SPECTRAL OPTIMIZATION FOR ROBIN LAPLACIAN IN DOMAINS WITHOUT CUT LOCI

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Dedicated to the memory of our friend and colleague Hagen Neidhardt

ABSTRACT. In this paper we deal with spectral optimization for the Robin Laplacian on a family of planar domains without cut loci, namely a fixed-width strip built over a smooth closed curve and the exterior of a convex set with a smooth boundary. We show that if the curve length is kept fixed, the first eigenvalue referring to the fixed-width strip is for any value of the Robin parameter maximized by a circular annulus. Furthermore, we prove that the second eigenvalue in the exterior of a convex domain Ω corresponding to a negative Robin parameter does not exceed the analogous quantity for a disk whose boundary has a curvature larger or equal to the maximum of that for $\partial\Omega$.

1. Introduction

1.1. **Motivation and the problem description.** Relations between geometry and spectral properties have a rather long history. They are a trademark topic of mathematical physics at least since the celebrated Faber and Krahn proof [F23, K24] of Lord Rayleigh's conjecture [R1877] about the shape of the drum that produces the lowest tone. While the original interest focused on problems with Dirichlet or Neumann boundary conditions, more recently the attention shifted to mixed boundary conditions. Spectral optimization for the Robin Laplacian was the topic of a number of studies in the last few years and it still offers many challenging open problems, see the reviews [BFK17, L19] and the references therein.

It has to be said that spectral optimization can mean both the upper and lower bounds. While in the mentioned Faber and Krahn result the circle minimizes the principal eigenvalue, the Dirichlet Laplacian on non-simply connected domains can exhibit the opposite effect when the full symmetry makes this eigenvalue maximal [EHL99, HKK01]. Moreover, this effect is robust, we note that a similar result holds for a family of singular Schrödinger operators with an attractive interaction supported by a closed planar curve [EHL06].

In the present paper we are going to deal with the optimization of the first and the second Robin eigenvalues on two particular classes of planar domains whose multiple connectedness follows from the absence of cut loci, specifically of loop-shaped curved strips and of exteriors of convex sets. In the first case, we are going to prove that the lowest Robin eigenvalue on such a curved strip of a fixed length of the inner

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boundary and a fixed width is maximized by that of the annulus. This result can be regarded as an extension of the indicated property of the Dirichlet Laplacian [EHL99, Thm 1a]. We stress that no restrictions are imposed on the Robin coefficient, it can be negative as well as positive, in other words, the strip boundary can be both repulsive and attractive. The proof of the claim relies on the min-max principle, an appropriate test function is constructed via transplantation of the ground state for the annular strip using the method of parallel coordinates. The width of the curved strip is restricted by the requirement that the cut locus set is empty which makes it possible to choose parallel coordinates well defined everywhere.

Our second result concerns an optimization of the second Robin eigenvalue in the exterior of a convex set under the assumption that the curvature κ of the boundary is non-negative and bounded from above by a fixed constant $\kappa_\circ > 0$. The exterior of a disk with the boundary curvature $\kappa_\circ$, in other words, with the radius $\kappa_\circ^{-1}$, turns out be the unique maximizer. This new result complements the optimization of the lowest Robin eigenvalue in the exterior of a bounded set considered recently in [KL17a, KL17b]. It is not yet clear whether the stated condition on the curvature can be replaced by a more standard perimeter-type constraint. In contrast to the previous case, the Robin coefficient is now assumed to be negative as otherwise the spectrum of the respective Laplacian is purely essential and coincides with $[0,\infty)$, making thus the spectral optimization question void. In the proof, we again make the advantage of the fact that the parallel coordinates are globally well defined. We apply the min-max principle to the span of the two transplanted eigenfunctions of the Robin Laplacian on the exterior of the disk corresponding to its first and second eigenvalues, respectively. Since the eigenfunction corresponding to the second eigenvalue is not radial, its transplantation is more involved and contains an additional geometric insight.

In the setting of bounded domains, it has been proved that the disk is a maximizer of the second Robin eigenvalue having a fixed area [FL18a] or a fixed perimeter [FL18b], provided that the negative boundary parameter lies in a specific interval. An analogous result has recently been proved in [GL19] for the third Robin eigenvalue with the maximizer being the union of two disks and with the negative boundary parameter again lying in a specific interval. In this context, we would like to emphasize that the optimization result for the second Robin eigenvalue in the present paper holds for *all negative values* of the boundary parameters.

1.2. **Geometric setting.** Since the domain geometry is crucial in our results, let us first recall the necessary notions and state the assumptions we are going to use.

Hypothesis 1. *Let a C^∞-smooth curve $\Sigma \subset \mathbb{R}^2$ be the boundary of a bounded, simply connected domain $\Omega \subset \mathbb{R}^2$. Let a circle $\mathcal{C} \subset \mathbb{R}^2$ be the boundary of a disk $\mathcal{B} \subset \mathbb{R}^2$. We denote by $L := |\Sigma|$ and $L_\circ := |\mathcal{C}|$ the lengths of Σ and $\mathcal{C}$, respectively.*

The mapping $\sigma\colon [0,L] \to \mathbb{R}^2$ provides the natural (counter-clockwise) parametrization of Σ with the tangential vector $\tau(s) := \sigma'(s)$ satisfying $|\tau(s)| = 1$. We denote by

$\kappa\colon [0,L]\to\mathbb{R}$ the signed curvature of Σ; the convention we adopt is that $\kappa\geq 0$ holds for convex Ω. Recall the *Frenet formula*

$$\tau'(s) = -\kappa\nu(s),$$

where ν is the outer unit normal vector to the domain Ω.

The object of our interest will be the curved strip built over Σ with the thickness $d\in(0,\infty]$, that is, the set

$$\Omega_d^{\mathrm{c}} := \left\{x\in\mathbb{R}^2\setminus\overline{\Omega}\colon \mathrm{dist}\,(x,\Sigma)<d\right\}. \tag{1.1}$$

The definition includes the unbounded domain $\Omega_\infty^{\mathrm{c}}$ identified with the exterior $\mathbb{R}^2\setminus\overline{\Omega}$ of Ω for which we will use the shorthand notation $\Omega^{\mathrm{c}} := \Omega_\infty^{\mathrm{c}}$. The boundary of Ω_d^{c} is therefore

$$\partial\Omega_d^{\mathrm{c}} = \begin{cases}\Sigma & \text{if } d=\infty,\\ \Sigma\cup\left\{x\in\mathbb{R}^2\setminus\overline{\Omega}\colon \mathrm{dist}\,(x,\Sigma)=d\right\} & \text{if } d<\infty.\end{cases}$$

In particular, $\partial\Omega_d^{\mathrm{c}}$ has two components for $d<\infty$ and, respectively, one component for $d=\infty$. If the curvature κ is sign-changing, we set $\kappa_- = \min\{\kappa,0\}$, and assume, in addition, that $d\in(0,\infty)$ satisfies the condition

$$d\|\kappa_-\|_\infty < 1. \tag{1.2}$$

The parallel coordinates $(s,t)\in[0,L)\times(0,d)$ on Ω_d^{c} [Hart64], alternatively dubbed Fermi or natural curvilinear, are under the assumption (1.2) everywhere well defined by the formula $\Omega_d^{\mathrm{c}}\ni x=\sigma(s)+t\nu(s)$. Since our setting is two-dimensional, it is useful to work with the complexified tangential and normal vectors

$$\mathbf{t}(s) = \tau_1(s)+\mathsf{i}\tau_2(s) \quad\text{and}\quad \mathbf{n}(s) = \nu_1(s)+\mathsf{i}\nu_2(s). \tag{1.3}$$

In this notation, the Frenet formula can be written in the complex form as

$$\mathbf{t}'(s) = -\kappa\mathbf{n}(s) = -\mathsf{i}\kappa\mathbf{t}(s). \tag{1.4}$$

1.3. **The Robin Laplacian on Ω_d^{c}.** For an arbitrary value of the coefficient $\alpha\in\mathbb{R}$, which characterizes the strength of the coupling to the boundary, we introduce the self-adjoint operator $\mathsf{H}_{\alpha,\Omega_d^{\mathrm{c}}}$ in the Hilbert space $L^2(\Omega_d^{\mathrm{c}})$ through its quadratic form

$$\mathfrak{h}_{\alpha,\Omega_d^{\mathrm{c}}}[u] := \|\nabla u\|^2_{L^2(\Omega_d^{\mathrm{c}};\mathbb{C}^2)} + \alpha\|u|_{\partial\Omega_d^{\mathrm{c}}}\|^2_{L^2(\partial\Omega_d^{\mathrm{c}})}, \qquad \mathrm{dom}\,\mathfrak{h}_{\alpha,\Omega_d^{\mathrm{c}}} = H^1(\Omega_d^{\mathrm{c}}),$$

where $H^1(\Omega_d^{\mathrm{c}})$ is the first-order L^2-based Sobolev space on Ω_d^{c}.

If $d<\infty$ the spectrum of $\mathsf{H}_{\alpha,\Omega_d^{\mathrm{c}}}$ is discrete and we denote by $\{\lambda_k^\alpha(\Omega_d^{\mathrm{c}})\}_{k\geq 1}$ its eigenvalues arranged in the non-decreasing order and repeated with multiplicities taken into account. The spectral properties of $\mathsf{H}_{\alpha,\Omega^{\mathrm{c}}}$ corresponding to $d=\infty$ are different; we recall them below:

(i) $\sigma_{\mathrm{ess}}(\mathsf{H}_{\alpha,\Omega^{\mathrm{c}}}) = [0,\infty)$.

(ii) $\#\sigma_{\rm d}(\mathsf{H}_{\alpha,\Omega^{\rm c}}) \geq 1$ for all $\alpha < 0$.

(iii) $\sigma_{\rm d}(\mathsf{H}_{\alpha,\Omega^{\rm c}}) = \varnothing$ for all $\alpha \geq 0$.

In analogy with the bounded domain case we denote by $\{\lambda_k^\alpha(\Omega^{\rm c})\}_{k\geq 1}$ the negative eigenvalues of $\mathsf{H}_{\alpha,\Omega^{\rm c}}$ arranged in the ascending order and repeated with the multiplicities taken into account. In the min-max spirit, this sequence is conventionally extended up to an infinite one by repeating the bottom of the essential spectrum $\inf \sigma_{\rm ess}(\mathsf{H}_{\alpha,\Omega^{\rm c}}) = 0$ infinitely many times.

1.4. **Main results.** Let us now state our main results. The first one concerns optimization of $\lambda_1^\alpha(\Omega_d^{\rm c})$ on curved strips of a fixed width.

Theorem 1.1. *Assume that Hypothesis 1 holds and that $L = L_\circ$. Let κ be the curvature of Σ. The strip-width $d \in (0,\infty]$ may be arbitrary if $\kappa \geq 0$, while for a sign-changing κ we assume that condition (1.2) is satisfied. Let the domains $\Omega_d^{\rm c}$ and $\mathcal{B}_d^{\rm c}$ be as in (1.1). Then for the lowest Robin eigenvalues on these domains the inequality*

$$\lambda_1^\alpha(\Omega_d^{\rm c}) \leq \lambda_1^\alpha(\mathcal{B}_d^{\rm c})$$

holds for any $\alpha \in \mathbb{R}$.

Let us add a few comments. The above inequality holds trivially in the Neumann case, $\alpha = 0$, since we have $\lambda_1^0(\Omega_d^{\rm c}) = \lambda_1^0(\mathcal{B}_d^{\rm c}) = 0$. In the limit $\alpha \to +\infty$ it implies the respective inequality for the Dirichlet Laplacians providing thus an alternative proof of Theorem 1a in [EHL99]. Furthermore, if $\alpha < 0$, $\kappa \geq 0$, and $d = \infty$, Theorem 1.1 reduces to the first claim of [KL17a, Thm. 1.3]. Note that the topological character of Ω manifested in the lack of simple connectedness mentioned in the introduction plays a role again: the annulus is always a maximizer here, even for $\alpha > 0$, while in the case of general bounded domains the disk is typically a maximizer for $\alpha < 0$ and a minimizer for $\alpha > 0$ under appropriate geometric constraints, *cf.* [FK15, AFK17, BFNT18] in the former case and [Bos86, Dan06] in the latter.

The proof of Theorem 1.1 will rely on the min-max principle with a suitable test function constructed through the transplantation of the radial ground-state eigenfunction for the annulus using the method of parallel coordinates.

Our second result concerns optimization of $\lambda_2^\alpha(\Omega^{\rm c})$ on unbounded exterior domains described above.

Theorem 1.2. *Assume that Hypothesis 1 holds. Let $\kappa\colon [0,L] \to \mathbb{R}$ and $\kappa_\circ \in \mathbb{R}_+$ be the curvatures of Σ and $\mathcal{C}$, respectively. Assume further that Ω is convex, that is, $\kappa \geq 0$, and that $\max \kappa \leq \kappa_\circ$ holds . Let the domains $\Omega^{\rm c}$ and $\mathcal{B}^{\rm c}$ be as in (1.1) with $d = \infty$. Then for the second Robin eigenvalues on these domains the inequality*

$$\lambda_2^\alpha(\Omega^{\rm c}) \leq \lambda_2^\alpha(\mathcal{B}^{\rm c}) \tag{1.5}$$

is valid for any $\alpha < 0$. If $\lambda_2^\alpha(\mathcal{B}^{\rm c}) < 0$ and the equality in (1.5) holds, the two domains are congruent, $\Omega \cong \mathcal{B}$.

The above theorem and monotonicity of $\lambda_2^\alpha(\mathcal{B}^c)$, with respect to $L_\circ$ shown in Proposition 2.2, below yield the following.

Corollary 1.3. *Assume that Hypothesis 1 holds and let $\kappa_\circ > 0$ be fixed. Then, for all $\alpha < 0$,*

$$\max_{\substack{\Omega \text{ convex} \\ \kappa \le \kappa_\circ}} \lambda_2^\alpha(\Omega^c) = \lambda_2^\alpha(\mathcal{B}^c), \tag{1.6}$$

where the maximum is taken over all convex smooth domains $\Omega \subset \mathbb{R}^2$ whose curvature satisfies $\max \kappa \le \kappa_\circ$ and where $\mathcal{B} \subset \mathbb{R}^2$ is a disk of the curvature $\kappa_\circ$.

We remark that the inequality (1.5) is nontrivial only if $\mathsf{H}_{\alpha,\mathcal{B}^c}$ has more than one negative eigenvalue. We also emphasize that, in contrast to Theorem 1.1 we have $L \ne L_\circ$ in general, in fact, it is easy to show that $L > L_\circ$ holds unless $\Omega \cong \mathcal{B}$. In order to prove Theorem 1.2, we apply the min-max principle transplanting to Ω^c the span of the two eigenfunctions of $\mathsf{H}_{\alpha,\mathcal{B}^c}$ corresponding to the eigenvalues $\lambda_1^\alpha(\mathcal{B}^c)$ and $\lambda_2^\alpha(\mathcal{B}^c)$, respectively. The ground-state is transplanted in a conventional way, however, the transplantation of the first excited state is a little more involved. We note that an eigenfunction corresponding to the second Robin eigenvalue on the exterior of a disk can be written in parallel coordinates on $\mathcal{B}^c$ as

$$v_\circ(s,t) = \phi(t)\exp\left(\frac{2\pi \mathsf{i}}{L_\circ}s\right).$$

Since $\exp\left(\frac{2\pi \mathsf{i}}{L_\circ}s\right)$ can be interpreted as the complexified tangent vector for $\mathcal{B}$, a natural way of transplantation of $v_\circ$ onto Ω^c would be

$$v_\star(s,t) = \phi(t)\mathbf{t}(s),$$

where $\mathbf{t}$ is the complexified tangent vector for Ω defined in (1.3).

2. Preliminaries

2.1. **The quadratic form $\mathfrak{h}_{\alpha,\Omega_d^c}$ in parallel coordinates.** Our first main tool is the representation of the quadratic form $\mathfrak{h}_{\alpha,\Omega_d^c}$ in the parallel coordinates on Ω_d^c. Using them, the inner product in the Hilbert space $L^2(\Omega_d^c)$ can be written as follows,

$$(u,v)_{L^2(\Omega_d^c)} = \int_0^d \int_0^L u(s,t)\overline{v(s,t)}\big(1+\kappa(s)t\big)\,\mathsf{d}s\,\mathsf{d}t.$$

It is well known that the gradient in these coordinates is expressed as

$$\nabla u = \frac{\tau(s)}{1+\kappa(s)t}\,\partial_s u + \nu(s)\partial_t u.$$

Consequently, the quadratic form $\mathfrak{h}_{\alpha,\Omega_d^c}$ can be written in the parallel coordinates as

$$\mathfrak{h}_{\alpha,\Omega_d^c}[u] = \int_0^d \int_0^L \left(\frac{|\partial_s u(s,t)|^2}{1+\kappa(s)t} + |\partial_t u(s,t)|^2(1+\kappa(s)t)\right)\mathsf{d}s\,\mathsf{d}t + \alpha\int_0^L |u(s,0)|^2\,\mathsf{d}s,$$
$$\mathrm{dom}\,\mathfrak{h}_{\alpha,\Omega_d^c} = \left\{u\colon \Sigma\times(0,d)\to\mathbb{C}\colon \int_0^d\int_0^L \left[\frac{|\partial_s u|^2}{1+\kappa t} + (|u|^2+|\partial_t u|^2)(1+\kappa t)\right]\mathsf{d}s\,\mathsf{d}t < \infty\right\}. \tag{2.1}$$

The above representation remains valid for $d = \infty$, provided that Ω is convex.

2.2. **Eigenfunctions in the radially symmetric case.** We also need properties of the eigenfunctions corresponding to the first and the second eigenvalue in the radially symmetric case. They are elementary but we describe them in the next two propositions, the proof of which are postponed to the appendices, in order to make the paper self-contained. Let us begin with the ground-state eigenfunction of the Robin annulus.

Proposition 2.1. *Assume that Hypothesis 1 holds. For any fixed $d > 0$ and any $\alpha \in \mathbb{R}$, or for $d = \infty$ and any $\alpha < 0$, the lowest eigenvalue $\lambda_1^\alpha(\mathcal{B}_d^{\mathrm{c}})$ of $\mathsf{H}_{\alpha,\mathcal{B}_d^{\mathrm{c}}}$ is simple and the corresponding eigenfunction can be written in the parallel coordinates on $\mathcal{B}_d^{\mathrm{c}}$ as*

$$u_\circ(s,t) = \psi(t),$$

with a given real-valued $\psi \in C^\infty([0,d])$ if $d < \infty$ and with $\psi \in C^\infty([0,\infty))$ satisfying

$$\int_0^\infty \left[\psi(t)^2 + \psi'(t)^2\right](1+t)\,\mathsf{d}t < \infty, \tag{2.2}$$

if $d = \infty$.

Consider next the first excited state of the Robin Laplacian in the exterior of a disk.

Proposition 2.2. *Assume that Hypothesis 1 holds. Then for any fixed $\alpha < 0$ such that $\#\sigma_{\mathrm{d}}(\mathsf{H}_{\alpha,\mathcal{B}^{\mathrm{c}}}) > 1$, the second eigenvalue $\lambda_2^\alpha(\mathcal{B}^{\mathrm{c}}) < 0$ of $\mathsf{H}_{\alpha,\mathcal{B}^{\mathrm{c}}}$ has multiplicity two and the respective eigenfunctions of $\mathsf{H}_{\alpha,\mathcal{B}^{\mathrm{c}}}$ can be written in parallel coordinates on $\mathcal{B}^{\mathrm{c}}$ as*

$$v_\circ^\pm(s,t) = \exp\left(\pm\frac{2\pi\mathsf{i}}{L_\circ}s\right)\phi(t), \qquad s \in [0, L_\circ),\ t \in [0,\infty),$$

with a given real-valued $\phi \in C^\infty([0,\infty))$ satisfying the integrability condition

$$\int_0^\infty \left[\phi(t)^2 + \phi'(t)^2\right](1+t)\,\mathsf{d}t < \infty. \tag{2.3}$$

Moreover, $\lambda_2^\alpha(\mathcal{B}^{\mathrm{c}})$ is a non-increasing function of $L_\circ$.

We remark that the functions ψ and ϕ in Propositions 2.1 and 2.2 can be explicitly expressed in terms of Bessel functions, however, this is not essential for our analysis.

3. Proofs of the main results

Now we are going to provide proofs of Theorems 1.1 and 1.2. Recall that the C^∞-smooth curve $\Sigma \subset \mathbb{R}^2$ is the boundary of a bounded, simply connected domain $\Omega \subset \mathbb{R}^2$, and the circle $\mathcal{C} \subset \mathbb{R}^2$ is the boundary of the disk $\mathcal{B} \subset \mathbb{R}^2$. The lengths of Σ and $\mathcal{C}$ are denoted by L and $L_\circ$, respectively. The curvature of Σ is denoted by κ and the curvature of $\mathcal{C}$ is a constant $\kappa_\circ > 0$.

3.1. **Proof of Theorem 1.1.** By assumption we have $L = L_\circ$ and we fix $d > 0$ satisfying the additional condition (1.2) in the case that κ is sign-changing. Furthermore, $\alpha \in \mathbb{R}$ is an arbitrary fixed number. The case $d = \infty$ is dealt with in [KL17a, Thm 1.3] and thus we may omit it here. By Proposition 2.1, there exists a function $\psi \in C^\infty([0,d])$ such that the ground-state $u_\circ \in C^\infty(\Omega_d^{\rm c})$ of $\mathsf{H}_{\alpha,\mathcal{B}_d^{\rm c}}$ can be written as $u_\circ(s,t) = \psi(t)$ in the parallel coordinates on $\mathcal{B}_d^{\rm c}$. Using it we define the test function $u_\star \in H^1(\Omega_d^{\rm c})$ in the parallel coordinates on the curved strip $\Omega_d^{\rm c}$ as follows,

$$u_\star(s,t) := \psi(t), \qquad s \in [0,L],\, t \in [0,d].$$

Using the representation of $\mathfrak{h}_{\alpha,\Omega_d^{\rm c}}$ in (2.1), applying the min-max principle and the total curvature identity $\int_0^L \kappa(s)\mathsf{d}s = 2\pi$ we obtain

$$\begin{aligned}
\lambda_1^\alpha(\Omega_d^{\rm c}) &\le \frac{\mathfrak{h}_{\alpha,\Omega_d^{\rm c}}[u_\star]}{\|u_\star\|^2_{L^2(\Omega_d^{\rm c})}} \\
&= \frac{\displaystyle\int_0^d\int_0^L \psi'(t)^2(1+\kappa(s)t)\,\mathsf{d}s\,\mathsf{d}t + \alpha\int_0^L\Big[|\psi(0)|^2 + |\psi(d)|^2(1+d\kappa(s))\Big]\mathsf{d}s}{\displaystyle\int_0^d\int_0^L \psi(t)^2(1+\kappa(s)t)\,\mathsf{d}s\,\mathsf{d}t} \\
&= \frac{\displaystyle\int_0^d \psi'(t)^2(L+2\pi t)\,\mathsf{d}t + \alpha L|\psi(0)|^2 + \alpha(L+2\pi d)|\psi(d)|^2}{\displaystyle\int_0^d \psi(t)^2(L+2\pi t)\,\mathsf{d}t} \\
&= \frac{\mathfrak{h}_{\alpha,\mathcal{B}_d^{\rm c}}[u_\circ]}{\|u_\circ\|^2_{L^2(\mathcal{B}_d^{\rm c})}} = \lambda_1^\alpha(\mathcal{B}_d^{\rm c}),
\end{aligned}$$

which yields the sought claim.

3.2. **Proof of Theorem 1.2.** In view of the convexity of Ω the curvature of Σ satisfies $\kappa \ge 0$ and by assumption $\max\kappa \le \kappa_\circ$ holds. Let us exclude the trivial case supposing that $\Omega \ncong \mathcal{B}$. Then we have $\min\kappa < \kappa_\circ$ which implies

$$L = \frac{L\kappa_\circ}{\kappa_\circ} > \frac{\displaystyle\int_0^L \kappa(s)\,\mathsf{d}s}{\kappa_\circ} = \frac{2\pi}{\kappa_\circ} = L_\circ. \tag{3.1}$$

We fix the 'width' $d = \infty$ and the coupling constant $\alpha < 0$. Without loss of generality, we may assume that $|\alpha|$ is large enough so that $\lambda_2^\alpha(\mathcal{B}^{\rm c}) < 0$ as otherwise the inequality (1.5) would trivially hold.

Step 1. *Test functions*. In view of Propositions 2.1 and 2.2, we can represent the eigenfunctions of $\mathsf{H}_{\alpha,\mathcal{B}^{\rm c}}$ corresponding to its simple first eigenvalue $\lambda_1^\alpha(\mathcal{B}^{\rm c})$ and the second eigenvalue $\lambda_2^\alpha(\mathcal{B}^{\rm c})$ of multiplicity two in parallel coordinates (s,t) on $\mathcal{B}^{\rm c}$ as

$$u_\circ(s,t) = \psi(t) \qquad \text{and} \qquad v_\circ^\pm(s,t) = \exp\left(\pm\frac{2\pi\mathsf{i}s}{L_\circ}\right)\phi(t), \tag{3.2}$$

where $\psi, \phi \in C^\infty([0,\infty))$ are real-valued and satisfy the integrability conditions (2.2) and (2.3), respectively. We introduce test functions $u_\star, v_\star \in H^1(\Omega^c)$ on Ω^c defining them in terms of the parallel coordinates as

$$u_\star(s,t) := \psi(t) \qquad \text{and} \qquad v_\star(s,t) := \mathsf{t}(s)\phi(t), \qquad s \in [0,L],\, t \in [0,\infty),$$

where $\mathsf{t}(s)$ is the complexified normal (1.3).

Step 2. *Orthogonality.* Next, we are going to show that $u_\star$ and $v_\star$ are orthogonal in $L^2(\Omega^c)$. To this aim, we observe that

$$\int_0^L \mathsf{t}(s)\,\mathsf{d}s = \int_0^L (\sigma_1'(s) + \mathsf{i}\sigma_2'(s))\,\mathsf{d}s = \sigma_1(L) + \mathsf{i}\sigma_2(L) - \sigma_1(0) - \mathsf{i}\sigma_2(0) = 0,$$

where the fact that Σ is a closed curve was employed. Furthermore, using the Frenet formula we get

$$\int_0^L \mathsf{t}(s)\kappa(s)\,\mathsf{d}s = \mathsf{i}\int_0^L \mathsf{t}'(s)\,\mathsf{d}s = \mathsf{i}(\mathsf{t}(L) - \mathsf{t}(0)) = 0,$$

where the closedness and smoothness of Σ were taken into account. Combining these two relations we infer that

(3.3)
$$\begin{aligned}(v_\star, u_\star)_{L^2(\Omega^c)} &= \int_0^\infty \int_0^L \psi(t)\phi(t)\mathsf{t}(s)(1+t\kappa(s))\,\mathsf{d}s\,\mathsf{d}t \\ &= \int_0^\infty \int_0^L \psi(t)\phi(t)\mathsf{t}(s)\,\mathsf{d}s\,\mathsf{d}t + \int_0^\infty \int_0^L t\psi(t)\phi(t)\mathsf{t}(s)\kappa(s)\,\mathsf{d}s\,\mathsf{d}t = 0.\end{aligned}$$

At the same time, we have

(3.4)
$$\mathfrak{h}_{\alpha,\Omega^c}[v_\star, u_\star] = \int_0^\infty \int_0^L \psi'(t)\phi'(t)\mathsf{t}(s)(1+t\kappa(s))\,\mathsf{d}s\,\mathsf{d}t + \alpha\psi(0)\phi(0)\int_0^L \mathsf{t}(s)\,\mathsf{d}s = 0.$$

Step 3. *Bounds on the Rayleigh quotients.* For a non-trivial function $u \in H^1(\Omega^c)$ we define

$$\mathsf{R}_{\alpha,\Omega^c}[u] := \frac{\mathfrak{h}_{\alpha,\Omega^c}[u]}{\|u\|^2_{L^2(\Omega^c)}}.$$

Using (2.1) and the total curvature identity $\int_0^L \kappa(s)\,\mathsf{d}s = 2\pi$, the Rayleigh quotient of the test function $u_\star$ defined in this way can be expressed as

$$\begin{aligned}
\mathsf{R}_{\alpha,\Omega^c}[u_\star] &= \frac{\displaystyle\int_0^\infty \int_0^L \psi'(t)^2(1+t\kappa(s))\,\mathsf{d}s\,\mathsf{d}t + \alpha L|\psi(0)|^2}{\displaystyle\int_0^\infty \int_0^L \psi(t)^2(1+t\kappa(s))\,\mathsf{d}s\,\mathsf{d}t} \\
&= \frac{\displaystyle\int_0^\infty \psi'(t)^2(L+2\pi t)\,\mathsf{d}t + \alpha L|\psi(0)|^2}{\displaystyle\int_0^\infty \psi(t)^2(L+2\pi t)\,\mathsf{d}t} \\
&= \frac{\displaystyle\int_0^\infty \psi'(t)^2\left(1+\frac{2\pi t}{L}\right)\mathsf{d}t + \alpha|\psi(0)|^2}{\displaystyle\int_0^\infty \psi'(t)^2\left(1+\frac{2\pi t}{L}\right)\mathsf{d}t}.
\end{aligned}$$

Furthermore, using the inequalities $\lambda_1^\alpha(\mathcal{B}^c) < 0$ and $L > L_\circ$, we get the following estimate

$$\mathsf{R}_{\alpha,\Omega^c}[u_\star] \le \frac{\displaystyle\int_0^\infty \psi'(t)^2\left(1+\frac{2\pi t}{L_\circ}\right)\mathsf{d}t + \alpha|\psi(0)|^2}{\displaystyle\int_0^\infty \psi'(t)^2\left(1+\frac{2\pi t}{L_\circ}\right)\mathsf{d}t} = \mathsf{R}_{\alpha,\mathcal{B}^c}[u_\circ] = \lambda_1^\alpha(\mathcal{B}^c). \tag{3.5}$$

Making use of (2.1), the total curvature identity and the Frenet formula (1.4), the Rayleigh quotient corresponding to the test function $v_\star$ is given by

$$\begin{aligned}
\mathsf{R}_{\alpha,\Omega^c}[v_\star] &= \frac{\displaystyle\int_0^\infty \int_0^L \phi'(t)^2(1+t\kappa(s))\,\mathsf{d}s\,\mathsf{d}t + \int_0^\infty \int_0^L \frac{\kappa^2(s)\phi(t)^2}{1+t\kappa(s)}\,\mathsf{d}s\,\mathsf{d}t + \alpha L|\phi(0)|^2}{\displaystyle\int_0^\infty \int_0^L \phi(t)^2(1+t\kappa(s))\,\mathsf{d}s\,\mathsf{d}t} \\
&= \frac{\displaystyle\int_0^\infty \phi'(t)^2(L+2\pi t)\,\mathsf{d}t + \int_0^\infty \int_0^L \frac{\kappa^2(s)\phi(t)^2}{1+t\kappa(s)}\,\mathsf{d}s\,\mathsf{d}t + \alpha L|\phi(0)|^2}{\displaystyle\int_0^\infty \phi(t)^2(L+2\pi t)\,\mathsf{d}t}.
\end{aligned}$$

Using further the strict monotonicity of the function

$$\mathbb{R}_+ \ni x \mapsto \frac{x^2}{1+tx}, \qquad t \ge 0,$$

in combination with the inequalities $L > L_\circ$, $\max\kappa \le \kappa_\circ$, and $\min\kappa < \kappa_\circ$, we get for the Rayleigh quotient corresponding to $v_\star$ the following estimate,

$$\begin{aligned}
\mathsf{R}_{\alpha,\Omega^c}[v_\star] &< \frac{\displaystyle\int_0^\infty \phi'(t)^2(L+2\pi t)\,\mathsf{d}t + L\int_0^\infty \frac{\kappa_\circ^2\phi(t)^2}{1+t\kappa_\circ}\,\mathsf{d}t + \alpha L|\phi(0)|^2}{\displaystyle\int_0^\infty \phi(t)^2(L+2\pi t)\,\mathsf{d}t}\\
&= \frac{\displaystyle\int_0^\infty \phi'(t)^2\left(1+\frac{2\pi t}{L}\right)\mathsf{d}t + \int_0^\infty \frac{\kappa_\circ^2\phi(t)^2}{1+t\kappa_\circ}\,\mathsf{d}t + \alpha|\phi(0)|^2}{\displaystyle\int_0^\infty \phi(t)^2\left(1+\frac{2\pi t}{L}\right)\mathsf{d}t}\\
&\le \frac{\displaystyle\int_0^\infty \phi'(t)^2\left(1+\frac{2\pi t}{L_\circ}\right)\mathsf{d}t + \int_0^\infty \frac{\kappa_\circ^2\phi(t)^2}{1+t\kappa_\circ}\,\mathsf{d}t + \alpha|\phi(0)|^2}{\displaystyle\int_0^\infty \phi(t)^2\left(1+\frac{2\pi t}{L_\circ}\right)\mathsf{d}t}\\
&= \mathsf{R}_{\alpha,\mathcal{B}^c}[v_\circ] = \lambda_2^\alpha(\mathcal{B}^c).
\end{aligned} \tag{3.6}$$

Step 4. *The min-max principle.* Any $w_\star \in \operatorname{span}\{u_\star, v_\star\}\setminus\{0\}$ can be represented as a linear combination $w_\star = pu_\star + qv_\star$ with $(p,q)\in\mathbb{C}^2_\times := \mathbb{C}^2\setminus\{(0,0)\}$. The following simple inequality,

$$\frac{a+b}{c+d} \le \max\left\{\frac{a}{c}, \frac{b}{d}\right\}, \tag{3.7}$$

holds obviously for any $a, b\in\mathbb{R}$ and $c, d > 0$. Applying the min-max principle, using the orthogonality relatons (3.3), (3.4), the bounds (3.5), (3.6), and the inequality (3.7) we get

$$\begin{aligned}
\lambda_2^\alpha(\Omega^c) &\le \max_{(p,q)\in\mathbb{C}^2_\times} \frac{\mathfrak{h}_{\alpha,\Omega^c}[pu_\star+qv_\star]}{\|pu_\star+qv_\star\|^2_{L^2(\Omega^c)}}\\
&= \max_{(p,q)\in\mathbb{C}^2_\times} \frac{|p|^2\mathfrak{h}_{\alpha,\Omega^c}[u_\star]+|q|^2\mathfrak{h}_{\alpha,\Omega^c}[v_\star]}{|p|^2\|u_\star\|^2_{L^2(\Omega^c)}+|q|^2\|v_\star\|^2_{L^2(\Omega^c)}} \le \max\left\{\mathsf{R}_{\alpha,\Omega^c}[u_\star], \mathsf{R}_{\alpha,\Omega^c}[v_\star]\right\} < \lambda_2^\alpha(\mathcal{B}^c),
\end{aligned}$$

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Appendix A. Proof of Proposition 2.1

The case $d=\infty$ was dealt with in [KL17a, Sec. 3]. Assume that $d<\infty$ and let $\alpha\in\mathbb{R}$ be arbitrary. For the sake of simplicity and without loss of generality, we also assume that $L_\circ = 2\pi$. In this case the curvilinear coordinates (s,t) essentially coincide with the polar coordinates. Using the complete family of orthogonal projections on $L^2(\mathcal{B}_d^c)$,

$$(\Pi_n u)(t) = \frac{1}{\sqrt{2\pi}}\mathsf{e}^{\mathsf{i}ns}\int_0^{2\pi} u(s,t)\,\mathsf{e}^{-\mathsf{i}ns}\,\mathsf{d}s, \qquad n\in\mathbb{Z},$$

one can decompose $\mathsf{H}_{\alpha,\mathcal{B}_d^c}$ into an orthogonal sum

$$\mathsf{H}_{\alpha,\mathcal{B}_d^c} = \bigoplus_{n\in\mathbb{Z}} \mathsf{H}^{[n]}_{\alpha,\mathcal{B}_d^c},$$

where the self-adjoint fiber operator $\mathsf{H}^{[n]}_{\alpha,\mathcal{B}_d^c}$ acts in the Hilbert space $L^2((0,d);(1+t)\,\mathsf{d}t)$ and corresponds to the quadratic form $(n\in\mathbb{Z})$

$$\begin{aligned}\mathfrak{h}^{[n]}_{\alpha,\mathcal{B}_d^c}[\psi] &= \int_0^d \left(|\psi'(t)|^2(1+t) + \frac{n^2|\psi(t)|^2}{1+t}\right)\mathsf{d}t + \alpha|\psi(0)|^2,\\ \operatorname{dom}\mathfrak{h}^{[n]}_{\alpha,\mathcal{B}_d^c} &= H^1((0,d)).\end{aligned}$$

Clearly the lowest eigenvalue of $\mathsf{H}^{[0]}_{\alpha,\mathcal{B}_d^c}$ is simple and strictly smaller than the lowest eigenvalues of $\mathsf{H}^{[n]}_{\alpha,\mathcal{B}_d^c}$ with $n\neq 0$. Thus, the ground-state of $\mathsf{H}_{\alpha,\mathcal{B}_d^c}$ is simple and depends on t variable only. The smoothness of the corresponding eigenfunction follows from standard elliptic regularity theory.

Appendix B. Proof of Proposition 2.2

Using the complete family of orthogonal projections on $L^2(\mathcal{B}^c)$

$$(\Pi_n u)(t) = \frac{1}{\sqrt{L_\circ}}\mathsf{e}^{\frac{2\pi i n s}{L_\circ}}\int_0^{L_\circ} u(s,t)\,\mathsf{e}^{-\frac{2\pi i n s}{L_\circ}}\,\mathsf{d}s, \qquad n\in\mathbb{Z},$$

One can again decompose $\mathsf{H}_{\alpha,\mathcal{B}^c}$ into an orthogonal sum

$$\mathsf{H}_{\alpha,\mathcal{B}^c} = \bigoplus_{n\in\mathbb{Z}} \mathsf{H}^{[n]}_{\alpha,\mathcal{B}^c},$$

where the fiber operators $\mathsf{H}^{[n]}_{\alpha,\mathcal{B}^c}$, $n\in\mathbb{Z}$, in the Hilbert space $L^2(\mathbb{R}_+;(1+\frac{2\pi t}{L_\circ})\,\mathsf{d}t)$ correspond to the quadratic forms

$$\begin{aligned}\mathfrak{h}^{[n]}_{\alpha,\mathcal{B}^c}[\psi] &= \int_0^\infty \left(|\psi'(t)|^2\left(1+\frac{2\pi t}{L_\circ}\right) + \frac{4\pi^2 n^2|\psi(t)|^2}{L_\circ+2\pi t}\right)\mathsf{d}t + \alpha|\psi(0)|^2,\\ \operatorname{dom}\mathfrak{h}^{[n]}_{\alpha,\mathcal{B}^c} &= \left\{\psi\colon \mathbb{R}_+\to\mathbb{C}\colon \psi,\psi'\in L^2(\mathbb{R}_+;\left(1+2\pi L_\circ^{-1}t\right)\,\mathsf{d}t)\right\}.\end{aligned}$$

It is easy to see that $\mathsf{H}^{[0]}_{\alpha,\mathcal{B}^c}$ has exactly one negative simple eigenvalue, which corresponds to the ground-state eigenvalue $\lambda_1^\alpha(\mathcal{B}^c)$ of $\mathsf{H}_{\alpha,\mathcal{B}^c}$. The first excited state eigenvalue $\lambda_2^\alpha(\mathcal{B}^c)$ corresponds to the lowest eigenvalues of the identical operators $\mathsf{H}^{[1]}_{\alpha,\mathcal{B}^c}$ and $\mathsf{H}^{[-1]}_{\alpha,\mathcal{B}^c}$. Moreover, the smoothness of ϕ follows from standard elliptic regularity theory.

Let $\mathcal{B}_1, \mathcal{B}_2$ be two disks with perimeters L_1 and L_2, respectively. Assume that $L_1 < L_2$. Then we obtain that

$$\begin{aligned}\lambda_2^\alpha(\mathcal{B}_1^{\rm c}) &= \inf_{\psi\in C_0^\infty([0,\infty))} \frac{\displaystyle\int_0^\infty \left(|\psi'(t)|^2\left(1+\frac{2\pi}{L_1}t\right) + \frac{1}{L_1}\frac{16\pi^2|\psi(t)|^2}{L_1+2\pi t}\right) \mathsf{d}t + \alpha|\psi(0)|^2}{\displaystyle\int_0^\infty |\psi(t)|^2\left(1+\frac{2\pi}{L_1}t\right)\mathsf{d}t} \\ &\geq \inf_{\psi\in C_0^\infty([0,\infty))} \frac{\displaystyle\int_0^\infty \left(|\psi'(t)|^2\left(1+\frac{2\pi}{L_2}t\right) + \frac{1}{L_2}\frac{16\pi^2|\psi(t)|^2}{L_2+2\pi t}\right) \mathsf{d}t + \alpha|\psi(0)|^2}{\displaystyle\int_0^\infty |\psi(t)|^2\left(1+\frac{2\pi}{L_2}t\right)\mathsf{d}t} \\ &= \lambda_2^\alpha(\mathcal{B}_2^{\rm c}).\end{aligned}$$

Hence, it follows that $\lambda_2^\alpha(\mathcal{B}^{\rm c})$ is a non-increasing function of its perimeter.

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