q^{-1}-Orthogonal Solutions of q^{-1}-Periodic Equations

F. I. Moxley

Abstract

In the year 1939, the Mathematician G.H. Hardy proved that the only functions f which satisfy the classical orthogonality relation

$$(1) \qquad \int_0^1 f(\lambda_m t) f(\lambda_n t) dt = 0, \quad m \neq n,$$

are the Bessel functions $J_\nu(t)$ under certain constraints, where $\nu > -1$ is the order of the Bessel function, and λ_m, λ_n are the zeros of the Bessel function. More recently, the Mathematician L.D. Abreu proved that if a function $f \in \mathcal{L}_q^2(0,1)$ is q-orthogonal with respect to its own zeros in the interval $(0,1)$, then it satisfies the q-orthogonality relation

$$(2) \qquad \int_0^1 f(\lambda_m t) f(\lambda_n t) d_q t = 0, \quad m \neq n,$$

where the q-integral is a Riemann-Stieltjes integral with respect to a step function having infinitely many points of increase at the points q^ℓ, with the step size at the point q^ℓ being q, $\forall\ \ell \in \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $0 < q < 1$. Following these developments, herein we present an equivalence class of entire q^{-1}-periodic functions satisfying the q^{-1}-orthogonality relation

$$(3) \qquad \int_0^1 f(\lambda_m t) f(\lambda_n t) d_{q^{-1}} t = 0, \quad m \neq n.$$

1. Introduction

The quantum calculus, otherwise known as the q-calculus [**1**], has been found to have a wide variety of interesting applications in computational number theory [**2**], and the theory of orthogonal polynomials [**3**–**5**], for example. As such, herein we investigate a class of entire functions that are q^{-1}-orthogonal with respect to their own zeros, and find that in this equivalence class, the only q^{-1}-periodic functions are nonzero constant-valued functions. It is well understood by the Fundamental Theorem of Algebra [**6**], that a nonzero constant function has no roots. Accordingly, this study aims to develop a novel approach to the field of q^{-1}-orthogonal polynomials [**7**], and the distribution of their zeros [**8**].

The paper is organized as follows: In Sec. 2 we introduce a class of entire functions, q^{-1}-orthogonal with respect to their own zeros, and demonstrate that the class is comprised of q^{-1}-periodic (i.e. constant) functions on the complex plane. Sec. 3 details the q^{-1}-Fourier series, and the completeness relations of the class. In Sec. 4, a first-order linear q^{-1}-difference equation is obtained for arriving at the value of the q^{-1}-periodic constant constituted by the class. Finally, concluding remarks are made in Sec. 5.

1.1. Preliminaries. If $q^{-1} \in \mathbb{R}$ is fixed, then a subset of $\mathbb{C}$ is named $\mathcal{A}$, and is also q^{-1}-geometric if $q^{-1}x \in \mathcal{A}$ whenever $x \in \mathcal{A}$. If $\mathcal{A} \subset \mathbb{C}$ is q^{-1}-geometric then it contains all geometric sequences $\{xq^{-\ell}\}_{\ell=0}^{\infty}$, where $x \in \mathcal{A}$ such that as $q \to 1$ then $\mathcal{A} \to \mathbb{C}$. Unless otherwise noted, herein $0 < q < 1$ [**9**].

Definition 1. A function f defined on the q^{-1}-geometric set $\mathcal{A}$, where $0 \in \mathcal{A}$, is said to be q-regular at infinity if there exists a constant $\mathcal{C}$ such that

$$\lim_{\ell\to\infty} f(xq^{-\ell}) = \mathcal{C}, \quad \forall\, x \in \mathcal{A}. \tag{4}$$

Definition 2. The Euler-Heine q^{-1}-difference operator [**10**, **11**], is defined by

$$\hat{\mathcal{D}}_{q^{-1}} f(x) := \frac{f(x) - f(q^{-1}x)}{x - q^{-1}x}, \quad \forall\, x \in \mathcal{A} \;/\; \{0\}. \tag{5}$$

If $0 \in \mathcal{A}$, the q-derivative at zero is defined for $|q| < 1$ by

$$\hat{\mathcal{D}}_{q^{-1}} f(0) := \lim_{\ell\to\infty} \frac{f(sq^{-\ell}) - f(0)}{sq^{-\ell}}, \quad \forall\, x \in \mathcal{A} \;/\; \{0\}. \tag{6}$$

The q^{-1}-derivative at zero is denoted as $f'(0)$, assuming the limit exists and is independent of x.

The q^{-1}-product rule is [**12**]

$$\hat{\mathcal{D}}_{q^{-1}}[f(x)g(x)] = f(q^{-1}x)\hat{\mathcal{D}}_{q^{-1}} g(x) + g(x)\hat{\mathcal{D}}_{q^{-1}} f(x), \tag{7}$$

and the q^{-1}-integral in the interval $(0, x)$ is

$$\int_0^x f(t) d_{q^{-1}} t = (1-q) \sum_{\ell=0}^{\infty} f(xq^{-\ell}) xq^{-\ell}. \tag{8}$$

Now let $1 \le p < \infty$, $x > 0$, and $\eta \in \mathbb{R}$. Also let $\mathcal{L}^p_{q^{-1},\eta}(0,x)$ be the space of all equivalence classes of functions satisfying

$$\int_0^x t^\eta |f(t)|^p d_{q^{-1}}t < \infty, \tag{9}$$

where two functions are defined as equivalent if they are equivalent on the sequence $\{xq^{-\ell} : \ell \in \mathbb{N}_0\}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Hence, f is a function in the Banach space $\mathcal{L}^p_{q^{-1},\eta}(0,x)$ with norm

$$||f||_{p,\eta,x} := \left(\int_0^x t^\eta |f(t)|^p d_{q^{-1}}t \right)^{\frac{1}{p}}. \tag{10}$$

For the case when $p = 2$, it can be seen that the inner product

$$\langle f, g \rangle := \int_0^x t^\eta f(t)\overline{g(t)} d_{q^{-1}}t, \tag{11}$$

is a separable Hilbert space, where $f, g \in \mathcal{L}^2_{q^{-1},\eta}(0,x)$. If $x = 1$, the resulting Hilbert space is $\mathcal{L}^2_{q^{-1},\eta}(0,1)$, and the function $f \in \mathcal{L}^2_{q^{-1},\eta}(0,1)$ is q^{-1}-orthogonal with respect to its own zeros in the interval $(0,1)$ if

$$\int_0^1 f(\lambda_m t) f(\lambda_n t) d_{q^{-1}}t = \sum_{\ell=0}^{\infty} f(\lambda_m q^{-\ell}) f(\lambda_n q^{-\ell}) q^{-\ell} = 0, \quad m \ne n. \tag{12}$$

Here, it should be pointed out that an orthonormal basis of $\mathcal{L}^2_{q^{-1},\eta}(0,x)$ is [**13**]

$$\varphi_n(t) = \begin{cases} \frac{1}{\sqrt{t^{\eta+1}(1-q)}}, & t = xq^{-\ell}, \quad \ell \in \mathbb{N}_0; \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

2. q^{-1}-Periodicity

Theorem 1. *If the class constituted by all entire functions f of order less than 1, or of order 1 and minimal type of the form*

$$f(x) = x^{\rho(x)} F(x), \tag{14}$$

where $f(0) = -1/2$, and $\rho(x)$ is given by the natural logarithmic relation [**14**]

$$\rho(x) = \frac{\log\left(-\frac{1}{2(1-x)\Gamma(1+x/2)}\right)}{\log(x)} > -\frac{1}{2}, \tag{15}$$

where Γ is the gamma function, and the entire function $F(x)$, with real but not necessarily positive zeros is

$$F(x) = \exp(cx) \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{x}{\lambda_n}\right) \exp\left(\frac{x}{\lambda_n}\right) \right\}, \tag{16}$$

where $c = \log(2\pi) - 1 - \gamma/2$, γ is the Euler-Mascheroni constant; if $F(x) \neq 0$ and $f(x)$ is q^{-1}-orthogonal with respect to its zeros; $\sum_n \lambda_n^{-1}$ is convergent, but not absolutely [**16**]*; then f has the q^{-1}-periodic representation*

$$f_{q^{-1}}(x) = \prod_{\ell=0}^{N-1} \frac{1}{q^{2\ell+1} + q^2}, \tag{17}$$

defined on the q^{-1}-geometric set $\mathcal{A}$, i.e., $f_{q^{-1}}(x)$ is constant in x.

Proof. The proof depends on two lemmas. If

$$\int_0^1 \{f(\lambda_n t)\}^2 d_{q^{-1}} t = (q^{-\ell})^{\eta+1}(1-q), \tag{18}$$

then the system

$$\varphi_n(t) = \frac{1}{\sqrt{(q^{-\ell})^{\eta+1}(1-q)}} f(\lambda_n t) \tag{19}$$

is orthonormal in $(0,1)$. The following Theorem 2 demonstrates the system $\varphi_n(t)$ is complete, independent of q^{-1}-orthogonality. □

Theorem 2. *If f satisfies the conditions of the previous Theorem 1, other than q^{-1}-orthogonality, g is q^{-1}-integrable, and*

$$\int_0^1 g(t) f(\lambda_n t) d_{q^{-1}} t = 0, \quad \forall\, n, \tag{20}$$

then $g(t) \equiv 0$.

Proof. Let $x = r\exp(i\theta)$, where θ is the complex argument, $i = \sqrt{-1}$, and

$$h(x) = \int_0^1 g(t) f(xt) d_{q^{-1}} t. \tag{21}$$

It is clear that

$$h(x) = x^{\rho(x)} H(x), \tag{22}$$

where $H(x)$ is an entire function. Here, we suppose that $F(x)$ is of order less than 1, when $H(x)$ is also of order less than 1. Since $h(\lambda_n) = 0\ \forall\, n$, it then follows that the ratio [**17**]

$$\chi(x) = \frac{h(x)}{f(x)} = \frac{H(x)}{F(x)} \tag{23}$$

is also an entire function of order less than 1. Along the imaginary axis $x = r\sin(\theta)$ it can be seen that $|\exp(cx)| = |\exp(x\lambda_n^{-1})| = 1 \; \forall \; n$, where again $c = \log(2\pi) - 1 - \gamma/2$, and

$$(24) \qquad \nu(x,t) = \left|\frac{F(xt)}{F(x)}\right| = \prod_{n=1}^{\infty}\left|\frac{\lambda_n - rt\sin(\theta)}{\lambda_n - r\sin(\theta)}\right|.$$

Here it should be pointed out that no factor exceeds 1, and the limit of each factor as $r \to \infty$ is simply t. Therefore $|\nu| \leq 1 \; \forall \; r, t$. Moreover, for every fixed value of $t < 1$, as $r \to \infty$ it can be seen that $\nu \to \infty$. As such,

$$(25) \qquad |\chi(x)| = \left|\int_0^1 g(t)\frac{F(xt)}{F(x)} d_{q^{-1}}t\right| \leq \int_0^1 |g(t)|\nu(x,t) d_{q^{-1}}t$$

is bounded, and tends to zero along the imaginary axis $x = r\sin(\theta)$. Furthermore, suppose that $\chi(x)$ makes an angle of π/α at the origin, and also along the imaginary axis. By denoting the bound on $\chi(x)$ as $\mathcal{B}$, such that along the imaginary axis

$$(26) \qquad |\chi(x)| \leq \mathcal{B},$$

then as $r \to \infty$, it can be seen that

$$(27) \qquad \chi(x) = \mathcal{O}\Big(\exp(\delta r^{\alpha})\Big)$$

for every positive δ, uniformly in the angle. It then follows that the boundedness holds in the region where f is entire and regular for $x = r\exp(i\theta)$. Without loss of generality, suppose that $\theta = \pm\pi/(2\alpha)$ for the two angles $(-\pi/(2\alpha), 0)$, and

$(0, \pi/(2\alpha))$. Also, by letting

$$F(x) = \exp(-\varepsilon x^{\alpha}) f(x) \tag{28}$$

it can be seen that $F(x)$ tends to zero on the real axis $x = r\cos(\theta)$, and therefore has an upper bound, denoted $\mathcal{B}'$. Then, by denoting

$$\mathcal{B}'' = \max(\mathcal{B}, \mathcal{B}'), \tag{29}$$

it can be seen that

$$|F(x)| = \left| \exp\left[-\varepsilon \Big(r\exp(i\theta) \Big)^{\alpha} \right] f(x) \right|, \tag{30}$$

where again $\theta = \pm\pi/(2\alpha)$. It then follows that throughout the angle, and along the imaginary axis $x = r\sin(\theta)$, that

$$|F(x)| \leq \mathcal{B}''. \tag{31}$$

Here, it should be pointed out that if $\mathcal{B}' \leq \mathcal{B}$, then $|F(x)|$ assumes the value $\mathcal{B}'$ at any point of the real axis $x = r\cos(\theta)$. Consequently $\mathcal{B}' = \mathcal{B}''$, $F(x)$ reduces to a constant, and $\mathcal{B} = \mathcal{B}''$. Otherwise $\mathcal{B}' < \mathcal{B}''$, such that $\mathcal{B} = \mathcal{B}''$ regardless. Thus,

$$|F(x)| \leq \mathcal{B}. \tag{32}$$

Accordingly,

$$(33)\qquad |f(x)| \leq \mathcal{B}|\exp(-\varepsilon x^{\alpha})|.$$

Taking $\varepsilon \to 0$ implies that $\mathcal{B} = 0$, since $\nu \to 0$ for every fixed $t < 1$ as $r \to \infty$. Therefore,

$$(34)\qquad \int_0^1 g(t)f(xt)d_{q^{-1}}t = 0.$$

However, we are interested in the class of functions of the form of Eq. (14), i.e.,

$$(35)\qquad f(x) = x^{\rho(x)} \sum_{\ell=0}^{\infty} a_\ell x^\ell,$$

where $a_\ell \neq 0$ for any ℓ. As such, we assume the following [**15**]:

(1) There exists a class of series, larger than that of series known classically as convergent, such that a *sum* corresponds to each series of that class;

(2) Let m and n, where $n < m$, be two positive integers. We then have the relation

$$(36)\qquad \frac{1-x^n}{1-x^m} = 1 - x^n + x^m - x^{n+m} + x^{2m} + \cdots .$$

At $x = 1$, we obtain the Euler series

$$(37)\qquad \frac{n}{m} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

which belongs to the class from assumption (1).

(3) Let $\mathcal{S}$ be the sum of the series $x^{\rho(x)}\sum_n a_n$ of the class, where $x^{\rho(x)}$ is given by Eq. (15). Then the series itself belongs to the class, and has the sum $x^{\rho(x)}\mathcal{S}$.

(4) If the series $a_0 + a_1 + \cdots + a_n + \cdots$ has the sum $\mathcal{S}$, then the series $a_1 + \cdots + a_n + \cdots$ itself has the sum $\mathcal{S} - a_0$. As such, it can be seen that

$$\begin{aligned} \mathcal{S} &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\ &= 1 - (1 - 1 + 1 - \cdots) \\ &= 1 - \mathcal{S}, \end{aligned} \tag{38}$$

from which we obtain $\mathcal{S} = 1/2$.

Hence,

$$\int_0^1 g(t)t^{\rho(xt)+n} d_{q^{-1}}t = 0, \quad \forall\, n, \tag{39}$$

and therefore $g(t) \equiv 0$. □

3. q^{-1}-Fourier Series

The q^{-1}-Fourier series of $f(xt)$ with respect to the system Eq. (13) is

$$\begin{aligned} f(xt) &\sim \sum_n a_n(x)\varphi_n(t) \\ &= \sum_n a_n(x)\frac{1}{\sqrt{(q^{-\ell})^{\eta+1}(1-q)}}, \end{aligned} \tag{40}$$

where the Fourier coefficient

$$
\begin{aligned}
a_n(x) &= \int_0^1 f(xt)\varphi_n(t)d_{q^{-1}}t \\
&= \frac{1}{\sqrt{(q^{-\ell})^{\eta+1}(1-q)}}\int_0^1 f(xt)f(\lambda_n t)d_{q^{-1}}t; \qquad (41)
\end{aligned}
$$

and by the Parseval completeness theorem [**19**], we obtain

$$
\begin{aligned}
\mathcal{P}(x,x') &= \int_0^1 f(xt)f(x't)d_{q^{-1}}t \\
&= \sum_{n=1}^{\infty} a_n(x)a_n(x'). \qquad (42)
\end{aligned}
$$

The following theorem gives the value of $a_n(x)$.

Theorem 3. *If the conditions of Theorem 1 are satisfied, and* $x \neq \lambda_n$, *then*

$$
\int_0^1 f(xt)f(\lambda_n t)d_{q^{-1}}t = \frac{(q^{-\ell})^{\eta+1}(1-q)}{f'(\lambda_n)} \cdot \frac{f(x)}{x-\lambda_n}. \qquad (43)
$$

Proof. First, supposing that $F(x)$ is of order less than 1, we write

$$
h(x) = \int_0^1 f(xt)f(\lambda_n t)d_{q^{-1}}t, \qquad (44a)
$$

$$
f_n(x) = \frac{f(x)}{x-\lambda_n}, \qquad (44b)
$$

$$
g(x) = \frac{h(x)}{f_n(x)}, \qquad (44c)
$$

$$
G(x) = \frac{g(x)}{x+1}. \qquad (44d)
$$

It then follows that g is an entire function of order less than 1; G is regular and of order less than 1 in the half-plane $r\cos(\theta) > 0$; and

$$G(x) = \frac{x-\lambda_n}{x+1}\int_0^1 \frac{f(xt)}{f(x)} f(\lambda_n t) d_{q^{-1}}t \tag{45}$$

is bounded, and goes to zero along the angle $\theta = \pm\pi/4$. It then follows in the quadrant between $\theta = \pm\pi/4$ that

$$g(x) = \mathcal{O}(|x|). \tag{46}$$

In a similar fashion, the same result follows for the remaining three quadrants in the complex plane $\mathbb{C}$. Obviously, g is linear and

$$h(x) = g(x) f_n(x) = \frac{ax+b}{x-\lambda_n} f(x). \tag{47}$$

However, G goes to zero along the angle $\theta = \pi/4$ such that $a = 0$, and

$$h(x) = \frac{b}{x-\lambda_n} f(x). \tag{48}$$

The constant b can be obtained by making $x \to \lambda_n$, to obtain Eq. (43). □

4. First-Order Linear q^{-1}-Difference Equation

From Eqs. (40), and (42)-(43) it follows that

$$\mathcal{P}(x, x') = \int_0^1 f(xt) f(x't) d_{q^{-1}}t = -f(x) f(x') \frac{\tau(x) - \tau(x')}{x - x'}, \tag{49}$$

where

$$(50)\qquad \tau(x)=\sum_{\ell=1}^{\infty}\frac{(q^{-\ell})^{\eta+1}(1-q)}{\{f'(\lambda_\ell)\}^2}\Big(\frac{1}{x-\lambda_\ell}+\frac{1}{\lambda_\ell}\Big),$$

such that $\tau(0)=0$. Eq. (49) will enable us to determine f. By making $x'\to 0$, it follows that

$$(51)\qquad \int_0^1 t^\eta f(xt)d_{q^{-1}}t=-f(x)\frac{\tau(x)}{x},$$

i.e.,

$$(52)\qquad \int_0^x u^\eta f(u)d_qu=-x^\eta f(x)\tau(x).$$

Hence,

$$(53)\qquad \tau'(0)=(q-1)q^{-\ell}[1+\eta(q^{-\ell}-1)].$$

Next, we write Eq. (49) in the form

$$(54)\qquad \int_0^x u^{\rho(u)}F(u)(x't)^{\rho(x't)}F(x't)d_qu=-x^{\rho(x)+1}F(x)(x')^{\rho(x')}F(x')\frac{\tau(x)-\tau(x')}{x-x'}.$$

Differentiating with respect to x', and evaluating at $x'=0$, it can be seen that

$$\frac{\partial}{\partial x'}(x't)^{\rho(x't)}F(x't)\bigg|_{x'=0} = -\frac{t}{4}(2+2c+\gamma), \tag{55a}$$

$$-xf(x)\frac{\partial}{\partial x'}(x')^{\rho(x')}F(x')\frac{\tau(x)-\tau(x')}{x-x'}\bigg|_{x'=0} = \frac{(2+2c+\gamma+2x^{-1})\tau(x)}{4}f(x) - \frac{\tau'(0)}{2}f(x). \tag{55b}$$

Using Eqs. (52)-(53), and letting $\eta = 1$ for brevity, we finally obtain the q^{-1}-integral equation for f, namely

$$\int_0^x uf(u)d_{q^{-1}}u = (1-q)q^{-2\ell}x^2 f(x). \tag{56}$$

By taking the q^{-1}-difference $\hat{\mathcal{D}}_{q^{-1}}$, and using the q^{-1}-integration by parts, i.e.,

$$\int_0^x g(t)\Big(\hat{D}_{q^{-1}}f(t)\Big)d_{q^{-1}}t + \int_0^x \Big(\hat{D}_{q^{-1}}g(t)\Big)f(q^{-1}t)d_{q^{-1}}t = [fg](x) - \lim_{\ell\to\infty}[fg](xq^{-\ell}), \tag{57}$$

it can be seen that

$$\hat{\mathcal{D}}_{q^{-1}}\int_0^x uf(u)d_{q^{-1}}u = xf(x) - \lim_{\ell\to\infty} xq^{-\ell}f(xq^{-\ell}), \tag{58}$$

and

$$\hat{\mathcal{D}}_{q^{-1}}[x^2 f(x)] = (\hat{\mathcal{D}}_{q^{-1}}x^2)f(x) + (q^{-1}x)^2\hat{\mathcal{D}}_{q^{-1}}f(x). \tag{59}$$

Hence, we arrive at the first-order linear q^{-1}-difference equation [**18**]

$$\hat{\mathcal{D}}_{q^{-1}} f(x) = \tilde{a}(x) f(x). \tag{60}$$

Carrying out the q^{-1}-difference $\hat{\mathcal{D}}_{q^{-1}}$ and upon making further simplifications,

$$f(x) = \left[\frac{q}{q + x\tilde{a}(x)(1-q)}\right] f(q^{-1}x), \tag{61}$$

where

$$\tilde{a}(x) = \frac{q - q^2(q^{2\ell} + q)}{(q-1)x}. \tag{62}$$

Repeating the above recurrence relation N times,

$$f(x) = f(x_0) \prod_{t=qx_0}^{x} \frac{q}{q + t\tilde{a}(t)(1-q)}. \tag{63}$$

As $N \to \infty$ with $0 < q < 1$, then $q^{-N} \to \infty$, and

$$\begin{aligned} f(x) &= f(q^{-N}x) \prod_{\ell=0}^{N-1} \frac{q}{q + xq^{-\ell}\tilde{a}(xq^{-\ell})(1-q)} \\ &= f(\infty) \prod_{\ell=0}^{N-1} \frac{1}{q^{2\ell+1} + q^2}. \end{aligned} \tag{64}$$

Since by Eq. (14) we have $f(\infty) = 1$, it can be seen in the classical limit where $q \to 1$ and $\mathcal{A} \to \mathbb{C}$ that $f(x) = 1/2 \;\forall\; x \in \mathbb{C}$.

5. Conclusion

By examining a class of entire first order q^{-1}-orthogonal functions $f \in \mathcal{L}^2_{q^{-1}}(0,1)$, it has been demonstrated that the class is indeed comprised of q^{-1}-periodic functions on the separable Hilbert space interval $(0,1)$. This was accomplished with the q^{-1}-Fourier series, and a q^{-1}-integral equation for obtaining the value of the q^{-1}-periodic constant constituted by the class.

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