ON THE SOLVABILITY OF SOME SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH ANOMALOUS DIFFUSION IN HIGHER DIMENSIONS

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Abstract: The article deals with the studies of the existence of solutions of a system of integro-differential equations in the case of the anomalous diffusion with the negative Laplace operator in a fractional power in $\mathbb{R}^d,\ d=4,5.$ The proof of the existence of solutions is based on a fixed point technique. Solvability conditions for non Fredholm elliptic operators in unbounded domains are used.

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1. Introduction

The present work deals with the studies of the existence of stationary solutions of the following system of integro-differential equations in $\mathbb{R}^d,\ d=4,5$

$$\frac{\partial u_m}{\partial t} = -D_m(-\Delta)^{s_m} u_m + \int_{\mathbb{R}^d} K_m(x-y) g_m(u(y,t))dy + f_m(x), \tag{1.1}$$

where $1 \leq m \leq N$ appearing in the cell population dynamics. Our method will work in the range of the powers of the negative Laplacians given by

$$\frac{3}{2} - \frac{d}{4} < s_m < 1, \quad 1 \leq m \leq N.$$

The space variable x in our problem is correspondent to the cell genotype, functions $u_m(x,t)$ describe the cell density distributions for various groups of cells as functions of their genotype and time,

$$u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

The right side of the system of equations (1.1) describes the evolution of cell densities by means of the cell proliferation, mutations and cell influx or efflux. The anomalous diffusion terms with positive coefficients D_m correspond to the change of genotype due to small random mutations, and the integral production terms describe large mutations. Functions $g_m(u)$ denote the rates of cell birth which depend on u (density dependent proliferation), and the kernels $K_m(x-y)$ express the proportions of newly born cells changing their genotype from y to x. We assume that they depend on the distance between the genotypes. The functions $f_m(x)$ stand for the influx or efflux of cells for different genotypes.

The operators $(-\Delta)^{s_m},\ 1 \leq m \leq N$ in problem (1.1) describe a particular case of the anomalous diffusion actively treated in the context of various applications in plasma physics and turbulence [7], [21], surface diffusion [14], [19], semiconductors [20] and so on. The anomalous diffusion can be understood as a random process of the particle motion characterized by the probability density distribution of jump length. The moments of this density distribution are finite in the case of the normal diffusion, but this is not the case for the anomalous diffusion. The asymptotic behavior at the infinity of the probability density function determines the value $s_m,\ 1 \leq m \leq N$ of the power of the negative Laplacian (see [18]). The operators $(-\Delta)^{s_m},\ 1 \leq m \leq N$ are defined by means of the spectral calculus. We consider the case of $\frac{3}{2} - \frac{d}{4} < s_m < 1,\ 1 \leq m \leq N$ in the present work. A similar system with the standard Laplacians in the diffusion terms was studied recently in [29]. We note that the restriction on the powers $s_m,\ 1 \leq m \leq N$ here is due to the solvability conditions of our problem.

Let us set here all $D_m = 1$ and show the existence of solutions of the system of equations for $\frac{3}{2} - \frac{d}{4} < s_m < 1$

$$-(-\Delta)^{s_m} u_m + \int_{\mathbb{R}^d} K_m(x-y) g_m(u(y)) dy + f_m(x) = 0, \tag{1.2}$$

with $1 \leq m \leq N,\ d = 4, 5$. We treat the case when the linear part of this operator does not satisfy the Fredholm property. As a consequence, the conventional methods of nonlinear analysis may not be applicable. We use the solvability conditions for the non Fredholm operators along with the method of contraction mappings.

Consider the equation

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either zero identically or converges to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of problem (1.3) contains the origin. As a consequence, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work deals with the studies of certain properties of the operators of this kind. Note that elliptic problems with non Fredholm operators were studied actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The Schrödinger type operators without Fredholm property were treated with the methods of the spectral and the scattering theory in [22], [26], [32]. Nonlinear non Fredholm elliptic problems were studied in [27] and [28]. The significant applications to the theory of reaction-diffusion type equations were developed in [9], [10]. The non Fredholm operators arise also when considering wave systems with an infinite number of localized traveling waves (see [1]). In particular, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is significantly different and the approach developed in these articles cannot be applied. Fredholm structures, topological invariants and their applications were discussed in [11]. Front propagation equations with anomalous diffusion were studied largely in recent years (see e.g. [23], [24]). The article [15] is devoted to the establishing of the imbedding theorems and the studies of the spectrum of a certain pseudodifferential operator. The form boundedness criterion for the relativistic Schrödinger operator was established in [16]. A new type of integral equations related to the co-area formula was considered in [17].

We set $K_m(x) = \varepsilon_m H_m(x)$, where $\varepsilon_m \geq 0$, such that

$$\varepsilon := \max_{1\leq m\leq N}\varepsilon_m, \quad s := \max_{1\leq m\leq N} s_m \tag{1.4}$$

with $\frac{3}{2} - \frac{d}{4} < s < 1$ and assume the following.

Assumption 1.1. *Let* $1 \leq m \leq N$ *and* $\frac{3}{2} - \frac{d}{4} < s_m < 1$*, where* $d = 4, 5$*. Let* $f_m(x) : \mathbb{R}^d \to \mathbb{R}$ *be nontrivial for a certain* m*. Let*

$$f_m(x) \in L^1(\mathbb{R}^d), \quad (-\Delta)^{\frac{3}{2}-s_m} f_m(x) \in L^2(\mathbb{R}^d).$$

We assume also that $H_m(x) : \mathbb{R}^d \to \mathbb{R}$*, such that*

$$H_m(x) \in L^1(\mathbb{R}^d), \quad (-\Delta)^{\frac{3}{2}-s_m} H_m(x) \in L^2(\mathbb{R}^d).$$

Furthermore,

$$H^2 := \sum_{m=1}^{N} \|H_m(x)\|^2_{L^1(\mathbb{R}^d)} > 0$$

and

$$Q^2 := \sum_{m=1}^{N} \|(-\Delta)^{\frac{3}{2}-s_m} H_m(x)\|^2_{L^2(\mathbb{R}^d)} > 0.$$

Let us choose here the space dimensions $d = 4, 5$, which is related to the solvability conditions for the linear Poisson type equation (4.1) stated in Lemma 4.1 below. For the applications, the space dimensions are not limited to $d = 4, 5$, since the space variable here corresponds to the cell genotype but not to the usual physical space. In $d = 1$ our problem was treated in [31] with all $0 < s_m = s < \frac{1}{4}$ based on the solvability conditions for the analog of (4.1) on the real line. In two dimensions our system was considered in [33] with $0 < s_m < \frac{1}{2}$, $1 \leq m \leq N$. In $d = 3$ our problem was studied in [30] with all $\frac{1}{4} < s_m = s < \frac{3}{4}$. As distinct from the situations in the lower dimensions $d = 1, 2$, in $\mathbb{R}^d$, $d = 3, 4, 5$ we are able to apply the Sobolev inequality for the fractional negative Laplacian (see Lemma 2.2 of [12], also [13]), namely

$$\|f_m(x)\|_{L^{\frac{2d}{d-6+4s_m}}(\mathbb{R}^d)} \leq c_{sob}\|(-\Delta)^{\frac{3}{2}-s_m} f_m(x)\|_{L^2(\mathbb{R}^d)}, \quad \frac{3}{2} - \frac{d}{4} < s_m < 1 \quad (1.5)$$

with $d = 4, 5$ and $1 \leq m \leq N$. By means of the Assumption 1.1 above along with the standard interpolation argument, we obtain that

$$f_m(x) \in L^2(\mathbb{R}^d), \quad d = 4, 5, \quad 1 \leq m \leq N \quad (1.6)$$

as well. We use the Sobolev spaces for the technical purposes with $0 < s \leq 1$, namely

$$H^{2s}(\mathbb{R}^d) := \{\phi(x) : \mathbb{R}^d \to \mathbb{R} \mid \phi(x) \in L^2(\mathbb{R}^d), \ (-\Delta)^s \phi \in L^2(\mathbb{R}^d)\}, \quad d = 4, 5$$

equipped with the norm

$$\|\phi\|^2_{H^{2s}(\mathbb{R}^d)} := \|\phi\|^2_{L^2(\mathbb{R}^d)} + \|(-\Delta)^s \phi\|^2_{L^2(\mathbb{R}^d)}. \quad (1.7)$$

For a vector vector function

$$u(x) = (u_1(x), u_2(x), ..., u_N(x))^T,$$

throughout the work we will use the norm

$$\|u\|^2_{H^3(\mathbb{R}^d, \mathbb{R}^N)} := \|u\|^2_{L^2(\mathbb{R}^d, \mathbb{R}^N)} + \sum_{m=1}^{N} \|(-\Delta)^{\frac{3}{2}} u_m\|^2_{L^2(\mathbb{R}^d)}, \quad (1.8)$$

where $d = 4, 5$ and

$$\|u\|^2_{L^2(\mathbb{R}^d, \mathbb{R}^N)} := \sum_{m=1}^{N} \|u_m\|^2_{L^2(\mathbb{R}^d)}.$$

Let us recall the Sobolev embedding in $\mathbb{R}^d$, $d = 4, 5$, namely

$$\|\phi\|_{L^\infty(\mathbb{R}^d)} \leq c_e \|\phi\|_{H^3(\mathbb{R}^d)}, \quad (1.9)$$

where $c_e > 0$ is the constant of the embedding. When all the nonnegative parameters ε_m vanish, we obtain the linear Poisson type equations

$$(-\Delta)^{s_m} u_m(x) = f_m(x), \quad 1 \leq m \leq N. \tag{1.10}$$

By means of Lemma 4.1 below under the given conditions each problem (1.10) possesses a unique solution

$$u_{0,m}(x) \in H^{2s_m}(\mathbb{R}^d), \quad \frac{3}{2} - \frac{d}{4} < s_m < 1, \quad 1 \leq m \leq N,$$

and no orthogonality relations for the right side of (1.10) are necessary here. Clearly,

$$(-\Delta)^{\frac{3}{2}} u_{0,m}(x) = (-\Delta)^{\frac{3}{2}-s_m} f_m(x) \in L^2(\mathbb{R}^d), \quad 1 \leq m \leq N$$

due to Assumption 1.1. We obtain that each linear equation (1.10) admits a unique solution $u_{0,m}(x) \in H^3(\mathbb{R}^d)$. Thus

$$u_0(x) := (u_{0,1}(x), u_{0,2}(x), ..., u_{0,N}(x))^T \in H^3(\mathbb{R}^d, \mathbb{R}^N).$$

Let us look for the resulting solution of nonlinear system of equations (1.2) as

$$u(x) = u_0(x) + u_p(x) \tag{1.11}$$

with

$$u_p(x) := (u_{p,1}(x), u_{p,2}(x), ..., u_{p,N}(x))^T.$$

Apparently, we easily derive the perturbative system of equations

$$(-\Delta)^{s_m} u_{p,m}(x) = \varepsilon_m \int_{\mathbb{R}^d} H_m(x-y) g_m(u_0(y) + u_p(y)) dy, \tag{1.12}$$

where $1 \leq m \leq N$, $\frac{3}{2} - \frac{d}{4} < s_m < 1$ and introduce a closed ball in our Sobolev space

$$B_\rho := \{u(x) \in H^3(\mathbb{R}^d, \mathbb{R}^N) \mid \|u\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{1.13}$$

Let us look for the solution of system (1.12) as the fixed point of the auxiliary nonlinear problem

$$(-\Delta)^{s_m} u_m(x) = \varepsilon_m \int_{\mathbb{R}^d} H_m(x-y) g_m(u_0(y) + v(y)) dy, \quad 1 \leq m \leq N, \tag{1.14}$$

with $\frac{3}{2} - \frac{d}{4} < s_m < 1$ in ball (1.13). For a given vector function $v(y)$ this is a system of equations with respect to $u(x)$. The left side of (1.14) contains the operators which do not satisfy the Fredholm property

$$(-\Delta)^{s_m} : H^{2s_m}(\mathbb{R}^d) \to L^2(\mathbb{R}^d). \tag{1.15}$$

The essential spectrum of (1.15) fills the nonnegative semi-axis $[0, +\infty)$. Therefore, such operator does not have a bounded inverse. The similar situation appeared in works [27] and [28] but as distinct from the present case, the problems studied there required orthogonality conditions. The fixed point technique was used in [25] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear equation there had the Fredholm property (see Assumption 1 of [25], also [8]). Let us introduce the closed ball in the space of N dimensions as

$$I := \{z \in \mathbb{R}^N \mid |z| \le c_e \|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + c_e\}, \quad d = 4,5 \tag{1.16}$$

and the closed ball D_M in the space of $C^2(I, \mathbb{R}^N)$ vector functions given by

$$\{g(z) := (g_1(z), g_2(z), ..., g_N(z)) \in C^2(I, \mathbb{R}^N) \mid \|g\|_{C^2(I,\mathbb{R}^N)} \le M\}, \tag{1.17}$$

with $M > 0$. Here the norms

$$\|g\|_{C^2(I,\mathbb{R}^N)} := \sum_{m=1}^{N} \|g_m\|_{C^2(I)}, \tag{1.18}$$

$$\|g_m\|_{C^2(I)} := \|g_m\|_{C(I)} + \sum_{n=1}^{N} \Big\| \frac{\partial g_m}{\partial z_n} \Big\|_{C(I)} + \sum_{n,l=1}^{N} \Big\| \frac{\partial^2 g_m}{\partial z_n \partial z_l} \Big\|_{C(I)}, \tag{1.19}$$

where $\|g_m\|_{C(I)} := \max_{z\in I}|g_m(z)|$. We make the following technical assumption on the nonlinear part of system (1.2).

Assumption 1.2. *Let $1 \le m \le N$. Assume that $g_m(z) : \mathbb{R}^N \to \mathbb{R}$, such that $g_m(0) = 0$ and $\nabla g_m(0) = 0$. It is also assumed that $g(z) \in D_M$ and it does not vanish identically in the ball I.*

We introduce the operator T_g, such that $u = T_g v$, where u is a solution of system (1.14). Our first main proposition is as follows.

Theorem 1.3. *Let Assumptions 1.1 and 1.2 hold. Then system (1.14) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all*

$$0 < \varepsilon \le \frac{\rho}{M(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2} \times$$

$$\times \left[\frac{H^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^{\frac{8s}{d}-2} d}{(d-4s)(2\pi)^{4S}} \left(\frac{|S^d|}{4S} \right)^{\frac{4S}{d}} + Q^2 \right]^{-\frac{1}{2}} \tag{1.20}$$

with $\varepsilon,\ s$ and S defined in (1.4) and (2.6). The unique fixed point $u_p(x)$ of this map T_g is the only solution of problem (1.12) in B_ρ.

Apparently, the resulting solution $u(x)$ of system (1.2) given by (1.11) will not be equal to zero identically because the influx/efflux terms $f_m(x)$ are nontrivial for a certain $1 \le m \le N$ and all $g_m(0)$ vanish as assumed. We will make use of the following elementary lemma.

Lemma 1.4. *Let $R \in (0, +\infty)$ and $d = 4, 5$. Consider the function*

$$\varphi(R) := \alpha R^{d-4s} + \frac{1}{R^{4s}}, \quad \frac{3}{2} - \frac{d}{4} < s < 1, \quad \alpha > 0.$$

It achieves the minimal value at $R^* := \left(\frac{4s}{\alpha(d-4s)}\right)^{\frac{1}{d}}$*, which is given by*

$$\varphi(R^*) = \left(\frac{\alpha}{4s}\right)^{\frac{4s}{d}} \frac{d}{(d-4s)^{\frac{d-4s}{d}}}.$$

Our second main statement is devoted to the continuity of the cumulative solution of system (1.2) given by formula (1.11) with respect to the nonlinear vector function g. We will use the following positive technical expression

$$\sigma := M(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1) \times$$

$$\times \left\{ \frac{H^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^{\frac{8s}{d}-2} d}{(d-4s)(2\pi)^{4S}} \left(\frac{|S^d|}{4S}\right)^{\frac{4S}{d}} + Q^2 \right\}^{\frac{1}{2}}. \tag{1.21}$$

Theorem 1.5. *Let $j = 1, 2$, the assumptions of Theorem 1.3 including inequality (1.20) hold, so that $u_{p,j}(x)$ is the unique fixed point of the map $T_{g_j} : B_\rho \to B_\rho$, which is a strict contraction for all ε which satisfy (1.20) and the resulting solution of system (1.2) with $g(z) = g_j(z)$ is*

$$u_j(x) := u_0(x) + u_{p,j}(x). \tag{1.22}$$

Then for all the values of ε satisfying inequality (1.20) the estimate

$$\|u_1 - u_2\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} \le \frac{\varepsilon\sigma}{M(1-\varepsilon\sigma)}(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)\|g_1 - g_2\|_{C^2(I,\mathbb{R}^N)} \tag{1.23}$$

is valid.

We turn our attention to the proof of our first main proposition.

2. The existence of the perturbed solution

Proof of Theorem 1.3. We choose arbitrarily a vector function $v(x) \in B_\rho$ and denote the terms involved in the integral expressions in the right side of system (1.14) as

$$G_m(x) := g_m(u_0(x) + v(x)), \quad 1 \le m \le N.$$

Let us use the standard Fourier transform throughout the article, namely

$$\widehat{\phi}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \phi(x) e^{-ipx} dx, \quad d = 4, 5. \tag{2.1}$$

Evidently, we have the estimate from above

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{(2\pi)^{\frac{d}{2}}} \|\phi(x)\|_{L^1(\mathbb{R}^d)}. \tag{2.2}$$

Let us apply (2.1) to both sides of system (1.14). This gives us

$$\widehat{u}_m(p) = \varepsilon_m (2\pi)^{\frac{d}{2}} \frac{\widehat{H}_m(p)\widehat{G}_m(p)}{|p|^{2s_m}}, \quad 1 \le m \le N, \quad d = 4, 5.$$

Thus we have the expression for the norm as

$$\|u_m\|^2_{L^2(\mathbb{R}^d)} = (2\pi)^d \varepsilon_m^2 \int_{\mathbb{R}^d} \frac{|\widehat{H}_m(p)|^2 |\widehat{G}_m(p)|^2}{|p|^{4s_m}} dp, \quad 1 \le m \le N \tag{2.3}$$

with $d = 4, 5$. As distinct from articles [27] and [28] with the standard Laplace operator in the diffusion term, here we do not try to control the norms

$$\left\| \frac{\widehat{H}_m(p)}{|p|^{2s_m}} \right\|_{L^\infty(\mathbb{R}^d)}, \quad 1 \le m \le N.$$

Instead, we estimate the right side of (2.3) using the analog of inequality (2.2) applied to functions H_m and G_m with $R \in (0, +\infty)$ as

$$(2\pi)^d \varepsilon_m^2 \Big[\int_{|p| \le R} \frac{|\widehat{H}_m(p)|^2 |\widehat{G}_m(p)|^2}{|p|^{4s_m}} dp + \int_{|p| > R} \frac{|\widehat{H}_m(p)|^2 |\widehat{G}_m(p)|^2}{|p|^{4s_m}} dp \Big] \le$$

$$\le \varepsilon_m^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} \left\{ \frac{|S^d|}{(2\pi)^d} \|G_m(x)\|^2_{L^1(\mathbb{R}^d)} \frac{R^{d-4s_m}}{d - 4s_m} + \frac{\|G_m(x)\|^2_{L^2(\mathbb{R}^d)}}{R^{4s_m}} \right\}. \tag{2.4}$$

Here and throughout the article S^d denotes the unit sphere in our d dimensional space centered at the origin and $|S^d|$ its Lebesgue measure. By virtue of norm definition (1.8) along with the triangle inequality and using the fact that $v(x) \in B_\rho$, we easily obtain

$$\|u_0 + v\|_{L^2(\mathbb{R}^d, \mathbb{R}^N)} \le \|u_0\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} + 1, \quad d = 4, 5.$$

Sobolev embedding (1.9) implies that

$$|u_0+v| \le c_e(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1).$$

Let the dot denote the scalar product of two vectors in $\mathbb{R}^N$. Evidently,

$$G_m(x)=\int_0^1 \nabla g_m(t(u_0(x)+v(x))).(u_0(x)+v(x))dt, \quad 1\le m\le N.$$

Using the ball I defined in (1.16) we easily derive

$$|G_m(x)| \le \sup_{z\in I}|\nabla g_m(z)||u_0(x)+v(x)| \le M|u_0(x)+v(x)|.$$

Hence,

$$\|G_m(x)\|_{L^2(\mathbb{R}^d)} \le M\|u_0+v\|_{L^2(\mathbb{R}^d,\mathbb{R}^N)} \le M(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1).$$

Apparently, for $t\in[0,1]$ and $1\le m,j\le N$, we can express

$$\frac{\partial g_m}{\partial z_j}(t(u_0(x)+v(x)))=\int_0^t \nabla\frac{\partial g_m}{\partial z_j}(\tau(u_0(x)+v(x))).(u_0(x)+v(x))d\tau.$$

This yields

$$\Big|\frac{\partial g_m}{\partial z_j}(t(u_0(x)+v(x)))\Big| \le \sup_{z\in I}\Big|\nabla\frac{\partial g_m}{\partial z_j}\Big||u_0(x)+v(x)| \le$$

$$\le \sum_{n=1}^N \Big\|\frac{\partial^2 g_m}{\partial z_n\partial z_j}\Big\|_{C(I)}|u_0(x)+v(x)|.$$

Thus,

$$|G_m(x)| \le |u_0(x)+v(x)|\sum_{n,j=1}^N \Big\|\frac{\partial^2 g_m}{\partial z_n\partial z_j}\Big\|_{C(I)}|u_{0,j}(x)+v_j(x)| \le M|u_0(x)+v(x)|^2,$$

such that

$$\|G_m(x)\|_{L^1(\mathbb{R}^d)} \le M\|u_0+v\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)} \le M(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2. \tag{2.5}$$

This enables us to obtain the upper bound for the right side of (2.4) as

$$\varepsilon_m^2 M^2\|H_m\|^2_{L^1(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2\times$$

$$\times\Bigg\{\frac{|S^d|(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2R^{d-4s_m}}{(2\pi)^d(d-4s_m)}+\frac{1}{R^{4s_m}}\Bigg\}$$

with $R \in (0, +\infty)$. Lemma 1.4 gives us the minimal value of the expression above. Therefore, $\|u_m\|^2_{L^2(\mathbb{R}^d)} \leq$

$$\leq {\varepsilon_m}^2 M^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^{2+\frac{8s_m}{d}} \left(\frac{|S^d|}{4s_m}\right)^{\frac{4s_m}{d}} \frac{d}{(d-4s_m)(2\pi)^{4s_m}}.$$

Let us define

$$\left(\frac{|S^d|}{4S}\right)^{\frac{4S}{d}} \frac{1}{(2\pi)^{4S}} := \max_{1\leq m\leq N} \left(\frac{|S^d|}{4s_m}\right)^{\frac{4s_m}{d}} \frac{1}{(2\pi)^{4s_m}}, \tag{2.6}$$

where $\frac{3}{2} - \frac{d}{4} < S < 1$. Thus

$$\|u\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)} \leq \varepsilon^2 M^2 H^2 (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^{2+\frac{8s}{d}} \frac{d}{d-4s} \left(\frac{|S^d|}{4S}\right)^{\frac{4S}{d}} \frac{1}{(2\pi)^{4S}}. \tag{2.7}$$

Clearly, (1.14) yields

$$(-\Delta)^{\frac{3}{2}} u_m(x) = \varepsilon_m (-\Delta)^{\frac{3}{2}-s_m} \int_{\mathbb{R}^d} H_m(x-y) G_m(y) dy, \quad 1 \leq m \leq N,$$

where $\frac{3}{2} - \frac{d}{4} < s_m < 1$. By means of the analog of upper bound (2.2) applied to function G_m along with (2.5) we arrive at

$$\|(-\Delta)^{\frac{3}{2}} u_m\|^2_{L^2(\mathbb{R}^d)} \leq \varepsilon_m^2 \|G_m\|^2_{L^1(\mathbb{R}^d)} \|(-\Delta)^{\frac{3}{2}-s_m} H_m\|^2_{L^2(\mathbb{R}^d)} \leq$$

$$\leq \varepsilon^2 M^2 (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^4 \|(-\Delta)^{\frac{3}{2}-s_m} H_m\|^2_{L^2(\mathbb{R}^d)}.$$

Hence,

$$\sum_{m=1}^{N} \|(-\Delta)^{\frac{3}{2}} u_m\|^2_{L^2(\mathbb{R}^d)} \leq \varepsilon^2 M^2 (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^4 Q^2. \tag{2.8}$$

Therefore, by virtue of the definition of the norm (1.8) along with inequalities (2.7) and (2.8) we derive the estimate from above for the norm $\|u\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}$ as

$$\varepsilon M (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2 \times$$

$$\times \left[\frac{H^2 (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^{\frac{8s}{d}-2} d}{(d-4s)(2\pi)^{4S}} \left(\frac{|S^d|}{4S}\right)^{\frac{4S}{d}} + Q^2 \right]^{\frac{1}{2}} \leq \rho \tag{2.9}$$

for all values of ε which satisfy (1.20), such that $u(x) \in B_\rho$ as well. Suppose for a certain $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of system (1.14). Clearly, their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^d, \mathbb{R}^N)$ satisfies

$$(-\Delta)^{s_m} w_m(x) = 0, \quad \frac{3}{2} - \frac{d}{4} < s_m < 1, \quad 1 \le m \le N.$$

Because each operator $(-\Delta)^{s_m}$ considered on the whole $\mathbb{R}^d$ does not possess any nontrivial square integrable zero modes, $w(x)$ vanishes identically in $\mathbb{R}^d$. Therefore, problem (1.14) defines a map $T_g : B_\rho \to B_\rho$ for all ε satisfying inequality (1.20).

Our goal is to demonstrate that this map is a strict contraction. Let us choose arbitrarily $v_{1,2}(x) \in B_\rho$. According to the argument above $u_{1,2} := T_g v_{1,2} \in B_\rho$ as well if ε satisfies (1.20). Evidently, by means of (1.14) we obtain for $1 \le m \le N$

$$(-\Delta)^{s_m} u_{1,m}(x) = \varepsilon_m \int_{\mathbb{R}^d} H_m(x-y) g_m(u_0(y) + v_1(y)) dy, \tag{2.10}$$

$$(-\Delta)^{s_m} u_{2,m}(x) = \varepsilon_m \int_{\mathbb{R}^d} H_m(x-y) g_m(u_0(y) + v_2(y)) dy \tag{2.11}$$

with all $\frac{3}{2} - \frac{d}{4} < s_m < 1$. We introduce

$$G_{1,m}(x) := g_m(u_0(x) + v_1(x)), \quad G_{2,m}(x) := g_m(u_0(x) + v_2(x)), \quad 1 \le m \le N$$

and apply the standard Fourier transform (2.1) to both sides of systems (2.10) and (2.11). This yields

$$\widehat{u}_{1,m}(p) = \varepsilon_m (2\pi)^{\frac{d}{2}} \frac{\widehat{H}_m(p) \widehat{G}_{1,m}(p)}{|p|^{2s_m}}, \quad \widehat{u}_{2,m}(p) = \varepsilon_m (2\pi)^{\frac{d}{2}} \frac{\widehat{H}_m(p) \widehat{G}_{2,m}(p)}{|p|^{2s_m}}.$$

Apparently,

$$\|u_{1,m} - u_{2,m}\|^2_{L^2(\mathbb{R}^d)} = \varepsilon_m^2 (2\pi)^d \int_{\mathbb{R}^d} \frac{|\widehat{H}_m(p)|^2 |\widehat{G}_{1,m}(p) - \widehat{G}_{2,m}(p)|^2}{|p|^{4s_m}} dp. \tag{2.12}$$

Obviously, the right side of (2.12) can be estimated from above using inequality (2.2) as

$$\varepsilon_m^2 (2\pi)^d \Bigg[\int_{|p| \le R} \frac{|\widehat{H}_m(p)|^2 |\widehat{G}_{1,m}(p) - \widehat{G}_{2,m}(p)|^2}{|p|^{4s_m}} dp +$$

$$+ \int_{|p| > R} \frac{|\widehat{H}_m(p)|^2 |\widehat{G}_{1,m}(p) - \widehat{G}_{2,m}(p)|^2}{|p|^{4s_m}} dp \Bigg] \le \varepsilon^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} \times$$

$$\times \left\{ \frac{\|G_{1,m}(x) - G_{2,m}(x)\|^2_{L^1(\mathbb{R}^d)}}{(2\pi)^d} \frac{|S^d| R^{d-4s_m}}{d - 4s_m} + \frac{\|G_{1,m}(x) - G_{2,m}(x)\|^2_{L^2(\mathbb{R}^d)}}{R^{4s_m}} \right\}$$

with $R \in (0, +\infty)$. Evidently, we have the identity for $1 \le m \le N$

$$G_{1,m}(x) - G_{2,m}(x) = \int_0^1 \nabla g_m(u_0(x) + tv_1(x) + (1-t)v_2(x)).(v_1(x) - v_2(x))dt.$$

Apparently, for $t \in [0, 1]$

$$\|v_2(x) + t(v_1(x) - v_2(x))\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} \le t\|v_1(x)\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} +$$

$$+(1-t)\|v_2(x)\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} \le \rho,$$

which implies that $v_2(x) + t(v_1(x) - v_2(x)) \in B_\rho$. Hence,

$$|G_{1,m}(x) - G_{2,m}(x)| \le \sup_{z \in I} |\nabla g_m(z)||v_1(x) - v_2(x)| \le M|v_1(x) - v_2(x)|,$$

so that

$$\|G_{1,m}(x) - G_{2,m}(x)\|_{L^2(\mathbb{R}^d)} \le M\|v_1 - v_2\|_{L^2(\mathbb{R}^d, \mathbb{R}^N)} \le M\|v_1 - v_2\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)}.$$

Clearly, we can express $\frac{\partial g_m}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x))$ for $1 \le m, j \le N$ as

$$\int_0^1 \nabla \frac{\partial g_m}{\partial z_j}(\tau[u_0(x) + tv_1(x) + (1-t)v_2(x)]).[u_0(x) + tv_1(x) + (1-t)v_2(x)]d\tau.$$

Therefore, for $t \in [0, 1]$

$$\Big|\frac{\partial g_m}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x))\Big| \le$$

$$\le \sum_{n=1}^N \Big\|\frac{\partial^2 g_m}{\partial z_n \partial z_j}\Big\|_{C(I)} (|u_0(x)| + t|v_1(x)| + (1-t)|v_2(x)|).$$

Hence, we obtain the upper bound for $G_{1,m}(x) - G_{2,m}(x)$ in the absolute value given by

$$M|v_1(x) - v_2(x)|\Big(|u_0(x)| + \frac{1}{2}|v_1(x)| + \frac{1}{2}|v_2(x)|\Big).$$

By means of the Schwarz inequality we derive the estimate from above for the norm $\|G_{1,m}(x) - G_{2,m}(x)\|_{L^1(\mathbb{R}^d)}$ as

$$M\|v_1 - v_2\|_{L^2(\mathbb{R}^d, \mathbb{R}^N)}\Big(\|u_0\|_{L^2(\mathbb{R}^d, \mathbb{R}^N)} + \frac{1}{2}\|v_1\|_{L^2(\mathbb{R}^d, \mathbb{R}^N)} + \frac{1}{2}\|v_2\|_{L^2(\mathbb{R}^d, \mathbb{R}^N)}\Big) \le$$

$$\le M\|v_1 - v_2\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)}(\|u_0\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} + 1). \tag{2.13}$$

Therefore, we arrive at the upper bound for the norm $\|u_{1,m}(x) - u_{2,m}(x)\|^2_{L^2(\mathbb{R}^d)}$ given by

$$\varepsilon^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} M^2 \|v_1 - v_2\|^2_{H^3(\mathbb{R}^d, \mathbb{R}^N)} \Big\{ \frac{(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2 |S^d| R^{d-4s_m}}{(2\pi)^d (d - 4s_m)} + \frac{1}{R^{4s_m}} \Big\}.$$

We minimize the expression above over $R \in (0, +\infty)$ by means of our Lemma 1.4, such that

$$\|u_{1,m}(x) - u_{2,m}(x)\|^2_{L^2(\mathbb{R}^d)} \le \varepsilon^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} M^2 \|v_1 - v_2\|^2_{H^3(\mathbb{R}^d,\mathbb{R}^N)} \times$$

$$\times (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^{\frac{8s_m}{d}} \left(\frac{|S^d|}{4s_m} \right)^{\frac{4s_m}{d}} \frac{d}{(2\pi)^{4s_m}(d - 4s_m)}.$$

Thus,

$$\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)} \le \varepsilon^2 H^2 M^2 \|v_1 - v_2\|^2_{H^3(\mathbb{R}^d,\mathbb{R}^N)} \times$$

$$\times (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^{\frac{8s}{d}} \frac{d}{(2\pi)^{4S}(d - 4s)} \left(\frac{|S^d|}{4S} \right)^{\frac{4S}{d}}. \tag{2.14}$$

By means of formulas (2.10) and (2.11) with $1 \le m \le N$ we have

$$(-\Delta)^{\frac{3}{2}} (u_{1,m}(x) - u_{2,m}(x)) = \varepsilon_m (-\Delta)^{\frac{3}{2} - s_m} \int_{\mathbb{R}^d} H_m(x - y)[G_{1,m}(y) - G_{2,m}(y)]dy.$$

Upper bounds (2.2) and (2.13) give us $\|(-\Delta)^{\frac{3}{2}} (u_{1,m}(x) - u_{2,m}(x))\|^2_{L^2(\mathbb{R}^d)} \le$

$$\le \varepsilon^2 \|G_{1,m} - G_{2,m}\|^2_{L^1(\mathbb{R}^d)} \|(-\Delta)^{\frac{3}{2} - s_m} H_m\|^2_{L^2(\mathbb{R}^d)} \le$$

$$\le \varepsilon^2 M^2 \|v_1 - v_2\|^2_{H^3(\mathbb{R}^d,\mathbb{R}^N)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2 \|(-\Delta)^{\frac{3}{2} - s_m} H_m\|^2_{L^2(\mathbb{R}^d)}.$$

Hence, $\sum_{m=1}^N \|(-\Delta)^{\frac{3}{2}} (u_{1,m}(x) - u_{2,m}(x))\|^2_{L^2(\mathbb{R}^d)} \le$

$$\le \varepsilon^2 M^2 \|v_1 - v_2\|^2_{H^3(\mathbb{R}^d,\mathbb{R}^N)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2 Q^2. \tag{2.15}$$

Inequalities (2.14) and (2.15) yield that the norm $\|u_1 - u_2\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}$ can be estimated from above by the expression $\varepsilon M (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1) \times$

$$\times \left\{ \frac{H^2 (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^{\frac{8s}{d} - 2} d}{(d - 4s)(2\pi)^{4S}} \left(\frac{|S^d|}{4S} \right)^{\frac{4S}{d}} + Q^2 \right\}^{\frac{1}{2}} \|v_1 - v_2\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}. \tag{2.16}$$

It can be easily verified that for all values of ε satisfying (1.20) the constant in the right side of (2.16) is less than one, such that the map $T_g : B_\rho \to B_\rho$ defined by system (1.14) is a strict contraction. Its unique fixed point $u_p(x)$ is the only solution of problem (1.12) in the ball B_ρ. The cumulative $u(x) \in H^3(\mathbb{R}^d, \mathbb{R}^N)$ given by (1.11) is a solution of system (1.2). Evidently, by means of (2.9) $u_p(x)$ tends to zero in the $H^3(\mathbb{R}^d, \mathbb{R}^N)$ norm as $\varepsilon \to 0$. ∎

We proceed to the proof of the second main statement of the article.

3. The continuity of the resulting solution

Proof of Theorem 1.5. Obviously, for all the values of ε satisfying (1.20)

$$u_{p,1} = T_{g_1} u_{p,1}, \quad u_{p,2} = T_{g_2} u_{p,2},$$

so that

$$u_{p,1} - u_{p,2} = T_{g_1} u_{p,1} - T_{g_1} u_{p,2} + T_{g_1} u_{p,2} - T_{g_2} u_{p,2}.$$

Hence,

$$\|u_{p,1} - u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} \le \|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}.$$

Upper bound (2.16) yields

$$\|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} \le \varepsilon\sigma \|u_{p,1} - u_{p,2}\|_{H^3(\mathbb{R}^2,\mathbb{R}^N)}$$

with σ defined in (1.21). We have $\varepsilon\sigma < 1$ since the map $T_{g_1} : B_\rho \to B_\rho$ is a strict contraction under our assumptions. Therefore,

$$(1 - \varepsilon\sigma)\|u_{p,1} - u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} \le \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}. \tag{3.1}$$

Apparently, for the fixed point $T_{g_2} u_{p,2} = u_{p,2}$. Let us denote $\xi(x) := T_{g_1} u_{p,2}$. For $1 \le m \le N$, we arrive at

$$(-\Delta)^{s_m} \xi_m(x) = \varepsilon_m \int_{\mathbb{R}^d} H_m(x-y) g_{1,m}(u_0(y) + u_{p,2}(y)) dy, \tag{3.2}$$

$$(-\Delta)^{s_m} u_{p,2,m}(x) = \varepsilon_m \int_{\mathbb{R}^d} H_m(x-y) g_{2,m}(u_0(y) + u_{p,2}(y)) dy, \tag{3.3}$$

with all $\frac{3}{2} - \frac{d}{4} < s_m < 1$. Let us introduce

$$G_{1,2,m}(x) := g_{1,m}(u_0(x) + u_{p,2}(x)), \quad G_{2,2,m}(x) := g_{2,m}(u_0(x) + u_{p,2}(x)).$$

We apply the standard Fourier transform (2.1) to both sides of formulas (3.2) and (3.3). This yields

$$\widehat{\xi}_m(p) = \varepsilon_m (2\pi)^{\frac{d}{2}} \frac{\widehat{H}_m(p) \widehat{G}_{1,2,m}(p)}{|p|^{2s_m}}, \quad \widehat{u}_{p,2,m}(p) = \varepsilon_m (2\pi)^{\frac{d}{2}} \frac{\widehat{H}_m(p) \widehat{G}_{2,2,m}(p)}{|p|^{2s_m}},$$

such that $\|\xi_m(x) - u_{p,2,m}(x)\|^2_{L^2(\mathbb{R}^d)} =$

$$= \varepsilon_m^2 (2\pi)^d \int_{\mathbb{R}^d} \frac{|\widehat{H}_m(p)|^2 |\widehat{G}_{1,2,m}(p) - \widehat{G}_{2,2,m}(p)|^2}{|p|^{4s_m}} dp. \tag{3.4}$$

Let us obtain the upper bound on the right side of (3.4) using (2.2) as

$$\varepsilon_m^2 (2\pi)^d \Bigg[\int_{|p|\leq R} \frac{|\widehat{H}_m(p)|^2 |\widehat{G}_{1,2,m}(p) - \widehat{G}_{2,2,m}(p)|^2}{|p|^{4s_m}} dp +$$

$$+ \int_{|p|>R} \frac{|\widehat{H}_m(p)|^2 |\widehat{G}_{1,2,m}(p) - \widehat{G}_{2,2,m}(p)|^2}{|p|^{4s_m}} dp \Bigg] \leq \varepsilon^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} \times$$

$$\times \left\{ \frac{|S^d|}{(2\pi)^d} \frac{\|G_{1,2,m} - G_{2,2,m}\|^2_{L^1(\mathbb{R}^d)} R^{d-4s_m}}{d - 4s_m} + \frac{\|G_{1,2,m} - G_{2,2,m}\|^2_{L^2(\mathbb{R}^d)}}{R^{4s_m}} \right\}$$

with $R \in (0, +\infty)$. Obviously, we can represent

$$G_{1,2,m}(x) - G_{2,2,m}(x) = \int_0^1 \nabla [g_{1,m} - g_{2,m}](t(u_0(x) + u_{p,2}(x))).(u_0(x) + u_{p,2}(x)) dt,$$

such that

$$|G_{1,2,m}(x) - G_{2,2,m}(x)| \leq \|g_{1,m} - g_{2,m}\|_{C^2(I)} |u_0(x) + u_{p,2}(x)|.$$

This implies

$$\|G_{1,2,m} - G_{2,2,m}\|_{L^2(\mathbb{R}^d)} \leq \|g_{1,m} - g_{2,m}\|_{C^2(I)} \|u_0 + u_{p,2}\|_{L^2(\mathbb{R}^d, \mathbb{R}^N)} \leq$$

$$\leq \|g_{1,m} - g_{2,m}\|_{C^2(I)} (\|u_0\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} + 1).$$

Let us make use of another representation formula with $1 \leq j \leq N$ and $t \in [0, 1]$, namely

$$\frac{\partial}{\partial z_j} (g_{1,m} - g_{2,m})(t(u_0(x) + u_{p,2}(x))) =$$

$$= \int_0^t \nabla \Big[\frac{\partial}{\partial z_j} (g_{1,m} - g_{2,m}) \Big] (\tau(u_0(x) + u_{p,2}(x))).(u_0(x) + u_{p,2}(x)) d\tau.$$

Therefore,

$$\Big| \frac{\partial}{\partial z_j} (g_{1,m} - g_{2,m})(t(u_0(x) + u_{p,2}(x))) \Big| \leq$$

$$\leq \sum_{n=1}^N \left\| \frac{\partial^2 (g_{1,m} - g_{2,m})}{\partial z_n \partial z_j} \right\|_{C(I)} |u_0(x) + u_{p,2}(x)|.$$

Hence,

$$|G_{1,2,m}(x) - G_{2,2,m}(x)| \le \|g_{1,m} - g_{2,m}\|_{C^2(I)} |u_0(x) + u_{p,2}(x)|^2,$$

so that

$$\|G_{1,2,m} - G_{2,2,m}\|_{L^1(\mathbb{R}^d)} \le \|g_{1,m} - g_{2,m}\|_{C^2(I)} \|u_0 + u_{p,2}\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)} \le$$

$$\le \|g_{1,m} - g_{2,m}\|_{C^2(I)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2. \tag{3.5}$$

This enables us to derive the upper bound for the norm $\|\xi_m - u_{p,2,m}\|^2_{L^2(\mathbb{R}^d)}$ as $\varepsilon^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2 \times$

$$\times \|g_{1,m} - g_{2,m}\|^2_{C^2(I)} \left[(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2 \frac{|S^d| R^{d-4s_m}}{(2\pi)^d (d - 4s_m)} + \frac{1}{R^{4s_m}} \right].$$

Let us minimize this expression over $R \in (0, +\infty)$ using Lemma 1.4. We arrive at the estimate from above $\|\xi_m(x) - u_{p,2,m}(x)\|^2_{L^2(\mathbb{R}^d)} \le$

$$\le \varepsilon^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^{2 + \frac{8s_m}{d}} \left(\frac{|S^d|}{4s_m} \right)^{\frac{4s_m}{d}} \frac{d \|g_{1,m} - g_{2,m}\|^2_{C^2(I)}}{(2\pi)^{4s_m} (d - 4s_m)}.$$

Therefore, $\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)} \le$

$$\le \varepsilon^2 H^2 (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^{2 + \frac{8s}{d}} \frac{d \|g_1 - g_2\|^2_{C^2(I,\mathbb{R}^N)}}{(d - 4s)(2\pi)^{4S}} \left(\frac{|S^d|}{4S} \right)^{\frac{4S}{d}}.$$

Formulas (3.2) and (3.3) with $1 \le m \le N$ yield

$$(-\Delta)^{\frac{3}{2}} \xi_m(x) = \varepsilon_m (-\Delta)^{\frac{3}{2} - s_m} \int_{\mathbb{R}^d} H_m(x - y) G_{1,2,m}(y) dy,$$

$$(-\Delta)^{\frac{3}{2}} u_{p,2,m}(x) = \varepsilon_m (-\Delta)^{\frac{3}{2} - s_m} \int_{\mathbb{R}^d} H_m(x - y) G_{2,2,m}(y) dy.$$

By means of (2.2) and (3.5) the norm $\|(-\Delta)^{\frac{3}{2}} (\xi_m(x) - u_{p,2,m}(x))\|^2_{L^2(\mathbb{R}^d)}$ can be estimated from above by

$$\varepsilon^2 \|G_{1,2,m} - G_{2,2,m}\|^2_{L^1(\mathbb{R}^d)} \|(-\Delta)^{\frac{3}{2} - s_m} H_m\|^2_{L^2(\mathbb{R}^d)} \le$$

$$\le \varepsilon^2 \|g_{1,m} - g_{2,m}\|^2_{C^2(I)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^4 \|(-\Delta)^{\frac{3}{2} - s_m} H_m\|^2_{L^2(\mathbb{R}^d)}.$$

Thus, $\displaystyle\sum_{m=1}^{N} \|(-\Delta)^{\frac{3}{2}} (\xi_m(x) - u_{p,2,m}(x))\|^2_{L^2(\mathbb{R}^d)} \le$

$$\le \varepsilon^2 \|g_1 - g_2\|^2_{C^2(I,\mathbb{R}^N)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^4 Q^2.$$

Hence, we arrive at $\|\xi(x)-u_{p,2}(x)\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} \leq \varepsilon\|g_1-g_2\|_{C^2(I,\mathbb{R}^N)}\times$

$$\times(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2\left[\frac{H^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^{\frac{8s}{d}-2}d}{(d-4s)(2\pi)^{4S}}\left(\frac{|S^d|}{4S}\right)^{\frac{4S}{d}}+Q^2\right]^{\frac{1}{2}}.$$

By virtue of (3.1), the norm $\|u_{p,1}-u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}$ can be bounded from above by

$$\frac{\varepsilon}{1-\varepsilon\sigma}(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2\times$$

$$\times\left[\frac{H^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^{\frac{8s}{d}-2}d}{(d-4s)(2\pi)^{4S}}\left(\frac{|S^d|}{4S}\right)^{\frac{4S}{d}}+Q^2\right]^{\frac{1}{2}}\|g_1-g_2\|_{C^2(I,\mathbb{R}^N)}.$$

We use formulas (1.21) and (1.22) to complete the proof of our theorem. ■

4. Auxiliary results

Let us formulate the solvability conditions for the linear Poisson type equation with a square integrable right side

$$(-\Delta)^s u = f(x), \quad x\in\mathbb{R}^d, \quad d=4,5, \quad 0<s<1. \tag{4.1}$$

This proposition was established in the one of the previous articles (see the part d) of Theorem 1.1 of [32]) by applying the standard Fourier transform (2.1) to both sides of problem (4.1).

Lemma 4.1. *Let* $0<s<1,\ f(x):\mathbb{R}^d\to\mathbb{R},\ d=4,5$ *and* $f(x)\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. *Then problem (4.1) possesses a unique solution* $u(x)\in H^{2s}(\mathbb{R}^d)$.

Note that in the lemma above we establish the solvability of equation (4.1) in $H^{2s}(\mathbb{R}^d),\ d=4,5$ for all values of the power of the negative Laplacian $0<s<1$ and no orthogonality conditions are imposed on the right side $f(x)$.

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