SOLVABILITY OF SOME INTEGRO-DIFFERENTIAL EQUATIONS WITH CONCENTRATED SOURCES

Vitali Vougalter

Department of Mathematics, University of Toronto
Toronto, Ontario, M5S 2E4, Canada
e-mail: vitali@math.toronto.edu

Abstract: The work deals with the existence of solutions of an integro-differential equation in the case of the normal diffusion and the influx/efflux term proportional to the Dirac delta function. The proof of the existence of solutions is based on a fixed point technique. Solvability conditions for non Fredholm elliptic operators in unbounded domains are used.

AMS Subject Classification: 35J05, 35P30, 47F05
Key words: integro-differential equations, Dirac delta function, non Fredholm operators, Sobolev spaces

1. Introduction

The present article is devoted to the existence of stationary solutions of the following nonlocal reaction-diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \int_{-\infty}^{\infty} K(x-y)g(w(y)u(y,t))dy + \alpha\delta(x), \quad \alpha \in \mathbb{R}, \quad \alpha \neq 0. \tag{1.1}$$

Here $w(x)$ is our cut-off function the assumptions on which will be formulated below. The problems of this kind appear in the cell population dynamics. The space variable x here corresponds to the cell genotype, $u(x,t)$ denotes the cell density as a function of their genotype and time. The right side of this problem describes the evolution of cell density via cell proliferation, mutations and cell influx. The diffusion term here corresponds to the change of genotype due to small random mutations, and the nonlocal term describes large mutations. Function $g(w(x)u(x))$ stands for the rate of cell birth which depends on u, w (density dependent proliferation), and the kernel $K(x-y)$ gives the proportion of newly born cells changing their genotype from y to x. We assume that it depends on the distance between the genotypes. Finally, the last term in the right side of this problem, which is proportional to the Dirac delta function denotes the influx/efflux of cells for different genotypes. A similar equation in one dimension in the case of the standard negative Laplacian raised to the power $0 < s < \frac{1}{4}$ in the diffusion term was investigated recently in [27]. Note that in [27] it was assumed that the influx/efflux term $f(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Therefore, in the present article we consider the more singular situation. In neuroscience, the integro-differential equations describe the nonlocal interaction of neurons (see [9] and the references therein).

Let us set $D = 1$ and establish the existence of solutions of the problem

$$\frac{d^2u}{dx^2} + \int_{-\infty}^{\infty} K(x-y)g(w(y)u(y))dy + \alpha\delta(x) = 0. \tag{1.2}$$

We will discuss the situation when the linear part of such operator fails to satisfy the Fredholm property. Consequently, conventional methods of nonlinear analysis may not be applicable. Let us use solvability conditions for non Fredholm operators along with the method of contraction mappings.

Consider the equation

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ either vanishes identically or converges to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of problem (1.3) contains the origin. Consequently, this operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work is devoted to the studies of certain properties of the operators of this kind. Note that elliptic problems with non Fredholm operators were treated actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [4], [5], [6], [7], [8]. The non Fredholm Schrödinger type operators were studied with the methods of the spectral and the scattering theory in [14], [18], [21], [22]. Fredholm structures, topological invariants and their applications were covered in [13]. The Laplace operator with drift from the point of view of non Fredholm operators was treated in [24] and linearized Cahn-Hilliard problems in [19] and [25]. Nonlinear non Fredholm elliptic equations were considered in [23] and [26]. Important applications to the theory of reaction-diffusion problems were developed in [11], [12]. Non Fredholm operators arise also when studying wave systems with an infinite number of localized traveling waves (see [2]). Standing lattice solitons in the discrete NLS equation with saturation were considered in [3]. In particular, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [4], [5], [6], [7], [8]). However, the case of $a \neq 0$ is considerably different and the approach developed in these articles cannot be adopted. The existence, stability and bifurcations of the solutions of the nonlinear partial differential equations involving Dirac delta function potentials were studied actively in [1], [15], [16].

We set $K(x) = \varepsilon \mathcal{K}(x)$, where $\varepsilon \geq 0$. When the nonnegative parameter ε vanishes, we obtain the linear Poisson equation

$$-\frac{d^2 u}{dx^2} = \alpha \delta(x). \tag{1.4}$$

Let us introduce the function

$$\xi_0(x) := \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{1.5}$$

Evidently, the solution of (1.4) vanishing at the minus infinity is given by $-\alpha\xi_0(x)$. Note that as distinct from the situation discussed in [27], such solution is unbounded and does not belong to $H^1(\mathbb{R})$. Suppose that the assumption below is satisfied.

Assumption 1.1. *Let $\mathcal{K}(x) : \mathbb{R} \to \mathbb{R}$ be nontrivial, such that $\mathcal{K}(x), x^2\mathcal{K}(x) \in L^1(\mathbb{R})$ and orthogonality relations (4.2) hold. Let us also assume that the cut-off function $w(x) : \mathbb{R} \to \mathbb{R}$ is such that $w(x)\xi_0(x)$ is nontrivial and $w(x)\xi_0(x) \in H^1(\mathbb{R})$. Moreover, $w(x) \in H^1(\mathbb{R})$ and for $\alpha \in \mathbb{R},\ \alpha \neq 0$ the inequality*

$$|\alpha| \leq \frac{1}{\|w(x)\xi_0(x)\|_{H^1(\mathbb{R})}} \tag{1.6}$$

holds.

It can be easily verified that $w(x) = e^{-|x|},\ x \in \mathbb{R}$ satisfies the assumptions above and therefore it can be used as our cut-off function. Note that in the argument of [27] we did not need to use such cut-off due to the more regular behaviour of the solution of the Poisson type equation. In the article we choose the space dimension $d = 1$, which is related to the solvability of the linear Poisson equation (1.4) discussed above. From the point of view of the applications, the space dimension is not restricted to $d = 1$ since the space variable corresponds to cell genotype but not to the usual physical space. Let us use the Sobolev space

$$H^1(\mathbb{R}) := \Big\{u(x) : \mathbb{R} \to \mathbb{R} \mid u(x) \in L^2(\mathbb{R}),\ \frac{du}{dx} \in L^2(\mathbb{R})\Big\}.$$

It is equipped with the norm

$$\|u\|^2_{H^1(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \Big\|\frac{du}{dx}\Big\|^2_{L^2(\mathbb{R})}. \tag{1.7}$$

Evidently, using the standard Fourier transform (2.1), this norm can be written as

$$\|u\|^2_{H^1(\mathbb{R})} = \|\widehat{u}(p)\|^2_{L^2(\mathbb{R})} + \|p\widehat{u}(p)\|^2_{L^2(\mathbb{R})}. \tag{1.8}$$

By means of the Sobolev inequality in one dimension (see e.g. Sect 8.5 of [17]), we have

$$\|u(x)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}}\|u(x)\|_{H^1(\mathbb{R})}. \tag{1.9}$$

We will be using the algebraic property of our Sobolev space, namely that for any $u(x), v(x) \in H^1(\mathbb{R})$

$$\|u(x)v(x)\|_{H^1(\mathbb{R})} \leq c_a \|u(x)\|_{H^1(\mathbb{R})}\|v(x)\|_{H^1(\mathbb{R})}, \tag{1.10}$$

where $c_a > 0$ is a constant. Upper bound (1.10) can be easily established, for instance via (1.9). Let us seek the resulting solution of nonlinear problem (1.2) as

$$u(x) = -\alpha\xi_0(x) + u_p(x). \tag{1.11}$$

Apparently, we derive the perturbative equation

$$-\frac{d^2u_p(x)}{dx^2} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y)g(w(y)[-\alpha\xi_0(y) + u_p(y)])dy. \tag{1.12}$$

Let us introduce a closed ball in our Sobolev space

$$B_\rho := \{u(x) \in H^1(\mathbb{R}) \mid \|u\|_{H^1(\mathbb{R})} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{1.13}$$

We seek the solution of equation (1.12) as the fixed point of the auxiliary nonlinear problem

$$-\frac{d^2u(x)}{dx^2} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y)g(w(y)[-\alpha\xi_0(y) + v(y)])dy \tag{1.14}$$

in ball (1.13). For a given function $v(y)$ this is an equation with respect to $u(x)$. The left side of (1.14) contains the operator $-\frac{d^2}{dx^2}$ acting on $L^2(\mathbb{R})$, which fails to satisfy the Fredholm property. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Therefore, this operator has no bounded inverse. The similar situation in the context of the integro-differential equations occurred also in articles [23] and [26]. The problems studied there also required the application of the orthogonality conditions. The contraction argument was used in [20] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear problem there satisfied the Fredholm property (see Assumption 1 of [20], also [10]). We introduce the interval on the real line

$$I := \Big[-\frac{1}{\sqrt{2}} - \frac{c_a}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})},\ \frac{1}{\sqrt{2}} + \frac{c_a}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big] \tag{1.15}$$

along with the closed ball in the space of $C_1(I)$ functions, namely

$$D_M := \{g(z) \in C_1(I) \mid \|g\|_{C_1(I)} \le M\}, \quad M > 0. \tag{1.16}$$

In this context the norm

$$\|g\|_{C_1(I)} := \|g\|_{C(I)} + \|g'\|_{C(I)}, \tag{1.17}$$

where $\|g\|_{C(I)} := \max_{z\in I}|g(z)|$. From the biological point of view, the rate of cell birth function is nonlinear and vanishes at the origin.

Assumption 1.2. *Let $g(z) : \mathbb{R} \to \mathbb{R}$, such that $g(0) = 0$. It is also assumed that $g(z) \in D_M$ and it does not vanish identically on the interval I.*

Note that in [27] the function $g(z)$ was assumed to be twice continuously differentiable on the corresponding interval I. We define the following positive technical expression

$$Q := \max\left\{\left\|\frac{\widehat{\mathcal{K}}(p)}{p^2}\right\|_{L^\infty(\mathbb{R})}, \quad \left\|\frac{\widehat{\mathcal{K}}(p)}{p}\right\|_{L^\infty(\mathbb{R})}\right\}. \tag{1.18}$$

Let us introduce the operator T_g, such that $u = T_g v$, where u is a solution of problem (1.14). Our first main proposition is as follows.

Theorem 1.3. *Let Assumptions 1.1 and 1.2 hold. Then problem (1.14) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all*

$$0 < \varepsilon \le \frac{\rho}{2\sqrt{\pi}QM(1 + c_a\|w(x)\|_{H^1(\mathbb{R})})}. \tag{1.19}$$

The unique fixed point $u_p(x)$ of this map T_g is the only solution of equation (1.12) in B_ρ.

Evidently, the resulting solution of problem (1.2) given by (1.11) will be nontrivial since $g(0) = 0$ and $\alpha \ne 0$ as assumed.

Our second main statement is about the continuity of the cumulative solution of equation (1.2) given by formula (1.11) with respect to the nonlinear function g. Let us define the following positive, auxiliary expression

$$\sigma := 2\sqrt{\pi}QMc_a\|w(x)\|_{H^1(\mathbb{R})}. \tag{1.20}$$

Theorem 1.4. *Let $j = 1, 2$, the assumptions of Theorem 1.3 are valid, such that $u_{p,j}(x)$ is the unique fixed point of the map $T_{g_j} : B_\rho \to B_\rho$, which is a strict* *contraction for all ε satisfying (1.19) and the resulting solution of equation (1.2) with $g(z) = g_j(z)$ is given by*

$$u_j(x) = -\alpha\xi_0(x) + u_{p,j}(x). \tag{1.21}$$

Then for all values of ε, which satisfy inequality (1.19) the upper bound

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R})} \le \frac{2\varepsilon\sqrt{\pi}Q(1 + c_a\|w(x)\|_{H^1(\mathbb{R})})}{1 - \varepsilon\sigma}\|g_1(z) - g_2(z)\|_{C_1(I)} \tag{1.22}$$

holds.

Let proceed to the proof of our first main proposition.

2. The existence of the perturbed solution

Proof of Theorem 1.3. We choose an arbitrary $v(x) \in B_\rho$ and designate the term involved in the integral expression in the right side of equation (1.14) as

$$G(x) := g(w(x)[-\alpha\xi_0(x) + v(x)]).$$

The standard Fourier transform is given by

$$\widehat{\phi}(p) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)e^{-ipx}dx. \tag{2.1}$$

Obviously, the inequality

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}}\|\phi(x)\|_{L^1(\mathbb{R})} \tag{2.2}$$

holds. Let us apply (2.1) to both sides of equation (1.14), which yields

$$\widehat{u}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{p^2}, \quad p\widehat{u}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{p}.$$

Therefore,

$$|\widehat{u}(p)| \le \varepsilon\sqrt{2\pi}Q|\widehat{G}(p)|, \quad |p\widehat{u}(p)| \le \varepsilon\sqrt{2\pi}Q|\widehat{G}(p)|, \tag{2.3}$$

where Q is given by (1.18). Note that under the given conditions $Q < \infty$ by virtue of Lemma 4.1 below. By means of (1.8) along with inequalities (2.3) we easily estimate the norm as

$$\|u(x)\|^2_{H^1(\mathbb{R})} \le 4\pi\varepsilon^2Q^2\|G(x)\|^2_{L^2(\mathbb{R})}. \tag{2.4}$$

It can be easily verified that for $v(x) \in B_\rho$, we have

$$|w(x)[-\alpha\xi_0(x) + v(x)]| \le \frac{1}{\sqrt{2}}\{1 + c_a\|w(x)\|_{H^1(\mathbb{R})}\}. \tag{2.5}$$

Indeed, the left side of (2.5) can be trivially bounded from above using inequalities (1.9) and (1.10) by

$$\frac{1}{\sqrt{2}}\{|\alpha|\|w(x)\xi_0(x)\|_{H^1(\mathbb{R})}+c_a\|w(x)\|_{H^1(\mathbb{R})}\|v(x)\|_{H^1(\mathbb{R})}\}. \tag{2.6}$$

Hence, (2.5) holds due to (1.6). Similarly, for $v(x)\in B_\rho$

$$\|w(x)[-\alpha\xi_0(x)+v(x)]\|_{L^2(\mathbb{R})}\le 1+c_a\|w(x)\|_{H^1(\mathbb{R})} \tag{2.7}$$

is valid via inequalities (1.6) and (1.10). Evidently,

$$G(x)=\int_0^{w(x)[-\alpha\xi_0(x)+v(x)]}g'(z)dz,$$

such that $|G(x)|\le$

$$\le \max_{z\in I}|g'(z)||w(x)[-\alpha\xi_0(x)+v(x)]|\le M|w(x)[-\alpha\xi_0(x)+v(x)]|, \tag{2.8}$$

where the interval I is defined in (1.15). Therefore, by virtue of (2.8) along with (2.7) we derive

$$\|G(x)\|_{L^2(\mathbb{R})}\le M\|w(x)[-\alpha\xi_0(x)+v(x)]\|_{L^2(\mathbb{R})}\le M(1+c_a\|w(x)\|_{H^1(\mathbb{R})}). \tag{2.9}$$

By means of estimates (2.4) and (2.9) we arrive at

$$\|u(x)\|_{H^1(\mathbb{R})}\le 2\sqrt{\pi}\varepsilon QM(1+c_a\|w(x)\|_{H^1(\mathbb{R})})\le\rho \tag{2.10}$$

for all values of the parameter ε satisfying inequality (1.19), such that $u(x)\in B_\rho$ as well.

If for a certain $v(x)\in B_\rho$ there exist two solutions $u_{1,2}(x)\in B_\rho$ of equation (1.14), their difference $w(x):=u_1(x)-u_2(x)\in L^2(\mathbb{R})$ is a solution of the homogeneous problem

$$-\frac{d^2}{dx^2}w(x)=0.$$

Since the negative second derivative operator considered on the whole real line does not possess any nontrivial square integrable zero modes, $w(x)\equiv 0$ on $\mathbb{R}$. Thus, equation (1.14) defines a map $T_g: B_\rho\to B_\rho$ for all values of ε satisfying bound (1.19).

We will establish that under the given conditions this map is a strict contraction. Let us choose arbitrarily $v_{1,2}(x)\in B_\rho$. The argument above implies that $u_{1,2}:=T_g v_{1,2}\in B_\rho$ as well when ε satisfies (1.19). By means of (1.14) we have

$$-\frac{d^2}{dx^2}u_1(x)=\varepsilon\int_{-\infty}^{\infty}\mathcal{K}(x-y)g(w(y)[-\alpha\xi_0(y)+v_1(y)])dy, \tag{2.11}$$

$$-\frac{d^2}{dx^2}u_2(x) = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y)g(w(y)[-\alpha\xi_0(y)+v_2(y)])dy. \tag{2.12}$$

Let us define

$$G_1(x) := g(w(x)[-\alpha\xi_0(x)+v_1(x)]), \quad G_2(x) := g(w(x)[-\alpha\xi_0(x)+v_2(x)])$$

and apply the standard Fourier transform (2.1) to both sides of equations (2.11) and (2.12). This yields

$$\widehat{u_1}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G_1}(p)}{p^2}, \quad \widehat{u_2}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G_2}(p)}{p^2},$$

such that

$$\widehat{u_1}(p) - \widehat{u_2}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)[\widehat{G_1}(p) - \widehat{G_2}(p)]}{p^2},$$

$$p[\widehat{u_1}(p) - \widehat{u_2}(p)] = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)[\widehat{G_1}(p) - \widehat{G_2}(p)]}{p}.$$

Therefore,

$$|\widehat{u_1}(p) - \widehat{u_2}(p)| \le \varepsilon\sqrt{2\pi}Q|\widehat{G_1}(p) - \widehat{G_2}(p)|,$$

$$|p[\widehat{u_1}(p) - \widehat{u_2}(p)]| \le \varepsilon\sqrt{2\pi}Q|\widehat{G_1}(p) - \widehat{G_2}(p)|$$

holds. This allows us to estimate the norm using (1.8) as

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R})}^2 = \int_{-\infty}^{\infty} |\widehat{u_1}(p) - \widehat{u_2}(p)|^2 dp + \int_{-\infty}^{\infty} |p(\widehat{u_1}(p) - \widehat{u_2}(p))|^2 dp \le$$

$$\le 4\pi\varepsilon^2 Q^2 \|G_1(x) - G_2(x)\|_{L^2(\mathbb{R})}^2,$$

such that

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R})} \le 2\sqrt{\pi}\varepsilon Q \|G_1(x) - G_2(x)\|_{L^2(\mathbb{R})}. \tag{2.13}$$

Evidently, we can express

$$G_1(x) - G_2(x) = \int_{w(x)[-\alpha\xi_0(x)+v_2(x)]}^{w(x)[-\alpha\xi_0(x)+v_1(x)]} g'(z)dz.$$

Hence, we obtain

$$|G_1(x) - G_2(x)| \le \max_{z\in I}|g'(z)||w(x)(v_1(x) - v_2(x))| \le M|w(x)(v_1(x) - v_2(x))|$$

and trivially derive the upper bound for the norm via (1.10) as

$$\|G_1(x) - G_2(x)\|_{L^2(\mathbb{R})} \le M\|w(x)(v_1(x) - v_2(x))\|_{H^1(\mathbb{R})} \le$$

$$\leq Mc_a \|w(x)\|_{H^1(\mathbb{R})} \|v_1(x) - v_2(x)\|_{H^1(\mathbb{R})}. \tag{2.14}$$

By virtue of (2.13) along with (2.14) we arrive at

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R})} \leq 2\sqrt{\pi}\varepsilon Q M c_a \|w(x)\|_{H^1(\mathbb{R})} \|v_1(x) - v_2(x)\|_{H^1(\mathbb{R})}. \tag{2.15}$$

Clearly,

$$\frac{\rho}{2\sqrt{\pi}QM(1 + c_a\|w(x)\|_{H^1(\mathbb{R})})} < \frac{1}{2\sqrt{\pi}QMc_a\|w(x)\|_{H^1(\mathbb{R})}}.$$

Then, by means of (1.19) for our parameter ε we have

$$0 < \varepsilon < \frac{1}{2\sqrt{\pi}QMc_a\|w(x)\|_{H^1(\mathbb{R})}}.$$

Therefore, the constant in the right side of inequality (2.15) is less than one, which yields that our map $T_g : B_\rho \to B_\rho$ defined by equation (1.14) is a strict contraction for all the values of ε, which satisfy (1.19). Its unique fixed point $u_p(x)$ is the only solution of problem (1.12) in the ball B_ρ. It easily follows from (2.10) that $\|u_p(x)\|_{H^1(\mathbb{R})} \to 0$ as $\varepsilon \to 0$. The resulting $u(x)$ given by (1.11) is a solution of equation (1.2). ■

Let us turn our attention to establishing the second main result of our article.

3. The continuity of the resulting solution

Proof of Theorem 1.4. Evidently, for all the values of ε satisfying inequality (1.19), we have

$$u_{p,1} = T_{g_1}u_{p,1}, \quad u_{p,2} = T_{g_2}u_{p,2}.$$

Hence

$$u_{p,1} - u_{p,2} = T_{g_1}u_{p,1} - T_{g_1}u_{p,2} + T_{g_1}u_{p,2} - T_{g_2}u_{p,2}.$$

Therefore,

$$\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R})} \leq \|T_{g_1}u_{p,1} - T_{g_1}u_{p,2}\|_{H^1(\mathbb{R})} + \|T_{g_1}u_{p,2} - T_{g_2}u_{p,2}\|_{H^1(\mathbb{R})}.$$

By means of bound (2.15), we have

$$\|T_{g_1}u_{p,1} - T_{g_1}u_{p,2}\|_{H^1(\mathbb{R})} \leq \varepsilon\sigma\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R})}$$

with σ given by (1.20). Note that $\varepsilon\sigma < 1$ because the map $T_{g_1} : B_\rho \to B_\rho$ under the given conditions is a strict contraction. Thus, we derive

$$(1 - \varepsilon\sigma)\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R})} \leq \|T_{g_1}u_{p,2} - T_{g_2}u_{p,2}\|_{H^1(\mathbb{R})}. \tag{3.1}$$

Clearly, for our fixed point $T_{g_2} u_{p,2} = u_{p,2}$. Let us denote $r(x) := T_{g_1} u_{p,2}$. We arrive at

$$-\frac{d^2 r(x)}{dx^2} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g_1(w(y)[-\alpha\xi_0(y) + u_{p,2}(y)])dy, \tag{3.2}$$

$$-\frac{d^2 u_{p,2}(x)}{dx^2} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g_2(w(y)[-\alpha\xi_0(y) + u_{p,2}(y)])dy. \tag{3.3}$$

Let us denote

$$G_{1,2}(x) := g_1(w(x)[-\alpha\xi_0(x)+u_{p,2}(x)]),\ G_{2,2}(x) := g_2(w(x)[-\alpha\xi_0(x)+u_{p,2}(x)]).$$

We apply the standard Fourier transform (2.1) to both sides of problems (3.2) and (3.3) above. This yields

$$\widehat{r}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G}_{1,2}(p)}{p^2}, \quad \widehat{u}_{p,2}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G}_{2,2}(p)}{p^2},$$

such that

$$\widehat{r}(p) - \widehat{u}_{p,2}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)}{p^2}[\widehat{G}_{1,2}(p) - \widehat{G}_{2,2}(p)],$$

$$p[\widehat{r}(p) - \widehat{u}_{p,2}(p)] = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)}{p}[\widehat{G}_{1,2}(p) - \widehat{G}_{2,2}(p)].$$

This enables us to obtain the upper bounds

$$|\widehat{r}(p) - \widehat{u}_{p,2}(p)| \le \varepsilon\sqrt{2\pi}Q|\widehat{G}_{1,2}(p) - \widehat{G}_{2,2}(p)|, \tag{3.4}$$

$$|p[\widehat{r}(p) - \widehat{u}_{p,2}(p)]| \le \varepsilon\sqrt{2\pi}Q|\widehat{G}_{1,2}(p) - \widehat{G}_{2,2}(p)|. \tag{3.5}$$

By virtue of (3.4) we derive

$$\|\widehat{r}(p) - \widehat{u}_{p,2}(p)\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |\widehat{r}(p) - \widehat{u}_{p,2}(p)|^2 dp \le$$

$$\le 2\pi\varepsilon^2 Q^2 \|G_{1,2}(x) - G_{2,2}(x)\|^2_{L^2(\mathbb{R})}. \tag{3.6}$$

Similarly, using (3.5) we arrive at

$$\|p[\widehat{r}(p) - \widehat{u}_{p,2}(p)]\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |p[\widehat{r}(p) - \widehat{u}_{p,2}(p)]|^2 dp \le$$

$$\le 2\pi\varepsilon^2 Q^2 \|G_{1,2} - G_{2,2}\|^2_{L^2(\mathbb{R})}. \tag{3.7}$$

By means of (1.8) along with (3.6) and (3.7) we estimate the norm as

$$\|r(x) - u_{p,2}(x)\|^2_{H^1(\mathbb{R})} = \|\widehat{r}(p) - \widehat{u}_{p,2}(p)\|^2_{L^2(\mathbb{R})} + \|p[\widehat{r}(p) - \widehat{u}_{p,2}(p)]\|^2_{L^2(\mathbb{R})} \le$$

$$\le 4\pi\varepsilon^2 Q^2 \|G_{1,2}(x) - G_{2,2}(x)\|^2_{L^2(\mathbb{R})},$$

such that

$$\|r(x)-u_{p,2}(x)\|_{H^1(\mathbb{R})}\leq 2\sqrt{\pi}\varepsilon Q\|G_{1,2}(x)-G_{2,2}(x)\|_{L^2(\mathbb{R})}. \tag{3.8}$$

Obviously

$$G_{1,2}(x)-G_{2,2}(x)=\int_0^{w(x)[-\alpha\xi_0(x)+u_{p,2}(x)]}[g_1'(z)-g_2'(z)]dz.$$

Therefore

$$|G_{1,2}(x)-G_{2,2}(x)|\leq \max_{z\in I}|g_1'(z)-g_2'(z)||w(x)[-\alpha\xi_0(x)+u_{p,2}(x)]|\leq$$

$$\leq\|g_1-g_2\|_{C_1(I)}|w(x)[-\alpha\xi_0(x)+u_{p,2}(x)]|,$$

which allows us to estimate the norm using (2.7) as

$$\|G_{1,2}(x)-G_{2,2}(x)\|_{L^2(\mathbb{R})}\leq\|g_1-g_2\|_{C_1(I)}\|w(x)[-\alpha\xi_0(x)+u_{p,2}(x)]\|_{L^2(\mathbb{R})}\leq$$

$$\leq\|g_1-g_2\|_{C_1(I)}(1+c_a\|w(x)\|_{H^1(\mathbb{R})}). \tag{3.9}$$

By means of (3.8) along with (3.9) we derive

$$\|r(x)-u_{p,2}(x)\|_{H^1(\mathbb{R})}\leq 2\sqrt{\pi}\varepsilon Q\|g_1-g_2\|_{C_1(I)}(1+c_a\|w(x)\|_{H^1(\mathbb{R})}). \tag{3.10}$$

Let us use (3.1) and (3.10) to obtain

$$\|u_{p,1}(x)-u_{p,2}(x)\|_{H^1(\mathbb{R})}\leq\frac{2\sqrt{\pi}\varepsilon Q(1+c_a\|w(x)\|_{H^1(\mathbb{R})})}{1-\varepsilon\sigma}\|g_1-g_2\|_{C_1(I)}. \tag{3.11}$$

By virtue of (1.21) along with (3.11) estimate (1.22) is valid. ■

4. Auxiliary results

Below we obtain the conditions under which the expression Q defined above in (1.18) is finite. Let us denote the inner product as

$$(f(x),g(x))_{L^2(\mathbb{R})}:=\int_{-\infty}^{\infty}f(x)\bar{g}(x)dx, \tag{4.1}$$

with a slight abuse of notations when the functions involved in (4.1) are not square integrable, like for example the ones present in the first orthogonality relation in (4.2) of Lemma 4.1 below. Indeed, if $f(x)\in L^1(\mathbb{R})$ and $g(x)$ is bounded , then the integral in the right side of (4.1) makes sense. The proof of Lemma 4.1 was partially presented in the part b) of Lemma A1 of [23]. We provide it here for the convenience of the readers.

Lemma 4.1. *Let $\mathcal{K}(x) : \mathbb{R} \to \mathbb{R}$ be nontrivial, such that $\mathcal{K}(x) \in L^1(\mathbb{R})$ and $x^2\mathcal{K}(x) \in L^1(\mathbb{R})$. Then $Q < \infty$ if and only if the orthogonality conditions*

$$(\mathcal{K}(x), 1)_{L^2(\mathbb{R})} = 0, \quad (\mathcal{K}(x), x)_{L^2(\mathbb{R})} = 0 \tag{4.2}$$

hold.

Proof. Let us first estimate the norm as

$$\|x\mathcal{K}(x)\|_{L^1(\mathbb{R})} = \int_{|x|\le 1} |x||\mathcal{K}(x)|dx + \int_{|x|>1} |x||\mathcal{K}(x)|dx \le$$

$$\le \|\mathcal{K}(x)\|_{L^1(\mathbb{R})} + \|x^2\mathcal{K}(x)\|_{L^1(\mathbb{R})} < \infty$$

as assumed. Therefore, the expression in the left side of the second orthogonality relation in (4.2) is well defined. Let us show that if $\frac{\widehat{\mathcal{K}}(p)}{p^2} \in L^\infty(\mathbb{R})$ then we have $\frac{\widehat{\mathcal{K}}(p)}{p} \in L^\infty(\mathbb{R})$ as well. Evidently,

$$\frac{\widehat{\mathcal{K}}(p)}{p} = \frac{\widehat{\mathcal{K}}(p)}{p}\chi_{\{|p|\le 1\}} + \frac{\widehat{\mathcal{K}}(p)}{p}\chi_{\{|p|>1\}}. \tag{4.3}$$

Here and below χ_A will denote the characteristic function of a set $A \subseteq \mathbb{R}$. Obviously, the second term in the right side of (4.3) can be bounded above in the absolute value using (2.2) by

$$|\widehat{\mathcal{K}}(p)| \le \frac{1}{\sqrt{2\pi}}\|\mathcal{K}(x)\|_{L^1(\mathbb{R})} < \infty$$

due to the one of our assumptions. The first term in the right side of (4.3) can be estimated from above in the absolute value as

$$\left|\frac{\widehat{\mathcal{K}}(p)}{p^2}\right|\chi_{\{|p|\le 1\}}|p| \le \left\|\frac{\widehat{\mathcal{K}}(p)}{p^2}\right\|_{L^\infty(\mathbb{R})} < \infty$$

as assumed. Hence, $\frac{\widehat{\mathcal{K}}(p)}{p}$ is bounded as well. Clearly,

$$\frac{\widehat{\mathcal{K}}(p)}{p^2} = \frac{\widehat{\mathcal{K}}(p)}{p^2}\chi_{\{|p|\le 1\}} + \frac{\widehat{\mathcal{K}}(p)}{p^2}\chi_{\{|p|>1\}}. \tag{4.4}$$

Using (2.2), we easily derive

$$\left|\frac{\widehat{\mathcal{K}}(p)}{p^2}\chi_{\{|p|>1\}}\right| \le |\widehat{\mathcal{K}}(p)| \le \frac{1}{\sqrt{2\pi}}\|\mathcal{K}(x)\|_{L^1(\mathbb{R})} < \infty$$

via the one of our assumptions. Let us express

$$\widehat{\mathcal{K}}(p) = \widehat{\mathcal{K}}(0) + p\frac{d\widehat{\mathcal{K}}}{dp}(0) + \int_0^p \left(\int_0^s \frac{d^2\widehat{\mathcal{K}}(q)}{dq^2} dq \right) ds.$$

Therefore, the first term in the right side of (4.4) is given by

$$\left[\frac{\widehat{\mathcal{K}}(0)}{p^2} + \frac{\frac{d\widehat{\mathcal{K}}}{dp}(0)}{p} + \frac{\int_0^p \left(\int_0^s \frac{d^2\widehat{\mathcal{K}}(q)}{dq^2} dq \right) ds}{p^2} \right] \chi_{\{|p|\le 1\}}. \tag{4.5}$$

From the definition of the standard Fourier transform (2.1) it can be easily deduced that

$$\left| \frac{d^2\widehat{\mathcal{K}}(p)}{dp^2} \right| \le \frac{1}{\sqrt{2\pi}} \|x^2\mathcal{K}(x)\|_{L^1(\mathbb{R})}.$$

Thus,

$$\left| \frac{\int_0^p \left(\int_0^s \frac{d^2\widehat{\mathcal{K}}(q)}{dq^2} dq \right) ds}{p^2} \chi_{\{|p|\le 1\}} \right| \le \frac{1}{2\sqrt{2\pi}} \|x^2\mathcal{K}(x)\|_{L^1(\mathbb{R})} < \infty$$

as assumed. By means of definition (2.1), we easily obtain

$$\widehat{\mathcal{K}}(0) = \frac{1}{\sqrt{2\pi}}(\mathcal{K}(x), 1)_{L^2(\mathbb{R})}, \quad \frac{d\widehat{\mathcal{K}}}{dp}(0) = -\frac{i}{\sqrt{2\pi}}(\mathcal{K}(x), x)_{L^2(\mathbb{R})},$$

which enables us to write the sum of the first two terms in (4.5) as

$$\left[\frac{(\mathcal{K}(x), 1)_{L^2(\mathbb{R})}}{\sqrt{2\pi}p^2} - i\frac{(\mathcal{K}(x), x)_{L^2(\mathbb{R})}}{\sqrt{2\pi}p} \right] \chi_{\{|p|\le 1\}}. \tag{4.6}$$

Evidently, expression (4.6) is bounded if and only if orthogonality conditions (4.2) hold. ∎

Note that as distinct from the present article, the argument of [27] does not rely on the orthogonality relations.

References

[1] R. Adami, D. Noja, *Exactly solvable models and bifurcations: the case of the cubic NLS with a δ or a δ' interaction in dimension one*, Math. Model. Nat. Phenom., **9** (2014), No.5, 1–16.

[2] G.L. Alfimov, E.V. Medvedeva, D.E. Pelinovsky, *Wave Systems with an Infinite Number of Localized Traveling Waves*, Phys. Rev. Lett., **112** (2014), 054103, 5pp.

[3] G.L. Alfimov, A.S. Korobeinikov, C.J. Lustri, D.E. Pelinovsky, *Standing lattice solitons in the discrete NLS equation with saturation*, Nonlinearity, **32** (2019), No.9, 3445–3484.

[4] C. Amrouche, V. Girault, J. Giroire, *Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator: an approach in weighted Sobolev spaces*, J. Math. Pures Appl., **76** (1997), No.1, 55–81.

[5] C. Amrouche, F. Bonzom, Mixed exterior Laplace's problem, J. Math. Anal. Appl., **338** (2008), 124–140.

[6] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Holderienne pour des opérateurs différentiels elliptiques dans R^n*, J. Math. Pures Appl., **72** (1993), No.1, 105–119.

[7] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Hölderienne pour le problème extérieur de Dirichlet*, Comm. Partial Differential Equations, **26** (2001), No. 1-2, 315-334.

[8] N. Benkirane, *Propriété d'indice en théorie Holderienne pour des opérateurs elliptiques dans R^n*, CRAS, **307**, Série I (1988), 577–580.

[9] S. Coombes, P. Beim Graben, R. Potthast, J. Wright, eds. *Neural Fields: Theory and Applications*, Berlin Heidelberg: Springer, (2014).

[10] S. Cuccagna, D. Pelinovsky, V. Vougalter, *Spectra of positive and negative energies in the linearized NLS problem*, Comm. Pure Appl. Math., **58** (2005), No. 1, 1–29.

[11] A. Ducrot, M. Marion, V. Volpert, *Systemes de réaction-diffusion sans propriété de Fredholm*, CRAS, **340** (2005), 659–664.

[12] A. Ducrot, M. Marion, V. Volpert, *Reaction-diffusion problems with non Fredholm operators*, Advances Diff. Equations, **13** (2008), No. 11-12, 1151–1192.

[13] M. Efendiev, *Fredholm structures, topological invariants and applications*. AIMS Series on Differential Equations & Dynamical Systems, **3**. American Institute of Mathematical Sciences (AIMS), Springfield, MO (2009), 205 pp.

[14] M. Efendiev, V. Vougalter, *Solvability in the sense of sequences for some fourth order non-Fredholm operators*, J. Differential Equations, **271** (2021), No. 1, 280-300.

[15] R. Fukuizumi, L. Jeanjean, *Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential*, Discrete Contin. Dyn. Syst., **21** (2008), No. 1, 121–136.

[16] J. Holmer, J. Marzuola, M. Zworski, *Fast soliton scattering by delta impurities,* Comm. Math. Phys, **274** (2007), No. 1, 187–216.

[17] E. Lieb, M. Loss, *Analysis. Graduate Studies in Mathematics*, **14**, American Mathematical Society, Providence (1997).

[18] V. Volpert. Elliptic partial differential equations. Volume 1. Fredholm theory of elliptic problems in unbounded domains. Birkhauser, 2011.

[19] V. Volpert, V. Vougalter. *On the solvability conditions for a linearized Cahn-Hilliard equation,* Rend. Istit. Mat. Univ. Trieste, **43** (2011), 1–9.

[20] V. Vougalter, *On threshold eigenvalues and resonances for the linearized NLS equation*, Math. Model. Nat. Phenom., **5** (2010), No. 4, 448–469.

[21] V. Vougalter, V. Volpert. *On the solvability conditions for some non Fredholm operators,* Int. J. Pure Appl. Math., **60** (2010), No. 2, 169–191.

[22] V. Vougalter, V. Volpert, *Solvability conditions for some non-Fredholm operators*, Proc. Edinb. Math. Soc. (2), **54** (2011), No.1, 249–271

[23] V. Vougalter, V. Volpert. *On the existence of stationary solutions for some non-Fredholm integro-differential equations*, Doc. Math., **16** (2011), 561–580.

[24] V. Vougalter, V. Volpert. *On the solvability conditions for the diffusion equation with convection terms,* Commun. Pure Appl. Anal., **11** (2012), No. 1, 365–373.

[25] V. Vougalter, V. Volpert. *Solvability conditions for a linearized Cahn-Hilliard equation of sixth order,* Math. Model. Nat. Phenom., **7** (2012), No. 2, 146–154.

[26] V. Vougalter, V. Volpert. *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems,* Anal. Math. Phys., **2** (2012), No.4, 473–496.

[27] V. Vougalter, V. Volpert. *Solvability of some integro- differential equations with anomalous diffusion.* Regularity and stochasticity of nonlinear dynamical systems, Nonlinear Syst. Complex., **21** (2018), Springer, Cham, 1–17.