Extremised Action Functionals On Manifolds

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Abstract

In this brief note we consider the construction of action functionals on differentiable manifolds and discuss the validity of available variational methods that might be used for their extremisation. We describe the 1-dimensional case only and note that consideration of the various maps required to define a Lagrangian results in extremum equations different to the Euler-Lagrange equations.

I. Introduction

We know that the correct objects to integrate on manifolds are differential k-forms [1]. This of course makes it rather straightforward to define functionals by integrals of k-forms. For example, for 1-forms, 2-forms, and 3-forms, that describe curves, surfaces, and solids, the corresponding definite integrals are lengths, areas and volumes, and so forth similarly for higher dimensional forms.

For action functionals defined in terms of integrals of *functions* on manifolds (0-forms), which are used broadly throughout much of physics [2, 3, 4, 5], a distinction between coordinates used for the parameterisation of so-called 'generalised coordinates', and coordinates that are assumed to be used for purposes of integration is crucial. When performing the variational calculus in particular, we find that it is imperative to be aware of the structure of maps on which the various objects depend.

Let us briefly review how we write the 1-form integral in the 1-dimensional case. Given a 1-form, ω, we compute the length of the curve it describes by the integral [6]

$$s = \int_{\mathcal{M}} \omega \,, \tag{1}$$

where the integration region $\mathcal{M} \sim [a, b] \subset \mathbb{R}$. We can introduce the map ϕ to define a coordinate x

$$\begin{aligned} \phi \,:\, \mathcal{M} &\to X \subset \mathbb{R} \\ p &\mapsto x_p \in X \,, \end{aligned}$$

with which the 1-form can be expanded in components as $\omega = w(x)\,\mathrm{d}x$, so the curve length is

$$s = \int_{\mathcal{M}} w(x)\,\mathrm{d}x \,. \tag{2}$$

Since $x \in \mathbb{R}$, we can think of the component $w(x)$ as a function of the real numbers x_p and write this explicitly as $w \in \mathcal{F}(\mathbb{R})$.

Now, suppose we have a function L in the manifold function space (rather than the real numbers), $L \in \mathcal{F}(\mathcal{M})$, and that we wish to define an action functional as its integral. How do we integrate the function L in a coordinate independent manner? This can be achieved if we first product the function

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with the volume element $\star 1$, where $\star$ is the Hodge star [7]. The resulting product is a *volume-form*. This is integrable [1] and may be used as an integrand, so we can define the action functional as

$$S = \int_{\mathcal{M}} L \star 1 \,. \tag{3}$$

In mechanics and in describing physical systems in general, it is important to be able to determine dynamics in a coordinate independent manner. It is thus reasonable to expect that, if we suppose the function embodies some kind of dynamical variable constraints, it should exist on a manifold and not, for example, on a particular map from a manifold to a set of real numbers. In light of this, equation (3) is a satisfactory starting point for an action functional.

Now, if we suppose that L itself is defined in terms of *generalised coordinates* q and derivative q', which are necessary in order to be able to acquire any dynamical behaviour, then it will be helpful to introduce a coordinate for the purposes of parameterising q and for defining its derivative, q'. Let us use the coordinate x that we defined earlier via the map ϕ.

What we have just described is what is usually referred to as the 'Lagrangian' function [2], which can be written showing dependencies as $L = L(x, q(x), q'(x))$. Thus we may write the action as

$$S = \int_{\mathcal{M}} L(x, q(x), q'(x)) \star 1 \,. \tag{4}$$

Note that we have $q : X \to Q \subset \mathbb{R}$ and $q' : X \to Q' \subset \mathbb{R}$, so that $q, q' \in \mathcal{F}(\mathbb{R})$. That is, the q and q' depend on coordinates x, but L itself is still coordinate independent.

If, as is ordinarily done [2, 3, 4, 5, 8, 9], we had instead started with the definition

$$A = \int_{\mathcal{M}} L(y, q(y), q'(y)) \, \mathrm{d}y \,, \tag{5}$$

we would see that $L(y, q(y), q'(y))$ is *not a function on* $\mathcal{M}$, it is rather a function on $\mathbb{R}$ and may be interpreted as the component of a 1-form in coordinate system y with the differential $\mathrm{d}y$. This 1-form is altogether a different object, and not what we originally set out to define the action by. With the differences clarified and the case made for using (4) instead, let us now explore the consequences.

II. Extremising The Action Functionals

What options do we have to vary the action in (4)? We can of course vary q. But we can also see that now we have the option to vary x, too. Compare this to the paradigmatical definition in (5), where a variation of y simply amounts to changing the coordinate of integration. Since differential forms are by construction coordinate independent, this cannot induce variation of the action in (5).

Using the construction presented in (4), we find in fact that we have $S = S[q, x]$. That is, S is now a functional of the *two* functions, q and x. Let us try to extremise S via variations of both of these.

Variation of x

For brevity we present the infinitesimal treatment of variations. Varying $x \to x + \epsilon f$ for infinitesimal ϵ and arbitrary f to $\mathcal{O}(\epsilon)$ we get

$$\begin{aligned} S[q,x] \to S[q, x+\epsilon f] &= \int_{\mathcal{M}} L(x+\epsilon f, q(x+\epsilon f), q'(x+\epsilon f)) \star 1 \\ &= \int_{\mathcal{M}} \left[L(x, q(x), q'(x)) + \epsilon f \left(\frac{\partial L}{\partial x} + q' \frac{\partial L}{\partial q} + q'' \frac{\partial L}{\partial q'} \right) \right] \star 1 \,, \end{aligned}$$

so that

$$\delta S[q,x] = \int_{\mathcal{M}} \epsilon f \left(\frac{\partial L}{\partial x} + q' \frac{\partial L}{\partial q} + q'' \frac{\partial L}{\partial q'} \right) \star 1 \,.$$

Thus for arbitrary f, if S is at an extremum (whereby we assert that $\delta S = 0$), then by the *fundamental lemma* of variational calculus [9], we have that

$$\frac{\partial L}{\partial x} + q' \frac{\partial L}{\partial q} + q'' \frac{\partial L}{\partial q'} = \frac{\mathrm{d} L_q(x)}{\mathrm{d}x} = 0 \,. \tag{6}$$

where we have defined $L_q \equiv L(x, q(x), q'(x))$. These are the extremising equations (or what are often called *equations of motion* for dynamical systems), and are somewhat different to the Euler-Lagrange equations. Before discussing implications of this, let us explore the alternative method of variation.

Variation of q

Varying $q \to q + \epsilon\eta$ for infinitesimal ϵ also induces a variation in the derivative $q' \to q' + \epsilon\eta'$, and for arbitrary η to $\mathcal{O}(\epsilon)$ the varied action is

$$\begin{aligned} S[q,x] \to S[q+\epsilon\eta, x] &= \int_{\mathcal{M}} L(x, q+\epsilon\eta, q'+\epsilon\eta')) \star 1 \\ &= \int_{\mathcal{M}} \left[L(x, q(x), q'(x)) + \epsilon \left(\eta \frac{\partial L}{\partial q} + \eta' \frac{\partial L}{\partial q'} \right) \right] \star 1 \,, \end{aligned}$$

and thus

$$\delta S[q,x] = \int_{\mathcal{M}} \epsilon \left(\eta \frac{\partial L}{\partial q} + \eta' \frac{\partial L}{\partial q'} \right) \star 1 \,. \tag{7}$$

This integral must be valid for all possible integration coordinates, and the term in brackets must be a function on the manifold. One might argue that it is possible to select integration coordinate x so that we can continue in the usual manner with integration by parts and discard the boundary term (if we set $\eta|_{\partial\mathcal{M}} = 0$) to obtain the Euler-Lagrange equations.

However, like q, the variation $\eta \in \mathcal{F}(\mathbb{R})$ is a function of the real numbers x_p. Thus $\eta|_{\partial\mathcal{M}}$ is fixed by coordinates x, it is not fixed at any points $p \in \mathcal{M}$, but rather at some real numbers $x_{p(a)}$ and $x_{p(b)}$, where we have called the boundary points of the manifold $p(a)$ and $p(b)$.

The map ϕ however is arbitrary, and since we can rederive (7) using a different map

$$\begin{aligned} \phi' \,:\, \mathcal{M} &\to Z \subset \mathbb{R} \\ p &\mapsto z_p \in Z \,, \end{aligned}$$

then $\eta|_{\partial\mathcal{M}}$ must *also* be fixed at the numbers $z_{p(a)} \neq x_{p(a)}$ and $z_{p(b)} \neq x_{p(b)}$. Since the same argument applies for all possible coordinate maps, then η must be fixed *everywhere*, and thus η is not arbitrary and may not be used to generate variations in S suitable for applications of the fundamental lemma of variational calculus.

III. Conclusions

We have seen that actions formulated as in (4) can only be varied through the function x. Compared to writing an action as a differential form such as in (5), we unsurprisingly obtain different equations of motion for any given function $L = L(t, q(t), q'(t))$, on $\mathcal{M}$ or on $\mathbb{R}$. Reassuringly however, it seems quite generally that the correct equations of motion are obtained from the 'function implementation' of the action if the *sum* of 'energies' is taken to define the Lagrangian: $L \equiv T + V$, where T is the kinetic energy and V is potential energy, rather than the '1-form implementation', where the *difference* of energies defines the Lagrangian instead: $L \equiv T - V$ (see example in Ref. [10]).

When possible, it is easy to redefine Lagrangians for simple systems to ensure the correct equations of motion extremise the action, and if the Lagrangian is modified appropriately, we generally do not see modified equations of motion. However, for the Einstein-Hilbert action of gravity for example, a modification of the field equations arises [10].

References

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