coordinates*, and

$$Q_i(t, x, \dot{x}) = \frac{\partial \varphi^k(t, x)}{\partial x^i} F_k\left(t, \varphi(t, x), \frac{\partial \varphi(t, x)}{\partial x^k}\dot{x}^k + \frac{\partial \varphi(t, x)}{\partial t}\right)$$

are called the *generalized forces*. Further we will not use this term and continue to call Q_i the *active forces*.

Introduce the functions

$$\Psi^q(t, x, \dot{x}) = \Omega_j^q(t, x)\dot{x}^j + B^q(t, x), \qquad q = 1, \dots, \nu, \tag{19}$$

where

$$\Omega_j^q = \frac{\partial \varphi^u(t, x)}{\partial x^j}\omega_u^q(t, \varphi(t, x)), \qquad B^q = \beta^q(t, \varphi(t, x)) + \omega_j^q(t, \varphi(t, x))\frac{\partial \varphi^j(t, x)}{\partial t}.$$

Formulas (19) are obtained by insertion of $y = \varphi(t, x)$ into (4). Thus, for fixed t the differential forms $\Omega^q = \Omega_j^q dx^j$ are the result of the operation "pull-back" applied to the narrowing of the forms ω^q on $M(t)$.

Theorem 6. *The differential forms Ω^q are linearly independent for all* (t, x).

Proof. Since φ is an embedding, the linear operator

$$\delta\varphi := \frac{\partial\varphi}{\partial x} : T_xX \to T_{\varphi(t,x)}, \qquad M(t) = \bigcap_{j=1}^{\mu} \ker \delta f^j$$

is an isomorphism.

Assume that there is a nontrivial linear combination of Ω^q that is equal to zero

$$\lambda_q\Omega^q = \lambda_q\omega^q \circ \delta\varphi = 0,$$

and not all λ_q are zeros. But this means that

$$\bigcap_{j=1}^{\mu} \ker \delta f^j \subset \ker(\lambda_q\omega^q).$$

Therefore, $\lambda_q\omega^q$ is the linear combination of the forms δf^j. However, this is impossible for the independent forms (17).

Theorem 6 is proved. □

Thus, we will consider the system with the configurational space X, Lagrangian (1), active forces Q_i, and ideal constraints

$$\Psi^q(t, x, \dot{x}) = 0, \qquad q = 1, \ldots, \nu. \tag{20}$$

The corresponding virtual displacement space has the form

$$\Lambda(t, x) = \big\{\delta x \in T_xX \mid \Omega^q_j(t, x)\delta x^j = 0,\ q = 1, \ldots, \nu\big\}.$$

The number of degrees of freedom of this system is equal to $r - \nu = s - \nu - \mu$.

By Theorem 2, for the general equation of dynamics

$$([L]_i - Q_i)\delta x^i = 0, \qquad \delta x \in \Lambda(t, x), \tag{21}$$

with constraints (20). Theorem 4 and 5 hold from which the generalized forces of reactions of the ideal constraints are recovered, and so Theorem 3 is proved.

If the differential constraints (20) are absent and there are only constraints (16), i.e. $\Lambda(t, x) = T_xX$; then (21) turns out to the Lagrange equation of the second kind $[L]_i - Q_i = 0$, $i = 1, \ldots, r$, on the manifold X.

Note that $\delta\varphi\ :\ \Lambda(t, x) \to \Delta(t, \varphi(t, x))$ is the isomorphism.

By Theorem 1,

$$([\mathcal{L}]_i - F_i)\delta y^i = ([L]_j - Q_j)\delta x^j, \qquad \delta y^i = \frac{\partial\varphi^i(t, x)}{\partial x^j}\delta x^j, \qquad \delta x \in \Lambda(t, x).$$

From this formula we derive

Theorem 7. *If some function $x(t)$ is a solution of* (20), (21) *for all $\delta x \in \Lambda(t, x(t))$ then $y(t) = \varphi(t, x(t))$ is the solution of* (5), (10), *and* (16) *for all $\delta y \in \Delta(t, y(t))$.*

Conversely, suppose that $y(t)$ is the solution of the system of equations for all $\delta y \in \Delta(t, y(t))$. Since for every t the mapping φ is the diffeomorphism to the corresponding image; therefore, the equation

$$y(t) = \varphi(t, x) \tag{22}$$

determines $x = x(t)$ uniquely, and $x(t)$ is a smooth function. In fact, select a subsystem of r functionally independent equations from (22) and apply to it the Implicit Function Theorem.

Theorem 8. *If $y(t)$ is a solution to* (5), (10), *and* (16) *for all $\delta y \in \Delta(t, y(t))$ then there exists a unique function $x(t)$ that is the solution of* (20), (21) *for all $\delta x \in \Lambda(t, x(t))$ and $y(t) = \varphi(t, x(t))$.*

4. THE D'ALEMBERT–LAGRANGE PRINCIPLE FOR A SYSTEM OF MATERIAL POINTS

Consider a system of the material points of masses $m_1,\dots,m_N$ with the position vectors $\mathbf{r}_1,\dots,\mathbf{r}_N$ with respect to some inertial coordinate system XYZ. Represent the position vector as $\mathbf{r}_k=(X_k,Y_k,Z_k)$.

Assume that at every point m_k the force

$$\mathbf{F}_k=\mathbf{F}_k\big(t,\mathbf{r}_1,\dots,\mathbf{r}_N,\dot{\mathbf{r}}_1,\dots,\dot{\mathbf{r}}_N\big),\qquad k=1,\dots,N,$$

acts so that the motion equations of the whole system of mass points are of the form

$$m_k\ddot{\mathbf{r}}_k=\mathbf{F}_k,\qquad k=1,\dots,N. \tag{23}$$

The functions $\mathbf{F}_i$ are defined on $(t_1,t_2)\times TY$, where TY is the tangent bundle of some domain

$$Y\subset\mathbb{R}^{3N},\qquad (\mathbf{r}_1,\dots,\mathbf{r}_N)\in Y.$$

Note that $TY=Y\times\mathbb{R}^{3N}$.

Thus, the configurational space of the system is the domain Y with the coordinates

$$y=\big(y^1,\dots,y^s\big)=\big(X_1,Y_1,Z_1,\dots,X_N,Y_N,Z_N\big),\qquad s=3N,$$

respectively; and

$$\mathbf{F}_k=(F_k^X,F_k^Y,F_k^Z),\qquad \big(F_1,\dots,F_s\big)=\big(F_1^X,F_1^Y,F_1^Z,dots,F_N^X,F_N^Y,F_N^Z\big).$$

System (23) can be rewritten in the form (3):

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\mathbf{r}}_k}-\frac{\partial T}{\partial\mathbf{r}_k}=\mathbf{F}_k,\qquad k=1,\dots,N,$$

where

$$T=\frac{1}{2}\sum_{i=1}^{N}m_i|\dot{\mathbf{r}}_i|^2$$

is the kinetic energy of the system.

Thus, we can apply the above developed theory for studying (23).

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