FORWARD-BACKWARD BOUNDED SOLUTIONS TO DIFFERENTIAL INCLUSIONS

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ABSTRACT. It is shown that some class of differential inclusions has solutions that are defined and bounded for all real values of independent variable. Applications to dynamics are considered.

1. Introduction

Discontinuous differential equations often arise in applied mathematics: one of the traditional examples is the motion of a body in presence of dry friction [8]; more recent sources of interest come from control theory and games theory. In important article [7] the author studies and compares solutions in sense of [8] with other notions, due to Krasowskii and Hermes, and with the classical ones (Newton and Carathodory solutions). Other remarkable work was done in [3], [5] and [4] (see also the reference therein).

In the context of control theory, other types of solutions (Euler solutions) have been successfully employed (see Ancona et al [2], Malisoff et al [12]).

In the present article we focus our attention on discontinuous differential equations of type which usually arises in mechanics of systems with the Coulomb (dry) friction. Such systems are described by Filippov's construction presented in [8].

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We study the systems of the Newton second law type, that is the systems of the second order equations. The Coulomb friction generates discontinuities in velocities and generally it does not spoil dependence on time and spatial variables. So that we impose conditions which carry out such a specialization.

We show that in some sense unstable system with Coulomb friction has solutions defined for all real time and these solutions are bounded in both directions of time. Herewith the Coulomb friction mollifies the system in the direction of positive time and destabilizes it in the negative time direction.

The main effect can be described as follows. Assume we have a mechanical system

$$\ddot{x}=-\frac{\partial V}{\partial x}(x),\quad x\in\mathbb{R}^m$$

and the potential energy $V=V(x)$ attains maximum at a point $\tilde{x}\in\mathbb{R}^m$. It is well known that the equilibrium $x(t)\equiv\tilde{x}$ is unstable. Now let us perturbate the system:

$$\ddot{x}=-\frac{\partial V}{\partial x}(x)+g(t,x,\dot{x}).\tag{1.1}$$

It turns out that if we impose certain conditions on $g,\quad V$ then system (1.1) has a solution $x(t)$ that is defined for all $t\in\mathbb{R}$ and $\sup_{t\in\mathbb{R}}|x(t)|<\infty$. This holds true even if the term g is a discontinuous function of $\dot{x}$, the Coulomb friction for example.

2. Definitions and The Statement of the Problem

Let μ stand for the standard Lebesgue measure in

$$\mathbb{R}^m=\{x=(x^1,\ldots,x^m)\},\quad d\mu_x=dx^1\ldots dx^m.$$

By $(x,y)=\sum_{k=1}^m x^iy^i$ we denote the standard Euclidean inner product and $|x|$ stands for the corresponding norm.

Let

$$B_r(x_0)=\{x\in\mathbb{R}^m\mid |x-x_0|<r\}$$

stand for the open ball of radius $r>0$ with center at x_0;

$$\partial B_r(x_0)=\{x\in\mathbb{R}^m\mid |x-x_0|=r\}.$$

Let $\operatorname{conv}U,\quad U\subset\mathbb{R}^m$ stand for the closed convex hull of U.

Let $M\subset\mathbb{R}^m$ be an open domain. Introduce a domain

$$G=\mathbb{R}\times M\times\mathbb{R}^m.$$

Consider a mapping

$$f:G\to\mathbb{R}^m,\quad f=f(t,x,y).$$

For each $(t,x)\in\mathbb{R}\times M$ the mapping $f(t,x,\cdot)$ is measurable in y; for almost all y the functions

$$f(t,x,y),\quad d_xf(t,x,y),\quad d^2_{xx}f(t,x,y)$$

are continuous in $(t,x)\in\mathbb{R}\times M$.

Moreover, the following hypotheses hold:

A: for any compact set $K\subset\mathbb{R}\times M$ there exists a positive constant c_K such that for almost all y and for all $(t,x)\in K$ it follows that

$$|d_xf(t,x,y)|+|d^2_{xx}f(t,x,y)|+|f(t,x,y)|<c_K;$$

B: for all small enough $\varepsilon>0$, for any compact interval $I\subset\mathbb{R}$ and for any compact set $K\subset M$ there exists $\delta>0$ such that for almost all y and for all

$$x\in K,\quad t',t''\in I$$

the following implication holds:

$$|t'-t''|<\delta\Longrightarrow|f(t',x,y)-f(t'',x,y)|<\varepsilon;$$

C: for all small enough $\varepsilon>0$, for any t and for any $\tilde{x}\in M$ there exists $\delta>0$ such that for almost all y we have

$$f(t,B_\delta(\tilde{x}),y)\subset B_\varepsilon(f(t,\tilde{x},y)).$$

The main object of our study is the following initial value problem

$$\ddot{x}=f(t,x,\dot{x}).\tag{2.1}$$

Now we introduce a concept of generalized solution to this system. For briefness we will say "generalized solution" but it would be more accurate to call it "the solution in the sense of inclusions".

The following definition is a modified version of one in [8]. This modification is physically reasonable.

Definition 1. *We shall say that a function $x(t)\in C^1((t_1,t_2),\mathbb{R}^m)$ is a generalized solution to problem (2.1) if*

1) $\dot{x}(t)$ is an absolutely continuous function in (t_1,t_2);

2) for almost all $t\in(t_1,t_2)$ the following inclusion holds

$$\ddot{x}(t)\in\bigcap_{r>0}\bigcap_N\operatorname{conv} f\Big(t,x(t),B_r\big(\dot{x}(t)\big)\backslash N\Big).\tag{2.2}$$

Here $\bigcap_N$ stands for the intersection over all measure-null sets:

$$N\subset\mathbb{R}^m,\quad\mu(N)=0.$$

Recall that an absolutely continuous function has derivative almost everywhere and this derivative is locally Lebesgue integrable [11].

Remark 1. *If the function f is continuous in G then the set in the right-hand side of (2.2) consists of the single element $\{f(t,x(t),\dot x(t))\}$.*

3. The Main Theorem

Introduce a function $F\in C^4(M)$ and two sets

$$D_c=\{x\in M\mid F(x)<c\},\quad \hat D_c=\{x\in M\mid F(x)=c\}.$$

Here c is a constant.

Suppose that

d1: the closure of D_c in $\mathbb{R}^m$ is a compact set and it is contained in M:

$$\overline{D_c}\subset M;$$

d2: there exists a homeomorphism $\psi:\overline{B_1(0)}\to D_c\cup\hat D_c$ such that

$$\psi(\partial B_1(0))=\hat D_c.$$

Theorem 1. *Assume that* **A**, **B**, **C**, **d1**, **d2** *are satisfied. Let a form $d^2F(x)$ be positive definite or positive semi-definite for all $x\in\hat D_c$; and for almost all*

$$(t,x,y)\in\mathbb{R}\times\hat D_c\times\mathbb{R}^m$$

it follows that

$$dF(x)[f(t,x,y)]>0.$$

Then equation (2.1) has a generalized solution $x(t)\in C(\mathbb{R},\mathbb{R}^m)$, and for all t one has

$$x(t)\in\overline{D}_c.$$

Remark 2. *The theorem remains valid if we replace condition* **d2** *with the following one: the set D_c does not admit a continuous retraction $\overline{D}_c\to\partial D_c$ and $\partial D_c=\hat D_c$.*

If $f\in C^2(G,\mathbb{R}^m)$ and all the above conditions except **C** *are satisfied then the theorem remains valid and $x(t)$ is a solution in the classical sense.*

Note also that the function $\dot x(t)$ is not obliged to be bounded in $\mathbb{R}$.

Theorem 1 is proved in section 6; section 5 contains auxiliary facts.

4. System with Potential Forces and Discontinuous Perturbation

Let F,f be the same functions as in sections 2, 3 and all the above conditions, particularly **A**, **B**, **C**, **d1**, **d2**, are fulfilled.

Consider a system

$$\ddot x=\frac{\partial F}{\partial x}(x)+f(t,x,\dot x). \tag{4.1}$$

Here the function F plays a role of potential energy taken with opposite sign.

Theorem 2. *Let the form $d^2F(x)$ be positive definite or positive semi-definite for all $x \in \hat{D}_c$; and for almost all*

$$(t, x, y) \in \mathbb{R} \times \hat{D}_c \times \mathbb{R}^m$$

it follows that

$$|dF(x)|^2 + dF(x)[f(t, x, y)] > 0.$$

Then equation (4.1) has a generalized solution $x(t) \in C(\mathbb{R}, \mathbb{R}^m)$, and for all t one has

$$x(t) \in \overline{D}_c.$$

Indeed, introduce a function $\eta \in C^\infty(\mathbb{R}^m)$ such that $\eta(x) = 1$ provided $x \in \overline{D_c}$ and $\operatorname{supp} \eta \subset M$ is a compact set.

Consider a system

$$\ddot{x} = \eta(x)\frac{\partial F}{\partial x}(x) + f(t, x, \dot{x}).$$

By theorem 1 this system has a solution $x(t)$ and this solution belongs to $\overline{D_c}$ for all time. Thus $x(t)$ is a solution to system (4.1) also.

5. Regular Statement

5.1. **Forward Bounded Solutions.** We use the following smooth result.

Let

$$H = \mathbb{R}_+ \times M \times \mathbb{R}^m, \quad \mathbb{R}_+ = \{t \geq 0\}.$$

Consider a system

$$\ddot{x} = a(t, x, \dot{x}), \quad a \in C^2(H, \mathbb{R}^m). \tag{5.1}$$

Theorem 3 ([13])**.** *Assume that the conditions* **d1**, **d2** *hold and for any compact set*

$$K \subset \mathbb{R}_+ \times M$$

there exists a positive constant C_K such that for all

$$(t, x, y) \in K \times \mathbb{R}^m$$

one has

$$|a(t, x, y)| < C_K.$$

Assume also that for all $x \in \hat{D}_c$ a quadratic form $d^2F(x)$ is positive definite or positive semi-definite; and for all

$$(t, x, y) \in \mathbb{R}_+ \times \hat{D}_c \times \mathbb{R}^m$$

it follows that

$$dF(x)[a(t,x,y)] > 0.$$

Then equation (5.1) has a solution $x(t) \in C^4(\mathbb{R}_+, \mathbb{R}^m)$*, and for all* $t \geq 0$ *one has*

$$x(t) \in \overline{D}_c.$$

5.2. **Forward-Backward bounded solutions.** Consider a system

$$\ddot{x} = b(t, x, \dot{x}). \tag{5.2}$$

The functions $b = b(t,x,y)$,

$$d_y b(t,x,y), \quad d_x b(t,x,y), \quad d^2_{xy} b(t,x,y), \quad d^2_{yy} b(t,x,y), \quad d^2_{xx} b(t,x,y)$$

are continuous in G.

Let the following hypotheses hold.

H1: For any compact set

$$K \subset \mathbb{R} \times M$$

there exists positive constant $\check{c}_K$ such that for all

$$(t,x,y) \in K \times \mathbb{R}^m$$

one has

$$|d_x b(t,x,y)| + |d^2_{xy} b(t,x,y)| + |d^2_{yy} b(t,x,y)| + |d^2_{xx} b(t,x,y)| \\ + |b(t,x,y)| + |d_y b(t,x,y)| < \check{c}_K;$$

H2: for all small enough $\varepsilon > 0$, for any compact interval $I \subset \mathbb{R}$ and for any compact set $K \subset M$ there exists $\delta > 0$ such that for all $(x,y) \in K \times \mathbb{R}^m$ and for all $t', t'' \in I$ the following implication holds:

$$|t' - t''| < \delta \Longrightarrow |b(t',x,y) - b(t'',x,y)| < \varepsilon.$$

Theorem 4. *Assume that the conditions* **H1**, **H2**, **d1**, **d2** *hold.*

Assume also that for all $x \in \hat{D}_c$ *a quadratic form* $d^2F(x)$ *is positive definite or positive semi-definite; and for all*

$$(t,x,y) \in \mathbb{R} \times \hat{D}_c \times \mathbb{R}^m$$

it follows that

$$dF(x)[b(t,x,y)] > 0.$$

Then equation (5.2) has a solution $x(t) \in C^2(\mathbb{R}, \mathbb{R}^m)$ *such that*

$$x(t) \in \overline{D}_c$$

for any $t \in \mathbb{R}$.

5.2.1. *Proof of theorem 4.* Let $\chi_k(t),\quad k\in\mathbb{N}$ stand for the indicator:

$$\chi_k(t)=1\quad\text{provided}\quad t\in(-k,k],$$

and $\chi_k(t)=0$ otherwise.

Pick a function $\psi\in C^\infty(\mathbb{R})$ such that

$$\operatorname{supp}\psi\subset(-1,1),\quad \psi\ge 0,\quad \int_{\mathbb{R}}\psi(t)dt=1.$$

Introduce a sequence of functions

$$\delta_l(t)=l\psi(lt),\quad \operatorname{supp}\delta_l\subset(-1/l,1/l),\quad l\in\mathbb{N}.$$

Introduce a function

$$b_k(t,x,y)=\sum_{j\in\mathbb{Z}}b(t+2kj,x,y)\chi_k(t+2kj)$$

and a function

$$b_{kl}(t,x,y)=\int_{\mathbb{R}}b_k(s,x,y)\delta_l(s-t)ds.$$

It is clear all the functions b_k, b_{kl} are $2k-$ periodic in t and

$$b_k(t,x,y)=b(t,x,y),\quad t\in(-k,k]. \tag{5.3}$$

Moreover $b_{kl}\in C^2(G,\mathbb{R}^m)$.

For any $\varepsilon>0$, for any $T>0$ and for any compact set $K\subset M$ there exists $L>0$ such that for all $k\ge 2T$,

$$t\in[-T,T]\subset[-k/2,k/2],\quad (x,y)\in K\times\mathbb{R}^m$$

the following implication holds

$$l>L\Longrightarrow|b_{kl}(t,x,y)-b(t,x,y)|<\varepsilon. \tag{5.4}$$

This follows from the hypothesis **H2** and formula (5.3).

Consider the following system

$$\ddot x=b_{kl}(t,x,\dot x). \tag{5.5}$$

This system satisfies all the conditions of theorem 3. Indeed, let $(t,x,y)\in\mathbb{R}\times\hat D_c\times\mathbb{R}^m$ then

$$\begin{aligned}&dF(x)[b_{kl}(t,x,y)]\\&\quad=\int_{\mathbb{R}}\sum_{j\in\mathbb{Z}}dF(x)[b(s+2kj,x,y)]\chi_k(s+2kj)\delta_l(s-t)ds>0;\end{aligned}$$

and for any compact set $K\subset\mathbb{R}\times M$ it follows that

$$|b_{kl}(t,x,y)|\le\check c_K,\quad (t,x,y)\in K\times\mathbb{R}^m. \tag{5.6}$$

Thus system (5.5) has a solution $x_{kl}\in C^4(\mathbb{R}_+,\mathbb{R}^m)$ and

$$x_{kl}(t)\in\overline{D_c},\quad t\ge 0.$$

Introduce functions

$$z_{kli}(t)=x_{kl}(t+2ki),\quad i\in\mathbb{N}.$$

All these functions are the solutions to (5.5) and

$$z_{kli}\in C^4([-2ki,\infty),\mathbb{R}^m);\qquad z_{kli}(t)\in\overline{D_c},\quad t\ge -2ki. \tag{5.7}$$

Due to (5.6) for $k,i,l\in\mathbb{N}$ and fixed $T\le k/2$ we have

$$|\ddot z_{kli}(t)|\le \check c_{[-T,T]\times\overline{D_c}},\quad t\in[-T,T]. \tag{5.8}$$

Let $H^2([-T,T],\mathbb{R}^m)$ stand for the Sobolev space of functions $u(t)$ such that

$$u,\dot u,\ddot u\in L^2([-T,T],\mathbb{R}^m).$$

Recall that $H^2([-T,T],\mathbb{R}^m)$ is a Hilbert space with the inner product

$$(a,b)_{H^2[-T,T]}=\int_{-T}^{T}\big((a(s),b(s))+(\ddot a(s),\ddot b(s))\big)ds.$$

Another inner product

$$\langle a,b\rangle_{H^2[-T,T]}=\int_{-T}^{T}\big((a(s),b(s))+(\dot a(s),\dot b(s))+(\ddot a(s),\ddot b(s))\big)ds$$

gives the same topology in $H^2([-T,T],\mathbb{R}^m)$.

Recall also that there is a compact embedding

$$H^2([-T,T],\mathbb{R}^m)\subset C^1([-T,T],\mathbb{R}^m). \tag{5.9}$$

Here and below we refer [1] for the properties of the Sobolev spaces.

Lemma 1. *The sequence $Z_p:=z_{ppp}$ contains a subsequence Z_{p_s} that is convergent to a function $z_*\in C^1(\mathbb{R},\mathbb{R}^m)$ in $C^1([-T,T],\mathbb{R}^m)$ for any $T>0$.*

The function z_* is such that

$$z_*(t)\in\overline{D_c},\quad t\in\mathbb{R}.$$

Proof of the lemma 1. Fix $T=1$. From formulas (5.8), (5.7), (5.9) it follows that there exists a subsequence Z_{p_q} that is convergent in $C^1([-1,1],\mathbb{R}^m)$. By the same reason this subsequence contains a subsequence that is convergent in $C^1([-2,2],\mathbb{R}^m)$ etc.

The diagonal argument finishes the proof.

The function z_* is the announced solution to system (5.2).

Indeed, to see this fix $T>0$ and for $t\in[-T,T]$ write

$$\dot Z_{p_s}(t)=\dot Z_{p_s}(0)+\int_0^t b_{p_sp_s}(\xi,Z_{p_s}(\xi),\dot Z_{p_s}(\xi))d\xi.$$

To pass to the limit in this equality let

$$\begin{aligned}\int_0^t b_{p_sp_s}(\xi,Z_{p_s}(\xi),\dot Z_{p_s}(\xi))d\xi-\int_0^t b(\xi,z_*(\xi),\dot z_*(\xi))d\xi\\ =\int_0^t\big(b_{p_sp_s}(\xi,Z_{p_s}(\xi),\dot Z_{p_s}(\xi))-b(\xi,Z_{p_s}(\xi),\dot Z_{p_s}(\xi))\big)d\xi\\ +\int_0^t\big(b(\xi,Z_{p_s}(\xi),\dot Z_{p_s}(\xi))-b(\xi,z_*(\xi),\dot z_*(\xi))\big)d\xi.\end{aligned}\tag{5.10}$$

By (5.4) we have

$$|b_{p_sp_s}(\xi,Z_{p_s}(\xi),\dot Z_{p_s}(\xi))-b(\xi,Z_{p_s}(\xi),\dot Z_{p_s}(\xi))|\to 0$$

uniformly in $\xi\in[-T,T]$.

On the other hand the difference

$$|b(\xi,Z_{p_s}(\xi),\dot Z_{p_s}(\xi))-b(\xi,z_*(\xi),\dot z_*(\xi))|\to 0$$

vanishes pointwise in $\xi\in[-T,T]$ and

$$|b(\xi,Z_{p_s}(\xi),\dot Z_{p_s}(\xi))-b(\xi,z_*(\xi),\dot z_*(\xi))|\le 2\check c_{[-T,T]\times\overline{D_c}}.$$

Thus the last integral in (5.10) also tends to zero by the dominated convergence theorem.

Theorem 4 is proved.

6. Proof of Theorem 1

Pick a function $\varphi\in C^\infty(\mathbb{R}^m)$ such that

$$\operatorname{supp}\varphi\subset B_1(0),\quad \varphi\ge 0,\quad \int_{\mathbb{R}^m}\varphi(x)d\mu_x=1.$$

Introduce a sequence of functions

$$\delta_k(x)=k^m\varphi(kx),\quad \operatorname{supp}\delta_k\subset B_{1/k}(0),\quad k\in\mathbb{N}$$

and put

$$f_k(t,x,y)=\int_{\mathbb{R}^m} f(t,x,z)\delta_k(z-y)d\mu_z.\tag{6.1}$$

Constructions of type (6.1) are usually employed in approximation theory. Intuitively speaking, the sequence f_k approximates the function f in some sense. Nevertheless, in this section we do not use such an argument and we do not refer any approximation theorems.

We use only one convolution property: $f_k(t,\cdot,\cdot)\in C^2(M\times\mathbb{R}^m,\mathbb{R}^m)$.

Observe that the following system

$$\ddot{x} = f_k(t, x, \dot{x}) \tag{6.2}$$

satisfies all the conditions of theorem 4. Indeed, **H2** follows from **B**;

$$|f_k(t, x, y)| \le \int_{\mathbb{R}^m} c_K \delta_k(z - y) d\mu_z \le c_K; \tag{6.3}$$

and

$$dF(x)[f_k(t, x, y)] = \int_{\mathbb{R}^m} dF(x)[f(t, x, z)]\delta_k(z - y) d\mu_z > 0$$

provided $(t, x) \in \mathbb{R} \times \hat{D}_c$.

Thus theorem 4 supplies each system (6.2) with the solution $x_k(t)$.

From the properties of the Lebesgue integral for all t we have

$$\begin{aligned} \ddot{x}_k(t) &= f_k(t, x_k(t), \dot{x}_k(t)) \\ &\in \bigcap_N \operatorname{conv} f\Big(t, x_k(t), B_{\frac{1}{k}}\big(\dot{x}_k(t)\big)\backslash N\Big). \end{aligned} \tag{6.4}$$

Here the intersection is taken over all measure-null sets:

$$N \subset \mathbb{R}^m, \quad \mu(N) = 0.$$

Indeed,

$$\begin{aligned} f_k(t, x_k(t), \dot{x}_k(t)) &= \int_{\mathbb{R}^m} f(t, x_k(t), z)\delta_k(z - \dot{x}_k(t)) d\mu_z \\ &= \int_{\mathbb{R}^m \backslash N} f(t, x_k(t), z)\delta_k(z - \dot{x}_k(t)) d\mu_z \\ &\in \operatorname{conv} f\Big(t, x_k(t), B_{\frac{1}{k}}\big(\dot{x}_k(t)\big)\backslash N\Big). \end{aligned}$$

Lemma 2. *For any $T > 0$ the sequence $\{x_k\}$ is bounded in*

$$H^2([-T, T], \mathbb{R}^m),$$

that is

$$\sup_k \|x_k\|_{H^2[-T,T]} < \infty.$$

Proof of lemma 2. First, notice that

$$\{x_k(t)\} \subset \overline{D_c}$$

for all $t \in \mathbb{R}$ and k. Thus by formula (6.3) we get

$$|\ddot{x}_k(t)| \le c_{[-T,T]\times\overline{D_c}}, \quad t \in [-T, T].$$

The lemma is proved.

Lemma 3. *The sequence* $\{x_k\}$ *contains a subsequence* $\{x_{k_s}\}$ *that is convergent to a function*

$$x_* \in H^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^m)$$

in the following sense:
for any $T>0$ *one has*

$$\|x_{k_s}-x_*\|_{C^1[-T,T]}\to 0,$$

and $x_{k_s}\to x_*$ *weakly in* $H^2([-T,T],\mathbb{R}^m)$.

Proof of lemma 3. Recall a theorem.

Theorem 5 ([10])**.** *A Banach space* $(Y,\|\cdot\|_Y)$ *is reflexive iff any bounded sequence in* Y *contains a weakly convergent subsequence.*

Take an increasing sequence $T_n\to\infty$.

Since the embedding $H^2([-T_n,T_n],\mathbb{R}^m)\subset C^1([-T_n,T_n],\mathbb{R}^m)$ is compact and the space $H^2([-T_n,T_n],\mathbb{R}^m)$ is reflexive, we can choose a subsequence $\{x^{(1)}_{k_j}\}\subset\{x_k\}$ that is convergent in $C^1([-T_1,T_1],\mathbb{R}^m)$ and weakly convergent in

$$H^2([-T_1,T_1],\mathbb{R}^m).$$

The sequence $\{x^{(1)}_{k_j}\}$ contains a subsequence $\{x^{(2)}_{k_j}\}$ that is convergent in $C^1([-T_2,T_2],\mathbb{R}^m)$ and weakly convergent in $H^2([-T_2,T_2],\mathbb{R}^m)$ etc. The diagonal sequence is convergent in the desired manner.

The lemma is proved.

Note that $x_*(t)\in\overline{D_c},\quad t\in\mathbb{R}$.

Choose a constant $\rho>0$ such that for any $x_0\in\overline{D_c}$ it follows that $\overline{B_r(x_0)}\subset M,\quad r\le\rho$.

Introduce sets

$$U(t,r,r')=\bigcap_{\mu(N)=0}\operatorname{conv} f\Big(t,B_r\big(x_*(t)\big),B_{r'}\big(\dot x_*(t)\big)\backslash N\Big)\subset\mathbb{R}^m,\quad r<\rho.$$

By lemma 5 (see below) the sets $U(t,r,r')$ are nonvoid for almost all t. The sets $U(t,r,r')$ are closed and convex as an intersection of closed convex sets.

The sets $U(t,r,r')$ are uniformly bounded relative

$$t\in[-T,T]\quad,0<r\le\rho,\quad r'>0.$$

Inddded,

$$z\in U(t,r,r')\Longrightarrow |z|\le c_{[-T,T]\times S_r},\quad S_r=\overline{\bigcup_{x\in D_c}B_r(x)}\subset M. \qquad (6.5)$$

Let $W^T(r,r')\subset L^2([-T,T],\mathbb{R}^m)$ stand for a set of functions $u(t)\in L^2([-T,T],\mathbb{R}^m)$ such that for almost all $t\in[-T,T]$ one has

$$u(t)\in U(t,r,r').$$

Lemma 4. *The sets $W^T(r,r')$ are closed bounded and convex.*

Proof of lemma 4. Convexity is evident. Prove that the sets are closed. Indeed, let a sequence $\{w_i\}\subset W^T(r,r')$ converges to w in $L^2([-T,T],\mathbb{R}^m)$. Then it contains a subsequence $\{w_{i_j}\}$ that is convergent to w almost everywhere [9]. Thus $w(t)\in U(t,r,r')$ for almost all t.

The boundedness follows from formula (6.5).

The lemma is proved.

Lemma 5. *For any $T,r'>0,\quad r\in(0,\rho]$ there exists a number J such that*

$$k_s>J\Longrightarrow \ddot x_{k_s}\in W^T(r,r').$$

Proof of lemma 5. Choose J such that

1) $J>2/r'$;

2) $k_s>J\Longrightarrow \|x_{k_s}-x_*\|_{C^1[-T,T]}<\min\{r,r'/2\}$.

Take any $k_s>J$ it is clear

$$x_{k_s}(t)\in B_r(x_*(t)). \tag{6.6}$$

The following inclusion holds for all $t\in[-T,T]$

$$B_{\frac{1}{k_s}}(\dot x_{k_s}(t))\subset B_{r'}(\dot x_*(t)). \tag{6.7}$$

Indeed, Let $\xi\in B_{\frac{1}{k_s}}(\dot x_{k_s}(t))$ then

$$|\xi-\dot x_*(t)|\le|\xi-\dot x_{k_s}(t)|+|\dot x_{k_s}(t)-\dot x_*(t)|<\frac{1}{k_s}+\frac{r'}{2}<r'.$$

The assertion of the lemma follows from formulas (6.6), (6.7), (6.4).

The lemma is proved.

Lemma 6. *For any $T,r'>0,\quad r\in(0,\rho]$ one has*

$$\ddot x_*\in W^T(r,r').$$

Indeed, by lemma 3 the sequence $\{\ddot x_{k_s}\}$ is weakly convergent to $\ddot x_*$ in $L^2([-T,T],\mathbb{R}^m)$. The set $W^T(r,r')$ is convex and closed thus it is weakly closed [6]. This proves the lemma.

The result of lemma 6 can explicitly be formulated as follows. For almost all t we have

$$\ddot x_*(t)\in\bigcap_{r>0}\bigcap_{r'>0}\bigcap_{\mu(N)=0}\operatorname{conv} f\Big(t,B_r\big(x_*(t)\big),B_{r'}\big(\dot x_*(t)\big)\backslash N\Big). \tag{6.8}$$

According to **C** for any t we can choose sequences

$$\delta_q, \varepsilon_q \to 0, \quad \mathbb{N} \ni q \to \infty$$

and a set $\mathcal{N}$, $\mu(\mathcal{N}) = 0$ such that for all q and for all $y \in \mathbb{R}^m \backslash \mathcal{N}$ the following inclusion holds

$$f\big(t, B_{\delta_q}(x_*(t)), y\big) \subset B_{\varepsilon_q}\big(f(t, x_*(t), y)\big). \tag{6.9}$$

Lemma 7. *For any $r' > 0$ and for any*

$$N, \quad \mathcal{N} \subset N, \quad \mu(N) = 0 \tag{6.10}$$

the following equality holds

$$\bigcap_{q\in\mathbb{N}} \operatorname{conv} f\Big(t, B_{\delta_q}\big(x_*(t)\big), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big) = \operatorname{conv} f\Big(t, x_*(t), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big). \tag{6.11}$$

Obviously, formula (6.11) can be rewritten as follows

$$\bigcap_{r>0} \operatorname{conv} f\Big(t, B_r\big(x_*(t)\big), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big) = \operatorname{conv} f\Big(t, x_*(t), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big). \tag{6.12}$$

Moreover, observe that

$$\bigcap_{\mu(N)=0} \bigcap_{r>0} \operatorname{conv} f\Big(t, B_r\big(x_*(t)\big), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big) = \widetilde{\bigcap} \bigcap_{r>0} \operatorname{conv} f\Big(t, B_r\big(x_*(t)\big), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big),$$

and

$$\bigcap_{\mu(N)=0} \operatorname{conv} f\Big(t, x_*(t), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big) = \widetilde{\bigcap} \operatorname{conv} f\Big(t, x_*(t), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big),$$

where $\bigcap_{\mu(N)=0}$ means the intersection over all measure-null sets N and $\widetilde{\bigcap}$ means the intersection over all N such that (6.10).

So that formula (6.12) implies

$$\bigcap_{\mu(N)=0} \bigcap_{r>0} \operatorname{conv} f\Big(t, B_r\big(x_*(t)\big), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big) = \bigcap_{\mu(N)=0} \operatorname{conv} f\Big(t, x_*(t), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big).$$

Thus formula (6.8) takes the form

$$\ddot{x}_*(t) \in \bigcap_{r'>0} \bigcap_{\mu(N)=0} \operatorname{conv} f\Big(t, x_*(t), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big).$$

This proves theorem 1.

Proof of lemma 7. From formula (6.9) for all $y \in B_{r'}\big(\dot{x}_*(t)\big)\backslash N$ we have

$$f\Big(t, B_{\delta_q}\big(x_*(t)\big), y\Big) \subset B_{\varepsilon_q}\Big(f\big(t, x_*(t), y\big)\Big).$$

So that the set

$$\begin{aligned} &f\Big(t, B_{\delta_q}\big(x_*(t)\big), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big) \\ &= \bigcup_{y\in B_{r'}\left(\dot{x}_*(t)\right)\backslash N} f\Big(t, B_{\delta_q}\big(x_*(t)\big), y\Big) \\ &\subset \bigcup_{y\in B_{r'}\left(\dot{x}_*(t)\right)\backslash N} B_{\varepsilon_q}\Big(f\big(t, x_*(t), y\big)\Big) \end{aligned}$$

is contained in an ε_q- neighbourhood of the set

$$f\Big(t, x_*(t), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big).$$

Thus we obtain

$$\bigcap_{q\in\mathbb{N}} f\Big(t, B_{\delta_q}\big(x_*(t)\big), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big) = \overline{f\Big(t, x_*(t), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big)}. \tag{6.13}$$

Observe that the closed convex hull of a set $Q \subset \mathbb{R}^m$ is the intersection over all closed half-spaces that contain Q. That is

$$\operatorname{conv} Q = \bigcap_{\xi\in\mathbb{R}^m} P_\xi, \quad P_\xi = \{u \in \mathbb{R}^m \mid (u, \xi) \le \sup_{x\in Q}(x, \xi)\}.$$

Observe also that

$$\operatorname{conv} \overline{Q} = \operatorname{conv} Q. \tag{6.14}$$

We have

$$\bigcap_{q\in\mathbb{N}} \operatorname{conv} f\Big(t, B_{\delta_q}\big(x_*(t)\big), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\Big) = \bigcap_{q\in\mathbb{N}} \bigcap_{\xi\in\mathbb{R}^m} P_{q\xi}, \tag{6.15}$$

where

$$\begin{aligned} P_{q\xi} = \Big\{u \in \mathbb{R}^m \Big| &(u, \xi) \\ &\le \sup\big\{(x, \xi) \mid x \in f\big(t, B_{\delta_q}\big(x_*(t)\big), B_{r'}\big(\dot{x}_*(t)\big)\backslash N\big)\big\}\Big\}. \end{aligned}$$

Changing the order of intersections in the right-hand side of formula (6.15) and by formula (6.13) we get

$$\bigcap_{q\in\mathbb{N}}\mathrm{conv}\, f\Big(t, B_{\delta_q}\big(x_*(t)\big), B_{r'}\big(\dot x_*(t)\big)\backslash N\Big)$$
$$=\mathrm{conv}\,\overline{f\Big(t, x_*(t), B_{r'}\big(\dot x_*(t)\big)\backslash N\Big)}.$$

Now formula (6.11) follows from (6.14).

The lemma is proved.

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