PERSISTENCE OF PULSES FOR SOME REACTION-DIFFUSION EQUATIONS

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Abstract: We prove the persistence under a perturbation of stationary pulse solutions of certain reaction-diffusion type equations on the real line and compute the asymptotic approximations of such pulses to the leading order in the parameter of the perturbation.

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1. Introduction

Consider the problem,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au^2(1-u) - \sigma(x)u, \quad x \in \mathbb{R}, \quad t \geq 0, \tag{1.1}$$

where $a > 0$ is a constant. Equations of this kind are crucial in the study of population dynamics (see e.g. [7]). In this context, $u(x,t)$ denotes the density of a population at time t in location x, the diffusion term describes its migration, the second term in the right side is the reproduction rate, and the last term is the mortality rate. In the case of sexual reproduction, the reproduction rate is proportional to the second power of the population density and to the available resources $(1-u)$, given by the difference of the rate of production of resources and the rate of their consumption. Stationary solutions of equation (1.1) are solutions to

$$\frac{d^2 u}{dx^2} + au^2(1-u) - \sigma(x)u = 0, \quad x \in \mathbb{R}. \tag{1.2}$$

Stationary solutions of reaction-diffusion equations decaying at infinities are called *pulses*. Their existence along with stability and other related issues have been actively studied for both local and nonlocal models in recent years, for instance, in [1], [2], [3], [4], [5], [6], [8], [12], [13]. In the present article, we only focus on the persistence of pulses and their asymptotic approximations to the leading order of the small parameter when a perturbation is applied to equation (1.1). Similar studies in the context of standing solitary waves of the nonlinear Schrödinger equation, when a perturbation is applied to either a scalar potential involved in it or to the nonlinear term, were exploited in [9].

The unperturbed stationary problem in our case is given by

$$\frac{d^2 w_0}{dx^2} + a w_0^2 (1 - w_0) - \sigma_0(x) w_0 = 0, \quad x \in \mathbb{R}. \tag{1.3}$$

In the article we will consider the space $H^2(\mathbb{R})$ equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \left\| \frac{d^2 u}{dx^2} \right\|^2_{L^2(\mathbb{R})}. \tag{1.4}$$

By virtue of the standard Sobolev embedding, we have

$$\|u\|_{L^\infty(\mathbb{R})} \le c_e \|u\|_{H^2(\mathbb{R})}, \tag{1.5}$$

where $c_e > 0$ is the constant of the embedding. We first make the following assumption on the parameters of our unperturbed problem along with its solution that we are going to consider.

Assumption 1. *Let the constant $a > 0$ and the function*

$$\sigma_0(x) \in C^\infty(\mathbb{R}), \ \lim_{x \to \pm\infty} \sigma_0(x) = \delta > 0.$$

We also assume that equation (1.3) admits a pulse solution $w_0(x) > 0, \ x \in \mathbb{R}$, satisfying

$$w_0(x) \in C^\infty(\mathbb{R}) \cap H^2(\mathbb{R}), \ \lim_{x \to \pm\infty} w_0(x) = 0.$$

When a perturbation is applied to our stationary problem (1.3), we arrive at

$$\frac{d^2 w}{dx^2} + a w^2 (1 - w) - [\sigma_0(x) + \varepsilon \sigma_1(x)] w = 0, \quad x \in \mathbb{R}. \tag{1.6}$$

Assumption 2. *Let the parameter $\varepsilon \ge 0$ and the nontrivial function*

$$\sigma_1(x) \in C^\infty(\mathbb{R}), \ \lim_{x \to \pm\infty} \sigma_1(x) = 0. \tag{1.7}$$

We seek solutions of equation (1.6) in the form

$$w(x) = w_0(x) + w_p(x). \tag{1.8}$$

Then, by means of (1.6) along with (1.3), we get

$$L_0 w_p(x) = a(1-3w_0(x))w_p^2(x) - aw_p^3(x) - \varepsilon\sigma_1(x)(w_0(x)+w_p(x)), \tag{1.9}$$

where

$$L_0 = -\frac{d^2}{dx^2} + a(3w_0^2(x) - 2w_0(x)) + \sigma_0(x) : H^2(\mathbb{R}) \to L^2(\mathbb{R}). \tag{1.10}$$

Under Assumption 1, it is easy to see that the essential spectrum of L_0 is

$$\sigma_{ess}(L_0) = [\delta, +\infty). \tag{1.11}$$

If $\sigma_0(x)$ were a constant function on the real line, then the operator L_0 would have a zero mode $\frac{dw_0}{dx}$, which easily follows by differentiating both sides of equation (1.3). However, in the present article we assume the function $\sigma_0(x)$ to be generic such that operator (1.10) would have a trivial kernel.

Assumption 3. *The kernel* $\ker(L_0) = \{0\}$.

By means of (1.11) along with Assumption 3, the operator $L_0^{-1} : L^2(\mathbb{R}) \to H^2(\mathbb{R})$ is bounded, that is,

$$\|L_0^{-1}\| < \infty. \tag{1.12}$$

Let us denote a closed ball in the Sobolev space $H^2(\mathbb{R})$ as

$$B_\rho := \{u \in H^2(\mathbb{R}) \mid \|u\|_{H^2(\mathbb{R})} \le \rho\}, \quad \rho > 0. \tag{1.13}$$

We look for solutions of problem (1.9) as fixed points of the auxiliary nonlinear equation

$$L_0 u(x) = a(1-3w_0(x))v^2(x) - av^3(x) - \varepsilon\sigma_1(x)(w_0(x)+v(x)). \tag{1.14}$$

For a given function $v(x)$ this is an equation with respect to $u(x)$. We mention that similar ideas for problems involving non-Fredholm operators in their left sides have been exploited in [10] and [11]. We introduce the operator T such that $u = Tv$, where u is a solution of equation (1.14). Our main result is as follows.

Theorem 4. *Let Assumptions 1, 2 and 3 hold. Then equation (1.14) defines the map* $T : B_\rho \to B_\rho$, *which is a strict contraction for all* $0 < \rho < \rho^*$ *and* $0 < \varepsilon < \varepsilon^*$ *for some* $\rho^* > 0$ *and* $\varepsilon^* > 0$. *The unique fixed point* $w_p(x)$ *of this map* T *is the only solution of problem (1.9) in* B_ρ *such that*

$$w_p(x) = -\varepsilon L_0^{-1}[\sigma_1(x)w_0(x)] + O(\varepsilon^2). \tag{1.15}$$

Note that $O(\varepsilon^2)$ in the right side of formula (1.15) denotes the terms of the order ε^2 and higher in the $H^2(\mathbb{R})$ norm. The proof of Theorem 4 is given in the next section.

2. Proof of Theorem 4

First of all, we establish the uniqueness of solutions of problem (1.14). Suppose that there is a $v \in B_\rho$ such that (1.14) has two different solutions $u_1, u_2 \in B_\rho$. Then their difference $\psi(x) := u_1(x) - u_2(x) \in H^2(\mathbb{R})$ solves the homogeneous problem

$$L_0\psi = 0. \tag{2.16}$$

By means of Assumption 3, equation (2.16) admits only a trivial solution, a contradiction. This proves the uniqueness of solutions of (1.14).

Next, for arbitrary $v(x) \in B_\rho$, we estimate the right side of problem (1.14) in the absolute value from above by

$$[a(1 + 3\|w_0\|_{L^\infty(\mathbb{R})})\|v\|_{L^\infty(\mathbb{R})}+$$

$$+a\|v\|^2_{L^\infty(\mathbb{R})} + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}]|v(x)| + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}|w_0(x)|. \tag{2.17}$$

Note that $\sigma_1(x) \in L^\infty(\mathbb{R})$ due to Assumption 2. By means of the Sobolev embedding (1.5), expression (2.17) can be bounded from above by

$$[a(1 + 3c_e\|w_0\|_{H^2(\mathbb{R})})c_e\|v\|_{H^2(\mathbb{R})}+$$

$$+ac_e^2\|v\|^2_{H^2(\mathbb{R})} + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}]|v(x)| + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}|w_0(x)|. \tag{2.18}$$

The fact that $v(x) \in B_\rho$ yields the upper bound for (2.18) given by

$$[a(1 + 3c_e\|w_0\|_{H^2(\mathbb{R})})c_e\rho+$$

$$+ac_e^2\rho^2 + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}]|v(x)| + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}|w_0(x)|. \tag{2.19}$$

Therefore, from (1.14) we easily deduce that

$$\|u\|_{H^2(\mathbb{R})} \le \|L_0^{-1}\|\{[a(1 + 3c_e\|w_0\|_{H^2(\mathbb{R})})c_e\rho+$$

$$+ac_e^2\rho^2 + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}]\rho + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}\|w_0\|_{H^2(\mathbb{R})}\}. \tag{2.20}$$

Apparently, the estimate

$$\|L_0^{-1}\|\{ac_e(1 + 3c_e\|w_0\|_{H^2(\mathbb{R})})\rho^2+$$

$$+ac_e^2\rho^3 + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}(\rho + \|w_0\|_{H^2(\mathbb{R})})\} \le \rho \tag{2.21}$$

can be achieved for all $\rho > 0$ and $\varepsilon > 0$ small enough. Note that the upper bound on the values of $\varepsilon > 0$ here will depend on ρ. This means that

$$\|u\|_{H^2(\mathbb{R})} \le \rho, \tag{2.22}$$

that is, $u \in B_\rho$ as well. Hence the problem (1.14) defines the map $T : B_\rho \to B_\rho$ for all $\rho > 0$ and $\varepsilon > 0$ sufficiently small.

Our goal is to establish that this map is a strict contraction. In fact, let v_1, $v_2 \in B_\rho$. The argument above gives us $u_1 = Tv_1$, $u_2 = Tv_2 \in B_\rho$ as well. By means of equation (1.14) we obtain

$$L_0 u_1(x) = a(1 - 3w_0(x))v_1^2(x) - av_1^3(x) - \varepsilon\sigma_1(x)(w_0(x) + v_1(x)), \tag{2.23}$$

$$L_0 u_2(x) = a(1 - 3w_0(x))v_2^2(x) - av_2^3(x) - \varepsilon\sigma_1(x)(w_0(x) + v_2(x)). \tag{2.24}$$

Formulas (2.23) and (2.24) give us

$$L_0(u_1(x) - u_2(x)) = (v_1(x) - v_2(x))\times$$

$$\times\{a(1-3w_0(x))(v_1(x)+v_2(x))-a(v_1^2(x)+v_1(x)v_2(x)+v_2^2(x))-\varepsilon\sigma_1(x)\}. \tag{2.25}$$

We estimate the right side of equality (2.25) from above in the absolute value by

$$|v_1(x) - v_2(x)|\{a(1 + 3\|w_0\|_{L^\infty(\mathbb{R})})(\|v_1\|_{L^\infty(\mathbb{R})} + \|v_2\|_{L^\infty(\mathbb{R})})+$$

$$+a(\|v_1\|^2_{L^\infty(\mathbb{R})} + \|v_1\|_{L^\infty(\mathbb{R})}\|v_2\|_{L^\infty(\mathbb{R})} + \|v_2\|^2_{L^\infty(\mathbb{R})}) + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}\}. \tag{2.26}$$

With the help of the Sobolev embedding (1.5), expression (2.26) can be bounded from above by

$$|v_1(x) - v_2(x)|\{ac_e(1 + 3c_e\|w_0\|_{H^2(\mathbb{R})})(\|v_1\|_{H^2(\mathbb{R})} + \|v_2\|_{H^2(\mathbb{R})})+$$

$$+ac_e^2(\|v_1\|^2_{H^2(\mathbb{R})} + \|v_1\|_{H^2(\mathbb{R})}\|v_2\|_{H^2(\mathbb{R})} + \|v_2\|^2_{H^2(\mathbb{R})}) + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}\}. \tag{2.27}$$

The fact that v_1, $v_2 \in B_\rho$ gives us the upper bound for (2.27) as

$$|v_1(x) - v_2(x)|\{2ac_e(1 + 3c_e\|w_0\|_{H^2(\mathbb{R})})\rho + 3ac_e^2\rho^2 + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}\}. \tag{2.28}$$

Therefore, by means of (2.25) we arrive at

$$\begin{aligned}&\|u_1 - u_2\|_{H^2(\mathbb{R})}\\ \le&\|L_0^{-1}\|\{2ac_e(1 + 3c_e\|w_0\|_{H^2(\mathbb{R})})\rho + 3ac_e^2\rho^2 + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}\}\|v_1 - v_2\|_{H^2(\mathbb{R})}.\end{aligned} \tag{2.29}$$

Evidently, the bound

$$\|L_0^{-1}\|\{2ac_e(1 + 3c_e\|w_0\|_{H^2(\mathbb{R})})\rho + 3ac_e^2\rho^2 + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}\} < 1 \tag{2.30}$$

can be attained for all $\rho > 0$ and $\varepsilon > 0$ sufficiently small. Therefore, the map $T : B_\rho \to B_\rho$ defined by equation (1.14) is a strict contraction. Its unique fixed point $w_p(x)$ is the only solution of problem (1.9) in the ball B_ρ. Note that the function $w_p(x)$ is nontrivial for $\varepsilon > 0$, which easily follows from equation (1.9), since by means of Assumptions 1 and 2 the intersection of supports supp$\sigma_1(x) \cap$ supp$w_0(x)$ is a set of nonzero Lebesgue measure on the real line. Clearly, the resulting solution

$w(x)$ of problem (1.6) given by formula (1.8) belongs to $H^2(\mathbb{R})$. Let the radius of the ball B_ρ be small enough, namely,

$$\rho < \|w_0\|_{H^2(\mathbb{R})}. \tag{2.31}$$

Then by means of (1.8) along with (2.31) via the triangle inequality we have

$$\|w\|_{H^2(\mathbb{R})} \geq \|w_0\|_{H^2(\mathbb{R})} - \|w_p\|_{H^2(\mathbb{R})} \geq \|w_0\|_{H^2(\mathbb{R})} - \rho > 0 \tag{2.32}$$

and hence $w(x)$ is nontrivial as well.

Finally, we finish the proof by obtaining the asymptotics for the function $w_p(x)$ to the leading order in the parameter ε. Clearly, (1.9) yields

$$w_p(x) = {L_0}^{-1}[a(1-3w_0(x))w_p^2(x)-$$

$$-aw_p^3(x) - \varepsilon\sigma_1(x)w_p(x)] - \varepsilon {L_0}^{-1}[\sigma_1(x)w_0(x)]. \tag{2.33}$$

Apparently, the leading term in the small parameter ε in the right side of (2.33) is given by

$$-\varepsilon {L_0}^{-1}[\sigma_1(x)w_0(x)]. \tag{2.34}$$

Clearly, (2.34) can be estimated from above in the $H^2(\mathbb{R})$ norm by

$$\varepsilon\|L_0^{-1}\|\|\sigma_1(x)\|_{L^\infty(\mathbb{R})}\|w_0(x)\|_{H^2(\mathbb{R})} < \infty \tag{2.35}$$

under Assumptions 1 and 2 along with (1.12). Evidently, we have the upper bound

$$|a(1-3w_0(x))w_p^2(x) - aw_p^3(x) - \varepsilon\sigma_1(x)w_p(x)| \leq a(1+3\|w_0\|_{L^\infty(\mathbb{R})})\times$$

$$\times\|w_p\|_{L^\infty(\mathbb{R})}|w_p(x)| + a\|w_p\|^2_{L^\infty(\mathbb{R})}|w_p(x)| + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}|w_p(x)|. \tag{2.36}$$

By means of the Sobolev embedding (1.5), the right side of inequality (2.36) can be estimated from above by

$$[ac_e(1+3c_e\|w_0\|_{H^2(\mathbb{R})})\|w_p\|_{H^2(\mathbb{R})} + ac_e^2\|w_p\|^2_{H^2(\mathbb{R})} + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}]|w_p(x)|. \tag{2.37}$$

Therefore, the remaining term in the right side of (2.33) can be bounded from above in the $H^2(\mathbb{R})$ norm by

$$\|L_0^{-1}\|[ac_e(1+3c_e\|w_0\|_{H^2(\mathbb{R})})\|w_p\|_{H^2(\mathbb{R})}+$$

$$+ac_e^2\|w_p\|^2_{H^2(\mathbb{R})} + \varepsilon\|\sigma_1\|_{L^\infty(\mathbb{R})}]\|w_p\|_{H^2(\mathbb{R})} = O(\varepsilon^2), \tag{2.38}$$

namely, the identity (1.15) holds. ■

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