Quantum Dynamics Generated by Long-Range Interactions for Lattice Fermions and Quantum Spins

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Abstract

We study the macroscopic dynamics of fermion and quantum-spin systems with long-range, or mean-field, interactions, which is equivalent to an intricate combination of classical and short-range quantum dynamics. In this paper we focus on the *quantum* part of the long-range macroscopic dynamics. Its classical part is studied in a companion paper. Altogether, the results obtained are far beyond previous ones and required the development of a suitable mathematical framework. The entanglement of classical and quantum worlds is noteworthy, opening new theoretical perspectives, and is shown here to be a consequence of the highly non-local character of long-range, or mean-field, interactions.

Dedicated to V.A. Zagrebnov for his important contributions to the mathematics of quantum many-body theory.

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Contents

1 Introduction

Following [1], we pursue our study on macroscopic dynamics of fermion and quantum-spin systems with long-range, or mean-field, interactions. In [1] we only focus on the classical part of this dynamics. In the current paper we study its quantum part in detail. The results obtained are far beyond previous ones because the permutation-invariance of lattice-fermion or quantum-spin systems is *not* required:

- The short-range part of the corresponding Hamiltonian is very general since only a sufficiently strong polynomial decay of its interactions and a translation invariance are necessary.
- The long-range part is also very general, being an infinite sum (over n) of mean-field terms of order $n \in \mathbb{N}$. In fact, even for permutation-invariant systems, the class of long-range, or mean-field, interactions we are able to handle is much larger than what was previously studied.
- The initial state is only required to be periodic. By [1, Proposition 2.2], observe that the set of all such initial states is weak*-dense within the set of all even states, the physically relevant ones.

For an exhaustive historical discussion on fermion or quantum-spin systems with long-range, or mean-field, interactions we refer to [1, Section 1]. Here, we add several observations concerning the physical relevance of long-range interactions in Physics.

The most general form of a translation-invariant model for fermions (with spin set S) in a cubic box $\Lambda_L \doteq \{\mathbb{Z} \cap [-L, L]\}^3$ of volume $|\Lambda_L|$, $L \in \mathbb{N}$, with a quartic (in the fermionic fields) gauge-invariant, translation-invariant interaction is given in momentum space by

$$\begin{aligned} H &= \sum_{k\in\Lambda_L^*,\ \mathrm{s}\in\mathrm{S}} (\varepsilon_k - \mu)\, \tilde{a}_k^* \tilde{a}_k \\ &\quad + \frac{1}{|\Lambda_L|} \sum_{\substack{k,k',q\in\Lambda_L^* \\ \mathrm{s}_1,\mathrm{s}_2,\mathrm{s}_3,\mathrm{s}_4\in\mathrm{S}}} g_{\mathrm{s}_1,\mathrm{s}_2,\mathrm{s}_3,\mathrm{s}_4}(k,k',q)\, \tilde{a}_{k+q,\mathrm{s}_1}^* \tilde{a}_{k'-q,\mathrm{s}_2}^* \tilde{a}_{k',\mathrm{s}_3} \tilde{a}_{k,\mathrm{s}_4} \,. \end{aligned} \tag{1}$$

See [2, Eq. (2.1)]. Here, Λ_L^* is the reciprocal lattice of quasi-momenta (periodic boundary conditions) associated with Λ_L and the operator $\tilde{a}_{k,\mathrm{s}}^*$ (respectively $\tilde{a}_{k,\mathrm{s}}$) creates (respectively annihilates) a fermion with spin $\mathrm{s} \in \mathrm{S}$ and (quasi-) momentum $k \in \Lambda_L^*$. The function ε_k represents the kinetic energy of a fermion with (quasi-) momentum k and the real number μ is the chemical potential. The last term of (1) corresponds to a general translation-invariant two-body interaction written in the (quasi-) momentum space.

One important example of a lattice-fermion system with a long-range interaction is given in the scope of the celebrated BCS theory – proposed in the late 1950s (1957) to explain conventional type I superconductors. The lattice version of this theory is obtained from (1) by taking $\mathrm{S} \doteq \{\uparrow, \downarrow\}$ and imposing

$$g_{\mathrm{s}_1,\mathrm{s}_2,\mathrm{s}_3,\mathrm{s}_4}\left(k, k', q\right) = \delta_{k,-k'}\delta_{\mathrm{s}_1,\uparrow}\delta_{\mathrm{s}_2,\downarrow}\delta_{\mathrm{s}_3,\downarrow}\delta_{\mathrm{s}_4,\uparrow} f\left(k, -k, q\right)$$

for some function f: It corresponds to the so-called (reduced) BCS Hamiltonian

$$\mathrm{H}_\Lambda^{BCS} \doteq \sum_{k\in\Lambda_L^*} \left(\varepsilon_k - \mu\right) \left(\tilde{a}_{k,\uparrow}^* \tilde{a}_{k,\uparrow} + \tilde{a}_{k,\downarrow}^* \tilde{a}_{k,\downarrow}\right) - \frac{1}{|\Lambda_L|} \sum_{k,q\in\Lambda_L^*} \gamma_{k,q} \tilde{a}_{k,\uparrow}^* \tilde{a}_{-k,\downarrow}^* \tilde{a}_{-q,\downarrow} \tilde{a}_{q,\uparrow} \ , \qquad (2)$$

where $\gamma_{k,q}$ is a positive[1] function. Because of the term $\delta_{k,-k'}$, the interaction of this model has a long-range character, in position space. The simple choice $\gamma_{k,q} \doteq \gamma > 0$ in (2) is still very interesting since, even when $\varepsilon_k = 0$, the BCS Hamiltonian qualitatively displays most of basic properties of real conventional type I superconductors. See, e.g., [3, Chapter VII, Section 4]. Written in the x-space, the BCS interaction in (2) is, in this case, equal to

$$-\frac{\gamma}{|\Lambda_L|} \sum_{k,q\in\Lambda_L^*} \tilde{a}_{k,\uparrow}^* \tilde{a}_{-k,\downarrow}^* \tilde{a}_{-q,\downarrow} \tilde{a}_{q,\uparrow} = -\frac{\gamma}{|\Lambda_L|} \sum_{x,y\in\Lambda_L} a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow} \ , \qquad (3)$$

the operators $a_{x,\mathrm{s}}^*$,$a_{x,\mathrm{s}}$ being respectively the creation and annihilation operators of a fermion with spin $\mathrm{s} \in \{\uparrow, \downarrow\}$ at lattice site $x \in \Lambda_L$. The right-hand side of the equality explicitly shows the long-range character of the interaction. It is a mean-field interaction since

$$\frac{1}{|\Lambda_L|} \sum_{x,y\in\Lambda_L} a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow} = \sum_{y\in\Lambda_L} \left(\frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} a_{x,\uparrow}^* a_{x,\downarrow}^* \right) a_{y,\downarrow} a_{y,\uparrow} \ .$$

The (reduced) BCS Hamiltonian with $\gamma_{k,q} \doteq \gamma > 0$ is an important, albeit very elementary, example of the far more general case treated in this paper.

Long-range, or mean-field, effective models are essential in condensed matter physics to study, from microscopic considerations, macroscopic phenomena like superconductivity. They come from different approximations or Ansätze, like the choice $\gamma_{k,q} \doteq \gamma > 0$. The general form of the (effective) BCS interaction in (2) comes from the celebrated Fröhlich electron-phonon interactions. What's more, they are possibly not merely effective interactions.

Long-range, or mean-field, models captures surprisingly well many phenomena in condensed matter physics. For instance, recall that the BCS interaction (3) allows us to qualitatively display most of basic properties of conventional superconductors [3, Chapter VII, Section 4]. Ergo, one could wonder whether such interactions may have a more fundamental physical relevance. Such a question is usually not addressed, because these interactions seem to break the spacial locality of Einstein's relativity. For instance, the BCS interaction (3) can be seen as a kinetic term for fermion pairs that can hop from $y \in \Lambda_L$ to *any* other lattice site $x \in \Lambda_L$, for each $L \in \mathbb{N}$.

[1] The positivity of $\gamma_{k,q}$ imposes constraints on the choice of the function f.

This non-locality property is reminiscent of the inherent non-locality of quantum mechanics, highlighted by Einstein, Podolsky and Rosen with the celebrated EPR paradox. Philosophically, this general issue challenges causality, in its local sense, as well as the notion of a material object[2]. In [5], Einstein says the following:

"*If one asks what, irrespective of quantum mechanics, is characteristic of the world of ideas of physics, one is first of all struck by the following: the concepts of physics relate to a real outside world... it is further characteristic of these physical objects that they are thought of as a range in a space-time continuum. An essential aspect of this arrangement of things in physics is that they lay claim, at a certain time, to an existence independent of one another, provided these objects "are situated in different parts of space".*

The following idea characterizes the relative independence of objects far apart in space (A and B): external influence on A has no direct influence on B...

There seems to me no doubt that those physicists who regard the descriptive methods of quantum mechanics as definitive in principle would react to this line of thought in the following way: they would drop the requirement... for the independent existence of the physical reality present in different parts of space; they would be justified in pointing out that the quantum theory nowhere makes explicit use of this requirement.

I admit this, but would point out: when I consider the physical phenomena known to me, and especially those who are being so successfully encompassed by quantum mechanics, I still cannot find any fact anywhere which would make it appear likely that (that) requirement will have to be abandoned.

I am therefore inclined to believe that the description of quantum mechanics... has to be regarded as an incomplete and indirect description of reality, to be replaced at some later date by a more complete and direct one."

The debate on non-locality in Physics, experimentally shown, refers to the existence of quantum entanglement, used in quantum information theory. For a discussion on locality and realism in quantum mechanics, see, e.g., [6] by A. Aspect, who is one of the main initiators of experimental studies on quantum entanglement, in the beginning of the 1980s.

The non-locality of long-range, or mean-field, interactions like the BCS interaction[3] (3) can be seen as an instance of the (controversial) intrinsic non-locality of quantum physics. Long-range interactions are thus usually not considered by the physics community as being fundamental interactions, in order to avoid polemics. We partially agree with this position and see long-range interactions as possibly resulting from (more fundamental) interactions with (bosonic) mediators, like phonons in conventional superconductivity.

Nonetheless, a long-range interaction like (3), being quantum mechanical, does not refer to an actuality (in Aristotle's sense), *but only to a potentiality*. Physical properties of any (energy-conserving) physical system do not just depend on its Hamiltonian but also on its state which accounts for the "environmental" part of the system: This situation is analogous to the epigenetics[4] showing that the DNA sequence is only a set of constraints and potentialities, the physical realizations of which depend on the history and environment of the corresponding organism. For instance, in a lattice-fermion system described by the so-called (reduced) BCS Hamiltonian with $\gamma_{k,q} = \gamma$, pairs of particles may (almost)

[2]According to the spatio-temporal identity of classical mechanics, the same physical object cannot be at the same time on two distinct points of the phase space. This refers to Leibniz's Principle of Identity of Indiscernibles [4, p. 1]. The spatio-temporal identity of classical mechanics is questionable in quantum mechanics. See, e.g., [4].

[3]The strength of the BCS interaction (3) between two points of the space does not decay at large distances.

[4]Quoting [7]: "*Epigenetics is typically defined as the study of heritable changes in gene expression that are not due to changes in DNA sequence. Diverse biological properties can be affected by epigenetic mechanisms: for example, the morphology of flowers and eye colour in fruitflies.*"

never hop in arbitrarily large distances if the state[5] ρ of the corresponding system is such that

$$\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\sum_{x,y\in\Lambda_L}\rho\left(a_{x,\uparrow}^*a_{x,\downarrow}^*a_{y,\downarrow}a_{y,\uparrow}\right)=0\ .$$

This is the case for equilibrium states of this model at sufficiently high temperatures. It is thus too reductive to a priori eliminate such interactions from "fundamental" Hamiltonians of physical systems.

On the top of that, as is well-known, the thermodynamic limit of mean-field dynamics is representation-dependent. This is basically Haag's original argument proposed in 1962 [8] for the BCS model. In fact, the description of the full dynamics requires an extended quantum framework [9], which is an intricate combination of classical and quantum dynamics, as observed by Bóna already thirty years ago [10]. The paper [9] shows the emergence of classical mechanics defined from Poisson brackets on state spaces without necessarily a disappearance of the quantum world, offering a general formal mathematical framework to understand physical phenomena with macroscopic quantum coherence. In the context of lattice-fermion systems, it is explained in detail in [1]. Such an entanglement of classical and quantum worlds is noteworthy, opening new theoretical perspectives, and is here a direct consequence of the highly non-local character of long-range, or mean-field, interactions.

To conclude, our main results are Proposition 4.2, Theorem 4.3 and Corollary 4.5. These results are non-trivial mathematical statements resulting from a combination of ergodic decompositions of periodic states [11, Chapter 4] with the theory of direct integrals [12] and Lieb-Robinson bounds [13, Section 4.3].

The paper is organized as follows: Section 2 explains the algebraic formulation of lattice-fermion systems with short- and long-range interactions and we make explicit the problem of the thermodynamic limit of their associated dynamics. Like in [1, 11], note that we prefer to use, from now on, the term "long-range" instead of "mean-field", since the latter can refer to different scalings. Our description of long-range dynamics requires the mathematical framework of [9], which is thus presented in Section 3. Section 4 describes long-range dynamics, which are based on self-consistency equations [1, Theorem 6.5]. Observe that we shortly explain the classical part of the long-range dynamics in Section 4.1, while its quantum part is given in detail in Section 4.2, which gathers the main results of the paper. All proofs are postponed to Section 5. Finally, Section 6 is a detailed appendix on the theory of direct integrals of measurable families of Hilbert spaces, operators, von Neumann algebras and C^*-algebra representations. This appendix is useful to make the paper self-contained and is of pedagogical interest for the non-expert. It will additionally be useful in future applications of the general results presented here to the study of KMS states of lattice-fermion or quantum-spin systems with long-range interactions.

In this paper, we only focus on lattice-fermion systems which are, from a technical point of view, slightly more difficult than quantum-spin systems, because of a non-commutativity issue at different lattice sites. However, all the results presented here hold true for quantum-spin systems, via obvious modifications.

Notation 1.1
(i) *A norm on a generic vector space $\mathcal{X}$ is denoted by $\|\cdot\|_{\mathcal{X}}$ and the identity mapping of $\mathcal{X}$ by $\mathbf{1}_{\mathcal{X}}$. The space of all bounded linear operators on $(\mathcal{X},\|\cdot\|_{\mathcal{X}})$ is denoted by $\mathcal{B}(\mathcal{X})$. The unit element of any algebra $\mathcal{X}$ is denoted by $\mathfrak{1}$, provided it exists. The scalar product of any Hilbert space $\mathcal{H}$ is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$.*
(ii) *For any topological space $\mathcal{X}$ and normed space $(\mathcal{Y},\|\cdot\|_{\mathcal{Y}})$, $C(\mathcal{X};\mathcal{Y})$ denotes the space of continuous maps from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{X}$ is a locally compact topological space, then $C_b(\mathcal{X};\mathcal{Y})$ denotes the*

[5] I.e., a positive and normalized continuous functional on the CAR algebra.

Banach space of bounded continuous mappings from $\mathcal{X}$ to $\mathcal{Y}$ along with the topology of uniform convergence.
(iii) *The notion of an automorphism depends on the structure of the corresponding domain. In algebra, a ($*$-) automorphism acting on a $*$-algebra is a bijective $*$-homomorphism from this algebra to itself. In topology, an automorphism acting on a topological space is a self-homeomorphism, that is, a homeomorphism of the space to itself.*

2 Algebraic Formulation of Lattice-Fermion Systems

The mathematical framework used here is the same than in [1], including the notation. We thus give it in a concise way and refer to [1] for more details.

2.1 CAR Algebra of Lattices

2.1.1 Background Lattice

Let $\mathfrak{L} \doteq \mathbb{Z}^d$ for some fixed $d \in \mathbb{N}$ and $\mathcal{P}_f \subseteq 2^{\mathfrak{L}}$ be the set of all non-empty finite subsets of $\mathfrak{L}$. In order to define the thermodynamic limit, for simplicity, we use the cubic boxes

$$\Lambda_L \doteq \{(x_1, \ldots, x_d) \in \mathfrak{L} : |x_1|, \ldots, |x_d| \leq L\} \subseteq \mathfrak{L} , \qquad L \in \mathbb{N} , \tag{4}$$

as a so-called van Hove net. We also use a positive-valued symmetric function $\mathbf{F} : \mathfrak{L}^2 \to (0, 1]$ with maximum value $\mathbf{F}(x, x) = 1$ for all $x \in \mathfrak{L}$ and satisfying

$$\|\mathbf{F}\|_{1,\mathfrak{L}} \doteq \sup_{y\in\mathfrak{L}} \sum_{x\in\mathfrak{L}} \mathbf{F}(x, y) < \infty \tag{5}$$

and

$$\mathbf{D} \doteq \sup_{x,y\in\mathfrak{L}} \sum_{z\in\mathfrak{L}} \frac{\mathbf{F}(x, z)\, \mathbf{F}(z, y)}{\mathbf{F}(x, y)} < \infty . \tag{6}$$

Explicit examples of such a function are given in [1, Section 3.1].

2.1.2 The CAR C^*-Algebra

For any subset $\Lambda \subseteq \mathfrak{L}, \mathcal{U}_\Lambda$ is the separable universal unital C^*-algebra generated by elements $\{a_{x,\mathrm{s}}\}_{x\in\Lambda,\mathrm{s}\in\mathrm{S}}$ satisfying the canonical anti-commutation relations (CAR), S being some finite set of spins. Note that we use the notation $\mathcal{U} \equiv \mathcal{U}_{\mathfrak{L}}$ and the subspace

$$\mathcal{U}_0 \doteq \bigcup_{\Lambda\in\mathcal{P}_f} \mathcal{U}_\Lambda \tag{7}$$

is a dense $*$-algebra of $\mathcal{U}$. Elements of $\mathcal{U}_0$ are called local elements. The (real) Banach subspace of all self-adjoint elements of $\mathcal{U}$ is denoted by $\mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U}$.

2.1.3 Even Elements

The condition

$$\sigma(a_{x,\mathrm{s}}) = -a_{x,\mathrm{s}}, \qquad x \in \Lambda,\ \mathrm{s} \in \mathrm{S} , \tag{8}$$

defines a unique $*$-automorphism σ of the C^*-algebra $\mathcal{U}$. The subspace

$$\mathcal{U}^+ \doteq \{A \in \mathcal{U} : A = \sigma(A)\} \subseteq \mathcal{U} \tag{9}$$

is the C^*-subalgebra of so-called even elements. $\mathcal{U}^+$ should be seen as more fundamental than $\mathcal{U}$ in Physics, because of the local causality in quantum field theory.

Note that the fact that the local causality in quantum field theory can be invoked to see $\mathcal{U}^+$ as being more fundamental than $\mathcal{U}$ in Physics does not prevent us from considering long-range interactions as possibly fundamental interactions, as explained in the introduction. The choice of $\mathcal{U}^+$ only compel us to consider (local) observables satisfying the local causality as measurable physical quantities, the full energy of lattice Fermi systems with short-range or long-range interactions being generally inaccessible in infinite volume. In fact, the long-range part yields possibly non-vanishing background fields, in the spirit of the Higgs mechanism of quantum field theory, in a given representation of the observable algebra, which is fixed by the initial state.

2.1.4 Translation Automorphisms

Translations refer to the group homomorphism $x \mapsto \alpha_x$ from $(\mathbb{Z}^d, +)$ to the group of $*$-automorphisms of $\mathcal{U}$, which is uniquely defined by the condition

$$\alpha_x(a_{y,\mathrm{s}}) = a_{y+x,\mathrm{s}} \ , \quad y \in \mathfrak{L}, \ \mathrm{s} \in \mathrm{S} \ . \tag{10}$$

This group homomorphism is used below to define the notion of (space) periodicity of states as well as the translation invariance of interactions of lattice-fermion systems.

2.2 State Space

2.2.1 Full State Space

The state space associated with $\mathcal{U}$ is defined by

$$E \doteq \{\rho \in \mathcal{U}^* : \rho \geq 0, \ \rho(\mathfrak{1}) = 1\} \ . \tag{11}$$

As explained in Section 5.1, E is a metrizable and weak*-compact convex subset of the dual space $\mathcal{U}^*$. It is also the state space of the classical dynamics studied in [1,9]. By the Krein-Milman theorem [14, Theorem 3.23], E is the weak*-closure of the convex hull of the (non-empty) set $\mathcal{E}(E)$ of its extreme points, which turns out to be weak*-dense [1,9]:

$$E = \overline{\mathrm{co}\mathcal{E}\left(E\right)} = \overline{\mathcal{E}(E)} \ . \tag{12}$$

All state spaces we define below have this peculiar geometrical structure.

We define $C\left(E;E\right)$ to be the set of weak*-continuous functions from the state space E to itself, endowed with the topology of uniform convergence. In other words, any net $(f_j)_{j\in J} \subseteq C\left(E;E\right)$ converges to $f \in C\left(E;E\right)$ whenever

$$\lim_{j\in J} \max_{\rho\in E} |f_j(\rho)(A) - f(\rho)(A)| = 0 \ , \qquad \text{for all } A \in \mathcal{U} \ . \tag{13}$$

We denote by $\mathrm{Aut}\left(E\right) \subsetneq C\left(E;E\right)$ the subspace of all automorphisms of E, i.e., element of $C\left(E;E\right)$ with weak*-continuous inverse. As is usual, we identify constant functions on $\mathbb{R}$ with elements of the codomain of such a function:

$$E \subseteq C\left(\mathbb{R};E\right) \qquad \text{and} \qquad \mathrm{Aut}\left(E\right) \subseteq C\left(\mathbb{R};\mathrm{Aut}\left(E\right)\right) \ . \tag{14}$$

2.2.2 Even States

The physically relevant set of states is the metrizable and weak*-compact convex set of *even* states defined by

$$E^{+} \doteq \{\rho \in E : \rho \circ \sigma = \rho\} \ , \tag{15}$$

σ being the unique $*$-automorphism of $\mathcal{U}$ satisfying (8). The set $\mathcal{E}(E^{+})$ of extreme points of E^{+} is also a weak*-dense subset of E^{+}:

$$E^{+} = \overline{\mathrm{co}\mathcal{E}\left(E^{+}\right)} = \overline{\mathcal{E}(E^{+})} \ .$$

See [1, Proposition 2.1].

2.2.3 Periodic States

For $\vec{\ell} \in \mathbb{N}^{d}$, consider the subgroup $(\mathbb{Z}^{d}_{\vec{\ell}}, +) \subseteq (\mathbb{Z}^{d}, +)$, where

$$\mathbb{Z}^{d}_{\vec{\ell}} \doteq \ell_1 \mathbb{Z} \times \cdots \times \ell_d \mathbb{Z} \ . \tag{16}$$

Any state $\rho \in E$ satisfying $\rho \circ \alpha_x = \rho$ for all $x \in \mathbb{Z}^{d}_{\vec{\ell}}$ is called $\mathbb{Z}^{d}_{\vec{\ell}}$*-invariant* on $\mathcal{U}$ or $\vec{\ell}$*-periodic*, α_x being the unique $*$-automorphism of $\mathcal{U}$ satisfying (10). Translation-invariant states refer to $(1, \cdots, 1)$-periodic states. For any $\vec{\ell} \in \mathbb{N}^{d}$, the metrizable and weak*-compact convex set

$$E_{\vec{\ell}} \doteq \left\{\rho \in E : \rho \circ \alpha_x = \rho \quad \text{for all } x \in \mathbb{Z}^{d}_{\vec{\ell}}\right\} \tag{17}$$

is called the $\vec{\ell}$-periodic-state space. By [11, Lemma 1.8], periodic states are even and, by [1, Proposition 2.3], the set

$$E_{\mathrm{p}} \doteq \bigcup_{\vec{\ell} \in \mathbb{N}^{d}} E_{\vec{\ell}} \tag{18}$$

of all periodic states is a weak*-dense subset of even states. For any $\vec{\ell} \in \mathbb{N}^{d}$, the set $\mathcal{E}(E_{\vec{\ell}})$ of extreme points of $E_{\vec{\ell}}$ is a weak*-dense G_δ subset of $E_{\vec{\ell}}$:

$$E_{\vec{\ell}} = \overline{\mathrm{co}\mathcal{E}(E_{\vec{\ell}})} = \overline{\mathcal{E}(E_{\vec{\ell}})} \ , \qquad \vec{\ell} \in \mathbb{N}^{d} \ . \tag{19}$$

In fact, up to an affine homeomorphism, for any $\vec{\ell} \in \mathbb{N}^{d}$, $E_{\vec{\ell}}$ is the so-called Poulsen simplex [11, Theorem 1.12]. This property is well-known and also holds true for lattice quantum spin systems [15, Example 4.3.26 and discussions p. 464].

2.3 Banach Spaces of Short-Range Interactions

2.3.1 Complex Interactions

A (complex) *interaction* is a mapping $\Phi : \mathcal{P}_f \to \mathcal{U}^{+}$ such that $\Phi_\Lambda \in \mathcal{U}_\Lambda$ for all $\Lambda \in \mathcal{P}_f$. The set $\mathcal{V}$ of all interactions can be naturally endowed with the structure of a complex vector space and the involution

$$\Phi \mapsto \Phi^{*} \doteq (\Phi^{*}_{\Lambda})_{\Lambda \in \mathcal{P}_f} \ . \tag{20}$$

An interaction Φ is, by definition, self-adjoint if $\Phi = \Phi^{*}$. The set of all self-adjoint interactions forms a real subspace of the space of all interactions.

2.3.2 Short-Range Interactions

The separable Banach space of short-range interactions is defined by

$$\mathcal{W} \doteq \{\Phi \in \mathcal{V} : \|\Phi\|_{\mathcal{W}} < \infty\} \tag{21}$$

with the norm of $\mathcal{W}$ being defined, from the positive-valued symmetric function $\mathbf{F}$ of Section 2.1, by

$$\|\Phi\|_{\mathcal{W}} \doteq \sup_{x,y\in\mathfrak{L}} \sum_{\Lambda\in\mathcal{P}_f,\ \Lambda\supseteq\{x,y\}} \frac{\|\Phi_\Lambda\|_{\mathcal{U}}}{\mathbf{F}(x,y)} \ . \tag{22}$$

The (real) Banach subspace of all self-adjoint interactions is denoted by $\mathcal{W}^{\mathbb{R}} \subsetneq \mathcal{W}$, similar to $\mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U}$.

2.3.3 Translation-Invariant Interactions

By definition, the interaction Φ is translation-invariant if

$$\Phi_{\Lambda+x} = \alpha_x(\Phi_\Lambda) \ , \qquad x\in\mathbb{Z}^d,\ \Lambda\in\mathcal{P}_f \ , \tag{23}$$

where

$$\Lambda + x \doteq \{y + x \in \mathfrak{L} : y \in \Lambda\} \ . \tag{24}$$

Here, $\{\alpha_x\}_{x\in\mathbb{Z}^d}$ is the family of (translation) $*$-automorphisms of $\mathcal{U}$ defined by (10). We then denote by $\mathcal{W}_1 \subsetneq \mathcal{W}$ the (separable) Banach subspace of translation-invariant, short-range interactions on $\mathfrak{L}$.

2.3.4 Finite-Range Interactions

For any $\Lambda \in \mathcal{P}_f$, we define the closed subspace[6]

$$\mathcal{W}_\Lambda \doteq \{\Phi \in \mathcal{W}_1 : \Phi_{\mathcal{Z}} = 0 \text{ whenever } \mathcal{Z} \nsubseteq \Lambda,\ \mathcal{Z} \ni 0\} \tag{25}$$

of finite-range translation-invariant interactions. Note that, for any $\Lambda\in\mathcal{P}_f$ and $\vec{\ell}\in\mathbb{N}^d$,

$$\mathcal{W}_\Lambda \subseteq \left\{\Phi \in \mathcal{W}_1 : \mathfrak{e}_{\Phi,\vec{\ell}} \in \mathcal{U}_{\Lambda^{(\vec{\ell})}}\right\} \subseteq \mathcal{W}_1 \ , \tag{26}$$

where, for any $\Phi\in\mathcal{W}$ and $\vec{\ell}\in\mathbb{N}^d$,

$$\mathfrak{e}_{\Phi,\vec{\ell}} \doteq \frac{1}{\ell_1\cdots\ell_d} \sum_{x=(x_1,\ldots,x_d),\ x_i\in\{0,\ldots,\ell_i-1\}} \ \sum_{\mathcal{Z}\in\mathcal{P}_f,\ \mathcal{Z}\ni x} \frac{\Phi_{\mathcal{Z}}}{|\mathcal{Z}|} \tag{27}$$

and

$$\Lambda^{(\vec{\ell})} \doteq \cup\{\Lambda + x : x = (x_1,\ldots,x_d),\ x_i\in\{0,\ldots,\ell_i-1\}\} \in \mathcal{P}_f \ . \tag{28}$$

From Equations (5) and (22), observe that

$$\|\mathfrak{e}_{\Phi,\vec{\ell}}\|_{\mathcal{U}} \leq \|\mathbf{F}\|_{1,\mathfrak{L}} \|\Phi\|_{\mathcal{W}} \ , \qquad \Phi\in\mathcal{W},\ \vec{\ell}\in\mathbb{N}^d. \tag{29}$$

[6]This follows from the continuity and linearity of the mappings $\Phi \mapsto \Phi_{\mathcal{Z}}$ for all $\mathcal{Z}\in\mathcal{P}_f$.

2.4 Banach Space of Long-Range Models

2.4.1 Self-Adjoint Measures on Interactions

Let $\mathbb{S}$ be the unit sphere of $\mathcal{W}_1$. For any $n \in \mathbb{N}$ and any finite signed Borel measure $\mathfrak{a}$ on the Cartesian product $\mathbb{S}^n$ (endowed with its product topology), we define the finite signed Borel measure $\mathfrak{a}^*$ to be the pushforward of $\mathfrak{a}$ through the automorphism

$$\left(\Psi^{(1)}, \ldots, \Psi^{(n)}\right) \mapsto ((\Psi^{(n)})^*, \ldots, (\Psi^{(1)})^*) \in \mathbb{S}^n \tag{30}$$

of $\mathbb{S}^n$ as a topological space. A finite signed Borel measure $\mathfrak{a}$ on $\mathbb{S}^n$ is, by definition, self-adjoint whenever $\mathfrak{a}^* = \mathfrak{a}$. For any $n \in \mathbb{N}$, the real Banach space of self-adjoint, finite, signed Borel measures on $\mathbb{S}^n$ endowed with the norm

$$\|\mathfrak{a}\|_{\mathcal{S}(\mathbb{S}^n)} \doteq |\mathfrak{a}|(\mathbb{S}^n) , \qquad n \in \mathbb{N} , \tag{31}$$

is denoted by $\mathcal{S}(\mathbb{S}^n)$.

2.4.2 Sequences of Self-Adjoint Measures on Interactions

Endowed with point-wise operations, $\mathcal{S}$ is the real Banach space of all sequences $\mathfrak{a} \equiv (\mathfrak{a}_n)_{n\in\mathbb{N}}$ of self-adjoint, finite signed Borel measures $\mathfrak{a}_n \in \mathcal{S}(\mathbb{S}^n)$, along with the norm

$$\|\mathfrak{a}\|_{\mathcal{S}} \doteq \sum_{n\in\mathbb{N}} n^2 \|\mathbf{F}\|_{1,\mathfrak{L}}^{n-1} \|\mathfrak{a}_n\|_{\mathcal{S}(\mathbb{S}^n)} , \qquad \mathfrak{a} \equiv (\mathfrak{a}_n)_{n\in\mathbb{N}} \in \mathcal{S} . \tag{32}$$

Recall that $\mathbf{F} : \mathfrak{L}^2 \to (0, 1]$ is any positive-valued symmetric function with maximum value $\mathbf{F}(x, x) = 1$ for all $x \in \mathfrak{L}$, satisfying Equations (5)-(6).

2.4.3 Long-Range Models

The separable Banach space of long-range models is defined by

$$\mathcal{M} \doteq \left\{\mathfrak{m} \in \mathcal{W}^{\mathbb{R}} \times \mathcal{S} : \|\mathfrak{m}\|_{\mathcal{M}} < \infty\right\} , \tag{33}$$

where the norm of $\mathcal{M}$ is defined from (22) and (32) by

$$\|\mathfrak{m}\|_{\mathcal{M}} \doteq \|\Phi\|_{\mathcal{W}} + \|\mathfrak{a}\|_{\mathcal{S}} , \qquad \mathfrak{m} \doteq (\Phi, \mathfrak{a}) \in \mathcal{M} . \tag{34}$$

The spaces $\mathcal{W}^{\mathbb{R}}$ and $\mathcal{S}$ are canonically seen as subspaces of $\mathcal{M}$, i.e.,

$$\mathcal{W}^{\mathbb{R}} \subseteq \mathcal{M} \qquad \text{and} \qquad \mathcal{S} \subseteq \mathcal{M} . \tag{35}$$

In particular, $\Phi \equiv (\Phi, 0) \in \mathcal{M}$ for $\Phi \in \mathcal{W}^{\mathbb{R}}$. Similar to (25), we define the subspaces

$$\mathcal{M}_\Lambda \doteq \mathcal{W}^{\mathbb{R}} \times \mathcal{S}_\Lambda \subseteq \mathcal{M} , \qquad \Lambda \in \mathcal{P}_f, \tag{36}$$

where, for any $\Lambda \in \mathcal{P}_f$,

$$\mathcal{S}_\Lambda \doteq \{(\mathfrak{a}_n)_{n\in\mathbb{N}} \in \mathcal{S} : \forall n \in \mathbb{N},\ |\mathfrak{a}_n|(\mathbb{S}^n) = |\mathfrak{a}_n|((\mathbb{S} \cap \mathcal{W}_\Lambda)^n)\} . \tag{37}$$

Note that

$$\mathcal{M}_0 \doteq \bigcup_{L\in\mathbb{N}} \mathcal{M}_{\Lambda_L}$$

is a dense subspace of $\mathcal{M}$.

Long-range models $\mathfrak{m} \doteq (\Phi, \mathfrak{a})$ are not necessarily translation-invariant, because of its short-range component Φ which may be non-translation-invariant, and we define

$$\mathcal{M}_1 \doteq \left(\mathcal{W}_1 \cap \mathcal{W}^{\mathbb{R}}\right) \times \mathcal{S} \subsetneq \mathcal{M} \tag{38}$$

to be the Banach space of all translation-invariant long-range models.

2.5 Local Energies

2.5.1 Local Energy Elements of Short-Range Interactions

The local energy elements of any complex interaction $\Phi \in \mathcal{W}$ are defined by

$$U_L^{\Phi} \doteq \sum_{\Lambda \subseteq \Lambda_L} \Phi_{\Lambda} \in \mathcal{U}_{\Lambda_L} \cap \mathcal{U}^+ \ , \qquad L \in \mathbb{N} \ . \tag{39}$$

Note that

$$\left\| U_L^{\Phi} \right\|_{\mathcal{U}} \leq |\Lambda_L| \, \|\mathbf{F}\|_{1,\mathfrak{L}} \, \|\Phi\|_{\mathcal{W}} \ , \qquad L \in \mathbb{N}, \ \Phi \in \mathcal{W} \ . \tag{40}$$

By [1, Proposition 3.2], for any $\vec{\ell} \in \mathbb{N}^d$, $\vec{\ell}$-periodic state $\rho \in E_{\vec{\ell}}$ (17) and translation-invariant complex interaction $\Phi \in \mathcal{W}_1$,

$$\lim_{L \to \infty} \frac{\rho\left(U_L^{\Phi}\right)}{|\Lambda_L|} = \rho(\mathfrak{e}_{\Phi,\vec{\ell}})$$

with $\mathfrak{e}_{\Phi,\vec{\ell}}$ being the even observable defined by (27).

2.5.2 Local Energy Elements of Long-Range Models

The local Hamiltonians of any model $\mathfrak{m} \doteq (\Phi, \mathfrak{a}) \in \mathcal{M}$ are the (well-defined) self-adjoint elements

$$U_L^{\mathfrak{m}} \doteq U_L^{\Phi} + \sum_{n \in \mathbb{N}} \frac{1}{|\Lambda_L|^{n-1}} \int_{\mathbb{S}^n} U_L^{\Psi^{(1)}} \cdots U_L^{\Psi^{(n)}} \mathfrak{a}_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right) \ , \qquad L \in \mathbb{N} \ . \tag{41}$$

Note that $U_L^{(\Phi,0)} \doteq U_L^{\Phi}$ for $\Phi \in \mathcal{W}^{\mathbb{R}}$ (cf. (35)) and straightforward estimates yield the bound

$$\|U_L^{\mathfrak{m}}\|_{\mathcal{U}} \leq |\Lambda_L| \, \|\mathbf{F}\|_{1,\mathfrak{L}} \, \|\mathfrak{m}\|_{\mathcal{M}} \ , \qquad L \in \mathbb{N} \ , \tag{42}$$

by Equations (31)-(34) and (40).

2.6 Dynamical Problem Associated with Long-Range Interactions

2.6.1 Local Dynamics on the CAR Algebra

<u>Local derivations:</u> The local (symmetric) derivations $\{\delta_L^{\mathfrak{m}}\}_{L \in \mathbb{N}} \subsetneq \mathcal{B}(\mathcal{U})$ associated with any model $\mathfrak{m} \in \mathcal{M}$ are defined by

$$\delta_L^{\mathfrak{m}}(A) \doteq i \left[U_L^{\mathfrak{m}}, A\right] \doteq i \left(U_L^{\mathfrak{m}} A - A U_L^{\mathfrak{m}}\right) \ , \qquad A \in \mathcal{U}, \ L \in \mathbb{N} \ . \tag{43}$$

Note that $\delta_L^{(\Phi,0)} \equiv \delta_L^{\Phi}$ for $L \in \mathbb{N}$ and $\Phi \in \mathcal{W}^{\mathbb{R}}$ (cf. (35)).

<u>Local non-autonomous dynamics:</u> Let $\mathfrak{m} \in C(\mathbb{R}; \mathcal{M})$ be a continuous function from $\mathbb{R}$ to the Banach space $\mathcal{M}$. Then, for any $L \in \mathbb{N}$, there is a unique (fundamental) solution $(\tau_{t,s}^{(L,\mathfrak{m})})_{s,t \in \mathbb{R}}$ in $\mathcal{B}(\mathcal{U})$ to the (finite-volume) non-autonomous evolution equations

$$\forall s,t \in \mathbb{R}: \qquad \partial_s \tau_{t,s}^{(L,\mathfrak{m})} = -\delta_L^{\mathfrak{m}(s)} \circ \tau_{t,s}^{(L,\mathfrak{m})} \ , \qquad \tau_{t,t}^{(L,\mathfrak{m})} = \mathbf{1}_{\mathcal{U}} \ , \tag{44}$$

and

$$\forall s,t \in \mathbb{R}: \qquad \partial_t \tau_{t,s}^{(L,\mathfrak{m})} = \tau_{t,s}^{(L,\mathfrak{m})} \circ \delta_L^{\mathfrak{m}(t)} \ , \qquad \tau_{s,s}^{(L,\mathfrak{m})} = \mathbf{1}_{\mathcal{U}} \ . \tag{45}$$

In these two equations, $\mathbf{1}_{\mathcal{U}}$ refers to the identity mapping of $\mathcal{U}$. Note also that, for any $L \in \mathbb{N}$ and $\mathfrak{m} \in C(\mathbb{R}; \mathcal{M})$, $(\tau_{t,s}^{(L,\mathfrak{m})})_{s,t \in \mathbb{R}}$ is a continuous two-parameter family of $*$-automorphisms of $\mathcal{U}$ that satisfies the (reverse) cocycle property

$$\forall s,r,t \in \mathbb{R}: \qquad \tau_{t,s}^{(L,\mathfrak{m})} = \tau_{r,s}^{(L,\mathfrak{m})} \tau_{t,r}^{(L,\mathfrak{m})} \ .$$

Again, $\tau_{t,s}^{(L,(\Psi,0))} \equiv \tau_{t,s}^{(L,\Psi)}$ for $s,t \in \mathbb{R}$, $L \in \mathbb{N}$ and $\Psi \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$ (cf. (35)).

2.6.2 Dynamical Problem at Infinite Volume

For any $\Psi \in C(\mathbb{R};\mathcal{W}^{\mathbb{R}})$, $(\tau_{t,s}^{(L,\Psi)})_{s,t\in\mathbb{R}}$, $L \in \mathbb{N}$, converges strongly, uniformly for s,t on compacta, to a strongly continuous two-parameter family $(\tau_{t,s}^{\Psi})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{U}$, as stated in [1, Proposition 3.7]. The main aim of this paper is to make sense of the thermodynamic limit

$$\lim_{L\to\infty} \tau_{t,s}^{(L,\mathfrak{m})}(A) \ , \qquad s,t\in\mathbb{R},\ A\in\mathcal{U} \ ,$$

for any $\mathfrak{m} \in C(\mathbb{R};\mathcal{M}_\Lambda)$, where $\mathcal{M}_\Lambda$ is the space of long-range models defined by (36)-(37) for a fixed $\Lambda \in \mathcal{P}_f$. This cannot be done within the C^*-algebra $\mathcal{U}$, as explained in [1, Section 4.3], but by using appropriate representations of $\mathcal{U}$.

3 State-Dependent Interactions and Dynamics

The long-range dynamics takes place in the space $C(E;\mathcal{U})$ of $\mathcal{U}$-valued weak*-continuous functions on the metrizable compact space E, a quantum C^*-algebra of continuous functions on states. As explained in [1, Section 6.6], it is used to construct the infinite-volume limit of non-autonomous dynamics of time-dependent long-range models within a cyclic representation associated with some periodic state. We show that, generically at infinite-volume, the long-range dynamics is equivalent to an intricate combination of classical and short-range quantum dynamics. Similar to [9], the existence of both dynamics will be a (non-trivial) consequence of the well-posedness of a self-consistency problem, see [1, Theorem 6.5].

3.1 The Quantum C^*-Algebra of Continuous Functions on States

3.1.1 Quantum Algebra

Endowed with the point-wise $*$-algebra operations inherited from $\mathcal{U}$, $C(E;\mathcal{U})$ is the unital C^*-algebra denoted by

$$\mathfrak{U} \equiv \mathcal{U}_{\mathfrak{L}} \doteq \left(C\left(E;\mathcal{U}\right), +, \cdot_{\mathbb{C}}, \times, ^*, \|\cdot\|_{\mathfrak{U}}\right) \ . \tag{46}$$

The unique C^*-norm $\|\cdot\|_{\mathfrak{U}}$ is the supremum norm:

$$\|f\|_{\mathfrak{U}} \doteq \max_{\rho\in E} \|f(\rho)\|_{\mathcal{U}} \ , \qquad f\in\mathfrak{U} \ . \tag{47}$$

The (real) Banach subspace of all $\mathcal{U}^{\mathbb{R}}$-valued functions is denoted by $\mathfrak{U}^{\mathbb{R}} \subsetneq \mathfrak{U}$. We identify the primordial C^*-algebra $\mathcal{U}$ with the subalgebra of constant functions of $\mathfrak{U}$, i.e., $\mathcal{U} \subseteq \mathfrak{U}$.

3.1.2 Local Elements

Similar to (7), we define the $*$-subalgebras

$$\mathfrak{U}_\Lambda \doteq \{f\in\mathfrak{U} : f(E) \subseteq \mathcal{U}_\Lambda\} \ , \qquad \Lambda\in\mathcal{P}_f \ , \tag{48}$$

and

$$\mathfrak{U}_0 \doteq \bigcup_{\Lambda\in\mathcal{P}_f} \mathfrak{U}_\Lambda \subseteq \mathfrak{U} \ , \tag{49}$$

which is a dense $*$-subalgebra of $\mathfrak{U}$. See [1, Section 5.3].

3.1.3 Even Elements

The $*$-automorphism σ of $\mathcal{U}$ uniquely defined by (8) naturally induces a $*$-automorphism Ξ of $\mathfrak{U}$ defined by

$$[\Xi(f)](\rho) \doteq \sigma(f(\rho)) , \qquad \rho \in E,\ f \in \mathfrak{U} . \tag{50}$$

The set

$$\mathfrak{U}^{+} \doteq \{f \in \mathfrak{U} : f = \Xi(f)\} = \left\{f \in \mathfrak{U} : f(E) \subseteq \mathcal{U}^{+}\right\} \subseteq \mathfrak{U} \tag{51}$$

of all even $\mathcal{U}$-valued continuous functions is a C^*-subalgebra of $\mathfrak{U}$. Compare with (9).

3.1.4 Translation Automorphisms

The $*$-automorphisms α_x, $x \in \mathbb{Z}^d$, of $\mathcal{U}$ uniquely defined by (10) naturally induce a group homomorphism $x \mapsto \mathrm{A}_x$ from $(\mathbb{Z}^d, +)$ to the group of $*$-automorphisms of $\mathfrak{U}$, defined by

$$[\mathrm{A}_x(f)](\rho) \doteq \alpha_x(f(\rho)) , \qquad \rho \in E,\ f \in \mathfrak{U},\ x \in \mathbb{Z}^d . \tag{52}$$

These $*$-automorphisms represent the translation group in $\mathfrak{U}$.

3.2 The Classical C^*-Algebra of Continuous Functions on States

3.2.1 Classical Algebra

Endowed with point-wise vector space operations and complex conjugation, $C(E;\mathbb{C})$ is the unital commutative C^*-algebra denoted by

$$\mathfrak{C} \doteq \left(C(E;\mathbb{C}), +, \cdot_{\mathbb{C}}, \times, \overline{(\cdot)}, \|\cdot\|_{\mathfrak{C}}\right) , \tag{53}$$

where the corresponding (unique) C^*-norm is

$$\|f\|_{\mathfrak{C}} \doteq \max_{\rho \in E} |f(\rho)| , \qquad f \in \mathfrak{C} . \tag{54}$$

The (real) Banach subspace of all real-valued functions of $\mathfrak{C}$ is denoted by $\mathfrak{C}^{\mathbb{R}} \subsetneq \mathfrak{C}$. The C^*-algebra $\mathfrak{C}$ is separable, E being metrizable and compact.

The classical dynamics takes place in $\mathfrak{C}$. This unital commutative C^*-algebra is identified with the subalgebra of functions of $\mathfrak{U}$ whose values are multiples of the unit $\mathfrak{1} \in \mathcal{U}$. In other words, we have the canonical inclusion $\mathfrak{C} \subseteq \mathfrak{U}$.

3.2.2 Poisson Bracket for Polynomial Functions

Elements of the (separable and unital) C^*-algebra $\mathcal{U}$ naturally define continuous affine functions $\hat{A} \in \mathfrak{C}$ by

$$\hat{A}(\rho) \doteq \rho(A) , \qquad \rho \in E,\ A \in \mathcal{U} . \tag{55}$$

This mapping $A \mapsto \hat{A}$ is a linear isometry from $\mathcal{U}^{\mathbb{R}}$ to $\mathfrak{C}^{\mathbb{R}}$. We denote by

$$\mathfrak{C}_{\mathcal{U}_0} \doteq \mathbb{C}[\{\hat{A} : A \in \mathcal{U}_0\}] \subseteq \mathfrak{C} \tag{56}$$

the subalgebras of polynomials in the elements of $\{\hat{A} : A \in \mathcal{U}_0\}$, with complex coefficients. Note that $\mathfrak{C}_{\mathcal{U}_0}$ is dense in $\mathfrak{C}$, i.e., $\mathfrak{C} = \overline{\mathfrak{C}_{\mathcal{U}_0}}$ (the Stone-Weierstrass theorem).

In [1, Section 5.2] we define a Poisson bracket

$$\{\cdot,\cdot\} : \mathfrak{C}_{\mathcal{U}_0} \times \mathfrak{C}_{\mathcal{U}_0} \to \mathfrak{C} ,$$

i.e., a skew-symmetric biderivation satisfying the Jacobi identity. This Poisson bracket can be extended to any continuously differentiable real-valued functions on the state space E. For more details, see [1, Section 5.2] as well as [9, Section 3]. In fact, it is not really used in this paper and is only shortly mentioned in order to establish a connection between the results of this paper and those of [1].

3.3 Banach Spaces of State-Dependent Short-Range Interactions

3.3.1 Complex State-Dependent Interactions

As is done in Section 2.3, a state-dependent (complex) interaction is defined to be a mapping $\mathbf{\Phi} : \mathcal{P}_f \to \mathfrak{U}^+$ such that $\Phi_\Lambda \in \mathfrak{U}_\Lambda$ for any $\Lambda \in \mathcal{P}_f$. See Equations (48) and (51). The set $\mathfrak{V}$ of all state-dependent interactions is naturally endowed with the structure of a complex vector space and with the involution

$$\mathbf{\Phi} \mapsto \mathbf{\Phi}^* \doteq (\mathbf{\Phi}_\Lambda^*)_{\Lambda \in \mathcal{P}_f} \ . \tag{57}$$

Self-adjoint state-dependent interactions $\mathbf{\Phi}$ are, by definition, those satisfying $\mathbf{\Phi} = \mathbf{\Phi}^*$.

3.3.2 Short-Range State-Dependent Interactions

Similar to the Banach space $\mathcal{W}$ of short-range interactions, we define a Banach space

$$\mathfrak{W} \doteq \{\mathbf{\Phi} \in \mathfrak{V} : \|\mathbf{\Phi}\|_{\mathfrak{W}} < \infty\}$$

of state-dependent short-range interactions by using the norm

$$\|\mathbf{\Phi}\|_{\mathfrak{W}} \doteq \sup_{x,y \in \mathfrak{L}} \sum_{\Lambda \in \mathcal{P}_f,\, \Lambda \supseteq \{x,y\}} \frac{\|\mathbf{\Phi}_\Lambda\|_{\mathfrak{U}}}{\mathbf{F}(x,y)} \ .$$

Compare with (22). The (real) Banach subspace of all self-adjoint state-dependent interactions is denoted by $\mathfrak{W}^{\mathbb{R}} \subsetneq \mathfrak{W}$.

3.3.3 Approximating Interactions of Long-Range Models

In order to simplify the notation, for any $\mathbf{\Psi} \in C(\mathbb{R}; \mathfrak{W})$ and $\rho \in E$, $\mathbf{\Psi}(\rho) \in C(\mathbb{R}; \mathcal{W})$ stands for the time-dependent interaction defined by

$$\mathbf{\Psi}(\rho)(t) \doteq \mathbf{\Psi}(t; \rho) \ , \qquad \rho \in E,\ t \in \mathbb{R} \ . \tag{58}$$

For any $\Psi \in \mathcal{W}$ and $\rho \in E$, we define

$$\lfloor \rho; \Psi \rfloor_{\vec{\ell}} \doteq \Psi \tag{59}$$

and, for any $n \geq 2$ and all interactions $\Psi^{(1)}, \ldots, \Psi^{(n)} \in \mathcal{W}_1$,

$$\lfloor \rho; \Psi^{(1)}, \ldots, \Psi^{(n)} \rfloor_{\vec{\ell}} \doteq \sum_{m=1}^{n} \Psi^{(m)} \prod_{j \in \{1,\ldots,n\}, j \neq m} \rho(\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}) \in \mathcal{W}_1 \ , \qquad \rho \in E \ . \tag{60}$$

The pertinence of these objects is explained in [1, Section 6.4]. They yield an approximating (state-dependent, short-range) interaction, which is ubiquitous in the study of the infinite-volume dynamics of lattice-fermion (or quantum-spin) systems with long-range interactions.

Definition 3.1 (Non-autonomous approximating interactions)
For $\vec{\ell} \in \mathbb{N}^d$ and any continuous functions $\mathfrak{m} = (\Phi(t), \mathfrak{a}(t))_{t \in \mathbb{R}} \in C(\mathbb{R}; \mathcal{M})$, $\xi \in C(\mathbb{R}; E)$, we define the mapping $\Phi^{(\mathfrak{m},\xi)}$ from $\mathbb{R}$ to $\mathcal{W}^{\mathbb{R}}$ by

$$\Phi^{(\mathfrak{m},\xi)}(t) \doteq \Phi(t) + \sum_{n \in \mathbb{N}} \int_{\mathbb{S}^n} \lfloor \xi(t); \Psi^{(1)}, \ldots, \Psi^{(n)} \rfloor_{\vec{\ell}}\ \mathfrak{a}(t)_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right) \ , \qquad t \in \mathbb{R} \ .$$

If $\boldsymbol{\xi} \in C(\mathbb{R}; \mathrm{Aut}(E))$ then a mapping $\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}$ from $\mathbb{R}$ to $\mathfrak{W}^{\mathbb{R}}$ is defined, for any $\rho \in E$ and $t \in \mathbb{R}$, by

$$\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}(t; \rho) \doteq \Phi(t) + \sum_{n \in \mathbb{N}} \int_{\mathbb{S}^n} \lfloor \boldsymbol{\xi}(t; \rho); \Psi^{(1)}, \ldots, \Psi^{(n)} \rfloor_{\vec{\ell}}\ \mathfrak{a}(t)_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right) \ .$$

Recall that $\mathrm{Aut}(E) \subsetneq C(E; E)$ is the subspace of all automorphisms of E, i.e., weak*-continuous state-valued functions over E with weak*-continuous inverse. The topology in $\mathrm{Aut}(E)$ is the one of $C(E; E)$, that is, the topology of uniform convergence (cf. (13)). Recall also the identifications (14). By [1, Lemma 6.4], note finally that, for any $\mathfrak{m} \in C(\mathbb{R}; \mathcal{M})$ and $\xi \in C(\mathbb{R}; E)$,

$$\left\| \Phi^{(\mathfrak{m},\xi)}(t) \right\|_{\mathcal{W}} \leq \| \mathfrak{m}(t) \|_{\mathcal{M}} , \qquad t \in \mathbb{R} . \tag{61}$$

3.4 State-Dependent Quantum Dynamics

Limit derivations: Using that $\mathfrak{U}_0 = \mathrm{span}\{\mathfrak{C}\mathcal{U}_0\}$, we define the symmetric derivations $\boldsymbol{\delta}^{\boldsymbol{\Phi}}$ associated with any $\boldsymbol{\Phi} \in \mathfrak{W}$ on the dense subset $\mathfrak{U}_0$ (49) by

$$\left[\boldsymbol{\delta}^{\boldsymbol{\Phi}}(fA) \right](\rho) \doteq f(\rho)\, \delta^{\boldsymbol{\Phi}(\rho)}(A) , \qquad \rho \in E,\ f \in \mathfrak{C},\ A \in \mathcal{U}_0 .$$

The right-hand side of the above equation defines an element of $\mathfrak{U}$, by [1, Corollary 3.5]. If $\Phi \in \mathfrak{W}^{\mathbb{R}}$ then the symmetric derivation $\boldsymbol{\delta}^{\boldsymbol{\Phi}}$ is (norm-) closable. See [1, discussions after Definition 6.1].

State-dependent dynamics: Any $\boldsymbol{\Psi} \in C(\mathbb{R}; \mathfrak{W}^{\mathbb{R}})$ determines a two-parameter family $\mathfrak{T}^{\boldsymbol{\Psi}} \equiv (\mathfrak{T}^{\boldsymbol{\Psi}}_{t,s})_{s,t \in \mathbb{R}}$ of $*$-automorphisms of $\mathfrak{U}$ defined by

$$\left[\mathfrak{T}^{\boldsymbol{\Psi}}_{t,s}(f) \right](\rho) \doteq \tau^{\boldsymbol{\Psi}(\rho)}_{t,s}(f(\rho)) , \qquad \rho \in E,\ f \in \mathfrak{U},\ s,t \in \mathbb{R} , \tag{62}$$

where the two-parameter family $(\tau^{\Psi}_{t,s})_{s,t \in \mathbb{R}}$ of $*$-automorphisms of $\mathcal{U}$ is the strong limit of $(\tau^{(L,\Psi)}_{t,s})_{s,t \in \mathbb{R}}$, $L \in \mathbb{N}$, which is defined by (44)-(45). See [1, Proposition 3.7]. The right-hand side of (62) defines an element of $\mathfrak{U}$. In fact, by [1, Proposition 6.4], for any $\boldsymbol{\Psi} \in C(\mathbb{R}; \mathfrak{W}^{\mathbb{R}})$, $\mathfrak{T}^{\boldsymbol{\Psi}} \equiv (\mathfrak{T}^{\boldsymbol{\Psi}}_{t,s})_{s,t \in \mathbb{R}}$ is a strongly continuous two-parameter family of $*$-automorphisms of $\mathfrak{U}$, which is the unique solution in $\mathcal{B}(\mathfrak{U})$ to the non-autonomous evolution equation

$$\forall s,t \in \mathbb{R}: \qquad \partial_t \mathfrak{T}^{\boldsymbol{\Psi}}_{t,s} = \mathfrak{T}^{\boldsymbol{\Psi}}_{t,s} \circ \boldsymbol{\delta}^{\boldsymbol{\Psi}(t)} , \qquad \mathfrak{T}^{\boldsymbol{\Psi}}_{s,s} = \mathbf{1}_{\mathfrak{U}} , \tag{63}$$

in the strong sense on the dense subspace $\mathfrak{U}_0 \subseteq \mathfrak{U}$, $\mathbf{1}_{\mathfrak{U}}$ being the identity mapping of $\mathfrak{U}$. It satisfies, in particular, the reverse cocycle property:

$$\forall s,r,t \in \mathbb{R}: \qquad \mathfrak{T}^{\boldsymbol{\Psi}}_{t,s} = \mathfrak{T}^{\boldsymbol{\Psi}}_{r,s} \mathfrak{T}^{\boldsymbol{\Psi}}_{t,r} . \tag{64}$$

4 Long-Range Dynamics

4.1 Classical Part of Long-Range Dynamics

We show that, generically, long-range (or mean-field) dynamics are equivalent to intricate combinations of classical *and* short-range quantum dynamics. The existence of both dynamics is a (non-trivial) consequence of the well-posedness of the self-consistency problem of [1, Theorem 6.5]:

Theorem 4.1 (Self-consistency equations)
Fix $\Lambda \in \mathcal{P}_f$ and $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda)$. There is a unique $\boldsymbol{\varpi}^{\mathfrak{m}} \in C(\mathbb{R}^2; \mathrm{Aut}(E))$ such that

$$\boldsymbol{\varpi}^{\mathfrak{m}}(s,t;\rho) = \rho \circ \tau^{\boldsymbol{\Phi}^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(\alpha,\cdot))}(\rho)}_{t,s} \big|_{\alpha=s} , \qquad s,t \in \mathbb{R} ,$$

with the strongly continuous two-parameter family $(\tau^{\Psi}_{t,s})_{s,t \in \mathbb{R}}$ being, at fixed $s,t \in \mathbb{R}$, the strong limit of the local dynamics $(\tau^{(L,\Psi)}_{t,s})_{s,t \in \mathbb{R}}$ defined by (44)-(45) for any $\Psi \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$. See [1, Proposition 3.7]. Note that, above, we use the notation (58).

The continuous family $\boldsymbol{\varpi}^{\mathfrak{m}}$ of Theorem 4.1 yields a family $(V_{t,s}^{\mathfrak{m}})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{C} \doteq C(E;\mathbb{C})$ defined by

$$V_{t,s}^{\mathfrak{m}}(f) \doteq f \circ \boldsymbol{\varpi}^{\mathfrak{m}}(s,t) \ , \qquad f \in \mathfrak{C},\ s,t \in \mathbb{R} \ . \tag{65}$$

It is a strongly continuous two-parameter family defining a classical dynamics on the classical C^*-algebra $\mathfrak{C}$ of continuous complex-valued functions on states, defined by (53)-(54).

This classical dynamics is a Feller evolution, as explained in [1, Section 6.5]. Additionally, the classical flow $\boldsymbol{\varpi}^{\mathfrak{m}}(s,t)$, $s,t\in\mathbb{R}$, conserves the even-state space E^+. Thus, $V_{t,s}^{\mathfrak{m}}$ can be seen as either a mapping from $C(E^+;\mathbb{C})$ to itself or from $C(E\backslash E^+;\mathbb{C})$ to itself:

$$V_{t,s}^{\mathfrak{m}}(f|_{E^+}) \doteq \left(V_{t,s}^{\mathfrak{m}}f\right)|_{E^+} \ , \qquad V_{t,s}^{\mathfrak{m}}\left(f|_{E\backslash E^+}\right) \doteq \left(V_{t,s}^{\mathfrak{m}}f\right)|_{E\backslash E^+} \ , \qquad f \in \mathfrak{C},\ s,t \in \mathbb{R} \ .$$

Recall that E^+ is the (physically relevant) weak*-compact convex set of even states defined by (15).

If $\mathbf{F}$ decays sufficiently fast, as $|x-y| \to \infty$, then, using the local classical energy functions [1, Definition 6.8] associated with $\mathfrak{m} \in \mathcal{M}$, that is, the functions

$$\mathrm{h}_L^{\mathfrak{m}} \doteq \widehat{U_L^{\Phi}} + \sum_{n\in\mathbb{N}} \frac{1}{|\Lambda_L|^{n-1}} \int_{\mathbb{S}^n} \widehat{U_L^{\Psi^{(1)}}} \cdots \widehat{U_L^{\Psi^{(n)}}}\ \mathfrak{a}(t)_n \left(\mathrm{d}\Psi^{(1)},\ldots,\mathrm{d}\Psi^{(n)}\right) \ , \qquad L \in \mathbb{N} \ ,$$

(see (55)), we prove in [1, Theorem 6.10] that, for any $s,t\in\mathbb{R}$, $f \in \mathfrak{C}_{\mathcal{U}_0}$ and $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda \cap \mathcal{M}_1)$,

$$\partial_t V_{t,s}^{\mathfrak{m}}(f)|_{E_{\mathrm{p}}} = \lim_{L\to\infty} V_{t,s}^{\mathfrak{m}}\left(\{\mathrm{h}_L^{\mathfrak{m}(t)}, f\}\right)|_{E_{\mathrm{p}}}$$

while, for any $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda)$,

$$\partial_s V_{t,s}^{\mathfrak{m}}(f)|_{E_{\mathrm{p}}} = -\lim_{L\to\infty} \{\mathrm{h}_L^{\mathfrak{m}(s)}, V_{t,s}^{\mathfrak{m}}(f)\}|_{E_{\mathrm{p}}} \ ,$$

where $\{\cdot,\cdot\}$ is the Poisson bracket referred to in Section 3.2.2. All limits have to be understood in the point-wise sense on the metrizable weak*-dense (in E^+) convex set E_{p} of periodic states defined by (18). These non-trivial statements are proven from Lieb-Robinson bounds for multi-commutators derived in [13]. In the autonomous situation, we obtain the usual (autonomous) dynamics of classical mechanics written in terms of Poisson brackets (see, e.g., [16, Proposition 10.2.3]), i.e., *Liouville's equation*. See [1, Corollary 6.11].

4.2 Quantum Part of Long-Range Dynamics

The classical part of the dynamics of lattice-fermion systems with long-range interactions, which is defined within the classical C^*-algebra $\mathfrak{C}$ defined by (53)-(54), is shown to result from the solution to the self-consistency equation of Theorem 4.1.

As soon as only the classical part of the long-range dynamics is concerned, there is no need to assume any additional property on initial states. However, for the quantum part, we need periodic states as initial states. Note that the set of all periodic states is still a weak*-dense subset of the physically relevant space E^+ of all even states, by [1, Proposition 2.3].

Fix from now on $\vec{\ell} \in \mathbb{N}^d$ and consider the metrizable and weak*-compact convex set $E_{\vec{\ell}}$ of $\vec{\ell}$-periodic states defined by (17), with $\mathcal{E}(E_{\vec{\ell}})$ being its (non-empty) set of extreme points. By the Choquet theorem [11, Theorem 10.18], for any state $\rho \in E_{\vec{\ell}}$,

$$\rho(A) = \int_{E_{\vec{\ell}}} \hat{\rho}(A)\,\mu_\rho(\mathrm{d}\hat{\rho}) = \int_{\mathcal{E}(E_{\vec{\ell}})} \hat{\rho}(A)\,\mu_\rho(\mathrm{d}\hat{\rho}) \ , \qquad A \in \mathcal{U} \ , \tag{66}$$

where $\mu_\rho \equiv \mu_\rho^{(\vec{\ell})}$ is the (unique) orthogonal probability measure of Theorem 5.1. Then, any state $\rho \in E_{\vec{\ell}} \subseteq \mathcal{U}^*$ naturally extends to a state on the quantum C^*-algebra $\mathfrak{U}$ of continuous $\mathcal{U}$-valued functions on states, also denoted by $\rho \in \mathfrak{U}^*$, via the definition

$$\rho(f) \doteq \int_{\mathcal{E}(E_{\vec{\ell}})} \hat{\rho}(f(\hat{\rho}))\, \mu_\rho(\mathrm{d}\hat{\rho}) \ , \qquad f \in \mathfrak{U} \, . \tag{67}$$

Cf. Definition 5.18 with $\mathcal{X} = \mathcal{U}$ and $F = E_{\vec{\ell}}$. Note that this extension equals $\mu_\rho \circ \Xi$, where μ_ρ is a seen as a state on $\mathfrak{C}$ and Ξ is the conditional expectation from $\mathfrak{U}$ to $\mathfrak{C} \subseteq \mathfrak{U}$ defined by

$$\Xi(f)(\rho) = \rho(f(\rho)) \ , \qquad \rho \in E \, .$$

Observe that this extension is $\vec{\ell}$-dependent: If $\vec{\ell}_1, \vec{\ell}_2 \in \mathbb{N}^d$ is such that $\mathbb{Z}^d_{\vec{\ell}_1} \subseteq \mathbb{Z}^d_{\vec{\ell}_2}$ (see (16)), then $E_{\vec{\ell}_2} \subseteq E_{\vec{\ell}_1}$, but an extreme state of $E_{\vec{\ell}_2}$ is not necessarily an extreme state of $E_{\vec{\ell}_1}$. Nevertheless, if $\rho \in E_{\vec{\ell}_2}$ then the unique probability measure $\mu_\rho^{(\vec{\ell}_1)}$ representing ρ in $E_{\vec{\ell}_1}$ is directly related to the unique one $\mu_\rho^{(\vec{\ell}_2)}$ representing ρ in $E_{\vec{\ell}_2}$. In fact, $\mu_\rho^{(\vec{\ell}_2)}$ is the pushforward of $\mu_\rho^{(\vec{\ell}_1)}$ through a natural space-averaging mapping $\mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}$ (see (87)). This is shown in Corollary 5.3. For any $\rho \in E_{\vec{\ell}_2}$ and any function $f \in \mathfrak{U}$ satisfying

$$\Xi(f) \circ \mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}(\hat{\rho}) = \Xi(f)(\hat{\rho}) \ , \qquad \hat{\rho} \in \mathcal{E}(E_{\vec{\ell}_1}) \, , \tag{68}$$

it follows that

$$\rho^{(\vec{\ell}_1)}(f) \doteq \int_{\mathcal{E}(E_{\vec{\ell}_1})} \hat{\rho}(f(\hat{\rho}))\, \mu_\rho^{(\vec{\ell}_1)}(\mathrm{d}\hat{\rho}) = \int_{\mathcal{E}(E_{\vec{\ell}_2})} \hat{\rho}(f(\hat{\rho}))\, \mu_\rho^{(\vec{\ell}_2)}(\mathrm{d}\hat{\rho}) \doteq \rho^{(\vec{\ell}_2)}(f) \ . \tag{69}$$

In general, $\rho^{(\vec{\ell}_1)}$ and $\rho^{(\vec{\ell}_2)}$ are two different extensions to $\mathfrak{U}$ of the state $\rho \in E_{\vec{\ell}_2} \subseteq E_{\vec{\ell}_1}$. In other words, the state of $\mathfrak{U}^*$ defined by (67) is $\vec{\ell}$-dependent. For instance, the set $E_{(1,\ldots,1)}$ of all translation-invariant states satisfies

$$E_{(1,\ldots,1)} \subseteq \bigcap_{\vec{\ell} \in \mathbb{N}^d} E_{\vec{\ell}}$$

and an arbitrary $\rho \in E_{(1,\ldots,1)}$ generally leads to a different extended state of $\mathfrak{U}^*$ for each $\vec{\ell} \in \mathbb{N}^d$.

A very nice characterization of cyclic representations of such an extension of a periodic state $\rho \in E_{\vec{\ell}} \subseteq \mathcal{U}^*$ is given in Theorem 5.19, the main assertions of which can be phrased as follows, for lattice fermion systems:

Proposition 4.2 (Cyclic representations of periodic states)
Fix $\vec{\ell} \in \mathbb{N}^d$ and $\rho \in E_{\vec{\ell}}$, seen as a state of either $\mathcal{U}^*$ or $\mathfrak{U}^*$.
(i) *Let $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ be any cyclic representation of $\rho \in \mathcal{U}^*$. Then, there exists a unique representation Π_ρ of $\mathfrak{U}$ on $\mathcal{H}_\rho$ such that $\Pi_\rho|_{\mathcal{U}} = \pi_\rho$ and $(\mathcal{H}_\rho, \Pi_\rho, \Omega_\rho)$ is a cyclic representation of $\rho \in \mathfrak{U}^*$.*
(ii) *Conversely, let $(\mathcal{H}_\rho, \Pi_\rho, \Omega_\rho)$ be any cyclic representation of $\rho \in \mathfrak{U}^*$. Then, $(\mathcal{H}_\rho, \Pi_\rho|_{\mathcal{U}}, \Omega_\rho)$ is a cyclic representation of $\rho \in \mathcal{U}^*$,*

$$[\Pi_\rho(\mathfrak{U})]'' = [\Pi_\rho(\mathcal{U})]'' \qquad \textit{and} \qquad [\Pi_\rho(\mathfrak{C})]'' \subseteq [\Pi_\rho(\mathcal{U})]' \cap [\Pi_\rho(\mathcal{U})]'' \, .$$

Proof. Apply Theorem 5.19 for $\mathcal{X} = \mathcal{U}$, $F = E$ and $\mu = \mu_\rho$, observing that μ_ρ is the unique Choquet measure, relative to the simplex $E_{\vec{\ell}}$, representing ρ, i.e., ρ is the barycenter of μ_ρ. As a measure on E, μ_ρ is an orthogonal measure. See Theorem 5.1. ∎

Note that the representation Π_ρ in Proposition 4.2 is of course $\vec{\ell}$-dependent. Recall now the following objects introduced above:

- For any $\Lambda \in \mathcal{P}_f$ and $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda)$, $\boldsymbol{\varpi}^{\mathfrak{m}} \in C(\mathbb{R}^2;\mathrm{Aut}(E))$ is the solution to the self-consistency equation of Theorem 4.1, proven in [1, Theorem 6.5].
- For any $\mathbf{\Psi} \in C(\mathbb{R};\mathfrak{W}^{\mathbb{R}})$, $(\mathfrak{T}^{\mathbf{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$ is the strongly continuous two-parameter family of $*$-automorphisms of $\mathfrak{U}$ defined by Equation (62).
- For any $\mathfrak{m} \in C(\mathbb{R};\mathcal{M})$ and each $\boldsymbol{\xi} \in C(\mathbb{R};\mathrm{Aut}(E))$, $\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}$ is the mapping from $\mathbb{R}$ to $\mathfrak{W}^{\mathbb{R}}$ of Definition 3.1. By (61), if $\mathfrak{m} \in C(\mathbb{R};\mathcal{M})$ and $\boldsymbol{\xi} \in C(\mathbb{R};\mathrm{Aut}(E))$ then $\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})} \in C(\mathbb{R};\mathfrak{W}^{\mathbb{R}})$.
- $\mathcal{M}_1$ is the Banach space of all translation-invariant long-range models defined by (38).

Using the orthogonality of the probability measures μ_ρ and the ergodicity of extreme states of $E_{\vec{\ell}}$, we obtain the existence of the quantum part of long-range dynamics, which is the main result of this paper:

Theorem 4.3 (Quantum part of long-range dynamics)
Fix $\Lambda \in \mathcal{P}_f$, $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda \cap \mathcal{M}_1)$, $\vec{\ell} \in \mathbb{N}^d$ and $\rho \in E_{\vec{\ell}}$. Let $(\mathcal{H}_\rho, \Pi_\rho, \Omega_\rho)$ be a cyclic representation of ρ, seen as a state (67) of $\mathfrak{U}^*$. Then, for any $s,t \in \mathbb{R}$ and $A \in \mathcal{U} \subseteq \mathfrak{U}$, in the σ-weak topology,

$$\lim_{L\to\infty} \pi_\rho\left(\tau^{(L,\mathfrak{m})}_{t,s}(A)\right) = \lim_{L\to\infty} \Pi_\rho\left(\tau^{(L,\mathfrak{m})}_{t,s}(A)\right) = \Pi_\rho\left(\mathfrak{T}^{\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(\alpha,\cdot))}}_{t,s}(A)\right)\Big|_{\alpha=s} \in \mathcal{B}(\mathcal{H}_\rho) \ . \tag{70}$$

If $\vec{\ell}_1, \vec{\ell}_2 \in \mathbb{N}^d$ is such that $\mathbb{Z}^d_{\vec{\ell}_1} \subseteq \mathbb{Z}^d_{\vec{\ell}_2}$ and $\rho \in E_{\vec{\ell}_2} \subseteq E_{\vec{\ell}_1}$ then the extension of the state of $\mathfrak{U}^*$ taken in Theorem 4.3, and thus the representation Π_ρ, depends on whether one sees ρ as an element of $E_{\vec{\ell}_2}$ or $E_{\vec{\ell}_1}$. However, the left-hand side of (70) does *not* depend on this choice, obviously. So the same is also true for the right-hand side of (70). This can directly be seen from (68)-(69) and the direct integral decomposition (72) of Π_ρ, by proving that the function

$$\rho \mapsto \mathfrak{T}^{\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(\alpha,\rho))}}_{t,s}(A)$$

of $\mathfrak{U}$ satisfies (68). See Lemma 5.2 (ii). This optional proof is not done here.

Because of Theorem 4.3, the state $\rho \circ \tau^{(L,\mathfrak{m})}_{t,s}$ converges in the weak*-topology to the restriction to $\mathcal{U}$ of the state

$$\rho_{t,s} \doteq \rho \circ \mathfrak{T}^{\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(\alpha,\cdot))}}_{t,s}|_{\alpha=s} \in \mathfrak{U}^* \ .$$

This restriction is thus, by definition, the ($\vec{\ell}$-periodic) state of the system at time $t \in \mathbb{R}$ when the state at initial time s is $\rho \in E_{\vec{\ell}}$. Since the exact time evolution of long-range order takes places in a C^*-algebra $\mathfrak{U}$ larger than $\mathcal{U}$, one expects that, in general, the mapping $\rho \mapsto \rho_{t,s}|_{\mathcal{U}}$ from $E_{\vec{\ell}}$ to itself does *not* preserve the entropy density of the initial state.

Before giving the proof of Theorem 4.3, we first explain its heuristics: Take $\Lambda \in \mathcal{P}_f$, $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda \cap \mathcal{M}_1)$, $\vec{\ell} \in \mathbb{N}^d$ and any extreme (or ergodic) state $\hat{\rho} \in \mathcal{E}(E_{\vec{\ell}}) \subseteq E_{\vec{\ell}}$. In this case, the (unique) probability measure $\mu_{\hat{\rho}}$ of Theorem 5.1 is the atomic one with the singleton $\{\hat{\rho}\}$ as its support. See (66). Then,

$$\Pi_{\hat{\rho}}(f) = \pi_{\hat{\rho}}(f(\hat{\rho})) \ , \qquad f \in \mathfrak{U} \ .$$

Compare with Equation (72) below. In particular, $\Pi_{\hat{\rho}}(\mathfrak{U}) = \pi_{\hat{\rho}}(\mathcal{U})$. Similarly, for any $s,t \in \mathbb{R}$,

$$\Pi_{\hat{\rho}}\left(\mathfrak{T}^{\mathbf{\Psi}}_{t,s}(f)\right) = \pi_{\hat{\rho}}\left(\tau^{\mathbf{\Psi}(\hat{\rho})}_{t,s}(f(\hat{\rho}))\right) \ , \qquad \mathbf{\Psi} \in C(\mathbb{R};\mathfrak{W}^{\mathbb{R}}) \ ,$$

using the notation (58). By Theorem 4.3, for any $s,t \in \mathbb{R}$ and $A \in \mathcal{U}$,

$$\pi_{\hat{\rho}}\left(\tau^{\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(\alpha,\cdot))}(\hat{\rho})}_{t,s}(A)\,|_{\alpha=s} - \tau^{(L,\mathfrak{m})}_{t,s}(A)\right)$$

σ-weak converges to 0, as $L \to \infty$. In fact, the derivation of this statement for extreme (ergodic) states is the starting point of the proof of Theorem 4.3 and corresponds to Theorem 5.8.

Now, if the periodic state is non-extreme, we use its Choquet decomposition (on extreme states), as stated in Theorem 5.1. To illustrate this, take, for instance, any state of the form

$$\rho_\lambda = (1-\lambda)\,\hat{\rho}_0 + \lambda\hat{\rho}_1 \ , \qquad \lambda \in (0,1) \ , \ \hat{\rho}_0 \neq \hat{\rho}_1 \in \mathcal{E}(E_{\vec{\ell}}) \ ,$$

i.e., ρ_λ is a non-trivial convex combination of two different extreme states of $E_{\vec{\ell}}$. Assume[7] that

$$\left(\mathcal{H}_{\rho_\lambda}, \pi_{\rho_\lambda}, \Omega_{\rho_\lambda}\right) = \left(\mathcal{H}_{\hat{\rho}_0} \oplus \mathcal{H}_{\hat{\rho}_1}, \pi_{\hat{\rho}_0} \oplus \pi_{\hat{\rho}_1}, \sqrt{1-\lambda}\Omega_{\hat{\rho}_0} \oplus \sqrt{\lambda}\Omega_{\hat{\rho}_1}\right)$$

is a cyclic representation of ρ_λ for $\lambda \in (0,1)$, where, as before, $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ is any cyclic representation of $\rho \in E_{\vec{\ell}}$. In this case, for any $f \in \mathfrak{U}$,

$$\Pi_{\rho_\lambda}(f) = \pi_{\hat{\rho}_0}\left(f\left(\hat{\rho}_0\right)\right) \oplus \pi_{\hat{\rho}_1}\left(f\left(\hat{\rho}_1\right)\right) \qquad \text{and} \qquad \Pi_{\rho_\lambda}(\mathfrak{U}) = \pi_{\hat{\rho}_0}(\mathcal{U}) \oplus \pi_{\hat{\rho}_1}(\mathcal{U}) \ .$$

Compare with Equation (72)[8] below. It follows that, for any $s,t \in \mathbb{R}$,

$$\Pi_{\rho_\lambda}\left(\mathfrak{T}_{t,s}^{\mathbf{\Psi}}(f)\right) = \pi_{\hat{\rho}_0}\left(\tau_{t,s}^{\mathbf{\Psi}(\hat{\rho}_0)}\left(f\left(\hat{\rho}_0\right)\right)\right) \oplus \pi_{\hat{\rho}_1}\left(\tau_{t,s}^{\mathbf{\Psi}(\hat{\rho}_1)}\left(f\left(\hat{\rho}_1\right)\right)\right) \ , \qquad \mathbf{\Psi} \in C(\mathbb{R};\mathfrak{W}^{\mathbb{R}}) \ .$$

By Theorem 4.3 applied to the extreme states $\hat{\rho}_0, \hat{\rho}_1 \in \mathcal{E}(E_{\vec{\ell}})$, for any $t \in \mathbb{R}$ and $A \in \mathcal{U}$,

$$\pi_{\hat{\rho}_0}\left(\tau_{t,s}^{\mathbf{\Phi}(\mathfrak{m},\varpi^{\mathfrak{m}}(\alpha,\cdot))(\hat{\rho}_0)}(A)\right) \oplus \pi_{\hat{\rho}_1}\left(\tau_{t,s}^{\mathbf{\Phi}(\mathfrak{m},\varpi^{\mathfrak{m}}(\alpha,\cdot))(\hat{\rho}_1)}(A)\right)|_{\alpha=s} - \pi_{\rho_\lambda}\left(\tau_{t,s}^{(L,\mathfrak{m})}(A)\right)$$

σ-weak converges to 0, as $L \to \infty$. The proof of Theorem 4.3 in the general case is essentially the same, except that one has to use the direct integral decomposition theory instead of finite direct sums. This theory for non-constant Hilbert spaces and von Neumann algebras as well as for GNS representations of a family of states is highly non-trivial, but it is a mature subject of mathematics. We review it in Section 6 and use this theory to prove below Theorem 4.3 in the general case:

Proof. Fix $\Lambda \in \mathcal{P}_f$, $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda \cap \mathcal{M}_1)$ and $\vec{\ell} \in \mathbb{N}^d$. For any $\rho \in E_{\vec{\ell}}$, there is a unique probability measure μ_ρ on $E_{\vec{\ell}}$ satisfying (66). By Theorem 5.1, this probability measure is orthogonal and supported in the Borel set $\mathcal{E}(E_{\vec{\ell}})$ (which is not a closed set). Therefore, by orthogonality of the measure μ_ρ (Theorem 5.1) and Theorem 5.19 (i) with $\mathcal{X} = \mathcal{U}$ and $F = E_{\vec{\ell}}$, a cyclic representation of $\rho \in \mathfrak{U}^*$ is given by $(\mathcal{H}_\rho^\oplus, \Pi_\rho^\oplus, \Omega_\rho^\oplus)$ with

$$\mathcal{H}_\rho^\oplus \doteq \int_{E_{\vec{\ell}}} \mathcal{H}_{\hat{\rho}}\mu_\rho\left(\mathrm{d}\hat{\rho}\right), \quad \Omega_\rho^\oplus \doteq \int_{E_{\vec{\ell}}} \Omega_{\hat{\rho}}\mu_\rho\left(\mathrm{d}\hat{\rho}\right) \tag{71}$$

and $\Pi_\rho^\oplus$ being the (direct integral) representation of $\mathfrak{U}$ on $\mathcal{H}_\rho^\oplus$ defined by

$$\Pi_\rho^\oplus(f) \doteq \int_{E_{\vec{\ell}}} \pi_{\hat{\rho}}(f(\hat{\rho}))\mu_\rho(\mathrm{d}\hat{\rho}) \ , \qquad f \in \mathfrak{U} \ , \tag{72}$$

where $(\mathcal{H}_{\hat{\rho}}, \pi_{\hat{\rho}}, \Omega_{\hat{\rho}})$ in all integrals are always the GNS representation of $\hat{\rho} \in E_{\vec{\ell}}$. Since μ_ρ is supported in the Borel set $\mathcal{E}(E_{\vec{\ell}})$ (Theorem 5.1), we can restrict all integrals of (71)-(72) to $\mathcal{E}(E_{\vec{\ell}})$. In particular, by (62),

$$\Pi_\rho^\oplus\left(\mathfrak{T}_{t,s}^{\mathbf{\Phi}(\mathfrak{m},\varpi^{\mathfrak{m}}(\alpha,\cdot))}(A)\,|_{\alpha=s} - \tau_{t,s}^{(L,\mathfrak{m})}(A)\right) = \int_{\mathcal{E}(E_{\vec{\ell}})} \pi_{\hat{\rho}}\left(\tau_{t,s}^{\mathbf{\Phi}(\mathfrak{m},\varpi^{\mathfrak{m}}(\alpha,\cdot))(\hat{\rho})}(A)\,|_{\alpha=s} - \tau_{t,s}^{(L,\mathfrak{m})}(A)\right)\mu\left(\mathrm{d}\hat{\rho}\right) \ .$$

[7] Note that the cyclicity of the representation is unclear for general (non-ergodic) states $\hat{\rho}_0, \hat{\rho}_1$. See Section 5.6.

[8] The representation $\Pi_{\rho_\lambda} \simeq \Pi_{\rho_\lambda}^\oplus$ is only defined up to some unitary equivalence. This detail is not important here.

Since $\tau_{t,s}^{\mathbf{\Phi}^{(\mathfrak{m},\varpi^{\mathfrak{m}}(s,\cdot))}(\hat{\rho})}$ and $\tau_{t,s}^{(L,\mathfrak{m})}$ are both $*$-automorphisms of $\mathcal{U}$, by Lebesgue's dominated convergence theorem together with Theorem 5.8 and (71), we arrive at the assertion of Theorem 4.3 for the representation $\Pi_{\rho}^{\oplus}$. By [15, Theorem 2.3.16], any cyclic representation $(\mathcal{H}_{\rho}, \Pi_{\rho}, \Omega_{\rho})$ of $\rho \in \mathfrak{U}^{*}$ is unitarily equivalent to $\Pi_{\rho}^{\oplus}$, similar to (144). This concludes the proof of Theorem 4.3. ■

Remark 4.4 (Subcentral decompositions of periodic states)
As explained after Corollary 5.14, $\pi_{\rho}^{\oplus}$ is a subcentral decomposition of the representation $\pi_{\rho_{\mu}}$, see Definition 6.26 (ii.1). By [11, Eq. (4.15)], the GNS representations of ergodic states are, in general, not factor representations. By Theorem 6.28 (ii), it follows that $\pi_{F}^{\oplus}$ is not, in general, the central decomposition of the representation $\pi_{\rho_{\mu}}$.

Note that, even if the finite-volume dynamics is autonomous, i.e., $\mathfrak{m} \in \mathcal{M}_{\Lambda} \cap \mathcal{M}_{1}$, the limit long-range dynamics is generally *non-autonomous*, as it can be seen from the next corollary:

Corollary 4.5 (From autonomous local dynamics to non-autonomous ones)
Fix $\Lambda \in \mathcal{P}_{f}$, $\mathfrak{m} \in \mathcal{M}_{\Lambda} \cap \mathcal{M}_{1}$, $\vec{\ell} \in \mathbb{N}^{d}$ and $\rho \in E_{\vec{\ell}}$ with cyclic representation $(\mathcal{H}_{\rho}, \Pi_{\rho}, \Omega_{\rho})$ seen as a state (67) of $\mathfrak{U}^{*}$. Then, for any $s,t \in \mathbb{R}$ and $A \in \mathcal{U} \subseteq \mathfrak{U}$, in the σ-weak topology,

$$\lim_{L\to\infty} \Pi_{\rho}\left(\tau_{t-s}^{(L,\mathfrak{m})}(A)\right) = \Pi_{\rho}\left(\mathfrak{T}_{t,s}^{\mathbf{\Phi}^{(\mathfrak{m},\varpi^{\mathfrak{m}}(\alpha,\cdot))}}(A)\right)\Big|_{\alpha=s} \in \mathcal{B}(\mathcal{H}_{\rho}) \ .$$

5 Technical Proofs

5.1 Cyclic Representations of Positive Functionals and Orthogonal Measures

The dual $\mathcal{U}^{*}$ of the C^{*}-algebra $\mathcal{U}$ is a locally convex space with respect to the weak*-topology, which is Hausdorff. Moreover, as $\mathcal{U}$ is separable, by [14, Theorem 3.16], the weak*-topology is metrizable on any weak*-compact subset of $\mathcal{U}^{*}$.

An important subset of $\mathcal{U}^{*}$ is the weak*-closed convex cone of positive functionals defined by

$$\mathcal{U}_{+}^{*} \doteq \bigcap_{A\in\mathcal{U}} \{\rho \in \mathcal{U}^{*} : \rho(A^{*}A) \geq 0\} \ . \tag{73}$$

Equivalently, $\rho \in \mathcal{U}_{+}^{*}$ iff $\|\rho\|_{\mathcal{U}^{*}} = \rho(\mathfrak{1})$. Additionally, any positive functional $\rho \in \mathcal{U}_{+}^{*}$ is hermitian, i.e., for all $A \in \mathcal{U}$, $\rho(A^{*}) = \overline{\rho(A)}$.

By the GNS construction, any positive functional $\rho \in \mathcal{U}_{+}^{*}$ has a cyclic representation $(\mathcal{H}_{\rho}, \pi_{\rho}, \Omega_{\rho})$: There exists a Hilbert space $\mathcal{H}_{\rho}$, a representation[9] π_{ρ} from $\mathcal{U}$ to the unital C^{*}–algebra $\mathcal{B}(\mathcal{H}_{\rho})$ of bounded operators on $\mathcal{H}_{\rho}$ and a cyclic vector[10] $\Omega_{\rho} \in \mathcal{H}_{\rho}$ for $\pi_{\rho}(\mathcal{U})$ such that

$$\rho(A) = \langle \Omega_{\rho}, \pi_{\rho}(A)\Omega_{\rho}\rangle_{\mathcal{H}_{\rho}} \ , \qquad A \in \mathcal{U} \ . \tag{74}$$

The representation π_{ρ} is faithful[11] if ρ is faithful, that is, if $\rho(A^{*}A) = 0$ implies $A = 0$. The triple $(\mathcal{H}_{\rho}, \pi_{\rho}, \Omega_{\rho})$ is unique up to unitary equivalence. See, e.g., [15, Theorem 2.3.16], which can trivially be extended to any positive functional. Two positive linear functionals $\rho_{1}, \rho_{2} \in \mathcal{U}_{+}^{*}$ are said to be *orthogonal* whenever

$$(\mathcal{H}_{\rho_{1}} \oplus \mathcal{H}_{\rho_{2}}, \pi_{\rho_{1}} \oplus \pi_{\rho_{2}}, \Omega_{\rho_{1}} \oplus \Omega_{\rho_{2}}) \tag{75}$$

is a cyclic representation for the positive functional $\rho_{1} + \rho_{2} \in \mathcal{U}_{+}^{*}$. As is usual, this orthogonality property is denoted by $\rho_{1} \perp \rho_{2}$. See, e.g., [15, Lemma 4.1.19 and Definition 4.1.20].

[9] I.e., it is a $*$–homomorphism from $\mathcal{U}$ to $\mathcal{B}(\mathcal{H}_{\rho})$. See, e.g., [15, Definition 2.3.2].
[10] I.e., $\mathcal{H}_{\rho}$ is the closure of (the linear span of) the set $\pi_{\rho}(\mathcal{U})\Omega_{\rho} \doteq \{\pi_{\rho}(A)\Omega_{\rho} : A \in \mathcal{U}\}$. See [15, p. 45].
[11] See for instance [15, Proposition 2.3.3].

The set of states on $\mathcal{U}$ is the subset of $\mathcal{U}_+^*$ defined by (11), that is,

$$E \doteq \{\rho \in \mathcal{U}^* : \rho \geq 0,\ \rho(\mathfrak{1}) = 1\} = \{\rho \in \mathcal{U}^* : \|\rho\|_{\mathcal{U}^*} = \rho(\mathfrak{1}) = 1\} .$$

Hence, E is a weak*-closed subset of the unit ball of $\mathcal{U}^*$ and, by the Banach-Alaoglu theorem, E is weak*-compact and metrizable. See, e.g., [14, Theorems 3.15-3.16].

Let Σ_E be the (Borel) σ-algebra generated by weak*-closed, or weak*-open, subsets of E. The set of all positive Radon measures on (E, Σ_E) is denoted by $\mathrm{M}(E)$. By weak*-compactness of E and the Riesz(-Markov) representation theorem, there is a one-to-one correspondence between positive functionals of $\mathfrak{C}^*$ and positive Radon measures on (E, Σ_E) and we write

$$\mu(f) = \int_E f(\rho)\, \mu(\mathrm{d}\rho) \ , \qquad f \in \mathfrak{C} \doteq C(E; \mathbb{C}) \ . \tag{76}$$

By metrizability of E, note additionally that any positive finite Borel measure on (E, Σ_E) is a positive Radon measure[12]. When $\mu(E) = 1$ we say that the positive Radon measure is normalized and μ is a probability measure. The subset of all probability measures on (E, Σ_E) is denoted by $\mathrm{M}_1(E)$.

For each $\mu \in \mathrm{M}(E)$, we define its restriction $\mu_{\mathfrak{B}} \in \mathrm{M}(E)$ to any Borel set $\mathfrak{B} \in \Sigma_E$ by

$$\mu_{\mathfrak{B}}(\mathfrak{B}_0) \doteq \mu(\mathfrak{B}_0 \cap \mathfrak{B}) \ , \qquad \mathfrak{B}_0 \in \Sigma_E \ . \tag{77}$$

Additionally, any convex and weak*-closed subset F of E defines a partial order $\prec_F$ in $\mathrm{M}(E)$: For any $\mu, \nu \in \mathrm{M}(E)$, $\mu \prec_F \nu$ if $\mu_F(f) \leq \nu_F(f)$ for all weak*-continuous convex functions $f : F \to \mathbb{R}$.

Each positive Radon measure $\mu \in \mathrm{M}(E)$, or equivalently a positive finite Borel measure, represents a positive functional $\rho_\mu \in \mathcal{U}_+^*$, which is defined by (76) for $f = \hat{A}$, that is,

$$\rho_\mu(A) \doteq \mu(\hat{A}) = \int_E \rho(A)\, \mu(\mathrm{d}\rho) \ , \qquad A \in \mathcal{U} \ . \tag{78}$$

Recall that $\rho \mapsto \hat{A}(\rho) \doteq \rho(A)$, as defined by (55), is an affine and weak*-continuous mapping from E to $\mathbb{C}$. The positive functional ρ_μ is called the *barycenter*[13] of $\mu \in \mathrm{M}(E)$. See, e.g., [17, Eq. (2.7) in Chapter I], [18, p. 1] or [11, Definition 10.15]. By [18, Propositions 1.1 and 1.2], barycenters are uniquely defined for all positive Radon measure in convex compact subsets of locally convex spaces and the mapping $\mu \mapsto \rho_\mu$ from $\mathrm{M}(E)$ to $\mathcal{U}_+^*$ is affine and weak*-continuous. Clearly, if $\mu \in \mathrm{M}_1(E)$ then $\rho_\mu \in E$. For any Borel subset $\mathfrak{B} \in \Sigma_E$, define by

$$\mathrm{M}^{(\rho)}(\mathfrak{B}) \doteq \left\{\mu \in \mathrm{M}(E) : \rho = \rho_\mu \quad \text{and} \quad \mu(\mathfrak{B}) = \mu(E)\right\} \ , \tag{79}$$

the set of all positive Radon measures representing ρ and with support within $\mathfrak{B} \subseteq E$.

As positive functional, the barycenter $\rho_\mu \in \mathcal{U}_+^*$ of any positive Radon measure $\mu \in \mathrm{M}(E)$ has a cyclic representation. A natural way to construct a triple $(\mathcal{H}_{\rho_\mu}, \pi_{\rho_\mu}, \Omega_{\rho_\mu})$ satisfying (74) for $\rho = \rho_\mu$ is to take the *direct integral* of cyclic representations $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ of $\rho \in E$ with respect to the measure $\mu \in \mathrm{M}(E)$:

$$\mathcal{H}_{\rho_\mu} \doteq \int_E \mathcal{H}_\rho \mu(\mathrm{d}\rho) \, , \quad \pi_{\rho_\mu} \doteq \int_E \pi_\rho \mu(\mathrm{d}\rho) \, , \quad \Omega_{\rho_\mu} \doteq \int_E \Omega_\rho \mu(\mathrm{d}\rho) \ . \tag{80}$$

See Section 5.6. However, Ω_{ρ_μ} is, in general, *not* a cyclic vector for $\pi_{\rho_\mu}(\mathcal{U})$, i.e., the subset $\pi_{\rho_\mu}(\mathcal{U})\Omega_{\rho_\mu}$ is generally not dense in $\mathcal{H}_{\rho_\mu}$. In particular, in this case, (80) is not spatially, or unitarily, equivalent to the cyclic representation of ρ_μ. A necessary and sufficient condition on the measure μ to get the

[12]In fact, for compact metrizable spaces, the Baire and Borel σ-algebras are the same.
[13]Other terminology existing in the literature: "x is represented by μ", "x is the resultant of μ".

cyclicity of Ω_{ρ_μ} is given by the orthogonality of the measure μ, in the following sense: The measure $\mu \in \mathrm{M}(E)$ is called *orthogonal* whenever $\rho_{\mu_{\mathfrak{B}}} \perp \rho_{\mu_{E\backslash\mathfrak{B}}}$ for *any* $\mathfrak{B} \in \Sigma_E$. See Definition 5.11 as well as [15, Definition 4.1.20] for more details. The set of all orthogonal measures on (E, Σ_E) is denoted by $\mathcal{O}(E)$.

For appropriate convex and weak*-closed subsets F of E, the orthogonality of a measure $\mu \in \mathrm{M}(E)$ can be directly related to its maximality with respect to the partial order $\prec_F$ in $\mathrm{M}^{(\rho_\mu)}(F)$. Examples of such a F are the subsets of periodic states discussed in the next section.

5.2 Ergodic Orthogonal Decomposition of Periodic States

As the state space E, for any $\vec{\ell} \in \mathbb{N}^d$, the set $E_{\vec{\ell}}$ of $\vec{\ell}$-periodic states defined by (17) is metrizable, weak*-compact and convex, with $\mathcal{E}(E_{\vec{\ell}})$ denoting its (non-empty) set of extreme points. See Equation (19). By metrizability of $E_{\vec{\ell}}$, $\mathcal{E}(E_{\vec{\ell}})$ is a Borel set[14]. Ergo, from the Choquet theorem [11, Theorem 10.18], each state $\rho \in E_{\vec{\ell}}$ is the barycenter of a probability measure μ_ρ which is supported on the set $\mathcal{E}(E_{\vec{\ell}})$ of extreme $\vec{\ell}$-periodic states:

Theorem 5.1 (Ergodic orthogonal decomposition of periodic states)
For any $\vec{\ell} \in \mathbb{N}^d$ and $\rho \in E_{\vec{\ell}}$, there is a unique probability measure $\mu_\rho \equiv \mu_\rho^{(\vec{\ell})} \in \mathrm{M}^{(\rho)}\left(E_{\vec{\ell}}\right)$ with $\mu_\rho\left(\mathcal{E}(E_{\vec{\ell}})\right) = 1$. Moreover, $\mu_\rho \in \mathcal{O}(E)$, i.e., it is an orthogonal measure on (E, Σ_E).

Proof. The first assertion corresponds to [11, Theorem 1.9]. In order to prove that $\mu_\rho \in \mathcal{O}(E)$, observe first that the set $\tilde{E}^+$ of all states on $\mathcal{U}^+$ (cf. (9)) can be identified with the even-state space E^+ defined by (15): A functional $\rho \in \mathcal{U}^*$ is even iff $\rho \circ \sigma = \rho$, with σ being the unique $*$-automorphism of the C^*-algebra $\mathcal{U}$ defined by (8). Any even functional $\rho \in \mathcal{U}^*$ can be seen as a functional $\tilde{\rho} = \rho|_{\mathcal{U}^+} \in (\mathcal{U}^+)^*$, by restriction. Conversely, any functional $\tilde{\rho} \in (\mathcal{U}^+)^*$ defines an even functional

$$\rho \doteq \tilde{\rho} \circ \left(\frac{\sigma + \mathbf{1}_{\mathcal{U}}}{2}\right) \in \mathcal{U}^*$$

on the C^*-algebra $\mathcal{U}$. Both mappings $\rho \mapsto \tilde{\rho}$ and $\tilde{\rho} \mapsto \rho$ are linear, weak*-continuous and order-preserving. Additionally, these mappings preserve states, i.e., $\rho \in \mathcal{U}^*$ is a state iff $\tilde{\rho} \in (\mathcal{U}^+)^*$ is a state. In particular, the mapping $\rho \mapsto \tilde{\rho}$ is bijective, by [1, proof of Proposition 2.1]. As a consequence, for any positive Radon measure $\mu \in \mathrm{M}(E)$ supported on E^+ with barycenter $\rho_\mu \in \mathcal{U}^*$, $\tilde{\rho}_\mu \in (\mathcal{U}^+)^*$ is the barycenter of the pushforward of the measure μ through the mapping $\rho \mapsto \tilde{\rho}$. Conversely, if $\tilde{\mu}$ is a positive Radon measure on the set $\tilde{E}^+$ of all states on $\mathcal{U}^+$ with barycenter $\tilde{\rho}_{\tilde{\mu}} \in (\mathcal{U}^+)^*$, then $\rho_{\tilde{\mu}}$ is the barycenter of the pushforward of the measure $\tilde{\mu}$ through the mapping $\tilde{\rho} \mapsto \rho$ and the pushforward of the measure $\tilde{\mu}$ is supported on E^+, because it is invariant under the pushforward through the mapping $\rho \mapsto \rho \circ \sigma$. By [15, Lemma 4.1.19], two positive linear functionals ρ_1, ρ_2 on any C^*-algebra (like $\mathcal{U}$ or $\mathcal{U}^+$) are orthogonal iff the zero functional is the unique positive functional on this C^*-algebra below ρ_1 and ρ_2. In particular, the pushforwards of positive Radon measures, associated with the mappings $\rho \mapsto \tilde{\rho}$ and $\tilde{\rho} \mapsto \rho$, preserve the orthogonality of such measures: The fact that the pushforward of positive Radon measures, associated with the mapping $\rho \mapsto \tilde{\rho}$, preserves orthogonality is clear, by the bijectivity, linearity and order-preserving property of this mapping combined with [15, Lemma 4.1.19]. To show the converse assertion, take any orthogonal positive Radon measure $\tilde{\mu}$ on $\tilde{E}^+$ and denote by $\mu \in \mathrm{M}(E)$ its pushforward through the mapping $\tilde{\rho} \mapsto \rho$. Fix any Borel set $\mathfrak{B} \in \Sigma_E$ and take any positive functional ω on $\mathcal{U}$ below the positive functionals $\rho_{\mu_{\mathfrak{B}}}, \rho_{\mu_{E\backslash\mathfrak{B}}} \in \mathcal{U}^*$, see (77). Then, the restriction to $\mathcal{U}^+$ of ω is lower than the restrictions to $\mathcal{U}^+$ of the states $\rho_{\mu_{\mathfrak{B}}}, \rho_{\mu_{E\backslash\mathfrak{B}}}$. Since μ is supported on E^+, for any $\mathfrak{B} \in \Sigma_E$,

$$\mu_{\mathfrak{B}} = \mu_{\mathfrak{B}\cap E^+} \qquad \text{and} \qquad \mu_{E\backslash\mathfrak{B}} = \mu_{E^+\backslash\mathfrak{B}} \,. \tag{81}$$

[14] It is even a G_δ set. See, e.g., [18, Proposition 1.3].

Since μ is the pushforward through the mapping $\tilde{\rho} \mapsto \rho$ of an *orthogonal* positive Radon measure $\tilde{\mu}$, by [15, Lemma 4.1.19] combined with the bijectivity, linearity and order-preserving property of the mapping, the unique *even* positive functional below $\rho_{\mu_{\mathfrak{B}}}$ and $\rho_{\mu_{E\backslash\mathfrak{B}}}$ is the zero functional. Suppose now that ω, not necessarily even, is a positive functional below $\rho_{\mu_{\mathfrak{B}}}$ and $\rho_{\mu_{E\backslash\mathfrak{B}}}$. In particular, because $\rho_{\mu_{\mathfrak{B}}}$ and $\rho_{\mu_{E\backslash\mathfrak{B}}}$ are even, $(\omega + \omega \circ \sigma)/2$ is an even positive functional below $\rho_{\mu_{\mathfrak{B}}}$ and $\rho_{\mu_{E\backslash\mathfrak{B}}}$. In particular,

$$\omega = -\omega \circ \sigma \qquad \text{and} \qquad \|\omega\|_{\mathcal{U}_+^*} = \omega(\mathfrak{1}) = 0 \; .$$

By [15, Lemma 4.1.19], the pushforward of positive Radon measures, associated with the mapping $\tilde{\rho} \mapsto \rho$, thus preserves orthogonality.

By [11, Lemma 1.8, Corollary 4.3], $E_{\vec{\ell}} \subseteq E^+ \equiv \tilde{E}^+$ for all $\vec{\ell} \in \mathbb{N}^d$. This identification is pivotal because, in contrast with $\mathcal{U}$, the even C^*-subalgebra $\mathcal{U}^+$ is asymptotically abelian since

$$\lim_{|x|\to\infty} [\alpha_x(A), B] = 0 \; , \qquad A \in \mathcal{U}^+, B \in \mathcal{U} \; . \tag{82}$$

As is usual, $[A, B] \doteq AB - BA$. Therefore, from [15, Propositions 4.3.3 and 4.3.7], it follows that μ_ρ, which is supported on $E_{\vec{\ell}} \subseteq E^+ \equiv \tilde{E}^+$, is an orthogonal measure, i.e., $\mu_\rho \in \mathcal{O}(E)$. ∎

By Theorem 5.1, $E_{\vec{\ell}}$ is a Choquet simplex and the mapping $\rho \mapsto \mu_\rho$ from $E_{\vec{\ell}}$ to $\mathrm{M}(E)$ ranges over orthogonal measures. The unique decomposition of any $\rho \in E_{\vec{\ell}}$ in terms of extreme states $\hat{\rho} \in \mathcal{E}(E_{\vec{\ell}})$, given in Theorem 5.1, is also called the *ergodic* decomposition of ρ: Fix $\vec{\ell} \in \mathbb{N}^d$ and define the space-averages of any element $A \in \mathcal{U}$ by

$$A_L \equiv A_{L,\vec{\ell}} \doteq \frac{1}{|\Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}}|} \sum_{x \in \Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}}} \alpha_x(A) \; , \qquad L \in \mathbb{N} \; . \tag{83}$$

Then, by definition, a $\vec{\ell}$–periodic state $\hat{\rho} \in E_{\vec{\ell}}$ is ($\vec{\ell}$–) ergodic iff, for all $A \in \mathcal{U}$,

$$\lim_{L\to\infty} \hat{\rho}(A_L^* A_L) = |\hat{\rho}(A)|^2 \; . \tag{84}$$

By [11, Theorem 1.16], any extreme state is ergodic and vice versa. To be more precise,

$$\mathcal{E}(E_{\vec{\ell}}) = \left\{ \hat{\rho} \in E_{\vec{\ell}} : \hat{\rho} \text{ is } \vec{\ell}\text{-ergodic} \right\} \; , \qquad \vec{\ell} \in \mathbb{N}^d \; . \tag{85}$$

Additionally, any extreme state $\hat{\rho} \in \mathcal{E}(E_{\vec{\ell}})$ is strongly clustering, i.e., for all $A, B \in \mathcal{U}$ and $x \in \mathbb{Z}^d$,

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}}|} \sum_{y \in \Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}}} \hat{\rho}(\alpha_x(A)\alpha_y(B)) = \hat{\rho}(A)\hat{\rho}(B) \; . \tag{86}$$

The ergodicity properties of extreme states has important consequences on the structure of the sets of periodic states. For instance, the weak*-density of the set $\mathcal{E}(E_{\vec{\ell}})$ of extreme points of $E_{\vec{\ell}}$ is proven by using that $\vec{\ell}$-ergodic states are extreme states of $E_{\vec{\ell}}$. See [11, Proof of Corollary 4.6]. If $\vec{\ell}_1, \vec{\ell}_2 \in \mathbb{N}^d$ is such that $\mathbb{Z}^d_{\vec{\ell}_1} \subseteq \mathbb{Z}^d_{\vec{\ell}_2}$ (see (16)), then $E_{\vec{\ell}_2} \subseteq E_{\vec{\ell}_1}$, but from (85) one checks that an extreme state of $E_{\vec{\ell}_2}$ is not necessarily an extreme state of $E_{\vec{\ell}_1}$, i.e., $\mathcal{E}(E_{\vec{\ell}_2}) \nsubseteq \mathcal{E}(E_{\vec{\ell}_1})$.

For $\vec{\ell}_j \doteq (\ell_{j,1}, \ldots, \ell_{j,d}) \in \mathbb{N}^d$, $j \in \{1, 2\}$, being such that $\mathbb{Z}^d_{\vec{\ell}_1} \subseteq \mathbb{Z}^d_{\vec{\ell}_2}$, we define the mapping $\mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}$ from E to itself by

$$\mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}(\rho) \doteq \frac{\ell_{2,1} \cdots \ell_{2,d}}{\ell_{1,1} \cdots \ell_{1,d}} \sum_{x=(x_1,\ldots,x_d),\ x_i \in \{0, \ell_{2,i}, 2\ell_{2,i}, \ldots, \ell_{1,i} - \ell_{2,i}\}} \rho \circ \alpha_x \; . \tag{87}$$

This transformation allows us to relate the sets $\mathcal{E}(E_{\vec{\ell}_1})$ and $\mathcal{E}(E_{\vec{\ell}_2})$ to each other:

Lemma 5.2 (From $\vec{\ell}_1$- to $\vec{\ell}_2$-periodic states)
Let $\vec{\ell}_j \doteq (\ell_{j,1}, \ldots, \ell_{j,d}) \in \mathbb{N}^d$, $j \in \{1,2\}$, be such that $\mathbb{Z}^d_{\vec{\ell}_1} \subseteq \mathbb{Z}^d_{\vec{\ell}_2}$ (see (16)). Then, the transformation $\mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}$ defined by (87) has the following properties:
(i) *It is weak*-continuous, $\mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}(\rho) = \rho$ for all $\rho \in E_{\vec{\ell}_2}$ and*

$$\mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}(E_{\vec{\ell}_1}) = E_{\vec{\ell}_2} .$$

(ii) *For all $\Phi \in \mathcal{W}_1$ and $\rho \in E_{\vec{\ell}_1}$,*

$$\rho(\mathfrak{e}_{\Phi,\vec{\ell}_1}) = \mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}(\rho)(\mathfrak{e}_{\Phi,\vec{\ell}_2}) .$$

(iii) *It maps extreme states of $E_{\vec{\ell}_1}$ to extreme states of $E_{\vec{\ell}_2}$ and*

$$\mathfrak{r}^{-1}_{\vec{\ell}_1,\vec{\ell}_2}(\mathcal{E}(E_{\vec{\ell}_2})) \cap E_{\vec{\ell}_1} = \mathcal{E}(E_{\vec{\ell}_1}) .$$

Proof. Fix all parameters of the lemma. Assertions (i)-(ii) are direct consequence of (87). See also Equation (27) defining the energy density observable $\mathfrak{e}_{\Phi,\vec{\ell}}$ of any $\Phi \in \mathcal{W}_1$. In order to prove Assertion (iii), first fix $\hat{\rho}_1 \in \mathcal{E}(E_{\vec{\ell}_1})$ and define $\hat{\rho}_2 \doteq \mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}(\hat{\rho}_1)$. Using (83) for $\vec{\ell} = \vec{\ell}_2$ as well as $\mathbb{Z}^d_{\vec{\ell}_1} \subseteq \mathbb{Z}^d_{\vec{\ell}_2}$, we compute that, for any $A \in \mathcal{U}$,

$$\hat{\rho}_2(|A_{L,\vec{\ell}_2}|^2) = \frac{1}{|\Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}_2}|^2} \sum_{x,y \in \Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}_2}} \hat{\rho}_1(\alpha_x(A^*)\alpha_y(A)) + o(1) , \tag{88}$$

as $L \to \infty$, which, combined with $\mathbb{Z}^d_{\vec{\ell}_1} \subseteq \mathbb{Z}^d_{\vec{\ell}_2}$, implies in turn that, as $L \to \infty$,

$$\hat{\rho}_2(|A_{L,\vec{\ell}_2}|^2) = \frac{1}{|\Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}_1}|^2} \sum_{x,y \in \Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}_1}} \hat{\rho}_1\left(\alpha_x(A^*_{\vec{\ell}_1,\vec{\ell}_2})\alpha_y(A_{\vec{\ell}_1,\vec{\ell}_2})\right) + o(1) \tag{89}$$

for any $A \in \mathcal{U}$, where

$$A_{\vec{\ell}_1,\vec{\ell}_2} \doteq \frac{\ell_{2,1}\cdots\ell_{2,d}}{\ell_{1,1}\cdots\ell_{1,d}} \sum_{x=(x_1,\ldots,x_d),\ x_i \in \{0,\ell_{2,i},2\ell_{2,i},\ldots,\ell_{1,i}-\ell_{2,i}\}} \alpha_x(A) . \tag{90}$$

Since $\hat{\rho}_1 \in \mathcal{E}(E_{\vec{\ell}_1})$, we can combine (89)-(90) with (83)-(85) to arrive at the limit

$$\lim_{L\to\infty} \hat{\rho}_2(|A_{L,\vec{\ell}_2}|^2) = |\hat{\rho}_1(A_{\vec{\ell}_1,\vec{\ell}_2})|^2 , \qquad A \in \mathcal{U} . \tag{91}$$

Observing from (87) that

$$\rho_1(A_{\vec{\ell}_1,\vec{\ell}_2}) = \mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}(\rho_1)(A) , \qquad A \in \mathcal{U},\ \rho_1 \in E_{\vec{\ell}_1} , \tag{92}$$

it follows from (91) and Assertion (i) that

$$\hat{\rho}_2 \doteq \mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}(\hat{\rho}_1) \in E_{\vec{\ell}_2}$$

is $\vec{\ell}_2$-ergodic and, thus, belongs to $\mathcal{E}(E_{\vec{\ell}_2})$, by (85). In other words, $\mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}$ maps extreme states of $E_{\vec{\ell}_1}$ to extreme states of $E_{\vec{\ell}_2}$. In particular,

$$\mathcal{E}(E_{\vec{\ell}_1}) \subseteq \mathfrak{r}^{-1}_{\vec{\ell}_1,\vec{\ell}_2}(\mathcal{E}(E_{\vec{\ell}_2})) \cap E_{\vec{\ell}_1} . \tag{93}$$

It remains to prove that

$$\mathfrak{r}^{-1}_{\vec{\ell}_1,\vec{\ell}_2}(\mathcal{E}(E_{\vec{\ell}_2})) \cap E_{\vec{\ell}_1} \subseteq \mathcal{E}(E_{\vec{\ell}_1}) . \tag{94}$$

We prove this by contradiction. Assume the existence of $\rho_1 \in E_{\vec{\ell}_1} \backslash \mathcal{E}(E_{\vec{\ell}_1})$ such that

$$\rho_2 \doteq \mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}(\rho_1) \in \mathcal{E}(E_{\vec{\ell}_2}). \tag{95}$$

Then, by (83)-(85), there is $A \in \mathcal{U}$ such that

$$\lim_{L\to\infty} \rho_1(A^*_{L,\vec{\ell}_1} A_{L,\vec{\ell}_1}) > |\rho_1(A)|^2 . \tag{96}$$

Now, if one computes $\rho_2(|A_{L,\vec{\ell}_2}|^2)$ as it is done in (88)-(90), then we infer from (96) and (92) that

$$\lim_{L\to\infty} \hat{\rho}_2(|A_{L,\vec{\ell}_2}|^2) > |\hat{\rho}_1(A_{\vec{\ell}_1,\vec{\ell}_2})|^2 = |\mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}(\rho_1)(A)|^2 = |\rho_2(A)|^2 .$$

By (83)-(85), it means that $\hat{\rho}_2 \notin \mathcal{E}(E_{\vec{\ell}_2})$, which contradicts (95). This yields Equation (94), which combined with (93) in turn implies Assertion (iii). ■

Corollary 5.3 (From $\vec{\ell}_1$- to $\vec{\ell}_2$-ergodic decompositions of periodic states)
Fix $\vec{\ell}_j \doteq (\ell_{j,1}, \ldots, \ell_{j,d}) \in \mathbb{N}^d$, $j \in \{1,2\}$, such that $\mathbb{Z}^d_{\vec{\ell}_1} \subseteq \mathbb{Z}^d_{\vec{\ell}_2}$ (see (16)). Let $\rho \in E_{\vec{\ell}_2} \subseteq E_{\vec{\ell}_1}$ and denote by $\mu_\rho^{(\vec{\ell}_1)} \in \mathrm{M}^{(\rho)}(E_{\vec{\ell}_1})$ the unique probability measure representing ρ in $E_{\vec{\ell}_1}$ with $\mu_\rho^{(\vec{\ell}_1)}(\mathcal{E}(E_{\vec{\ell}_1})) = 1$. The pushforward of $\mu_\rho^{(\vec{\ell}_1)}$ through the continuous mapping $\mathfrak{r}_{\vec{\ell}_1,\vec{\ell}_2}$ is the unique probability measure $\mu_\rho^{(\vec{\ell}_2)}$ representing ρ in $E_{\vec{\ell}_2}$ with $\mu_\rho^{(\vec{\ell}_2)}(\mathcal{E}(E_{\vec{\ell}_2})) = 1$.

Proof. The proof is a direct consequence of Theorem 5.1 and Lemma 5.2. ■

5.3 Strong Limit of Space Averages

Fix $\vec{\ell} \in \mathbb{N}^d$. The subject of this section is the limit $L \to \infty$ of the space averages (83) for any $A \in \mathcal{U}$, that is,

$$A_L \doteq \frac{1}{|\Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}}|} \sum_{x \in \Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}}} \alpha_x(A) , \qquad L \in \mathbb{N} . \tag{97}$$

As explained in [1, Section 4.3], this sequence does not generally converge in the C^*-algebra $\mathcal{U}$. Nevertheless, in a cyclic representation $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ of any $\vec{\ell}$-periodic state $\rho \in E_{\vec{\ell}}$ this limit exists in the sense of the strong operator topology of $\mathcal{B}(\mathcal{H}_\rho)$.

We thus fix $\rho \in E_{\vec{\ell}}$ for some $\vec{\ell} \in \mathbb{N}^d$. By $\vec{\ell}$-periodicity of ρ and [15, Corollary 2.3.17], there is a unique family of unitary operators $\{U_x\}_{x\in\mathbb{Z}^d_{\vec{\ell}}}$ in $\mathcal{B}(\mathcal{H}_\rho)$ with invariant vector Ω_ρ, i.e., $\Omega_\rho = U_x \Omega_\rho$ for any $x \in \mathbb{Z}^d_{\vec{\ell}}$, and

$$\pi_\rho(\alpha_x(A)) = U_x \pi_\rho(A) U_x^* , \qquad A \in \mathcal{U},\ x \in \mathbb{Z}^d_{\vec{\ell}} . \tag{98}$$

By the von Neumann ergodic theorem[15] [11, Theorem 4.2] (or [15, Proposition 4.3.4]), the space average

$$P^{(L)} \doteq \frac{1}{|\Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}}|} \sum_{x \in \Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}}} U_x \tag{99}$$

strongly converges, as $L \to \infty$, to an orthogonal projection in the commutant $[\{U_x\}_{x\in\mathbb{Z}^d_{\vec{\ell}}}]'$, which we denote by P_ρ. More precisely, P_ρ is the orthogonal projection on the closed subspace

$$\bigcap_{x \in \mathbb{Z}^d_{\vec{\ell}}} \{\psi \in \mathcal{H}_\rho \,:\, \psi = U_x(\psi)\} \supseteq \mathbb{C}\Omega_\rho \tag{100}$$

[15] It is also named the Alaoglu-Birkhoff mean ergodic theorem.

of all vectors of $\mathcal{H}_\rho$ that are invariant with respect to the family $\{U_x\}_{x\in\mathbb{Z}^d_{\vec{\ell}}}$ of unitaries. In particular, $\Omega_\rho = P_\rho\Omega_\rho$. With this definition, we can reformulate [11, Theorem 1.16] as follows: $\hat{\rho} \in \mathcal{E}(E_{\vec{\ell}})$ is an extreme state iff $P_{\hat{\rho}}$ is the orthogonal projection on the one-dimensional Hilbert subspace spanned by the cyclic vector $\Omega_{\hat{\rho}}$, i.e.,

$$\bigcap_{x\in\mathbb{Z}^d_{\vec{\ell}}} \{\psi \in \mathcal{H}_{\hat{\rho}} \,:\, \psi = U_x(\psi)\} = \mathbb{C}\Omega_{\hat{\rho}}. \tag{101}$$

See [11, Lemma 4.8]. Compare with (86) and (100).

We are now in a position to study the limit of space-averages (97) seen as bounded operators acting on $\mathcal{H}_\rho$ via the representation π_ρ.

Lemma 5.4 (Strong limit of space-averages)
Let $\vec{\ell} \in \mathbb{N}^d$ and $\rho \in E_{\vec{\ell}}$ be any $\vec{\ell}$-periodic state with cyclic representation $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$. For every element $A \in \mathcal{U}^+$, the sequence $(\pi_\rho(A_L))_{L\in\mathbb{N}}$ defined from (97) strongly converges in $\mathcal{B}(\mathcal{H}_\rho)$ to the element $A^\rho_\infty \in \pi_\rho(\mathcal{U})' \cap \pi_\rho(\mathcal{U})''$ uniquely defined by

$$A^\rho_\infty \pi_\rho(B)\,\Omega_\rho \doteq \pi_\rho(B)\,P_\rho \pi_\rho(A)\,\Omega_\rho\ , \qquad B \in \mathcal{U}\ .$$

Proof. Fix $A \in \mathcal{U}^+$, $\vec{\ell} \in \mathbb{N}^d$ and $\rho \in E_{\vec{\ell}}$ with $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ being its cyclic representation. Observe first that the uniformly bounded sequence $(\pi_\rho(A_L))_{L\in\mathbb{N}}$ strongly converges on $\mathcal{H}_\rho$ iff it strongly converges on the dense subspace $\pi_\rho(\mathcal{U})\Omega_\rho \subseteq \mathcal{H}_\rho$, by cyclicity of Ω_ρ. We meanwhile infer from (97)-(100) that, for all $B \in \mathcal{U}$ and $L \in \mathbb{N}$,

$$\pi_\rho(A_L)\,\pi_\rho(B)\,\Omega_\rho = \pi_\rho(B)\,P^{(L)}\pi_\rho(A)\Omega_\rho + \pi_\rho\left(\frac{1}{|\Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}}|}\sum_{x\in\Lambda_L\cap\mathbb{Z}^d_{\vec{\ell}}}[\alpha_x(A)\,, B]\right)\Omega_\rho\ , \tag{102}$$

where we recall that $[B, C] \doteq BC - CB$. By (82), for all $A \in \mathcal{U}^+$ and $B \in \mathcal{U}$,

$$\lim_{L\to\infty}\frac{1}{|\Lambda_L \cap \mathbb{Z}^d_{\vec{\ell}}|}\sum_{x\in\Lambda_L\cap\mathbb{Z}^d_{\vec{\ell}}}\|[\alpha_x(A)\,, B]\|_{\mathcal{U}} = 0$$

and $P^{(L)}$ strongly converges, as $L \to \infty$, to P_ρ. Consequently, we deduce from (102) the existence of a bounded operator $A^\rho_\infty \in \mathcal{B}(\mathcal{H}_\rho)$ such that

$$A^\rho_\infty \pi_\rho(B)\Omega_\rho = \pi_\rho(B)\,P_\rho\pi_\rho(A)\Omega_\rho = \lim_{L\to\infty}\pi_\rho(A_L)\,\pi_\rho(B)\,\Omega_\rho\ , \qquad B \in \mathcal{U}\ . \tag{103}$$

This equation uniquely defines $A^\rho_\infty \in \mathcal{B}(\mathcal{H}_\rho)$, by cyclicity of Ω_ρ. The limit operator A^ρ_∞ belongs to the commutant $\pi_\rho(\mathcal{U})'$ because Ω_ρ is a cyclic vector and Equation (103) implies that

$$[\pi_\rho(C)\,, A^\rho_\infty]\,\pi_\rho(B)\Omega_\rho = 0\ , \qquad B, C \in \mathcal{U}\ .$$

Further, A^ρ_∞ has to be an element of the bicommutant $\pi_\rho(\mathcal{U})''$, because it is the strong limit of elements of $\pi_\rho(\mathcal{U})$. ∎

The limit operator A^ρ_∞ of Lemma 5.4 takes a very simple form when ρ is an extreme $\vec{\ell}$-periodic state, i.e., $\rho \in \mathcal{E}(E_{\vec{\ell}})$.

Corollary 5.5 (Strong limit of space-averages for ergodic states)
Let $\vec{\ell} \in \mathbb{N}^d$ and $\hat{\rho} \in \mathcal{E}(E_{\vec{\ell}})$ with cyclic representation $(\mathcal{H}_{\hat{\rho}}, \pi_{\hat{\rho}}, \Omega_{\hat{\rho}})$. For every element $A \in \mathcal{U}$, the sequence $(\pi_\rho(A_L))_{L\in\mathbb{N}}$ defined from (97) strongly converges in $\mathcal{B}(\mathcal{H}_{\hat{\rho}})$ to

$$A^{\hat{\rho}}_\infty = \hat{\rho}(A)\,\mathbf{1}_{\mathcal{H}_{\hat{\rho}}}\ .$$

Proof. Since $\hat{\rho} \in \mathcal{E}(E_{\vec{\ell}})$ is an extreme $\vec{\ell}$-periodic state, by [11, Lemma 4.8], $P_{\hat{\rho}}$ is the orthogonal projection on the one-dimensional Hilbert subspace spanned by $\Omega_{\hat{\rho}}$. See Equation (101). In fact, this is directly related to the ergodicity of extreme states, see (83)-(85). By Lemma 5.4 and Equation (74), the assertion then follows. ■

Corollary 5.6 (Strong limit of local energies for ergodic states)
Let $\vec{\ell} \in \mathbb{N}^d$ and $\hat{\rho} \in \mathcal{E}(E_{\vec{\ell}})$ with cyclic representation $(\mathcal{H}_{\hat{\rho}}, \pi_{\hat{\rho}}, \Omega_{\hat{\rho}})$. For every translation-invariant interaction $\Phi \in \mathcal{W}_1$, the sequence

$$\left(\pi_{\hat{\rho}}\left(|\Lambda_L|^{-1} U_L^{\Phi}\right)\right)_{L\in\mathbb{N}}$$

defined by (4) and (39) strongly converges in $\mathcal{B}(\mathcal{H}_{\hat{\rho}})$ to $\hat{\rho}(\mathfrak{e}_{\Phi,\vec{\ell}})\mathbf{1}_{\mathcal{H}_{\hat{\rho}}}$, where $\mathfrak{e}_{\Phi,\vec{\ell}}$ is the even observable defined by (27).

Proof. Fix $\vec{\ell} \in \mathbb{N}^d$ and any translation-invariant interaction $\Phi \in \mathcal{W}_1$. By Equations (23)-(27) and (39),

$$\left\| \frac{U_L^{\Phi}}{|\Lambda_L|} - \frac{1}{|\Lambda_L \cap \mathbb{Z}_{\vec{\ell}}^d|} \sum_{y\in\Lambda_L\cap\mathbb{Z}_{\vec{\ell}}^d} \alpha_y(\mathfrak{e}_{\Phi,\vec{\ell}}) \right\|_{\mathcal{U}} \leq \sum_{x=(x_1,\cdots,x_d),\, x_i\in\{0,\cdots,\ell_i-1\}} \sum_{\Lambda\in\mathcal{P}_f,\Lambda\ni 0} \frac{\|\Phi_\Lambda\|_{\mathcal{U}}}{|\Lambda|} \times \frac{1}{|\Lambda_L|} \sum_{y\in\Lambda_L\cap\mathbb{Z}_{\vec{\ell}}^d} \left(1 - \mathbf{1}\left[x+y\in\Lambda_L\right]\mathbf{1}\left[\Lambda \subseteq (\Lambda_L - x - y)\right]\right) ,$$

where, by definition, $\mathbf{1}[p] = 1$ whenever the proposition p is true, otherwise $\mathbf{1}[p] = 0$. Since $\Phi \in \mathcal{W}_1$, by (22) and Lebesgue's dominated convergence theorem, we deduce from last inequality that

$$\lim_{L\to\infty} \left\| \frac{U_L^{\Phi}}{|\Lambda_L|} - \frac{1}{|\Lambda_L \cap \mathbb{Z}_{\vec{\ell}}^d|} \sum_{y\in\Lambda_L\cap\mathbb{Z}_{\vec{\ell}}^d} \alpha_y(\mathfrak{e}_{\Phi,\vec{\ell}}) \right\|_{\mathcal{U}} = 0 ,$$

using that

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{y\in\Lambda_L\cap\mathbb{Z}_{\vec{\ell}}^d} \left(1 - \mathbf{1}\left[x+y\in\Lambda_L\right]\mathbf{1}\left[\Lambda \subseteq (\Lambda_L - x - y)\right]\right) = 0$$

for all $x \in \mathfrak{L}$ and $\Lambda \in \mathcal{P}_f$. Therefore, the assertion is a direct consequence of Corollary 5.5 applied to $A = \mathfrak{e}_{\Phi,\vec{\ell}} \in \mathcal{U}$. ■

5.4 Commutator Estimates from Lieb-Robinson Bounds

In Section 5.5, we prove Theorem 4.3 for ergodic states. To this end, we employ the following uniform bound, which is a direct consequence of Lieb-Robinson bounds [13, Theorem 4.3]:

Lemma 5.7 (Commutator estimates from Lieb-Robinson bounds)
For any time-dependent model $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M})$, $\Phi \in \mathcal{W}$, times $s, t \in \mathbb{R}$, length $L \in \mathbb{N}$, subset $\Lambda \subseteq \Lambda_L$ and $A \in \mathcal{U}_\Lambda$,

$$\left\| \left[\tau_{t,s}^{(L,\mathfrak{m})}(A), U_L^{\Phi}\right] \right\|_{\mathcal{U}} \leq 2\,|\Lambda|\, \|A\|_{\mathcal{U}}\, \|\Phi\|_{\mathcal{W}}\, \mathrm{e}^{16\left(\mathbf{D}+2\|\mathbf{F}\|_{1,\mathfrak{L}}+1\right)\int_s^t \|\mathfrak{m}(\varsigma)\|_{\mathcal{M}} \mathrm{d}\varsigma}$$

with $(\tau_{t,s}^{(L,\mathfrak{m})})_{s,t\in\mathbb{R}}$ being the unique (fundamental) solution to (44)-(45), while U_L^{Φ} is the energy element defined by (39).

Proof. For any model $\mathfrak{m} \doteq (\Phi, \mathfrak{a}) \in \mathcal{M}$ and $L \in \mathbb{N}$, there is an interaction $\Phi^{(L,\mathfrak{m})}$ such that

$$U_L^{\mathfrak{m}} \doteq U_L^{\Phi^{(L,\mathfrak{m})}} \doteq \sum_{\Lambda \subseteq \Lambda_L} \Phi_\Lambda^{(L,\mathfrak{m})} \tag{104}$$

with Λ_L being the cubic box defined by (4) for $L \in \mathbb{N}$. Recall that $U_L^{\mathfrak{m}}$ is the local Hamiltonian of the model $\mathfrak{m} \in \mathcal{M}$, defined by (41) for $L \in \mathbb{N}$. Since, by (39),

$$U_L^{\Psi^{(1)}} \cdots U_L^{\Psi^{(n)}} = \sum_{\Lambda \in \mathcal{P}_f} \mathbf{1}\left[\Lambda \subseteq \Lambda_L\right] \sum_{\Lambda_1,\ldots,\Lambda_n \in \mathcal{P}_f : \Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n} \Psi_{\Lambda_1}^{(1)} \cdots \Psi_{\Lambda_n}^{(n)} ,$$

for any $n \in \mathbb{N}$ and $\Psi^{(1)}, \ldots, \Psi^{(n)} \in \mathcal{W}$, such an interaction $\Phi^{(L,\mathfrak{m})}$ is explicitly given by

$$\Phi_\Lambda^{(L,\mathfrak{m})} \doteq \Phi_\Lambda + \sum_{n \in \mathbb{N}} \frac{\mathbf{1}\left[\Lambda \subseteq \Lambda_L\right]}{|\Lambda_L|^{n-1}} \int_{\mathbb{S}^n} \mathfrak{a}_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right) \sum_{\Lambda_1,\ldots,\Lambda_n \in \mathcal{P}_f : \Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n} \Psi_{\Lambda_1}^{(1)} \cdots \Psi_{\Lambda_n}^{(n)} \tag{105}$$

for all $\Lambda \in \mathcal{P}_f$. See again (41) and (104).

Fix $L \in \mathbb{N}$. We next define the positive-valued symmetric function $\mathbf{F}_L : \Lambda_L^2 \to (0,1]$ by

$$\mathbf{F}_L(x,y) \doteq \frac{|\Lambda_L|}{|\Lambda_L|+1} \left(\mathbf{F}(x,y) + \frac{1}{|\Lambda_L|}\right) , \qquad x,y \in \Lambda_L ,$$

where $\mathbf{F} : \mathfrak{L}^2 \to (0,1]$ is some positive-valued symmetric function with maximum value $\mathbf{F}(x,x) = 1$ for all $x \in \mathfrak{L}$, satisfying (5)-(6). Note that $\mathbf{F}_L$ has also maximum value $\mathbf{F}_L(x,x) = 1$ for all $x \in \mathfrak{L}$. Then, a L-dependent seminorm for interactions Φ is defined by

$$\|\Phi\|_{\mathcal{W}_L} \doteq \sup_{x,y \in \Lambda_L} \sum_{\Lambda \subseteq \Lambda_L,\, \Lambda \supseteq \{x,y\}} \frac{\|\Phi_\Lambda\|_{\mathcal{U}}}{\mathbf{F}_L(x,y)} \leq \left(1 + |\Lambda_L|^{-1}\right) \|\Phi\|_{\mathcal{W}} . \tag{106}$$

Compare with Equation (22). This seminorm is in fact a norm in the finite-dimensional space $\mathcal{W}_L$ of interactions Φ supported on the cubic box Λ_L, i.e., $\Phi_\Lambda = 0$ whenever $\Lambda \nsubseteq \Lambda_L$. Compare with the Banach space $\mathcal{W}$ defined by (21)-(22). The (real) subspace of all self-adjoint interactions of $\mathcal{W}_L$ is denoted by $\mathcal{W}_L^{\mathbb{R}} \subsetneq \mathcal{W}_L$, similar to $\mathcal{W}^{\mathbb{R}} \subsetneq \mathcal{W}$. For any $\mathfrak{m} \doteq (\Phi, \mathfrak{a}) \in \mathcal{M}$, we compute from (105) and (106) that

$$\begin{aligned} \left\|\Phi^{(L,\mathfrak{m})}\right\|_{\mathcal{W}_L} \leq\ & \left(1 + |\Lambda_L|^{-1}\right) \|\Phi\|_{\mathcal{W}} + \sum_{n \in \mathbb{N}} \frac{1}{|\Lambda_L|^{n-1}} \int_{\mathbb{S}^n} |\mathfrak{a}_n| \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right) \\ & \sup_{x,y \in \Lambda_L} \sum_{\Lambda \subseteq \Lambda_L,\, \Lambda \supseteq \{x,y\}} \sum_{\Lambda_1,\ldots,\Lambda_n \in \mathcal{P}_f : \Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n} \frac{\|\Psi_{\Lambda_1}^{(1)}\|_{\mathcal{U}} \cdots \|\Psi_{\Lambda_n}^{(n)}\|_{\mathcal{U}}}{\mathbf{F}_L(x,y)} . \end{aligned} \tag{107}$$

Now, observe that, for any $\Psi^{(1)}, \Psi^{(2)} \in \mathbb{S}$ (the unit sphere in $\mathcal{W}$),

$$\begin{aligned} & \frac{1}{|\Lambda_L|} \sup_{x,y \in \Lambda_L} \sum_{\Lambda \subseteq \Lambda_L,\, \Lambda \supseteq \{x,y\}} \sum_{\Lambda_1,\Lambda_2 \in \mathcal{P}_f : \Lambda = \Lambda_1 \cup \Lambda_2} \frac{\|\Psi_{\Lambda_1}^{(1)}\|_{\mathcal{U}} \|\Psi_{\Lambda_2}^{(2)}\|_{\mathcal{U}}}{\mathbf{F}_L(x,y)} \\ \leq\ & \frac{2}{|\Lambda_L|} \sup_{x,y \in \Lambda_L} \sum_{\Lambda_1,\Lambda_2 \subseteq \Lambda_L : x \in \Lambda_1, y \in \Lambda_2} \frac{\|\Psi_{\Lambda_1}^{(1)}\|_{\mathcal{U}} \|\Psi_{\Lambda_2}^{(2)}\|_{\mathcal{U}}}{\mathbf{F}_L(x,y)} \\ & + \frac{\left\|\Psi^{(1)}\right\|_{\mathcal{W}_L}}{|\Lambda_L|} \sum_{\Lambda \subseteq \Lambda_L} \|\Psi_\Lambda^{(2)}\|_{\mathcal{U}} + \frac{\left\|\Psi^{(2)}\right\|_{\mathcal{W}_L}}{|\Lambda_L|} \sum_{\Lambda \subseteq \Lambda_L} \|\Psi_\Lambda^{(1)}\|_{\mathcal{U}} . \end{aligned} \tag{108}$$

Note additionally that

$$\frac{1}{|\Lambda_L|}\sum_{\Lambda\subseteq\Lambda_L}\|\Psi_\Lambda\|_{\mathcal{U}}\leq\|\mathbf{F}\|_{1,\mathfrak{L}}\ ,\qquad \Psi\in\mathbb{S}\ , \tag{109}$$

(compare with (40)) and, using

$$|\Lambda_L|\mathbf{F}_L(x,y)\geq\frac{|\Lambda_L|}{|\Lambda_L|+1}\ ,\qquad x,y\in\Lambda_L\ ,$$

one also verifies in a similar way that, for any $\Psi^{(1)}$, $\Psi^{(2)}\in\mathbb{S}$,

$$\frac{1}{|\Lambda_L|}\sup_{x,y\in\Lambda_L}\sum_{\Lambda_1,\Lambda_2\subseteq\Lambda_L:x\in\Lambda_1,y\in\Lambda_2}\frac{\|\Psi_{\Lambda_1}^{(1)}\|_{\mathcal{U}}\|\Psi_{\Lambda_2}^{(2)}\|_{\mathcal{U}}}{\mathbf{F}_L(x,y)}\leq 1+|\Lambda_L|^{-1}\ . \tag{110}$$

By combining (108) with (106), (109)-(110) and $\|\mathbf{F}\|_{1,\mathfrak{L}}\geq 1$, we deduce that

$$\sup_{x,y\in\Lambda_L}\sum_{\Lambda\subseteq\Lambda_L,\ \Lambda\supseteq\{x,y\}}\ \sum_{\Lambda_1,\Lambda_2\in\mathcal{P}_f:\Lambda=\Lambda_1\cup\Lambda_2}\frac{\|\Psi_{\Lambda_1}^{(1)}\|_{\mathcal{U}}\|\Psi_{\Lambda_2}^{(2)}\|_{\mathcal{U}}}{\mathbf{F}_L(x,y)}\leq 4\left(1+|\Lambda_L|^{-1}\right)\|\mathbf{F}\|_{1,\mathfrak{L}}$$

for any $\Psi^{(1)}$, $\Psi^{(2)}\in\mathbb{S}$. Using this estimate, as well as $n-2$ times Inequality (109), in order to bound the right-hand side of (107) from above, we obtain that

$$\left\|\Phi^{(L,\mathfrak{m})}\right\|_{\mathcal{W}_L}\leq\left(1+|\Lambda_L|^{-1}\right)\left(\|\Phi\|_{\mathcal{W}}+4\sum_{n\in\mathbb{N}}n^2\,\|\mathbf{F}\|_{1,\mathfrak{L}}^{n-1}\,\|\mathfrak{a}\|_{\mathcal{S}(\mathbb{S}^n)}\right)\ ,$$

(see (31)), which in turn implies that, for any $\mathfrak{m}\in\mathcal{M}$ and $L\in\mathbb{N}$,

$$\left\|\Phi^{(L,\mathfrak{m})}\right\|_{\mathcal{W}_L}\leq 4\left(1+|\Lambda_L|^{-1}\right)\|\mathfrak{m}\|_{\mathcal{M}}\leq 8\,\|\mathfrak{m}\|_{\mathcal{M}}\ , \tag{111}$$

according to (32) and (34). Since

$$\Phi^{(L,\mathfrak{m}_1)}-\Phi^{(L,\mathfrak{m}_2)}=\Phi^{(L,\mathfrak{m}_1-\mathfrak{m}_2)}\ ,\qquad \mathfrak{m}_1,\mathfrak{m}_2\in\mathcal{M},\ L\in\mathbb{N}\ ,$$

we thus deduce from (104), (105) and (111) that, for any $\mathfrak{m}\in C_b(\mathbb{R};\mathcal{M})$, there is $\Psi^{(L,\mathfrak{m})}\in C(\mathbb{R};\mathcal{W}_L^{\mathbb{R}})$ such that

$$\tau_{t,s}^{(L,\Psi^{(L,\mathfrak{m})})}\equiv\tau_{t,s}^{(L,(\Psi^{(L,\mathfrak{m})},0))}=\tau_{t,s}^{(L,\mathfrak{m})}\ ,\qquad L\in\mathbb{N}\ , \tag{112}$$

where $(\tau_{t,s}^{(L,\mathfrak{m})})_{s,t\in\mathbb{R}}$ is the unique (fundamental) solution to (44)-(45). By (5)-(6), note that, for all $L\in\mathbb{N}$,

$$\|\mathbf{F}_L\|_{1,\Lambda_L}\doteq\sup_{y\in\Lambda_L}\sum_{x\in\Lambda_L}\mathbf{F}_L(x,y)\leq 1+\|\mathbf{F}\|_{1,\mathfrak{L}} \tag{113}$$

and

$$\mathbf{D}_L\doteq\sup_{x,y\in\Lambda_L}\sum_{z\in\Lambda_L}\frac{\mathbf{F}_L(x,z)\,\mathbf{F}_L(z,y)}{\mathbf{F}_L(x,y)}\leq\mathbf{D}+2\,\|\mathbf{F}\|_{1,\mathfrak{L}}+1\ . \tag{114}$$

As a consequence, since $\Psi^{(L,\mathfrak{m})}\in C(\mathbb{R};\mathcal{W}_L^{\mathbb{R}})$, by (111) and (112), we can apply the Lieb-Robinson bounds [1, Proposition 3.8 with $\mathfrak{L}=\Lambda_L$] on the right-hand side of the inequality

$$\left\|\left[\tau_{t,s}^{(L,\mathfrak{m})}(A),U_L^{\Phi}\right]\right\|_{\mathcal{U}}\leq\sum_{\mathcal{Z}\subseteq\Lambda_L}\left\|\left[\tau_{t,s}^{(L,\mathfrak{m})}(A),\Phi_{\mathcal{Z}}\right]\right\|_{\mathcal{U}}$$

for any $\Lambda \subseteq \Lambda_L$ and element $A \in \mathcal{U}_\Lambda$ to get that

$$\left\| [\tau_{t,s}^{(L,\mathfrak{m})}(A), U_L^\Phi] \right\|_{\mathcal{U}} \leq 2\mathbf{D}_L^{-1} \|A\|_{\mathcal{U}} \left(\mathrm{e}^{16\mathbf{D}_L \int_s^t \|\mathfrak{m}(\varsigma)\|_{\mathcal{M}} \mathrm{d}\varsigma} - 1 \right) \sum_{x\in\Lambda} \sum_{y\in\Lambda_L} \sum_{\mathcal{Z}\subseteq\Lambda_L,\ \mathcal{Z}\supseteq\{y\}} \|\Phi_{\mathcal{Z}}\|_{\mathcal{U}} \mathbf{F}_L(x,y) \ .$$

By (22) and (113), we infer from the last inequality that

$$\left\| [\tau_{t,s}^{(L,\mathfrak{m})}(A), U_L^\Phi] \right\|_{\mathcal{U}} \leq 2\mathbf{D}_L^{-1} |\Lambda| \|A\|_{\mathcal{U}} \|\Phi\|_{\mathcal{W}} \left(\mathrm{e}^{16\mathbf{D}_L \int_s^t \|\mathfrak{m}(\varsigma)\|_{\mathcal{M}} \mathrm{d}\varsigma} - 1 \right) \|\mathbf{F}_L\|_{1,\mathfrak{L}}$$

for any $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M})$, $\Phi \in \mathcal{W}$, $s,t \in \mathbb{R}$, $L \in \mathbb{N}$, $\Lambda \in \mathcal{P}_f$ and element $A \in \mathcal{U}_\Lambda$. This yields the lemma, by (113)-(114) combined with rough estimates on $\mathbf{D}_L$. ■

5.5 Long-Range Dynamics on Ergodic States

In this section, we prove Theorem 4.3 for any ergodic state. To this end, we recall again some important objects and notation:

- For any $\Lambda \in \mathcal{P}_f$ and $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda)$, $\boldsymbol{\varpi}^{\mathfrak{m}} \in C(\mathbb{R}^2; \mathrm{Aut}(E))$ is the solution to the self-consistency equation of Theorem 4.1.
- For any $\Psi \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$, $(\tau_{t,s}^{\Psi})_{s,t\in\mathbb{R}}$ is the strongly continuous two-parameter family defined as the strong limit, for fixed s,t, of the local dynamics $(\tau_{t,s}^{(L,\Psi)})_{s,t\in\mathbb{R}}$ defined by (44)-(45). See [1, Proposition 3.7].
- For any $\mathbf{\Psi} \in C(\mathbb{R}; \mathfrak{W})$ and $\rho \in E$, $\mathbf{\Psi}(\rho) \in C(\mathbb{R}; \mathcal{W})$ stands for the time-dependent interaction defined by

  $$\mathbf{\Psi}(\rho)(t) \doteq \mathbf{\Psi}(t;\rho) \ , \qquad \rho \in E,\ t \in \mathbb{R} \ .$$

  This refers to Equation (58).
- For any $\mathfrak{m} \in C(\mathbb{R}; \mathcal{M})$ and each $\boldsymbol{\xi} \in C(\mathbb{R}; \mathrm{Aut}(E))$ (typically, $\boldsymbol{\xi} = \boldsymbol{\varpi}^{\mathfrak{m}}(\alpha, \cdot)$ at fixed $\alpha \in \mathbb{R}$), the approximating interaction $\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}$ is the mapping from $\mathbb{R}$ to $\mathfrak{W}^{\mathbb{R}}$ of Definition 3.1. By (61), if $\mathfrak{m} \in C(\mathbb{R}; \mathcal{M})$ and $\boldsymbol{\xi} \in C(\mathbb{R}; \mathrm{Aut}(E))$ then $\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}(\rho) \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$.
- $\mathcal{M}_1$ is the Banach space of all translation-invariant long-range models defined by (38).

Now, we are in a position to state the main theorem of this subsection:

Theorem 5.8 (Long-range dynamics on ergodic states)
Fix $\Lambda \in \mathcal{P}_f$, $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda \cap \mathcal{M}_1)$, $\vec{\ell} \in \mathbb{N}^d$ and $\hat{\rho} \in \mathcal{E}(E_{\vec{\ell}})$ with cyclic representation $(\mathcal{H}_{\hat{\rho}}, \pi_{\hat{\rho}}, \Omega_{\hat{\rho}})$. Then, for any $s,t \in \mathbb{R}$ and $A \in \mathcal{U} \subseteq \mathfrak{U}$, in the σ-weak topology,

$$\lim_{L\to\infty} \pi_{\hat{\rho}} \left(\tau_{t,s}^{(L,\mathfrak{m})}(A) \right) = \pi_{\hat{\rho}} \left(\tau_{t,s}^{\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(\alpha,\cdot))}(\hat{\rho})}(A) \right) \Big|_{\alpha=s} \in \mathcal{B}(\mathcal{H}_{\hat{\rho}}) \ .$$

Proof. Fix once and for all $\Lambda \in \mathcal{P}_f$, $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda \cap \mathcal{M}_1)$, $\vec{\ell} \in \mathbb{N}^d$ and $\hat{\rho} \in \mathcal{E}(E_{\vec{\ell}})$ with a cyclic representation denoted by $(\mathcal{H}_{\hat{\rho}}, \pi_{\hat{\rho}}, \Omega_{\hat{\rho}})$. By (26)-(28), we can assume without loss of generality that $\mathfrak{e}_{\Phi,\vec{\ell}} \in \mathcal{U}_\Lambda$. In order to simplify the notation, we denote

$$\daleth_{t,s} \doteq \tau_{t,s}^{\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(\alpha,\cdot))}(\hat{\rho})}\big|_{\alpha=s} \ , \qquad \alpha, s, t \in \mathbb{R} \ . \tag{115}$$

The proof is done in several steps:

Step 1: For any $s,t \in \mathbb{R}$, the sequence

$$\left\{ \pi_{\hat{\rho}} \left(\daleth_{t,s} (A) - \tau_{t,s}^{(L,\mathfrak{m})} (A) \right) \right\}_{L \in \mathbb{N}} \subseteq \mathcal{B} (\mathcal{H}_{\hat{\rho}})$$

is norm equicontinuous with respect to $A \in \mathcal{U}$ and we can consider, without loss of generality, only elements A within some dense set of $\mathcal{U}$, like the dense $*$-algebra $\mathcal{U}_0$ of local elements defined by (7). This sequence is also uniformly bounded by $2\|A\|_{\mathcal{U}}$ in the operator norm of $\mathcal{B}(\mathcal{H}_{\hat{\rho}})$ and hence, using [15, Proposition 2.4.2], we only need to prove the weak-operator convergence on any dense set of $\mathcal{H}_{\hat{\rho}}$, like

$$\{\pi_{\hat{\rho}} (B) \Omega_{\hat{\rho}} : B \in \mathcal{U}\} \subseteq \mathcal{H}_{\hat{\rho}} ,$$

in order to get the desired σ-weak convergence. Moreover, by [1, Proposition 3.7], we can replace $\daleth_{t,s}$ with the local dynamics

$$\daleth_{t,s}^{(L)} \doteq \tau_{t,s}^{(L,\mathbf{\Phi}^{(\mathfrak{m},\varpi^{\mathfrak{m}}(\alpha,\cdot))(\hat{\rho})})}|_{\alpha=s} , \qquad \alpha, s, t \in \mathbb{R}, \ L \in \mathbb{N} .$$

To summarize, at fixed $s,t \in \mathbb{R}$, it suffices to prove that

$$\begin{aligned} & \lim_{L\to\infty} \left\langle \pi_{\hat{\rho}} (B) \Omega_{\hat{\rho}}, \pi_{\hat{\rho}} \left(\daleth_{t,s}^{(L)} (A) - \tau_{t,s}^{(L,\mathfrak{m})} (A) \right) \pi_{\hat{\rho}} (B) \Omega_{\hat{\rho}} \right\rangle_{\mathcal{H}_{\hat{\rho}}} \\ = & \lim_{L\to\infty} \hat{\rho} \left(B^* \left(\daleth_{t,s}^{(L)} (A) - \tau_{t,s}^{(L,\mathfrak{m})} (A) \right) B \right) = 0 \end{aligned} \tag{116}$$

for all elements $A \in \mathcal{U}_0$ and $B \in \mathcal{U}$ in order to prove the theorem.

Step 2: By Duhamel's formula and Equations (43)-(45), for any $L \in \mathbb{N}$ and $s,t \in \mathbb{R}$,

$$\daleth_{t,s}^{(L)} - \tau_{t,s}^{(L,\mathfrak{m})} = \int_s^t \daleth_{u,s}^{(L)} \circ \left(\delta_L^{\mathbf{\Phi}^{(\mathfrak{m},\varpi^{\mathfrak{m}}(s,u))(\hat{\rho})}} - \delta_L^{\mathfrak{m}(u)} \right) \circ \tau_{t,u}^{(L,\mathfrak{m})} \mathrm{d}u , \tag{117}$$

where δ_L^{Φ} and $\delta_L^{\tilde{\mathfrak{m}}}$ are the bounded symmetric derivations defined by (43) for any model $\tilde{\mathfrak{m}} \in \mathcal{M}$ and short-range interaction $\Phi \equiv (\Phi, 0) \in \mathcal{W} \subseteq \mathcal{M}$. Observe from Equations (39), (41), (43) and Definition 3.1 together with explicit computations that, for any $L \in \mathbb{N}$, $u \in \mathbb{R}$ and $A \in \mathcal{U}$,

$$\begin{aligned} & \left(\delta_L^{\mathbf{\Phi}^{(\mathfrak{m},\varpi^{\mathfrak{m}}(s,u))(\hat{\rho})}} - \delta_L^{\mathfrak{m}(u)} \right) (A) \\ = & \sum_{n=2}^{\infty} \sum_{m=1}^{n} \int_{\mathbb{S}^n} \mathfrak{a}_n (u) \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)} \right) \\ & \qquad \left(\left(\prod_{j=1}^{m-1} \varpi^{\mathfrak{m}} (s,u;\hat{\rho}) (\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}) \right) \delta_L^{\Psi^{(m)}} (A) \left(\prod_{j=m+1}^{n} \varpi^{\mathfrak{m}} (s,u;\hat{\rho}) (\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}) \right) \right. \\ & \qquad \left. - \left(\prod_{j=1}^{m-1} \frac{U_L^{\Psi^{(j)}}}{|\Lambda_L|} \right) \delta_L^{\Psi^{(m)}} (A) \left(\prod_{j=m+1}^{n} \frac{U_L^{\Psi^{(j)}}}{|\Lambda_L|} \right) \right) , \end{aligned} \tag{118}$$

where

$$\prod_{j=1}^{m-1<j} (\cdot) \doteq 1 \doteq \prod_{j=m+1>n}^{n} (\cdot) .$$

Combining (118) with (117), we deduce that, for any $L \in \mathbb{N}$, $s,t \in \mathbb{R}$, $A \in \mathcal{U}_0$ and $B \in \mathcal{U}$,

$$\begin{aligned} & \left| \hat{\rho} \left(B^* \left(\daleth_{t,s}^{(L)} (A) - \tau_{t,s}^{(L,\mathfrak{m})} (A) \right) B \right) \right| \\ \leq & \sum_{n=2}^{\infty} \sum_{m=1}^{n} \int_s^t \mathrm{d}u \int_{\mathbb{S}^n} \mathfrak{a} (u)_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)} \right) \left| \mathbf{Y}_L^{(n,m)} \left(u; \Psi^{(1)}, \ldots, \Psi^{(n)} \right) \right| , \end{aligned} \tag{119}$$

where, for any integer $n \geq 2$, $m \in \{1, \ldots, n\}$, $L \in \mathbb{N}$, $u \in \mathbb{R}$ and $\Psi^{(1)}, \ldots, \Psi^{(n)} \in \mathbb{S}$,

$$\begin{aligned} &\mathbf{Y}_L^{(n,m)}\left(u; \Psi^{(1)}, \ldots, \Psi^{(n)}\right) \qquad (120)\\ &\doteq \hat{\rho}\left(B^* \beth_{u,s}^{(L)}\left(\left(\prod_{j=1}^{m-1} \boldsymbol{\varpi}^{\mathfrak{m}}(s,u;\hat{\rho})\left(\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}\right)\right)\delta_L^{\Psi^{(m)}} \circ \tau_{t,u}^{(L,\mathfrak{m})}(A)\left(\prod_{j=m+1}^{n} \boldsymbol{\varpi}^{\mathfrak{m}}(s,u;\hat{\rho})\left(\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}\right)\right)\right)B\right)\\ &\quad -\hat{\rho}\left(B^* \beth_{u,s}^{(L)}\left(\left(\prod_{j=1}^{m-1} \frac{U_L^{\Psi^{(j)}}}{|\Lambda_L|}\right)\delta_L^{\Psi^{(m)}} \circ \tau_{t,u}^{(L,\mathfrak{m})}(A)\left(\prod_{j=m+1}^{n} \frac{U_L^{\Psi^{(j)}}}{|\Lambda_L|}\right)\right)B\right). \end{aligned}$$

By (29) and (40),

$$\left|\boldsymbol{\varpi}^{\mathfrak{m}}(s,u;\hat{\rho})\left(\mathfrak{e}_{\Psi,\vec{\ell}}\right)\right| \leq \|\mathbf{F}\|_{1,\mathfrak{L}} \quad \text{and} \quad \left\||\Lambda_L|^{-1} U_L^{\Psi}\right\|_{\mathcal{U}} \leq \|\mathbf{F}\|_{1,\mathfrak{L}} \qquad (121)$$

for any $\Psi \in \mathbb{S}$, while, by Lemma 5.7,

$$\left\|\delta_L^{\Psi} \circ \tau_{t,u}^{(L,\mathfrak{m})}(A)\right\|_{\mathcal{U}} \leq 2\left|\mathcal{Z}\right| \|A\|_{\mathcal{U}}\, \mathrm{e}^{16\left(\mathbf{D}+2\|\mathbf{F}\|_{1,\mathfrak{L}}+1\right)\int_s^t \|\mathfrak{m}(\varsigma)\|_{\mathcal{M}}\mathrm{d}\varsigma} \qquad (122)$$

for any $\Psi \in \mathbb{S}$, $L \in \mathbb{N}$, $u \in [s,t]$, $\mathcal{Z} \subseteq \Lambda_L$ and $A \in \mathcal{U}_{\mathcal{Z}}$. Therefore, since $\beth_{u,s}^{(L)}$ is a $*$-automorphism of $\mathcal{U}$, for any integer $n \geq 2$, $m \in \{1, \ldots, n\}$, $L \in \mathbb{N}$, $u \in [s,t]$, $\Psi^{(1)}, \ldots, \Psi^{(n)} \in \mathbb{S}$, $\mathcal{Z} \subseteq \Lambda_L$ and $A \in \mathcal{U}_{\mathcal{Z}}$,

$$\left|\mathbf{Y}_L^{(n,m)}\left(u; \Psi^{(1)}, \ldots, \Psi^{(n)}\right)\right| \leq 4\left|\mathcal{Z}\right| \|A\|_{\mathcal{U}} \|B\|_{\mathcal{U}}^2 \|\mathbf{F}\|_{1,\mathfrak{L}}^{n-1}\, \mathrm{e}^{16\left(\mathbf{D}+2\|\mathbf{F}\|_{1,\mathfrak{L}}+1\right)\int_s^t \|\mathfrak{m}(\varsigma)\|_{\mathcal{M}}\mathrm{d}\varsigma}.$$

Since $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M})$, by Equations (31)-(32), (34) and (119), we deduce from the last estimate and Lebesgue's dominated convergence theorem that (116) follows from

$$\lim_{L\to\infty}\left|\mathbf{Y}_L^{(n,m)}\left(u; \Psi^{(1)}, \ldots, \Psi^{(n)}\right)\right| = 0 \qquad (123)$$

for $n \geq 2$, $m \in \{1, \ldots, n\}$, $u \in \mathbb{R}$, $\Psi^{(1)}, \ldots, \Psi^{(n)} \in \mathbb{S}$.

Step 3: For any integer $n \geq 2$, $k \in \{1, \ldots, n+1\}$, $l \in \{0, \ldots, n\}$, $k \leq l$ and $\Psi^{(k)}, \ldots, \Psi^{(l)} \in \mathbb{S}$, define

$$\boldsymbol{\Theta}_L^{(k,l)}\left(\Psi^{(k)}, \ldots, \Psi^{(l)}\right) \doteq \prod_{j=k}^{l} \boldsymbol{\varpi}^{\mathfrak{m}}(s,u;\hat{\rho})\left(\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}\right) - \prod_{j=k}^{l} \frac{U_L^{\Psi^{(j)}}}{|\Lambda_L|}. \qquad (124)$$

By Equations (120)-(122) for $A \in \mathcal{U}_0$, and since $\beth_{u,s}^{(L)}$ is a $*$-automorphism of $\mathcal{U}$, the limit we want to prove, i.e., (123), follows if we are able to show that

$$\lim_{L\to\infty}\left\|\pi_{\hat{\rho}} \circ \beth_{u,s}^{(L)}\left(\boldsymbol{\Theta}_L^{(k,l)}\left(\Psi^{(k)}, \ldots, \Psi^{(l)}\right)\right)\pi_{\hat{\rho}}(B)\,\Omega_{\hat{\rho}}\right\|_{\mathcal{H}_{\hat{\rho}}} = 0 \qquad (125)$$

for any integer $n \geq 2$, $k,l \in \{1, \ldots, n\}$ with $k \leq l$, $B \in \mathcal{U}$ and $\Psi^{(k)}, \ldots, \Psi^{(l)} \in \mathbb{S}$. Note that

$$\beth_{u,s} \circ \alpha_x = \alpha_x \circ \beth_{u,s}, \qquad x \in \mathbb{Z}^d,\ u,s \in \mathbb{R},$$

because $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_1)$. So, as $\hat{\rho} \in \mathcal{E}(E_{\vec{\ell}})$, we infer from [1, Proposition 3.7], Lemma 5.4, Corollary 5.6 and Equation (124) that

$$\begin{aligned} &\lim_{L\to\infty} \pi_{\hat{\rho}} \circ \beth_{u,s}^{(L)}\left(\boldsymbol{\Theta}_L^{(k,l)}\left(\Psi^{(k)}, \ldots, \Psi^{(l)}\right)\right)\pi_{\hat{\rho}}(B)\,\Omega_{\hat{\rho}}\\ &= \left(\prod_{j=k}^{l} \boldsymbol{\varpi}^{\mathfrak{m}}(s,u;\hat{\rho})\left(\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}\right) - \prod_{j=k}^{l} \hat{\rho} \circ \beth_{u,s}\left(\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}\right)\right)\pi_{\hat{\rho}}(B)\,\Omega_{\hat{\rho}} \end{aligned}$$

in the Hilbert space $\mathcal{H}_{\hat{\rho}}$, for any integer $n \geq 2$, $k,l \in \{1, \ldots, n\}$ with $k \leq l$, $B \in \mathcal{U}$ and $\Psi^{(k)}, \ldots, \Psi^{(l)} \in \mathbb{S}$. We finally invoke the self-consistency equations, that is, Theorem 4.1 (cf. (115)), to arrive from the last equality at Equation (125), which, by going backwards, in turn implies the theorem. ■

5.6 Direct Integrals of GNS Representations of Families of States

Theorem 4.3 is already proven for any ergodic state, by Theorem 5.8. In order to extend this result to all periodic states we need to decompose periodic states into ergodic states, as stated in Theorem 5.1. This leads to a technically convenient cyclic representation of each periodic state by using the direct integral of the GNS representation of ergodic states. In this subsection (and in the next one), we explain the direct integrals of GNS spaces in a general framework, since the particularities of the CAR algebra $\mathcal{U}$ are never used, apart from the general fact that it is a separable unital C^*-algebra.

Note additionally that this subsection is a collection of results that are rather standard. Nevertheless, it is important to present them in a coherent and self-contained manner because (i) we do not know any simple reference allowing the reader to get the relevant information in a concise way and (ii) the content of this subsection, in terms of results, notation, etc., is essential for Section 5.7. Proofs are also included here, making this subsection also useful for a full understanding by a non-expert reader.

Let $\mathcal{X}$ be any separable unital C^*-algebra and denote by $E \subseteq \mathcal{X}^*$ its set of states[16]. For any state $\rho \in E$, its GNS representation is denoted by the triplet $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ with the following definitions (see, e.g., [15, Section 2.3.3]):

<u>($\mathcal{H}$):</u> $\mathcal{L}_\rho \doteq \{X \in \mathcal{X} : \rho(X^*X) = 0\}$ is a closed left–ideal of $\mathcal{X}$ and $\mathcal{H}_\rho \doteq \overline{\mathcal{X}/\mathcal{L}_\rho}$ is the separable GNS Hilbert space with scalar product satisfying

$$\langle [X]_\rho, [Y]_\rho \rangle_{\mathcal{H}_\rho} = \rho\left(X^*Y\right) , \qquad [X]_\rho, [Y]_\rho \in \mathcal{X}/\mathcal{L}_\rho \subseteq \mathcal{H}_\rho . \tag{126}$$

<u>(π):</u> π_ρ is a representation of $\mathcal{X}$ on $\mathcal{B}(\mathcal{H}_\rho)$ uniquely defined by

$$\pi_\rho\left(A\right)[X]_\rho = [AX]_\rho \in \mathcal{X}/\mathcal{L}_\rho , \qquad A \in \mathcal{X},\ [X]_\rho \in \mathcal{X}/\mathcal{L}_\rho \subseteq \mathcal{H}_\rho . \tag{127}$$

<u>(Ω):</u> $\Omega_\rho \doteq [\mathfrak{1}]_\rho \in \mathcal{X}/\mathcal{L}_\rho \subseteq \mathcal{H}_\rho$ is a cyclic vector for $\pi_\rho(\mathcal{X})$, i.e., the set $\pi_\rho(\mathcal{X})\Omega_\rho$ is dense in $\mathcal{H}_\rho$.

We apply now the general theory discussed in Section 6 to the GNS objects (space, representation and cyclic vectors) of a separable C^*-algebra: Recall that E is compact and metrizable with respect to the weak* topology. Let Σ_E be the (Borel) σ-algebra generated by the weak* topology of E, like in Section 5.1. Pick any fixed (weak*) Borel subset $F \in \Sigma_E$ and denote by (F, Σ_F) the measurable space associated with the σ-algebra Σ_F generated by the weak* topology of F. Note that

$$\Sigma_F = \{F \cap \mathfrak{B} : \mathfrak{B} \in \Sigma_E\} .$$

(In Section 6, F refers to the set denoted by $\mathcal{Z}$.) Since compact metric spaces are complete and separable, the measurable space (F, Σ_F) is standard (Definition 6.13), whenever F is closed.

For any $F \in \Sigma_E$, $\mathcal{H}_F \doteq (\mathcal{H}_\rho)_{\rho \in F}$ is a family of separable GNS Hilbert spaces and, by the GNS construction, any element $X \in \mathcal{X}$ defines a vector field

$$v \doteq (v_\rho)_{\rho \in F} \doteq ([X]_\rho)_{\rho \in F} \tag{128}$$

over the family $\mathcal{H}_F$. Also, $\pi_F \doteq (\pi_\rho)_{\rho \in F}$ is a field of representations of $\mathcal{X}$ (Definition 6.22 (i)) on $\mathcal{H}_\rho$ for $\rho \in F$. Now, it suffices to use Theorems 6.5 and 6.8 to get the measurability of all these objects:

Lemma 5.9 (Measurability of GNS Hilbert spaces and representations)
Let $\mathcal{X}$ be a separable unital C^*-algebra and $F \in \Sigma_E$ any (weak) Borel subset of states. Then, $\mathcal{H}_F$ is measurable and there is a unique (equivalence class of) coherence α_F making, via (128), any countable (norm) dense set of $\mathcal{X}$ a sequence of α_F-measurable fields . Moreover, π_F is α_F-measurable.*

[16] I.e., continuous linear functionals $\rho \in \mathcal{X}^*$ which are positive, i.e., $\rho(A^*A) \geq 0$ for all $A \in \mathcal{X}$, and normalized, i.e., $\rho(\mathfrak{1}) = 1$.

Proof. Let $\{X^{(n)}\}_{n\in\mathbb{N}} \subseteq \mathcal{X}$ be any countable (norm) dense set. By the GNS construction, this set defines via (128) a countable dense subset of the GNS Hilbert space $\mathcal{H}_\rho$ for all $\rho \in E$. Hence, it defines a sequence of vector fields over $\mathcal{H}_F$, denoted by $(v^{(n)})_{n\in\mathbb{N}}$, where, for all $\rho \in E$, the set $\{v_\rho^{(n)}\}_{n\in\mathbb{N}}$ is dense in $\mathcal{H}_\rho$ and, in particular, total in this space. Since, for all $B \in \mathcal{X}$, the mapping $\rho \mapsto \rho(B)$ from F to $\mathbb{C}$ is weak*-continuous, we deduce from (126) that the mapping $\rho \mapsto \langle v_\rho^{(m)}, v_\rho^{(n)}\rangle_{\mathcal{H}_\rho}$ from F to $\mathbb{C}$ is Σ_F-measurable for all $m, n \in \mathbb{N}$. Thus, the sequence $(v^{(n)})_{n\in\mathbb{N}}$ of vector fields fulfills Conditions (a)-(b) of Theorem 6.5 and $\mathcal{H}_F$ is thus measurable.

We take the unique (up to an equivalence of coherences) coherence α_F for $\mathcal{H}_F$ such that $(v^{(n)})_{n\in\mathbb{N}}$ is a sequence of α_F-measurable fields, see Theorem 6.8 (i). Observing that the point-wise limit of a sequence of measurable functions is measurable, and using again (126) and Theorem 6.8 (i)-(ii), one checks that (the equivalence class of) the coherence α_F does not depend on the particular choice of the dense countable set $\{X^{(n)}\}_{n\in\mathbb{N}} \subseteq \mathcal{X}$ originally taken.

Finally, since $\{X^{(n)}\}_{n\in\mathbb{N}} \subseteq \mathcal{X}$ is any dense set, by (126)-(128), we have

$$\langle v_\rho^{(n)}, \pi_\rho(A) v_\rho^{(m)}\rangle_{\mathcal{H}_\rho} = \rho((X^{(n)})^* A X^{(m)}) , \qquad n, m \in \mathbb{N},\ A \in \mathcal{X} , \tag{129}$$

and the sequence $(v^{(n)})_{n\in\mathbb{N}}$ of vector fields satisfies Conditions (a)-(b) of Theorem 6.5. It follows from Theorem 6.8 and Definition 6.22 that π_ρ is α_F-measurable. ∎

In view of Section 5.1, take now any (weak*) closed set $F \in \Sigma_E$. The set of all positive Radon measures on (F, Σ_F) is denoted by $\mathrm{M}(F)$, each element of which corresponds (one-to-one) to a positive regular Borel measure. In fact, by separability of the C^*-algebra $\mathcal{X}$, E is metrizable and thus, any positive finite Borel measure on (E, Σ_E) is regular, as already explained for $\mathcal{X} = \mathcal{U}$.

On the one hand, by Lemma 5.9, one can construct a direct integral triplet $(\mathcal{H}_F^\oplus, \pi_F^\oplus, \Omega_F^\oplus)$ associated with each closed set $F \in \Sigma_E$ and any positive Radon measure $\mu \in \mathrm{M}(F)$:

<u>$(\mathcal{H}^\oplus)$:</u> $\mathcal{H}_F^\oplus$ is the direct integral Hilbert space associated with the measurable family $\mathcal{H}_F$. See Definition 6.9. Since $\mathcal{H}_F$ has a canonical (equivalence class of) coherence α_F, we use the notation

$$\mathcal{H}_F^\oplus \equiv \int_F^{\alpha_F} \mathcal{H}_\rho \mu(\mathrm{d}\rho) \equiv \int_F \mathcal{H}_\rho \mu(\mathrm{d}\rho) . \tag{130}$$

Similarly, we say that any vector field (respectively operator field) $v \doteq (v_\rho)_{\rho\in F}$ (respectively $A \doteq (A_\rho)_{\rho\in F}$) over $\mathcal{H}_F$ is measurable whenever it is α_F-measurable. When F is a closed set, (F, Σ_F, μ) is standard and hence, $\mathcal{H}_F^\oplus$ is in this case separable, by Theorem 6.14.

<u>$(\pi^\oplus)$:</u> $\pi_F^\oplus$ is the direct integral (representation) of the (α_F-) measurable representation field π_F, see (150)-(151). Similar to (130), we use the notation

$$\pi_F^\oplus \equiv \int_F^{\alpha_F} \pi_\rho \mu(\mathrm{d}\rho) \equiv \int_F \pi_\rho \mu(\mathrm{d}\rho) . \tag{131}$$

It is a representation of the separable unital C^*-algebra $\mathcal{X}$ on the direct integral Hilbert space $\mathcal{H}_F^\oplus$.

<u>$(\Omega^\oplus)$:</u> The vector $\Omega_F^\oplus$ is the element of the direct integral Hilbert space $\mathcal{H}_F^\oplus$ defined by

$$\Omega_F^\oplus \doteq \int_F^{\alpha_F} \Omega_\rho \mu(\mathrm{d}\rho) \equiv \int_F \Omega_\rho \mu(\mathrm{d}\rho) \in \mathcal{H}_F^\oplus . \tag{132}$$

This vector is well-defined because positive Radon measures on compact spaces are always finite. In contrast with the usual GNS representation, note that $\Omega_F^\oplus$ is generally not a cyclic vector for $\pi_F^\oplus(\mathcal{X})$.

On the other hand, as explained around (78), a positive Radon measure $\mu \in \mathrm{M}(F)$ represents a unique positive functional $\rho_\mu \in \mathcal{X}^*$, which is defined by

$$\rho_\mu(A) \doteq \int_F \rho(A)\,\mu(\mathrm{d}\rho) \ , \qquad A \in \mathcal{X} \ .$$

ρ_μ is called the barycenter of $\mu \in \mathrm{M}(F)$ and, as any positive functional of $\mathcal{X}^*$, it has a GNS representation. Clearly, the triplet $(\mathcal{H}_F^\oplus, \pi_F^\oplus, \Omega_F^\oplus)$ can be used to represent the positive functional ρ_μ, in the sense that

$$\rho_\mu(A) = \langle \Omega_F^\oplus, \pi_F^\oplus(A)\Omega_F^\oplus \rangle_{\mathcal{H}_F^\oplus} \ , \qquad A \in \mathcal{X} \ .$$

However, as already mentioned, $\Omega_F^\oplus$ is not necessarily a cyclic vector for $\pi_F^\oplus(\mathcal{X})$ and, in general, $(\mathcal{H}_F^\oplus, \pi_F^\oplus, \Omega_F^\oplus)$ is only quasi-equivalent (see [15, Theorem 2.4.26]) to any cyclic representation of ρ_μ:

Lemma 5.10 (Direct integral and GNS representations)
Let $\mathcal{X}$ be a separable unital C^*-algebra, $F \in \Sigma_E$ any (weak*) Borel subset of states and $\mu \in \mathrm{M}(F)$ a positive Radon measure with barycenter $\rho_\mu \in \mathcal{X}^*$ and associated cyclic representation $(\mathcal{H}_{\rho_\mu}, \pi_{\rho_\mu}, \Omega_{\rho_\mu})$. Then, $\pi_F^\oplus$ is quasi-equivalent to π_{ρ_μ}. It is equivalent to π_{ρ_μ} iff $\Omega_F^\oplus$ is a cyclic vector for $\pi_F^\oplus(\mathcal{X})$.

Proof. Fix all parameters of the lemma. Let P_F be the orthogonal projection on $\mathcal{H}_F^\oplus$ whose range $\mathrm{ran}P_F$ is the closure of the subspace $\pi_F^\oplus(\mathcal{X})\Omega_F^\oplus$:

$$\mathrm{ran}P_F \doteq \overline{\left\{\pi_F^\oplus(A)\Omega_F^\oplus : A \in \mathcal{X}\right\}} \subseteq \mathcal{H}_F^\oplus \ .$$

Clearly, $P_F \in [\pi_F^\oplus(\mathcal{X})]'$ and $P_F\Omega_F^\oplus = \Omega_F^\oplus$. Therefore, the mapping $A \mapsto \pi_F^\oplus(A)|_{\mathrm{ran}P_F}$ from $\mathcal{X}$ to $\mathcal{B}(P_F\mathcal{H}_F^\oplus)$ defines a representation $\tilde{\pi}_F^\oplus$ of $\mathcal{X}$ on the Hilbert space $P_F\mathcal{H}_F^\oplus$. Additionally, there is a $*$-isomorphism $\daleth$ from the von Neumann algebra $[\pi_F^\oplus(\mathcal{X})]''$ to $[\tilde{\pi}_F^\oplus(\mathcal{X})]''$ such that

$$\daleth\left(\pi_F^\oplus(A)\right) \doteq \pi_F^\oplus(A)|_{\mathrm{ran}P_F} = \tilde{\pi}_F^\oplus(A) \ , \qquad A \in \mathcal{X} \ .$$

This follows from the weak-operator continuity of $P_F(\cdot)P_F$. By [15, Theorem 2.4.26], the representations $\pi_F^\oplus$ and $\tilde{\pi}_F^\oplus$ are thus quasi-equivalent. On the other hand, by construction, $(P_F\mathcal{H}_F^\oplus, \tilde{\pi}_F^\oplus, \Omega_F^\oplus)$ is a cyclic representation for the positive functional ρ_μ and is thus spatially, or unitarily, equivalent to π_{ρ_μ}, by [15, Theorem 2.3.16] (which can trivially be extended to any positive functional of $\mathcal{X}^*$). ■

Unless $\Omega_F^\oplus$ is a cyclic vector for $\pi_F^\oplus(\mathcal{X})$, i.e., the orthogonal projection P_F of the last proof is the identity operator on $\mathcal{H}_F^\oplus$, the triplet $(\mathcal{H}_F^\oplus, \pi_F^\oplus, \Omega_F^\oplus)$ is not spatially equivalent to any cyclic representation of ρ_μ. In the following we discuss necessary and sufficient conditions on the positive Radon measure μ for $\Omega_F^\oplus$ to be cyclic, in order to have a direct integral decomposition of the cyclic representation of ρ_μ as $(\mathcal{H}_F^\oplus, \pi_F^\oplus, \Omega_F^\oplus)$.

For each $\mu \in \mathrm{M}(F)$, we define its restriction $\mu_{\mathfrak{B}} \in \mathrm{M}(F)$ to any Borel set $\mathfrak{B} \in \Sigma_F$ by

$$\mu_{\mathfrak{B}}(\mathfrak{B}_0) \doteq \mu(\mathfrak{B}_0 \cap \mathfrak{B}) \ , \qquad \mathfrak{B}_0 \in \Sigma_F \ .$$

See (77). If $\Omega_F^\oplus$ is a cyclic vector for $\pi_F^\oplus(\mathcal{X})$ then one easily checks that $\Omega_{\mathfrak{B}}^\oplus$ is also cyclic for $\pi_{\mathfrak{B}}^\oplus(\mathcal{X})$ and so, $(\mathcal{H}_{\mathfrak{B}}^\oplus, \pi_{\mathfrak{B}}^\oplus, \Omega_{\mathfrak{B}}^\oplus)$ is a cyclic representation of the barycenter $\rho_{\mu_{\mathfrak{B}}}$ of the restricted positive Radon measure $\mu_{\mathfrak{B}} \in \mathrm{M}(F)$. In particular, for all Borel sets $\mathfrak{B} \in \Sigma_F$,

$$\left(\mathcal{H}_{\rho_{\mu_{\mathfrak{B}}}} \oplus \mathcal{H}_{\rho_{\mu_{F\backslash\mathfrak{B}}}} \ , \ \pi_{\rho_{\mu_{\mathfrak{B}}}} \oplus \pi_{\rho_{\mu_{F\backslash\mathfrak{B}}}} \ , \ \Omega_{\rho_{\mu_{\mathfrak{B}}}} \oplus \Omega_{\rho_{\mu_{F\backslash\mathfrak{B}}}}\right) \tag{133}$$

is a cyclic representation of ρ_μ. This motivates the following definition:

Definition 5.11 (Orthogonal measures)
Let $\mathcal{X}$ be a separable unital C^*-algebra and $F \subseteq \mathcal{X}^*$ any weak*-closed subset of states. A positive Radon measure $\mu \in \mathrm{M}(F)$ is orthogonal whenever, for all Borel sets $\mathfrak{B} \in \Sigma_F$, (133) is a cyclic representation of its barycenter $\rho_\mu \in \mathcal{X}^*$, i.e., $\rho_{\mu_{\mathfrak{B}}} \perp \rho_{\mu_{F\backslash\mathfrak{B}}}$ (see around (75)).

As already explained, if $(\mathcal{H}_F^\oplus, \pi_F^\oplus, \Omega_F^\oplus)$ is a cyclic representation of ρ_μ then μ is an orthogonal measure. We prove below that this orthogonality property is also a sufficient condition for the cyclicity of $\Omega_F^\oplus$.

The positive Radon measures we are interested in concern those coming from the Choquet theorem [11, Theorem 10.18], which allow to decompose states of a compact convex set into extreme ones. Such measures are always probability measures, i.e., normalized positive Radon measures. The subset of all probability measures on (F, Σ_F) is denoted by $\mathrm{M}_1(F)$. So, for simplicity, we consider, from now on, only probability measures $\mu \in \mathrm{M}_1(F)$ on F.

Recall that $L^\infty(F,\mu)$ is the space of all (equivalence classes of) essentially bounded measurable complex-valued functions on F associated with the measure space (F, Σ_F, μ). The $\operatorname{ess\,sup}$ norm of $L^\infty(F,\mu)$ is denoted by $\|\cdot\|_\infty$. We give a first, very useful, lemma (cf. [15, Lemma 4.1.21]), similar to Theorem 6.18, which links $L^\infty(F,\mu)$ with the commutant $[\pi_{\rho_\mu}(\mathcal{X})]'$:

Lemma 5.12 (Bounded measurable functions and the GNS representation)
Let $\mathcal{X}$ be a separable unital C^*-algebra and $F \subseteq \mathcal{X}^*$ any weak*-closed subset of states. For any probability measure $\mu \in \mathrm{M}_1(F)$, there is a unique linear map $\varkappa_\mu : L^\infty(F,\mu) \to [\pi_{\rho_\mu}(\mathcal{X})]'$ such that

$$\int_F f(\rho)\rho(A)\mu(\mathrm{d}\rho) = \left\langle \Omega_{\rho_\mu}, \pi_{\rho_\mu}(A)\varkappa_\mu(f)\Omega_{\rho_\mu} \right\rangle_{\mathcal{H}_{\rho_\mu}} , \qquad A \in \mathcal{X} , \tag{134}$$

and $\|\varkappa_\mu(f)\|_{\mathcal{B}(\mathcal{H}_{\rho_\mu})} \leq \|f\|_\infty$ for all $f \in L^\infty(F,\mu)$, where $(\mathcal{H}_{\rho_\mu}, \pi_{\rho_\mu}, \Omega_{\rho_\mu})$ is any cyclic representation of the barycenter ρ_μ of μ. Additionally, $\varkappa_\mu$ is unital, positivity-preserving and, for all $f \in L^\infty(F,\mu)$, there is a unique $\varkappa_\mu(f) \in \mathcal{B}(\mathcal{H}_{\rho_\mu})$ satisfying (134).

Proof. Let $\mu \in \mathrm{M}_1(F)$ with $(\mathcal{H}_{\rho_\mu}, \pi_{\rho_\mu}, \Omega_{\rho_\mu})$ being a cyclic representation of its barycenter ρ_μ. Choose any (essentially) bounded function $f \in L^\infty(F,\mu)$. Observe that

$$(\pi_{\rho_\mu}(A)\Omega_{\rho_\mu}, \pi_{\rho_\mu}(B)\Omega_{\rho_\mu}) \mapsto \int_F f(\rho)\rho(A^*B)\mu(\mathrm{d}\rho) , \qquad A, B \in \mathcal{X} ,$$

defines a unique sesquilinear form on $\mathcal{H}_{\rho_\mu}$ bounded by $\|f\|_\infty$. Therefore, there is a unique $\varkappa_\mu(f) \in \mathcal{B}(\mathcal{H}_{\rho_\mu})$ such that $\|\varkappa_\mu(f)\|_{\mathcal{B}(\mathcal{H}_{\rho_\mu})} \leq \|f\|_\infty$ and satisfying (134). Clearly, if $f \geq 0$ then $\varkappa_\mu(f) \geq 0$ and $\varkappa_\mu(f) = 1$ when $f = 1$. Now, elementary computations show that $\varkappa_\mu(f) \in [\pi_{\rho_\mu}(\mathcal{X})]'$. See [15, Proof of Theorem 2.3.19] for more details. ■

If $(\mathcal{H}_F^\oplus, \pi_F^\oplus, \Omega_F^\oplus)$ is equivalent to any cyclic representation of ρ_μ then the mapping $\varkappa_\mu$ of Lemma 5.12 is nothing else as (up to unitary equivalence) the $*$-isomorphism from $L^\infty(F,\mu)$ to the abelian von Neumann algebra N_F of diagonalizable operators on $\mathcal{H}_F^\oplus$, as given in Theorem 6.18. In particular, $\varkappa_\mu$ has to be a $*$-homomorphism (and not only a linear map) whenever $(\mathcal{H}_F^\oplus, \pi_F^\oplus, \Omega_F^\oplus)$ is a cyclic representation of the barycenter ρ_μ of $\mu \in \mathrm{M}_1(F)$. We exploit now this observation to show that $(\mathcal{H}_F^\oplus, \pi_F^\oplus, \Omega_F^\oplus)$ is a cyclic representation of ρ_μ iff μ is an orthogonal measure. This is a consequence of the Tomita theorem:

Proposition 5.13 ("half" Tomita's theorem)
Let $\mathcal{X}$ be a separable unital C^*-algebra, $F \subseteq \mathcal{X}^*$ any weak*-closed subset of states and $\mu \in \mathrm{M}_1(F)$ any probability measure with barycenter ρ_μ and associated cyclic representation $(\mathcal{H}_{\rho_\mu}, \pi_{\rho_\mu}, \Omega_{\rho_\mu})$. If μ is an orthogonal measure, then the mapping $\varkappa_\mu$ of Lemma 5.12 is a $*$-isomorphism from $L^\infty(F,\mu)$ to $[\pi_{\rho_\mu}(\mathcal{X})]'$. In particular $\varkappa_\mu(L^\infty(F,\mu))$ is an abelian von Neumann subalgebra of the commutant $[\pi_{\rho_\mu}(\mathcal{X})]'$.

Proof. For completeness, we reproduce here the proof of [15, Proposition 4.1.22; (1) $\Rightarrow$ (2)]. Let $\mu \in \mathrm{M}_1(F)$ be an orthogonal probability measure with barycenter ρ_μ and corresponding cyclic representation $(\mathcal{H}_{\rho_\mu}, \pi_{\rho_\mu}, \Omega_{\rho_\mu})$. Then, by Definition 5.11 and [15, Theorem 2.3.16], for any Borel set $\mathfrak{B} \in \Sigma_F$, there is an orthogonal projection $P_\mathfrak{B} \in [\pi_{\rho_\mu}(\mathcal{X})]'$ acting on $\mathcal{H}_{\rho_\mu}$ such that, for all $A \in \mathcal{X}$,

$$\rho_{\mu_\mathfrak{B}}(A) = \left\langle \Omega_{\rho_\mu}, \pi_{\rho_\mu}(A) P_\mathfrak{B} \Omega_{\rho_\mu} \right\rangle_{\mathcal{H}_{\rho_\mu}} \qquad \text{and} \qquad \rho_{\mu_{F\backslash\mathfrak{B}}} = \left\langle \Omega_{\rho_\mu}, \pi_{\rho_\mu}(A)(\mathbf{1}_{\mathcal{H}_{\rho_\mu}} - P_\mathfrak{B}) \Omega_{\rho_\mu} \right\rangle_{\mathcal{H}_{\rho_\mu}} . \tag{135}$$

Therefore, if $\zeta_\mathfrak{B}$ denotes the characteristic function of any Borel set $\mathfrak{B} \in \Sigma_F$, then, by Lemma 5.12, $\varkappa_\mu(\zeta_\mathfrak{B}) = P_\mathfrak{B}$ is always an orthogonal projection and, for all $\mathfrak{B}_1, \mathfrak{B}_2 \in \Sigma_F$, $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \emptyset$,

$$\varkappa_\mu\left(\zeta_{\mathfrak{B}_1}\right) \varkappa_\mu\left(\zeta_{\mathfrak{B}_2}\right) = 0 . \tag{136}$$

This last equality comes from the fact that, whenever $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \emptyset$, $\zeta_{\mathfrak{B}_1} \leq 1 - \zeta_{\mathfrak{B}_2}$, leading by $\varkappa_\mu(1) = \mathbf{1}_{\mathcal{H}_{\rho_\mu}}$, linearity and positivity of $\varkappa_\mu$, to $\varkappa_\mu(\zeta_{\mathfrak{B}_1}) \leq \mathbf{1}_{\mathcal{H}_{\rho_\mu}} - \varkappa_\mu(\zeta_{\mathfrak{B}_2})$, in turn implying (136). Now, for any $\mathfrak{B}_1, \mathfrak{B}_2 \in \Sigma_F$, we can rewrite the characteristic functions $\zeta_{\mathfrak{B}_1}, \zeta_{\mathfrak{B}_2}$ as

$$\zeta_{\mathfrak{B}_1} = \zeta_{\mathfrak{B}_1}\zeta_{\mathfrak{B}_2} + \zeta_{\mathfrak{B}_1}(\zeta_F - \zeta_{\mathfrak{B}_2}) \qquad \text{and} \qquad \zeta_{\mathfrak{B}_2} = \zeta_{\mathfrak{B}_2}\zeta_{\mathfrak{B}_1} + \zeta_{\mathfrak{B}_2}(\zeta_F - \zeta_{\mathfrak{B}_1})$$

and use (136) for disjoint Borel subsets to deduce the equality

$$\varkappa_\mu(\zeta_{\mathfrak{B}_1})\varkappa_\mu(\zeta_{\mathfrak{B}_2}) = \varkappa_\mu(\zeta_{\mathfrak{B}_1}\zeta_{\mathfrak{B}_2}) , \qquad \mathfrak{B}_1, \mathfrak{B}_2 \in \Sigma_F . \tag{137}$$

All functions of $L^\infty(F, \mu)$ can be approximated in this Banach space by linear combination of characteristic functions and since $\varkappa_\mu$ is linear and a contractive mapping (Lemma 5.12), we deduce from (137) that $\varkappa_\mu$ is a $*$-homomorphism. Using now this property, one easily checks that $\varkappa_\mu(f)\Omega_{\rho_\mu} = 0$ iff $f = 0$. Since Ω_{ρ_μ} is a cyclic vector and $\varkappa_\mu(f) \in [\pi_{\rho_\mu}(\mathcal{X})]'$, it follows that $\varkappa_\mu$ is in fact a $*$-isomorphism. ∎

Proposition 5.13 is part of the Tomita theorem, which says that μ is an orthogonal measure iff the mapping $\varkappa_\mu$ of Lemma 5.12 is a $*$-homomorphism from $L^\infty(F, \mu)$ to $[\pi_{\rho_\mu}(\mathcal{X})]'$. See [15, Proposition 4.1.22]. In the case that $\varkappa_\mu(L^\infty(F, \mu))$ is an abelian von Neumann subalgebra of $[\pi_{\rho_\mu}(\mathcal{X})]'$, the corresponding direct integral decomposition of the representation π_{ρ_μ} is, as expected, $\pi_F^\oplus$, see (131). In fact, we have the following result, which refers to the Effros theorem [15, Theorem 4.4.9]:

Corollary 5.14 (Effros)
Let $\mathcal{X}$ be a separable unital C^*-algebra, $F \subseteq \mathcal{X}^*$ any weak*-closed subset of states and $\mu \in \mathrm{M}_1(F)$ any probability measure. μ is an orthogonal measure iff $\Omega_F^\oplus$ is a cyclic vector for $\pi_F^\oplus(\mathcal{X})$. In particular, if μ is orthogonal then $(\mathcal{H}_F^\oplus, \pi_F^\oplus, \Omega_F^\oplus)$ is a cyclic representation of its barycenter.

Proof. If $\Omega_F^\oplus$ is a cyclic vector for $\pi_F^\oplus(\mathcal{X})$ then, as already explained before Definition 5.11, μ has to be orthogonal. Now, suppose that the mapping $\varkappa_\mu$ of Lemma 5.12 is a $*$-homomorphism. In particular,

$$\varkappa_\mu(|f|^2) = \varkappa_\mu(f)^*\varkappa_\mu(f) , \qquad f \in L^\infty(F, \mu) . \tag{138}$$

Then, a simple computation using (138) and $\varkappa_\mu(f) \in [\pi_{\rho_\mu}(\mathcal{X})]'$ shows that, for all $A, B \in \mathcal{X}$ and $f \in L^\infty(F, \mu)$,

$$\left\|\pi_F^\oplus(B)\Omega_F^\oplus - \phi(f)\pi_F^\oplus(A)\Omega_F^\oplus\right\|_{\mathcal{H}_F^\oplus} = \left\|\pi_{\rho_\mu}(B)\Omega_{\rho_\mu} - \varkappa_\mu(f)\pi_{\rho_\mu}(A)\Omega_{\rho_\mu}\right\|_{\mathcal{H}_{\rho_\mu}} , \tag{139}$$

provided $\varkappa_\mu$ is a $*$-homomorphism, where

$$\phi(f) \doteq \int_F f(\rho)\mathbf{1}_{\mathcal{H}_\rho}\mu(\mathrm{d}\rho) \in \mathcal{B}\left(\mathcal{H}_F^\oplus\right) , \qquad f \in L^\infty(F, \mu) .$$

Now, by applying Lemma 6.11, observe that the set

$$\left\{\phi\left(f\right)\pi_{F}^{\oplus}\left(A\right)\Omega_{F}^{\oplus}:f\in L^{\infty}\left(F,\mu\right),\ A\in\mathcal{X}\right\}$$

is dense in $\mathcal{H}_{F}^{\oplus}$. Hence, if μ is orthogonal then, as $\Omega_{\rho_{\mu}}$ is cyclic for $\pi_{\rho_{\mu}}(\mathcal{X})$, we deduce from Proposition 5.13 and Equality (139) that $\Omega_{F}^{\oplus}$ is a cyclic vector for $\pi_{F}^{\oplus}(\mathcal{X})$. By Lemma 5.10, this implies in turn that $(\mathcal{H}_{F}^{\oplus},\pi_{F}^{\oplus},\Omega_{F}^{\oplus})$ is a cyclic representation of the barycenter ρ_{μ} of μ. ■

Therefore, Proposition 5.13, Corollary 5.14 and [15, Theorem 2.3.16] show, in a constructive way, that, for any cyclic representation $(\mathcal{H}_{\rho_{\mu}},\pi_{\rho_{\mu}},\Omega_{\rho_{\mu}})$ of an orthogonal probability measure $\mu\in\mathrm{M}_{1}(F)$, the von Neumann algebra $[\pi_{\rho_{\mu}}(\mathcal{X})]''$ is decomposable with respect to the abelian von Neumann subalgebra $\mathfrak{N}_{\rho_{\mu}}\doteq\varkappa_{\mu}(L^{\infty}\left(F,\mu\right))$ of the commutant $[\pi_{\rho_{\mu}}(\mathcal{X})]'$. See Definition 6.37. By Theorem 6.38, it means that

$$\mathfrak{N}_{\rho_{\mu}}\subseteq[\pi_{\rho_{\mu}}(\mathcal{X})]'\cap[\pi_{\rho_{\mu}}(\mathcal{X})]'' \tag{140}$$

while, by Corollary 6.36,

$$\left[\pi_{F}^{\oplus}(\mathcal{X})\right]''=\int_{F}[\pi_{\rho}(\mathcal{X})]''\,\mu(\mathrm{d}\rho)\ .$$

In this case, $\pi_{F}^{\oplus}$ is a so-called subcentral decomposition (of the representation $\pi_{\rho_{\mu}}$), see Definition 6.26 (ii.1).

5.7 C^{*}-Algebra of $\mathcal{X}$-valued Continuous Functions on States

We conclude with some properties of the C^{*}-algebra $C(F;\mathcal{X})$ of $\mathcal{X}$-valued weak*-continuous functions on any weak*-closed subset $F\subseteq E$ of states, where $\mathcal{X}$ is a separable unital C^{*}-algebra. Such a space is crucial to study the dynamics of long-range models at infinite volume. Similar to (46)-(47), $C(F;\mathcal{X})$ is endowed with the (point-wise) algebra operations inherited from $\mathcal{X}$. The unique C^{*}- norm of $C(F;\mathcal{X})$ is

$$\|f\|_{C(F;\mathcal{X})}\doteq\max_{\rho\in F}\|f\left(\rho\right)\|_{\mathcal{X}}\ ,\qquad f\in C\left(F;\mathcal{X}\right)\ .$$

We identify $\mathcal{X}$ with the subalgebra of constant functions of $C(F;\mathcal{X})$ and $C(F)\doteq C(F;\mathbb{C})$ with the subalgebra of functions whose values are multiples of the unit $\mathfrak{1}\in\mathcal{X}$. In other words,

$$\mathcal{X}\subseteq C\left(F;\mathcal{X}\right)\qquad\text{and}\qquad C\left(F\right)\doteq C\left(F;\mathbb{C}\right)\subseteq C\left(F;\mathcal{X}\right)\ .$$

In fact, $\mathcal{X}$ and $C(F)$ are C^{*}-subalgebras of $C(F;\mathcal{X})$ and the set $C(F)\cup\mathcal{X}$ generates this C^{*}-algebra:

Lemma 5.15 (Generation of $C(F;\mathcal{X})$ by elementary functions)
Let $\mathcal{X}$ be a separable unital C^{*}-algebra and $F\subseteq\mathcal{X}^{*}$ any weak*-closed subset of states. If $\mathcal{X}_{0}$ is a dense set of $\mathcal{X}$, then

$$C(F)\mathcal{X}_{0}\doteq\{fA:f\in C(F),\ A\in\mathcal{X}_{0}\}$$

is total in $C(F;\mathcal{X})$. If $\mathcal{X}_{0}$ is a $*$-subalgebra of $\mathcal{X}$ then $C(F)\mathcal{X}_{0}$ is a $*$-subalgebra of $C(F;\mathcal{X})$.

Proof. Use the density of $\mathcal{X}_{0}\subseteq\mathcal{X}$ as well as the weak*-compactness of F together with the existence of partitions of unity subordinated to any open cover of the metrizable (weak*-compact) space F. ■

Recall that Σ_{E} is the (Borel) σ-algebra generated by the weak* topology of the weak*-compact and metrizable space E of states. For any weak*-closed subset $F\in\Sigma_{E}$, Σ_{F} is the σ-algebra generated by the weak* topology of F. The GNS representation of any state $\rho\in E$ is denoted by $(\mathcal{H}_{\rho},\pi_{\rho},\Omega_{\rho})$. For any $F\in\Sigma_{E}$, $\mathcal{H}_{F}\doteq(\mathcal{H}_{\rho})_{\rho\in F}$ is a measurable family of separable GNS Hilbert spaces and $\pi_{F}\doteq(\pi_{\rho})_{\rho\in F}$ is a measurable field of GNS representations of $\mathcal{X}$ on $\mathcal{H}_{\rho}$ for $\rho\in F$, see Lemma 5.9. Similarly, for any weak*-closed subset $F\in\Sigma_{E}$ and all $f\in C(F;\mathcal{X})$, the bounded field $(\pi_{\rho}(f(\rho)))_{\rho\in F}$ of operators over $(\mathcal{H}_{\rho})_{\rho\in F}$ is also measurable:

Lemma 5.16 (Measurability of GNS representations applied to $\mathcal{X}$-valued functions)
Let $\mathcal{X}$ be a separable unital C^*-algebra and $F \subseteq \mathcal{X}^*$ any weak*-closed subset of states. Then, for all $f \in C(F;\mathcal{X})$, $(\pi_\rho(f(\rho)))_{\rho\in F}$ is a bounded (α_F-) measurable field of operators over $\mathcal{H}_F \doteq (\mathcal{H}_\rho)_{\rho\in F}$.

Proof. The proof is the same as the one proving that π_F is measurable. In particular, for any $f \in C(F;\mathcal{X})$, one uses (129) for $A = f(\rho)$ together with the weak*-continuity of functions of $C(F;\mathcal{X})$ and Theorem 6.8 (iii) to deduce the assertion. ■

For any $F \in \Sigma_E$ and any positive Radon measure $\mu \in \mathrm{M}(F)$, $\mathcal{H}_F^{\oplus}$ is the direct integral Hilbert space (130) associated with $\mathcal{H}_F$. The direct integral $\pi_F^{\oplus}$ of π_F, defined by (131), is a representation of $\mathcal{X}$ on the direct integral Hilbert space $\mathcal{H}_F^{\oplus}$. In the same way, for any weak*-closed subset F of states, we obtain from Lemma 5.16 a representation of $C(F;\mathcal{X})$ on $\mathcal{H}_F^{\oplus}$, that is, a $*$-homomorphism from $C(F;\mathcal{X})$ to $\mathcal{B}(\mathcal{H}_F^{\oplus})$. This representation is a natural extension of $\pi_F^{\oplus}$ from $\mathcal{X}$ to the C^*-algebra $C(F;\mathcal{X})$:

<u>$(\Pi^{\oplus})$:</u> $\Pi_F^{\oplus}$ is the (direct integral) representation of $C(F;\mathcal{X})$ on $\mathcal{H}_F^{\oplus}$ defined by

$$\Pi_F^{\oplus}(f) \doteq \int_F^{\alpha_F} \pi_\rho(f(\rho))\mu(\mathrm{d}\rho) \equiv \int_F \pi_\rho(f(\rho))\mu(\mathrm{d}\rho) , \qquad f \in C(F;\mathcal{X}) . \tag{141}$$

Compare with Equation (131) defining $\pi_F^{\oplus}$.

Recall that $\Omega_F^{\oplus} \in \mathcal{H}_F^{\oplus}$ is defined from the measurable family $\Omega_F \doteq (\Omega_\rho)_{\rho\in F}$ by the direct integral (132). By the Effros theorem (Corollary 5.14), if $\mu \in \mathrm{M}_1(F)$ is an orthogonal probability measure then $(\mathcal{H}_F^{\oplus}, \pi_F^{\oplus}, \Omega_F^{\oplus})$ is a cyclic representation of its barycenter ρ_μ and, by [15, Theorem 2.3.16], $(\mathcal{H}_F^{\oplus}, \pi_F^{\oplus}, \Omega_F^{\oplus})$ is equivalent to any cyclic representation of ρ_μ. In this case, $C(F;\mathcal{X})$ can be represented in the Hilbert space $\mathcal{H}_{\rho_\mu}$ of any cyclic representation of ρ_μ. More precisely, from the Tomita theorem (Proposition 5.13), we obtain the following assertion:

Proposition 5.17 (Orthogonal measures and representations of $C(F;\mathcal{X})$)
Let $\mathcal{X}$ be a separable unital C^*-algebra, $F \subseteq \mathcal{X}^*$ any weak*-closed subset of states and $\mu \in \mathrm{M}_1(F)$ any probability measure with barycenter ρ_μ and associated cyclic representation $(\mathcal{H}_{\rho_\mu}, \pi_{\rho_\mu}, \Omega_{\rho_\mu})$. If μ is an orthogonal measure, then there exists a unique representation Π_{ρ_μ} of $C(F;\mathcal{X})$ on $\mathcal{H}_{\rho_\mu}$ such that

$$\Pi_{\rho_\mu}|_{\mathcal{X}} = \pi_{\rho_\mu} \qquad \text{and} \qquad \Pi_{\rho_\mu}|_{C(F)} = \varkappa_\mu|_{C(F)}$$

with $\varkappa_\mu$ being the $*$-isomorphism from $L^\infty(F,\mu)$ to $[\pi_{\rho_\mu}(\mathcal{X})]'$ originally defined in Lemma 5.12. Additionally,

$$\left\langle \Omega_{\rho_\mu}, \Pi_{\rho_\mu}(f)\,\Omega_{\rho_\mu} \right\rangle_{\mathcal{H}_{\rho_\mu}} = \int_F \rho(f(\rho))\mu(\mathrm{d}\rho) , \quad f \in C(F;\mathcal{X}) . \tag{142}$$

Proof. Fix all parameters of the proposition. The representation $\Pi_F^{\oplus}$ of $C(F;\mathcal{X})$ on $\mathcal{H}_F^{\oplus}$ defined by (141) obviously satisfies $\Pi_F^{\oplus}|_{\mathcal{X}} = \pi_F^{\oplus}$, see (131). By (132) and (141),

$$\langle \Omega_F^{\oplus}, \Pi_F^{\oplus}(f)\Omega_F^{\oplus} \rangle_{\mathcal{H}_F^{\oplus}} = \int_F f(\rho)\rho(A)\mu(\mathrm{d}\rho) , \quad f \in C(F;\mathcal{X}) . \tag{143}$$

By Corollary 5.14 and [15, Theorem 2.3.16], if μ is an orthogonal measure then there is a unitary operator $\mathrm{U} : \mathcal{H}_F^{\oplus} \to \mathcal{H}_{\rho_\mu}$ such that $\mathrm{U}\Omega_F^{\oplus} = \Omega_{\rho_\mu}$ and $\pi_{\rho_\mu}(A) = \mathrm{U}\pi_F^{\oplus}(A)\mathrm{U}^*$ for any $A \in \mathcal{X}$. Let Π_{ρ_μ} be the representation of $C(F;\mathcal{X})$ on $\mathcal{H}_{\rho_\mu}$ defined by

$$\Pi_{\rho_\mu}(f) \doteq \mathrm{U}\Pi_F^{\oplus}(f)\,\mathrm{U}^* , \qquad f \in C(F;\mathcal{X}) . \tag{144}$$

Clearly, $\Pi_F^{\oplus}|_{\mathcal{X}} = \pi_F^{\oplus}$ yields $\Pi_{\rho_\mu}|_{\mathcal{X}} = \pi_{\rho_\mu}$ and, by (143), we also deduce Equation (142). In particular, for any $f \in C(F)$ and $A \in \mathcal{X}$,

$$\left\langle \Omega_{\rho_\mu}, \pi_{\rho_\mu}(A)\, \Pi_{\rho_\mu}(f)\, \Omega_{\rho_\mu} \right\rangle_{\mathcal{H}_{\rho_\mu}} = \int_F f(\rho)\rho(A)\mu(\mathrm{d}\rho) \ . \qquad (145)$$

By Lemma 5.12, $\Pi_{\rho_\mu}(f) = \varkappa_\mu(f)$ for any $f \in C(F)$. By Lemma 5.15, a representation of $C(F;\mathcal{X})$ on any Hilbert space is uniquely defined by its values on $\mathcal{X}$ and $C(F)$. Therefore, Π_{ρ_μ} is the unique representation which equals π_{ρ_μ} and $\varkappa_\mu$ on $\mathcal{X}$ and $C(F)$, respectively. ■

The support $\mathrm{supp}\mu$ of a positive Radon measure $\mu \in \mathrm{M}(F)$ is, by definition, the set

$$\{\rho \in F : \mu(V_\rho) > 0 \text{ for any weak}^*\text{-open neighborhood } V_\rho \text{ of } \rho\} \subseteq F \ .$$

In particular, it is weak*-compact. As Radon measures are (inner and outer) regular, $\mathrm{supp}\ \mu$ has full measure, i.e., $\mu(\mathrm{supp}\mu) = 1$. Therefore, one can assume, without loss of generality, that $F = \mathrm{supp}\mu$. In this case, if the state $\rho \in F$ is μ-almost everywhere faithful then the representations $\pi_F^{\oplus}$ and $\Pi_F^{\oplus}$ are faithful. If additionally $\mu \in \mathrm{M}_1(F)$ is an orthogonal probability measure on F, then, from Lemma 5.15, Proposition 5.17 and Equation (144), we deduce the existence of a unique $*$-isomorphism from the C^*-subalgebra of $\mathcal{B}(\mathcal{H}_{\rho_\mu})$ generated by $\pi_{\rho_\mu}(\mathcal{X}) \cup \varkappa_\mu(C(F))$ onto the C^*-algebra $C(F;\mathcal{X})$, satisfying

$$\varkappa_\mu(f)\pi_{\rho_\mu}(A) \mapsto fA$$

for all $f \in C(F)$ and $A \in \mathcal{X}$.

We conclude the section by observing that the barycenter of any probability measure on weak*-closed subset of states can naturally be extended to a state on the C^*-algebra $C(F;\mathcal{X})$:

Definition 5.18 (Extension of states on $\mathcal{X}$ to the whole C^*-algebra $C(F;\mathcal{X})$)
Let $\mathcal{X}$ be a separable unital C^*-algebra and $F \subseteq \mathcal{X}^*$ any weak*-closed subset of states. For any probability measure $\mu \in \mathrm{M}_1(F)$, its barycenter $\rho_\mu \in F$ can naturally be extended to a state on $C(F;\mathcal{X})$, again denoted by ρ_μ, via the definition

$$\rho_\mu(f) \doteq \int_F \rho(f(\rho))\mu(\mathrm{d}\rho) \ , \qquad f \in C(F;\mathcal{X}) \ .$$

Proposition 5.17 directly yields a natural characterization of cyclic representations of the extension to $C(F;\mathcal{X})$ of barycenters of orthogonal probability measures:

Theorem 5.19 (Cyclic representations of barycenters)
Let $\mathcal{X}$ be a separable unital C^*-algebra, $F \subseteq \mathcal{X}^*$ any weak*-closed subset of states and $\mu \in \mathrm{M}_1(F)$ any orthogonal probability measure with barycenter ρ_μ seen as a state of either $\mathcal{X}^*$ or $C(F;\mathcal{X})^*$.
(i) *$(\mathcal{H}_F^{\oplus}, \Pi_F^{\oplus}, \Omega_F^{\oplus})$ is a cyclic representation of $\rho_\mu \in C(F;\mathcal{X})^*$.*
(ii) *Let $(\mathcal{H}_{\rho_\mu}, \pi_{\rho_\mu}, \Omega_{\rho_\mu})$ be any cyclic representation of $\rho_\mu \in \mathcal{X}^*$. Then, there exists a unique representation Π_{ρ_μ} of $C(F;\mathcal{X})$ on $\mathcal{H}_{\rho_\mu}$ such that $\Pi_{\rho_\mu}|_{\mathcal{X}} = \pi_{\rho_\mu}$ and $(\Omega_{\rho_\mu}, \Pi_{\rho_\mu}, \mathcal{H}_{\rho_\mu})$ is a cyclic representation of $\rho_\mu \in C(F;\mathcal{X})^*$.*
(iii) *Conversely, let $(\mathcal{H}_{\rho_\mu}, \Pi_{\rho_\mu}, \Omega_{\rho_\mu})$ be any cyclic representation of $\rho_\mu \in C(F;\mathcal{X})^*$. Then, $(\mathcal{H}_{\rho_\mu}, \Pi_{\rho_\mu}|_{\mathcal{X}}, \Omega_{\rho_\mu})$ is a cyclic representation of $\rho_\mu \in \mathcal{X}^*$,*

$$[\Pi_{\rho_\mu}(C(F;\mathcal{X}))]'' = [\Pi_{\rho_\mu}(\mathcal{X})]'' \qquad \text{and} \qquad [\Pi_{\rho_\mu}(C(F))]'' \subseteq [\Pi_{\rho_\mu}(\mathcal{X})]' \cap [\Pi_{\rho_\mu}(\mathcal{X})]'' \ .$$

Proof. Fix all parameters of the theorem.
(i): By Corollary 5.14, $\Omega_F^{\oplus}$ is cyclic for the $*$-subalgebra

$$\pi_F^{\oplus}(\mathcal{X}) = \Pi_F^{\oplus}(\mathcal{X}) \subseteq \Pi_F^{\oplus}(C(F;\mathcal{X})) \ .$$

Therefore, it is also cyclic for $\Pi_F^{\oplus}(C(F;\mathcal{X}))$. Since $\Pi_F^{\oplus}$ is a representation of $C(F;\mathcal{X})$ on $\mathcal{H}_F^{\oplus}$, by (143), the first assertion follows.
(ii): Let $(\mathcal{H}_{\rho_\mu},\pi_{\rho_\mu},\Omega_{\rho_\mu})$ be any cyclic representation of $\rho_\mu \in \mathcal{X}^*$. By Proposition 5.17, there exists a unique representation Π_{ρ_μ} of $C(F;\mathcal{X})$ on $\mathcal{H}_{\rho_\mu}$ such that $\Pi_{\rho_\mu}|_{\mathcal{X}} = \pi_{\rho_\mu}$, $\Pi_{\rho_\mu}|_{C(F)} = \varkappa_\mu|_{C(F)}$ and, since Ω_{ρ_μ} is cyclic for the $*$- subalgebra $\pi_{\rho_\mu}(\mathcal{X})$, $(\Omega_{\rho_\mu},\Pi_{\rho_\mu},\mathcal{H}_{\rho_\mu})$ is a cyclic representation of $\rho_\mu \in C(F;\mathcal{X})^*$. See (142) and Definition 5.18. Assume now the existence of another representation $\tilde{\Pi}_{\rho_\mu}$ such that $\tilde{\Pi}_{\rho_\mu}|_{\mathcal{X}} = \pi_{\rho_\mu}$ and $(\Omega_{\rho_\mu},\tilde{\Pi}_{\rho_\mu},\mathcal{H}_{\rho_\mu})$ is a cyclic representation of $\rho_\mu \in C(F;\mathcal{X})^*$. Since

$$\tilde{\Pi}_{\rho_\mu}|_{\mathcal{X}} = \Pi_{\rho_\mu}|_{\mathcal{X}} = \pi_{\rho_\mu} ,$$

we deduce that

$$\left\langle \pi_{\rho_\mu}(A)\,\Omega_{\rho_\mu}, \left(\tilde{\Pi}_{\rho_\mu}(f) - \Pi_{\rho_\mu}(f)\right)\Omega_{\rho_\mu} \right\rangle_{\mathcal{H}_{\rho_\mu}} = 0 , \qquad f \in C(F),\ A \in \mathcal{X} .$$

By cyclicity of Ω_{ρ_μ} for $\pi_{\rho_\mu}(\mathcal{X})$, it follows that

$$\tilde{\Pi}_{\rho_\mu}(f) = \Pi_{\rho_\mu}(f) = \varkappa_\mu|_{C(F)} , \qquad f \in C(F) .$$

By Proposition 5.17, we conclude that $\tilde{\Pi}_{\rho_\mu} = \Pi_{\rho_\mu}$.
(iii): By Corollary 5.14, $(\mathcal{H}_F^{\oplus},\pi_F^{\oplus},\Omega_F^{\oplus})$ is a cyclic representation of $\rho_\mu \in \mathcal{X}^*$ and $(\mathcal{H}_F^{\oplus},\Pi_F^{\oplus},\Omega_F^{\oplus})$ is thus a cyclic representation of $\rho_\mu \in C(F;\mathcal{X})^*$, where $\Pi_F^{\oplus}$ is the representation of $C(F;\mathcal{X})$ on $\mathcal{H}_F^{\oplus}$ defined by (141). By [15, Theorem 2.3.16], any cyclic representation $(\mathcal{H}_{\rho_\mu},\Pi_{\rho_\mu},\Omega_{\rho_\mu})$ of $\rho_\mu \in C(F;\mathcal{X})^*$ is equivalent to $(\mathcal{H}_F^{\oplus},\pi_F^{\oplus},\Omega_F^{\oplus})$ and, since $\Omega_F^{\oplus}$ is cyclic for $\Pi_F^{\oplus}(\mathcal{X}) = \pi_F^{\oplus}(\mathcal{X})$, the unit vector Ω_{ρ_μ} is also cyclic for $\Pi_{\rho_\mu}(\mathcal{X})$. In particular, $(\mathcal{H}_{\rho_\mu},\Pi_{\rho_\mu}|_{\mathcal{X}},\Omega_{\rho_\mu})$ is a cyclic representation of $\rho_\mu \in \mathcal{X}^*$. Using the definition $\pi_{\rho_\mu} \doteq \Pi_{\rho_\mu}|_{\mathcal{X}}$, Equation (145) holds true, again by (143) and [15, Theorem 2.3.16]. Therefore, Π_{ρ_μ} must be the unique representation of Proposition 5.17. By Equation (140), $[\Pi_{\rho_\mu}(C(F))]''$ is an abelian von Neumann subalgebra of the center of $[\Pi_{\rho_\mu}(\mathcal{X})]''$. From Lemma 5.15 it follows that

$$[\Pi_{\rho_\mu}(C(F;\mathcal{X}))]'' = [\Pi_{\rho_\mu}(\mathcal{X}) \cup \Pi_{\rho_\mu}(C(F))]'' = [\Pi_{\rho_\mu}(\mathcal{X})]'' .$$

■

6 Appendix: Direct Integrals and Spatial Decompositions

In this section we review important aspects of the theory of direct integrals of measurable families of Hilbert spaces, operators, von Neumann algebras, and C^*-algebra representations, which are useful in the scope of the present work. Mathematical foundations of the theory go back to von Neumann in the pivotal paper [19][17], aiming to obtain factor decompositions of strongly closed operator algebras, i.e., von Neumann algebras.

Nowadays, constant-fiber spaces or algebras are much more popular[18] than the more general situation needed here, i.e., the non-constant fiber case, which were already introduced by von Neumann in [19]. Thus, for self-containedness of the paper and the reader's convenience, we concisely explain the general theory of direct integrals. For a more thorough exposition on the subject, as well as complete proofs, we refer to the monograph [12]. Indeed, in our opinion, the approach of [12], based on the notion of "*coherence*", is more intuitive, being more explicit, than other well-known mathematical expositions of direct integrals of separable Hilbert spaces. For another presentation of direct integrals, see, e.g., [15, Section 4.4, in particular 4.4.1].

[17] Note that this paper was already written in 1937-1938, but only published more than ten years later.

[18] For instance, the theory for constant-fiber Hilbert spaces is a standard tool to study Schrödinger operators with periodic potentials, as explained in [20, Section XIII.16].

Notation 6.1
For any set $\mathcal{Z}$, $\mathcal{T}_{\mathcal{Z}}$ always denotes a family $\mathcal{T}_{\mathcal{Z}} \doteq (\mathcal{T}_z)_{z\in\mathcal{Z}}$. If such a family is an operator or vector field, as defined above, we even omit the subscript $\mathcal{Z}$ to simplify expressions.

6.1 Measurable Families of Separable Hilbert Spaces

To start we fix some conventions. Through out Section 6, $\mathcal{H}$ (with decoration or not) always stands for either a separable (complex) Hilbert space or a family of such separable spaces. Recall that the σ-algebra generated by the norm topology coincides with the one generated by the weak topology of the Hilbert space $\mathcal{H}$. It is denoted here by $\mathfrak{F}_{\mathcal{H}}$. We say that a mapping from a measurable space $(\mathcal{Z}, \mathfrak{F})$ to $\mathcal{H}$ is *measurable* if it is $(\mathfrak{F}_{\mathcal{H}}, \mathfrak{F})$-measurable. Similarly, the weak-operator, σ-weak, strong (operator) and σ-strong topologies of the space $\mathcal{B}(\mathcal{H})$ of bounded (linear) operators acting on $\mathcal{H}$ all generate the same (Borel) σ-algebra, which we denote by $\mathfrak{F}_{\mathcal{B}(\mathcal{H})}$. Note, however, that the (Borel) σ-algebra generated by the norm topology of $\mathcal{B}(\mathcal{H})$ is generally strictly bigger than $\mathfrak{F}_{\mathcal{B}(\mathcal{H})}$. Again, we say that a mapping from $(\mathcal{Z}, \mathfrak{F})$ to $\mathcal{B}(\mathcal{H})$ is *measurable* if it is $(\mathfrak{F}_{\mathcal{B}(\mathcal{H})}, \mathfrak{F})$-measurable. $\mathbb{M}(\mathcal{Z}; \mathcal{H})$ and $\mathbb{M}(\mathcal{Z}; \mathcal{B}(\mathcal{H}))$ denote the spaces of measurable mappings from $(\mathcal{Z}, \mathfrak{F})$ to $\mathcal{H}$ and $\mathcal{B}(\mathcal{H})$, respectively. The following two lemmata are useful characterizations of $(\mathfrak{F}_{\mathcal{B}(\mathcal{H})}, \mathfrak{F})$- and $(\mathfrak{F}_{\mathcal{H}}, \mathfrak{F})$-mappings:

Lemma 6.2 (Characterization of measurable mappings)
Let $(\mathcal{Z}, \mathfrak{F})$ be a measurable space and $\mathfrak{T} \subseteq \mathcal{H}$ any total[19] family in a separable Hilbert space $\mathcal{H}$.
(i) *Any mapping $\varkappa : \mathcal{Z} \to \mathcal{H}$ is measurable iff the mapping $z \mapsto \langle v, \varkappa(z)\rangle_{\mathcal{H}}$ from $\mathcal{Z}$ to $\mathbb{C}$ is measurable for any $v \in \mathfrak{T}$. In this case, the mapping $z \mapsto \|\varkappa(z)\|_{\mathcal{H}}$ from $\mathcal{Z}$ to $\mathbb{R}$ is also measurable.*
(ii) *Any mapping $\varkappa : \mathcal{Z} \to \mathcal{B}(\mathcal{H})$ is measurable iff the mapping $z \mapsto \langle v, \varkappa(z)w\rangle_{\mathcal{H}}$ from $\mathcal{Z}$ to $\mathbb{C}$ is measurable for any $v, w \in \mathfrak{T}$. In this case, the mapping $z \mapsto \|\varkappa(z)\|_{\mathcal{B}(\mathcal{H})}$ from $\mathcal{Z}$ to $\mathbb{R}$ is also measurable.*

Proof. (i) refers to the fact that the measurability corresponds to the weak topology of the Hilbert space $\mathcal{H}$. Note that the mapping

$$z \mapsto \|\varkappa(z)\|_{\mathcal{H}} = \sup_{v\in\mathfrak{T}:\|v\|_{\mathcal{H}}=1} \langle v, \varkappa(z)\rangle_{\mathcal{H}}$$

from $\mathcal{Z}$ to $\mathbb{R}$ is also measurable, since $\mathcal{H}$ is separable and the supremum of a sequence of measurable functions is measurable. Similar arguments imply Assertion (ii). We omit the details. ■

Lemma 6.3 ($\mathbb{M}(\mathcal{Z}; \mathcal{H})$ as a $\mathbb{M}(\mathcal{Z}; \mathcal{B}(\mathcal{H}))$-module)
$\mathbb{M}(\mathcal{Z}; \mathcal{B}(\mathcal{H}))$ is a $*$-algebra and $\mathbb{M}(\mathcal{Z}; \mathcal{H})$ is a left $\mathbb{M}(\mathcal{Z}; \mathcal{B}(\mathcal{H}))$-module with respect to point-wise operations. In particular, for all $A, B \in \mathbb{M}(\mathcal{Z}; \mathcal{B}(\mathcal{H}))$ and all $\varphi \in \mathbb{M}(\mathcal{Z}; \mathcal{H})$, $A \cdot B, A^* \in \mathbb{M}(\mathcal{Z}; \mathcal{B}(\mathcal{H}))$ and $A\varphi \in \mathbb{M}(\mathcal{Z}; \mathcal{H})$, where $A \cdot B(z) \doteq A(z) \cdot B(z)$, $A^*(z) \doteq A(z)^*$ and $A\varphi(z) \doteq A(z)(\varphi(z))$ for $z \in \mathcal{Z}$.

Proof. As explained in [12, Chap. 2], the fact that $\mathbb{M}(\mathcal{Z}; \mathcal{B}(\mathcal{H}))$ is a $*$-algebra with respect to point-wise operations directly follows from Lemma 6.2 (ii), since the involution on $\mathcal{B}(\mathcal{H})$ is continuous in the weak-operator topology and the multiplication on $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ is jointly strongly continuous on bounded sets. The fact that $\mathbb{M}(\mathcal{Z}; \mathcal{H})$ is a left $\mathbb{M}(\mathcal{Z}; \mathcal{B}(\mathcal{H}))$-module is also clear. ■

It is important at this point to introduce the notion of measurable family of Hilbert spaces as defined in [12, Chap. 2].

[19] $\mathcal{H}$ is the norm closure of the linear hull of $\mathfrak{T}$.

Definition 6.4 (Measurable families of separable Hilbert spaces)
Let $(\mathcal{Z}, \mathfrak{F})$ be a measurable space. Then, $\mathcal{H}_{\mathcal{Z}} \doteq (\mathcal{H}_z)_{z\in\mathcal{Z}}$ is said to be a measurable family of Hilbert spaces if each subset

$$\mathcal{Z}_n \doteq \{z \in \mathcal{Z} : \dim \mathcal{H}_z = n\} \subseteq \mathcal{Z} , \qquad n \in \mathbb{N}_0 , \tag{146}$$

is measurable, i.e., $\mathcal{Z}_n \in \mathfrak{F}$ for all $n \in \mathbb{N}_0$.

Notice that, if $\mathcal{Z}_n$ is measurable for all $n \in \mathbb{N}_0$, then $\mathcal{Z}_\infty$ is also measurable[20].

Observe that, with point-wise operations, $\prod_{z\in\mathcal{Z}} \mathcal{H}_z$ and $\prod_{z\in\mathcal{Z}} \mathcal{B}(\mathcal{H}_z)$ are a vector space and a $*$-algebra, respectively. Elements of the vector space are named *vector fields* over the family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. Similarly, elements of the above $*$-algebra are *operator fields* over $\mathcal{H}_{\mathcal{Z}}$.

In the mathematical literature, measurable families of separable Hilbert spaces are often defined by the existence of a sequence of vector fields which is fiberwise total (or even dense) and whose fiberwise scalar products are measurable. See, e.g. [15, Definition 4.4.1B]. By [12, Proposition 8.1], this alternative definition is equivalent to Definition 6.4:

Theorem 6.5 (Measurable families of separable Hilbert spaces - Equivalent formulation)
Let $(\mathcal{Z}, \mathfrak{F})$ be a measurable space and $\mathcal{H}_{\mathcal{Z}}$ a family of separable Hilbert spaces. Then, $\mathcal{H}_{\mathcal{Z}}$ is measurable iff there is a sequence $(v^{(n)})_{n\in\mathbb{N}}$ of vector fields over $\mathcal{H}_{\mathcal{Z}}$ such that:
(a) *For all $m, n \in \mathbb{N}$, the mapping $z \mapsto \langle v_z^{(n)}, v_z^{(m)}\rangle_{\mathcal{H}_z}$ from $\mathcal{Z}$ to $\mathbb{C}$ is measurable.*
(b) *For each $z \in \mathcal{Z}$, the subset $\{v_z^{(n)}\}_{n\in\mathbb{N}}$ is total in $\mathcal{H}_z$.*

Proof. Let $(\mathcal{Z}, \mathfrak{F})$ be a measurable space and $\mathcal{H}_{\mathcal{Z}}$ a family of separable Hilbert spaces. In order to simplify the discussion of the proof, we assume, without loss of generality, that $\mathcal{Z}_0 = \emptyset$.

Let $\mathcal{H}_{\mathcal{Z}}$ be measurable and, for all $z \in \mathcal{Z}$, $(e_n^z)_{n=1}^{\dim \mathcal{H}_z}$ any orthonormal basis of the separable Hilbert space $\mathcal{H}_z$. Then define, for all $n \in \mathbb{N}$, $x_z^{(n)} \doteq e_n^z$ if $n \leq \dim \mathcal{H}_z$, and $x_z^{(n)} \doteq 0$ otherwise. With this definition, (b) with $v_z^{(n)} = x_z^{(n)}$ holds true, by construction. In order to prove (a), observe that, if $m \neq n$ then the mapping $z \mapsto \langle x_z^{(n)}, x_z^{(m)}\rangle_{\mathcal{H}_z}$ from $\mathcal{Z}$ to $\mathbb{C}$ is trivially measurable, for it is the zero function. Additionally, for all $n \in \mathbb{N}$, the mapping $z \mapsto \langle x_z^{(n)}, x_z^{(n)}\rangle_{\mathcal{H}_z}$ from $\mathcal{Z}$ to $\mathbb{C}$ is the characteristic function of the set $\{z \in \mathcal{Z} : \dim \mathcal{H}_z \geq n\}$ and is, hence, measurable, as $\mathcal{H}_{\mathcal{Z}}$ is a measurable family of Hilbert spaces. See Definition 6.4. Thus, there exists a sequence $(x^{(n)})_{n\in\mathbb{N}}$ of vector fields over $\mathcal{H}_{\mathcal{Z}}$ satisfying the following properties:

(ã) For all $m, n \in \mathbb{N}$, the mapping $z \mapsto \langle x_z^{(n)}, x_z^{(m)}\rangle_{\mathcal{H}_z}$ from $\mathcal{Z}$ to $\mathbb{C}$ is measurable.
($\tilde{\text{b}}$) For each $z \in \mathcal{Z}$, $(x_z^{(n)})_{n=1}^{\dim \mathcal{H}_z}$ is an orthonormal basis of $\mathcal{H}_z$ and $x_z^{(n)} = 0$ whenever $n > \dim \mathcal{H}_z$.

Conversely, assume the existence of a sequence $(v^{(n)})_{n\in\mathbb{N}}$ of vector fields over $\mathcal{H}_{\mathcal{Z}}$ satisfying Conditions (a)-(b). Apply next the Gram-Schmidt orthonormalization process to the total sequence $(v_z^{(n)})_{n\in\mathbb{N}}$ for each $z \in \mathcal{Z}$: $x_z^{(1)} \doteq \lceil v_z^{(1)} \rceil$ and, for all $n \in \mathbb{N}$, $n > 1$,

$$x_z^{(n)} \doteq \left\lceil v_z^{(n)} - \left\langle x_z^{(n-1)}, v_z^{(n)} \right\rangle_{\mathcal{H}_z} x_z^{(n-1)} - \cdots - \left\langle x_z^{(1)}, v_z^{(n)} \right\rangle_{\mathcal{H}_z} x_z^{(1)} \right\rceil ,$$

where, for all $w \in \mathcal{H}_z$, $\lceil w \rceil \doteq 0$ whenever $w = 0$ and $\lceil w \rceil \doteq \|w\|_{\mathcal{H}_z}^{-1} w$ otherwise. The new sequence $(x^{(n)})_{n\in\mathbb{N}}$ of vector fields over $\mathcal{H}_{\mathcal{Z}}$ satisfies, by construction, Conditions (ã)-($\tilde{\text{b}}$). In this case, by Lemma 6.2 (i), the mapping $z \mapsto \|x_z^{(n)}\|_{\mathcal{H}_z}$ from $\mathcal{Z}$ to $\mathbb{R}$ is measurable, while

$$\mathcal{Z}_n \doteq \{z \in \mathcal{Z} : \dim \mathcal{H}_z = n\} = \left\{ z \in \mathcal{Z} : \sum_{m\in\mathbb{N}} \|x_z^{(m)}\|_{\mathcal{H}_z} = n \right\} , \qquad n \in \mathbb{N} .$$

It follows that $\mathcal{H}_{\mathcal{Z}}$ is measurable, by Definition 6.4. ∎

[20] This results from the elementary facts that the complement of any measurable set is measurable while countable unions of measurable sets are measurable.

6.2 Coherences and Measurable Fields

We now introduce the notion of coherences and measurable fields. To this end, we denote by

$$\ell^2_\infty \doteq \left\{ (x_k)_{k\in\mathbb{N}} \subseteq \mathbb{C} : \sum_{k\in\mathbb{N}} |x_k|^2 < \infty \right\}$$

the Hilbert space of all square-summable sequences of complex numbers. For each integer $n \in \mathbb{N}$, let $\ell^2_n \subsetneq \ell^2_\infty$ be the subspace of sequences $(x_k)_{k\in\mathbb{N}} \in \ell^2_\infty$ such that $x_k = 0$ for all $k > n$.

Definition 6.6 (Coherences for families of separable Hilbert spaces)
Let $\mathcal{H}_\mathcal{Z} \doteq (\mathcal{H}_z)_{z\in\mathcal{Z}}$ be a family of separable Hilbert spaces. A family $\alpha_\mathcal{Z} = (\alpha_z)_{z\in\mathcal{Z}}$ is a coherence for $\mathcal{H}_\mathcal{Z}$ if, for each $z \in \mathcal{Z}$, α_z is a linear isometry[21] *from $\mathcal{H}_z$ into ℓ^2_∞ with range $\ell^2_{\dim \mathcal{H}_z}$.*

Note that the separability of each $\mathcal{H}_z$ is a necessary and sufficient condition for the existence of coherences. The concept of coherence is often omitted in the literature on direct integrals, but it is useful for it converts the case of fiber-dependent Hilbert spaces into the constant-fiber case.

We next introduce measurable fields with respect to some fixed coherence. To this end, recall that elements of the vector space $\prod_{z\in\mathcal{Z}} \mathcal{H}_z$ are named vector fields over the family $\mathcal{H}_\mathcal{Z}$ of separable Hilbert spaces. Similarly, elements of the $*$-algebra $\prod_{z\in\mathcal{Z}} \mathcal{B}(\mathcal{H}_z)$ are operator fields over $\mathcal{H}_\mathcal{Z}$. See also Notation 6.1.

Definition 6.7 (Measurability of fields and equivalence of coherences)
Let $(\mathcal{Z}, \mathfrak{F})$ be a measurable space and $\mathcal{H}_\mathcal{Z}$ a family of separable Hilbert spaces.
(i) *A vector field v (respectively operator field A) is called $\alpha_\mathcal{Z}$-measurable if the mapping $z \mapsto \alpha_z(v_z)$ from $\mathcal{Z}$ to ℓ^2_∞ (respectively $z \mapsto \alpha_z A_z \alpha_z^*$ from $\mathcal{Z}$ to $\mathcal{B}(\ell^2_\infty)$) is measurable.*
(ii) *Two coherences $\alpha_\mathcal{Z}$ and $\beta_\mathcal{Z}$ for $\mathcal{H}_\mathcal{Z}$ are equivalent if, for all vector fields v over $\mathcal{H}_\mathcal{Z}$, v is $\alpha_\mathcal{Z}$-measurable iff it is $\beta_\mathcal{Z}$-measurable.*

In this definition, ℓ^2_∞ and $\mathcal{B}(\ell^2_\infty)$ are seen as measurable spaces with respect to the σ-algebras $\mathfrak{F}_{\ell^2_\infty}$ and $\mathfrak{F}_{\mathcal{B}(\ell^2_\infty)}$, as defined above for any separable Hilbert space $\mathcal{H}$.

The $\alpha_\mathcal{Z}$-measurable vector (respectively operator) fields over $\mathcal{H}_\mathcal{Z}$ form a subspace of $\prod_{z\in\mathcal{Z}} \mathcal{H}_z$ (respectively $\prod_{z\in\mathcal{Z}} \mathcal{B}(\mathcal{H}_z)$). By Lemma 6.3, the $\alpha_\mathcal{Z}$-measurable operator fields over $\mathcal{H}_\mathcal{Z}$ even form a $*$-algebra. Moreover, by the same lemma, if v is a $\alpha_\mathcal{Z}$-measurable vector field and A is a $\alpha_\mathcal{Z}$-measurable operator field, then the mapping $z \mapsto A_z v_z$ from $\mathcal{Z}$ to $\prod_{z\in\mathcal{Z}} \mathcal{H}$ is again a $\alpha_\mathcal{Z}$-measurable vector field. By Lemma 6.2, if v, w are $\alpha_\mathcal{Z}$-measurable vector fields and A is a $\alpha_\mathcal{Z}$-measurable operator field then $z \mapsto \langle v_z, w_z \rangle_{\mathcal{H}_z}$ and $z \mapsto \|A_z\|_{\mathcal{B}(\mathcal{H}_z)}$ from $\mathcal{Z}$ to $\mathbb{C}$ are measurable functions[22].

Given a measurable space $(\mathcal{Z}, \mathfrak{F})$ and a measurable family $\mathcal{H}_\mathcal{Z}$ of separable Hilbert spaces, Theorem 6.5 yields a canonical procedure to construct coherences for $\mathcal{H}_\mathcal{Z}$:

Theorem 6.8 (Coherences associated with sequences of fields)
Let $(\mathcal{Z}, \mathfrak{F})$ be a measurable space and $\mathcal{H}_\mathcal{Z}$ a family of separable Hilbert spaces. Take any sequence $(v^{(n)})_{n\in\mathbb{N}}$ of vector fields over $\mathcal{H}_\mathcal{Z}$ satisfying Conditions (a)-(b) of Theorem 6.5. Then, one has:
(i) *Up to an equivalence of coherences, there is a unique coherence $\alpha_\mathcal{Z}$ for $\mathcal{H}_\mathcal{Z}$ such that $(v^{(n)})_{n\in\mathbb{N}}$ is a sequence of $\alpha_\mathcal{Z}$-measurable fields.*
(ii) *An arbitrary vector field w over $\mathcal{H}_\mathcal{Z}$ is $\alpha_\mathcal{Z}$-measurable iff, for all $n \in \mathbb{N}$, the mapping $z \mapsto \langle w_z, v_z^{(n)} \rangle_{\mathcal{H}_z}$ from $\mathcal{Z}$ to $\mathbb{C}$ is measurable.*
(iii) *Similar to (ii), if A is an operator field over $\mathcal{H}_\mathcal{Z}$ then it is $\alpha_\mathcal{Z}$-measurable iff, for all $n, m \in \mathbb{N}$, the mapping $z \mapsto \langle v_z^{(n)}, A_z v_z^{(m)} \rangle_{\mathcal{H}_z}$ from $\mathcal{Z}$ to $\mathbb{C}$ is measurable.*

[21] α_z is not defined as a linear isometry from $\mathcal{H}_z$ onto $\ell^2_{\dim \mathcal{H}_z}$ to avoid a z-dependent domain of its adjoint α_z^*.

[22] In order to prove the measurability of $z \mapsto \langle v_z, w_z \rangle_{\mathcal{H}_z}$, one also uses an orthonormal basis of ℓ^2_∞ together with the elementary facts that sums and products of measurable functions are measurable and the point-wise limit of a sequence of measurable complex-valued functions is also measurable.

Proof. For any sequence $(v^{(n)})_{n\in\mathbb{N}}$ of vector fields over $\mathcal{H}_{\mathcal{Z}}$ satisfying Conditions (a)-(b) of Theorem 6.5, one uses the Gram-Schmidt orthonormalization process to construct a sequence $(x^{(n)})_{n\in\mathbb{N}}$ of vector fields over $\mathcal{H}_{\mathcal{Z}}$ satisfying Conditions (ã)-($\tilde{\text{b}}$), as explained in the proof of Theorem 6.5. Let $\alpha_{\mathcal{Z}}$ be the unique coherence for $\mathcal{H}_{\mathcal{Z}}$ such that, for each $z \in \mathcal{Z}$ and $k \in \{1, 2, \ldots, \dim \mathcal{H}_z\}$, $\alpha_z x_z^{(n_k)} = e_k$ with n_k being the k-th natural number satisfying $x_z^{(n_k)} \neq 0$. Then, as one can easily check from Lemma 6.2, $(v^{(n)})_{n\in\mathbb{N}}$ is a sequence of $\alpha_{\mathcal{Z}}$-measurable fields and the coherence α_z we build satisfies (i)-(iii). ■

Theorem 6.8 gives a useful characterization for the set of $\alpha_{\mathcal{Z}}$-measurable fields of an implicitly defined coherence $\alpha_{\mathcal{Z}}$ for a measurable family of separable Hilbert spaces. Additionally, the proofs of Theorems 6.5 and 6.8 give an explicit, very natural, construction of coherences.

6.3 Direct Integrals of Measurable Families of Hilbert Spaces

Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a σ-finite measure space and $\alpha_{\mathcal{Z}}$ a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. Recall also Notation 6.1. Denote by

$$\tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus} \subseteq \prod_{z\in\mathcal{Z}} \mathcal{H}_z$$

the subspace of $\alpha_{\mathcal{Z}}$-measurable vector fields v over $\mathcal{H}_{\mathcal{Z}}$ for which the mapping $z \mapsto \|v_z\|_{\mathcal{H}_z}$ from $\mathcal{Z}$ to $\mathbb{C}$ belongs to the Hilbert space $L^2(\mathcal{Z}, \mu)$ of complex-valued functions that are square-integrable with respect to the σ-finite measure μ. A semi-inner-product on this space is naturally defined by

$$\langle v, w\rangle_{\tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus}} \doteq \int_{\mathcal{Z}} \langle v_z, w_z\rangle_{\mathcal{H}_z} \mu(\mathrm{d}z) , \qquad v, w \in \tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus} . \tag{147}$$

Then, as is usual, we define the seminorm

$$\|v\|_{\tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus}} \doteq \sqrt{\langle v, v\rangle_{\tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus}}} = \sqrt{\int_{\mathcal{Z}} \|v_z\|_{\mathcal{H}_z}^2 \mu(\mathrm{d}z)} , \qquad v \in \tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus} ,$$

and identify v and w whenever $\|v - w\|_{\tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus}} = 0$ to get a Hilbert space:

Definition 6.9 (Direct integrals of Separable Hilbert spaces)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a σ-finite measure space and $\alpha_{\mathcal{Z}}$ a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. The direct integral of $\mathcal{H}_{\mathcal{Z}}$ with respect to μ and $\alpha_{\mathcal{Z}}$, denoted by

$$\mathcal{H}_{\mathcal{Z}}^{\oplus} \equiv \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \mathcal{H}_z \mu(\mathrm{d}z) ,$$

is the Hilbert space of equivalence classes of elements of $\tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus}$, with inner product defined from (147). $\mathcal{H}_z$, $z \in \mathcal{Z}$, are named fiber Hilbert spaces.

There is a canonical mapping from $\tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus}$ to $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ defined by

$$v = (v_z)_{z\in\mathcal{Z}} \mapsto [v] \equiv \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} v_z \mu(\mathrm{d}z) . \tag{148}$$

To simplify notation, we often implicitly omit the distinction between $v \in \tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus}$ and the equivalence class $[v] \in \mathcal{H}_{\mathcal{Z}}^{\oplus}$.

The existence of coherences to define the direct integrals is very useful because it converts the study of non-constant fiber Hilbert spaces into the analysis of constant ones, in a natural way:

Lemma 6.10 (Conversion into direct integrals of constant fiber Hilbert spaces)
Let $(\mathcal{Z},\mathfrak{F},\mu)$ be a σ-finite measure space and $\alpha_{\mathcal{Z}}$ a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. For any $n\in\mathbb{N}_0\cup\{\infty\}$, let μ_n be the restriction to the measurable set $\mathcal{Z}_n$ of μ, see (146) by including the case $n=\infty$. Then, the mapping

$$\Upsilon_{\alpha_{\mathcal{Z}}}:\mathcal{H}_{\mathcal{Z}}^{\oplus}\equiv\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}}\mathcal{H}_z\mu(\mathrm{d}z)\rightarrow\bigoplus_{n\in\mathbb{N}_0\cup\{\infty\}}\int_{\mathcal{Z}_n}\ell_n^2\,\mu_n(\mathrm{d}z)\ ,$$

defined, for all $v\in\mathcal{H}_{\mathcal{Z}}^{\oplus}$, by

$$\Upsilon_{\alpha_{\mathcal{Z}}}\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}}v_z\mu(\mathrm{d}z)=\sum_{n\in\mathbb{N}_0\cup\{\infty\}}\int_{\mathcal{Z}_n}\alpha_z\left(v_z\right)\ \mu_n(\mathrm{d}z) \tag{149}$$

is a unitary mapping.

Proof. Fix all parameters of the lemma. For all $v\in\mathcal{H}_{\mathcal{Z}}^{\oplus}$, observe that

$$\|v\|_{\mathcal{H}_{\mathcal{Z}}^{\oplus}}^2\doteq\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}}\|v_z\|_{\mathcal{H}_z}^2\,\mu(\mathrm{d}z)=\sum_{n\in\mathbb{N}_0\cup\{\infty\}}\int_{\mathcal{Z}_n}\|\alpha_z\left(v_z\right)\|_{\ell_n^2}^2\ \mu_n(\mathrm{d}z)\ ,$$

using Definition 6.6 and Lebesgue's monotone convergence theorem. We thus deduce that $\Upsilon_{\alpha_{\mathcal{Z}}}$, as defined by (149), is a linear isometry. Additionally, any element

$$[w]\in\bigoplus_{n\in\mathbb{N}_0\cup\{\infty\}}\int_{\mathcal{Z}_n}\ell_n^2\,\mu_n(\mathrm{d}z)$$

is, by definition, a sequence $([w_n])_{n\in\mathbb{N}_0\cup\{\infty\}}$ with $[w_n]\in\int_{\mathcal{Z}_n}\ell_n^2\,\mu_n(\mathrm{d}z)$ for any $n\in\mathbb{N}_0\cup\{\infty\}$. Then, for any $n\in\mathbb{N}_0\cup\{\infty\}$ and any representative $(w_{n,z})_{z\in\mathcal{Z}_n}$ of the equivalence class $[w_n]$, define

$$v_z\doteq\alpha_z^*w_{n,z}\ ,\qquad z\in\mathcal{Z}_n,\ n\in\mathbb{N}_0\cup\{\infty\}\,,$$

and observe that the constructed element $(v_z)_{z\in\mathcal{Z}}\in\tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus}$ leads to a unique equivalence class

$$[v]\equiv\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}}v_z\mu(\mathrm{d}z)\in\mathcal{H}_{\mathcal{Z}}^{\oplus}\ ,$$

which satisfies $\Upsilon_{\alpha_{\mathcal{Z}}}[v]=[w]$. Therefore, $\Upsilon_{\alpha_{\mathcal{Z}}}$ is a surjective linear isometry between two Hilbert spaces, and thus a unitary mapping. ■

All the study of the general theory of direct integrals can be based on the well-known theory of constant fiber direct integrals. Additionally, by Lemma 6.10, two equivalent coherences $\alpha_{\mathcal{Z}}$ and $\beta_{\mathcal{Z}}$ for $\mathcal{H}_{\mathcal{Z}}$ (Definition 6.7 (ii)) clearly imply the same direct integral:

$$\mathcal{H}_{\mathcal{Z}}^{\oplus}\equiv\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}}\mathcal{H}_z\mu(\mathrm{d}z)=\int_{\mathcal{Z}}^{\beta_{\mathcal{Z}}}\mathcal{H}_z\mu(\mathrm{d}z)\ .$$

Note also that, similar to Theorem 6.5, there is at least one sequence of vector fields over $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ which is a total family in each fiber.

As is usual,

$$L^{\infty}\left(\mathcal{Z},\mu\right)\equiv L^{\infty}\left(\mathcal{Z},\mu;\mathbb{C}\right)$$

is the C^*-algebra of (equivalence classes of almost everywhere equal) measurable complex-valued functions on $\mathcal{Z}$ with

$$\|f\|_{L^{\infty}(\mathcal{Z},\mu)}\equiv\|f\|_{\infty}\doteq\operatorname*{ess\,sup}_{z\in\mathcal{Z}}|f\left(z\right)|<\infty$$

being the essential supremum of f associated with the (σ-finite) measure space $(\mathcal{Z}, \mathfrak{F}, \mu)$. As a Banach space, it is the (topological) dual space $L^1(\mathcal{Z}, \mu)^*$ of the Banach space

$$L^1(\mathcal{Z}, \mu) \equiv L^1(\mathcal{Z}, \mu; \mathbb{C})$$

of (equivalence classes of) complex-valued functions on $\mathcal{Z}$ that are absolutely integrable with respect to the (σ-finite) measure μ. Any element of $L^\infty(\mathcal{Z}, \mu)$ can also be seen[23] as a bounded operator acting, by the point-wise multiplication, on the Hilbert space

$$L^2(\mathcal{Z}, \mu) \equiv L^2(\mathcal{Z}, \mu; \mathbb{C})$$

of (equivalence class of almost everywhere equal) complex-valued functions on $\mathcal{Z}$ that are square-integrable with respect to the (σ-finite) measure μ.

Lemma 6.11 (Sequence of vector fields as fiberwise-total families)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a σ-finite measure space and $\alpha_{\mathcal{Z}}$ a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces.
(i) *There exists a sequence $(v^{(n)})_{n\in\mathbb{N}}$ in $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ such that $\{v_z^{(n)}\}_{n\in\mathbb{N}}$ is total in $\mathcal{H}_z$ for each $z \in \mathcal{Z}$.*
(ii) *Let $\{\varphi^{(i)}\}_{i\in I} \subseteq L^\infty(\mathcal{Z}, \mu) \equiv L^1(\mathcal{Z}, \mu)^*$ be a weak*-total family and $(v^{(n)})_{n\in\mathbb{N}}$ a sequence like in (i). Then the family of $\alpha_{\mathcal{Z}}$-measurable fields $\{\varphi^{(i)} v^{(n)}\}_{(i,n)\in I\times\mathbb{N}}$ is total in $\mathcal{H}_{\mathcal{Z}}^{\oplus}$. If, for all $z \in \mathcal{Z}$, $\{v_z^{(n)}\}_{n\in\mathbb{N}}$ is dense then $\{\varphi^{(i)} v^{(n)}\}_{(i,n)\in I\times\mathbb{N}}$ is dense in $\mathcal{H}_{\mathcal{Z}}^{\oplus}$.*

Proof. Fix all the assumptions of the lemma. Recall that $(e_n)_{n\in\mathbb{N}}$ is the canonical orthonormal basis of ℓ^2_∞. To prove (i), let $(x^{(n)})_{n\in\mathbb{N}}$ be the $\alpha_{\mathcal{Z}}$-measurable vector fields over $\mathcal{H}_{\mathcal{Z}}$ defined by $x_z^{(n)} \doteq \alpha_z^* e_n$ for each $z \in \mathcal{Z}$ and $n \in \mathbb{N}$. Since coherences are linear isometries from $\mathcal{H}_z$ into ℓ^2_∞ with range $\ell^2_{\dim \mathcal{H}_z}$ (Definition 6.6), the family $\{x_z^{(n)}\}_{n\in\mathbb{N}}$ is total in $\mathcal{H}_z$ for each $z \in \mathcal{Z}$, but $x^{(n)} \doteq (x_z^{(n)})_{z\in\mathcal{Z}}$, $n \in \mathbb{N}$, are not necessarily elements of $\tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus}$, for they are possibly non-square-integrable. Since μ is, by assumption, a σ-finite measure, there is a strictly positive measurable function f such that

$$\int_{\mathcal{Z}} f(z)\, \mu(\mathrm{d}z) < \infty \ .$$

Then, define $v_z^{(n)} = \sqrt{f(z)} x_z^{(n)}$ for each $z \in \mathcal{Z}$ and $n \in \mathbb{N}$ to arrive at Assertion (i).

In order to get Assertion (ii), it suffices to prove that any element $w \in \mathcal{H}_{\mathcal{Z}}^{\oplus}$ that is orthogonal to $\varphi^{(i)} v^{(n)}$ for all $(i, n) \in I \times \mathbb{N}$ must be zero. This is straightforward. See [12, Lemma 7.3] for more details. ■

The relation between different direct integrals relative to absolutely continuous measures is also very natural:

Lemma 6.12 (Sequence of vector fields as a total family in each fiber)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a σ-finite measure space and $\alpha_{\mathcal{Z}}$ a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. If $\tilde{\mu}$ is a σ-finite measure that is absolutely continuous with respect to μ, then

$$v_z \mapsto \left(\frac{\mathrm{d}\tilde{\mu}}{\mathrm{d}\mu}(z)\right)^{1/2} v_z$$

is a linear isometry from $\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \mathcal{H}_z \tilde{\mu}(\mathrm{d}z)$ to $\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \mathcal{H}_z \mu(\mathrm{d}z)$.

[23] In fact, $L^\infty(\mathcal{Z}, \mu)$ is $*$-isomorphic to an abelian von Neumann algebra.

Proof. The proof follows from direct computations. Note that the existence of the measurable function $\mathrm{d}\tilde{\mu}/\mathrm{d}\mu$ in $L^1(\mathcal{Z}, \mu)$, which defines the linear isometry, is a direct consequence of the Radon-Nikodym theorem. ■

Note finally that the direct integral $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ of Hilbert spaces, as defined above, is not necessarily separable. This very important property of a Hilbert space holds true when the measure space $(\mathcal{Z}, \mathfrak{F}, \mu)$ is *standard*, in the following sense:

Definition 6.13 (Standard measure spaces)
The measurable space $(\mathcal{Z}, \mathfrak{F})$ is standard if $\mathfrak{F}$ is the Borel σ-algebra of a polish space[24]*. The measure space $(\mathcal{Z}, \mathfrak{F}, \mu)$ is standard if it is σ-finite and $(\mathcal{Z}, \mathfrak{F})$ is standard as a measurable space.*

Standard measure spaces lead to the separability of direct integrals of families of separable Hilbert spaces:

Theorem 6.14 (Separability of direct integrals)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a standard measure space and $\alpha_{\mathcal{Z}}$ a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. Then, $\mathcal{H}_{\mathcal{Z}}^{\oplus} \equiv \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \mathcal{H}_z \mu(\mathrm{d}z)$ is separable.

Proof. By Lemma 6.10, we can assume without loss of generality that $\mathcal{H}_{\mathcal{Z}}$ is a family of constant separable Hilbert spaces, i.e., $\mathcal{H}_z = \mathcal{H}$ is fixed for all $z \in \mathcal{Z}$. In this special case, we can apply [12, Proposition 5.2] which states the existence of a unitary operator mapping $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ onto $L^2(\mathcal{Z}, \mu) \otimes \mathcal{H}$. It is well-known that $L^2(\mathcal{Z}, \mu)$ is separable when μ is a standard measure, see, e.g., [12, Corollary 5.3]. Since $\mathcal{H}$ is, by assumption, also a separable Hilbert space, the assertion follows. ■

6.4 Decomposable Operators

Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ still denote a σ-finite measure space and $\alpha_{\mathcal{Z}}$ a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. See also Notation 6.1. Let $A = (A_z)_{z \in \mathcal{Z}}$ be an $\alpha_{\mathcal{Z}}$-measurable operator field over $\mathcal{H}_{\mathcal{Z}}$. If A is μ-essentially bounded, i.e., the mapping $z \mapsto \|A_z\|_{\mathcal{B}(\mathcal{H}_z)}$ from $\mathcal{Z}$ to $\mathbb{C}$ belongs to $L^\infty(\mathcal{Z}, \mu)$, then there is a unique bounded operator acting on $\mathcal{H}_{\mathcal{Z}}^{\oplus}$, denoted by

$$\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z \mu(\mathrm{d}z) \in \mathcal{B}\left(\mathcal{H}_{\mathcal{Z}}^{\oplus}\right) ,$$

satisfying

$$\left(\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z \mu(\mathrm{d}z)\right) v = \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z v_z \, \mu(\mathrm{d}z) , \qquad v \in \mathcal{H}_{\mathcal{Z}}^{\oplus} .$$

(See also (148)). Operators of this type refer to *decomposable* operators:

Definition 6.15 (Decomposable operators)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a σ-finite measure space and $\alpha_{\mathcal{Z}}$ a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. $A \in \mathcal{B}(\mathcal{H}_{\mathcal{Z}}^{\oplus})$ is decomposable whenever there is a μ-essentially bounded, $\alpha_{\mathcal{Z}}$-measurable operator field $(A_z)_{z \in \mathcal{Z}}$ such that $A = \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z \mu(\mathrm{d}z)$. We denote by $M_{\mathcal{Z}} \subseteq \mathcal{B}(\mathcal{H}_{\mathcal{Z}}^{\oplus})$ the subspace of decomposable operators.

Important decomposable operators are the so-called *diagonalizable* ones:

[24] There is a metric $\mathfrak{d}$ on $\mathcal{Z}$ such that $(\mathcal{Z}, \mathfrak{d})$ is a separable and complete metric space and $\mathfrak{F}$ is the Borel σ–algebra associated with $\mathfrak{d}$.

Definition 6.16 (Diagonalizable operators)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a σ-finite measure space and $\alpha_{\mathcal{Z}}$ a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. $A \in \mathcal{B}(\mathcal{H}_{\mathcal{Z}}^{\oplus})$ is diagonalizable whenever there is $\varphi \in L^{\infty}(\mathcal{Z}, \mu)$ such that $A = \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \varphi(z) \mathbf{1}_{\mathcal{H}_z} \mu(\mathrm{d}z)$. We denote by $N_{\mathcal{Z}} \subseteq M_{\mathcal{Z}}$ the subspace of diagonalizable operators.

In order to explicitly characterize the subspaces $N_{\mathcal{Z}}$ and $M_{\mathcal{Z}}$ of operators, the existence of coherences to define the direct integral is very useful, by Lemma 6.10. First note the following fact:

Lemma 6.17 (Reduction to constant fiber Hilbert spaces)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a σ-finite measure space and $\alpha_{\mathcal{Z}}$ a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. Then,

$$M_{\mathcal{Z}} = \Upsilon_{\alpha_{\mathcal{Z}}}^{*} \bigoplus_{n \in \mathbb{N}_0 \cup \{\infty\}} M_{\mathcal{Z}}^{(n)} \Upsilon_{\alpha_{\mathcal{Z}}} \qquad \text{and} \qquad N_{\mathcal{Z}} = \Upsilon_{\alpha_{\mathcal{Z}}}^{*} \bigoplus_{n \in \mathbb{N}_0 \cup \{\infty\}} N_{\mathcal{Z}}^{(n)} \Upsilon_{\alpha_{\mathcal{Z}}}$$

with $\Upsilon_{\alpha_{\mathcal{Z}}}$ being the unitary mapping of Lemma 6.10, and where $M_{\mathcal{Z}}^{(n)}$ and $N_{\mathcal{Z}}^{(n)}$ are the subspaces of respectively decomposable and diagonalizable operators acting on the constant fiber direct integral $\int_{\mathcal{Z}_n} \ell_n^2 \, \mu(\mathrm{d}z)$ for each $n \in \mathbb{N}_0 \cup \{\infty\}$.

Proof. The proof is very similar to the one of Lemma 6.10 and we thus omit the details. See [12, p. 25]. ∎

Therefore, all the study of decomposable and diagonalizable operators can be based on the well-known theory of constant fiber direct integrals. We thus obtain the following result:

Theorem 6.18 (Structure of the subspace of diagonalizable operators)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a σ-finite measure space and $\alpha_{\mathcal{Z}}$ a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. $N_{\mathcal{Z}}$ is a von Neumann algebra on the Hilbert space $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ and the mapping

$$\varphi \mapsto \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \varphi(z) \mathbf{1}_{\mathcal{H}_z} \mu(\mathrm{d}z)$$

defines a $*$-isomorphism from the abelian von Neumann algebra $L^{\infty}(\mathcal{Z}, \mu) \subseteq \mathcal{B}(L^2(\mathcal{Z}, \mu))$ to $N_{\mathcal{Z}}$.

Proof. By Lemmata 6.10 and 6.17, we can assume without loss of generality that $\mathcal{H}_{\mathcal{Z}}$ is a family of constant separable Hilbert spaces, i.e., $\mathcal{H}_z = \mathcal{H}$ is fixed for all $z \in \mathcal{Z}$. Similar to the proof of Theorem 6.14, we apply again [12, Proposition 5.2] which directly implies that the set of diagonalizable operators acting on $\mathcal{H}_{\mathcal{Z}}^{\oplus} \equiv L^2(\mathcal{Z}, \mu) \otimes \mathcal{H}$ is a von Neumann algebra which is $*$-isomorphic to the abelian von Neumann algebra $L^{\infty}(\mathcal{Z}, \mu) \subseteq \mathcal{B}(L^2(\mathcal{Z}, \mu))$. ∎

The previous statement on diagonalizable operators has the following implication for the subspace $M_{\mathcal{Z}}$ of decomposable operators (cf. [12, Theorem 7.1 (iii)-(vii)]):

Theorem 6.19 (Structure of the subspace of decomposable operators)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a σ-finite measure space and $\alpha_{\mathcal{Z}}$ a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces.
(i) *$M_{\mathcal{Z}}$ is the commutant[25] of the abelian von Neumann algebra $N_{\mathcal{Z}}$, i.e., $M_{\mathcal{Z}} = N_{\mathcal{Z}}'$. In particular, $M_{\mathcal{Z}}$ is also a von Neumann algebra on the Hilbert space $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ and $M_{\mathcal{Z}}' = N_{\mathcal{Z}}$.*

[25] The commutant $\mathfrak{S}'$ of a set $\mathfrak{S} \subseteq \mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ being some Hilbert space) is, by definition, the set of all elements of $\mathcal{B}(\mathcal{H})$ that commute with all $A \in \mathfrak{S}$.

(ii) $A \doteq (A_z)_{z\in\mathcal{Z}} \mapsto \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z\mu(\mathrm{d}z)$ *defines a $*$-homomorphism from the $*$-algebra of the μ-essentially bounded and $\alpha_{\mathcal{Z}}$-measurable operator fields over $\mathcal{H}_{\mathcal{Z}}$ to $M_{\mathcal{Z}} \subseteq \mathcal{B}(\mathcal{H}_{\mathcal{Z}}^{\oplus})$ and*

$$\left\| \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z\mu(\mathrm{d}z) \right\|_{\mathcal{B}(\mathcal{H}_{\mathcal{Z}}^{\oplus})} = \operatorname*{ess\,sup}_{z\in\mathcal{Z}} \|A_z\|_{\mathcal{B}(\mathcal{H}_z)} .$$

(iii) *Let A, A_n, $n \in \mathbb{N}$, be essentially bounded $\alpha_{\mathcal{Z}}$-measurable fields of operators over $\mathcal{H}_{\mathcal{Z}}$ such that*

$$\operatorname*{ess\,sup}_{z\in\mathcal{Z}} \sup_{n\in\mathbb{N}} \|A_{n,z}\|_{\mathcal{B}(\mathcal{H}_z)} < \infty .$$

If $A_{n,z}$ converges in the strong operator topology of $\mathcal{B}(\mathcal{H}_z)$ to A_z μ-almost everywhere in $\mathcal{Z}$ when $n \to \infty$, then $\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_{n,z}\mu(\mathrm{d}z)$ tends to $\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z\mu(\mathrm{d}z)$ in the strong operator topology of $\mathcal{B}\left(\mathcal{H}_{\mathcal{Z}}^{\oplus}\right)$, as $n \to \infty$.
(iv) *Conversely, if $\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_{n,z}\mu(\mathrm{d}z)$ tends to $\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z\mu(\mathrm{d}z)$ in the strong operator topology when $n \to \infty$ then there is a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that $A_{n_k,z}$ converges in the strong operator topology to A_z μ-almost everywhere in $\mathcal{Z}$, as $k \to \infty$.*

Proof. (i) $M_{\mathcal{Z}} = N_{\mathcal{Z}}'$ is proven in [12, Theorem 6.2] for constant fiber direct integrals. Therefore, by Lemma 6.17 together with [12, Theorem 6.2],

$$N_{\mathcal{Z}}' = \Upsilon_{\alpha_{\mathcal{Z}}}^* \bigoplus_{n\in\mathbb{N}_0\cup\{\infty\}} \left(N_{\mathcal{Z}}^{(n)}\right)' \Upsilon_{\alpha_{\mathcal{Z}}} = M_{\mathcal{Z}}$$

which, combined with Theorem 6.18, implies Assertion (i). Note that $M_{\mathcal{Z}}' = N_{\mathcal{Z}}'' = N_{\mathcal{Z}}$ is a direct consequence of the celebrated bicommutant theorem [15, Theorem 2.4.11].

To prove Assertions (ii)-(iv) we can assume without loss of generality that $\mathcal{H}_{\mathcal{Z}}$ is a family of constant separable Hilbert spaces, by Lemmata 6.10 and 6.17. (ii) refers to [12, Proposition 6.1 (b)], which is straightforward to prove. (iii)-(iv) in the constant-fiber case is [12, Proposition 6.3]. In this case, one can again use [12, Proposition 5.2], i.e., $\mathcal{H}_{\mathcal{Z}}^{\oplus} \equiv L^2(\mathcal{Z},\mu) \otimes \mathcal{H}$. We omit the details. ■

By Theorems 6.18-6.19, if $(\mathcal{Z}, \mathfrak{F}, \mu)$ is standard then the space $N_{\mathcal{Z}}$ of diagonalizable operators is an abelian von Neumann over a separable Hilbert space. The following theorem says that this situation is universal, up to spatial isomorphisms[26]:

Theorem 6.20 (Abelian von Neumann algebras as spaces of diagonalizable operators)
Assume that $\mathfrak{N}$ is an abelian von Neumann algebra on a separable Hilbert space $\mathcal{H}$. Then, there is a standard measure space $(\mathcal{Z}, \mathfrak{F}, \mu)$, a $*$-isomorphism $\phi : L^\infty(\mathcal{Z},\mu) \to \mathfrak{N}$, a measurable family $\mathcal{H}_{\mathcal{Z}}$ of Hilbert spaces, a coherence $\alpha_{\mathcal{Z}}$ for $\mathcal{H}_{\mathcal{Z}}$ and a unitary mapping U from $\mathcal{H}$ to $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ such that

$$N_{\mathcal{Z}} = \mathrm{U}\mathfrak{N}\mathrm{U}^* \qquad \textit{and} \qquad \mathrm{U}\phi(f)\mathrm{U}^* = \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} f(z)\mathbf{1}_{\mathcal{H}_z}\mu(\mathrm{d}z) , \qquad f \in L^\infty(\mathcal{Z},\mu) .$$

Proof. Any abelian von Neumann algebra on a separable Hilbert space $\mathcal{H}$ is $*$-isomorphic to $L^\infty(\mathcal{Z},\mu)$ for some standard measure space $(\mathcal{Z}, \mathfrak{F}, \mu)$. This is a known result of the theory of abelian von Neumann algebras. See, e.g., [12, Proposition A.4]. The remaining part of the proof is not trivial and requires some rather long arguments. We thus refer to [12, Theorem 9.1] for a detailed proof. ■
This result motivates the following definition:

Definition 6.21 (Decomposition of Hilbert spaces via abelian von Neumann algebras)
Let $\mathfrak{N}$ be an abelian von Neumann algebra on a separable Hilbert space $\mathcal{H}$. We say that the direct integral Hilbert space $\mathcal{H}_{\mathcal{Z}}^{\oplus} \equiv \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \mathcal{H}_z\mu(\mathrm{d}z)$ is a decomposition of $\mathcal{H}$ with respect to the abelian von Neumann algebra $\mathfrak{N}$ whenever $(\mathcal{Z}, \mathfrak{F}, \mu)$ is a standard measure space, $\alpha_{\mathcal{Z}}$ is a coherence for a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces, and $N_{\mathcal{Z}}$ is spatially isomorphic to $\mathfrak{N}$.

[26] Von Neumann algebras $\mathfrak{M}_1, \mathfrak{M}_2$ over, respectively, the Hilbert spaces $\mathcal{H}_1$,$\mathcal{H}_2$ are spatially isomorphic iff there is a unitary map U from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $\mathfrak{M}_2 = \mathrm{U}\mathfrak{M}_1\mathrm{U}^*$.

By Theorem 6.20, a separable Hilbert space $\mathcal{H}$ admits a decomposition with respect to any abelian von Neumann algebra $\mathfrak{N}$ on $\mathcal{H}$. Moreover, the standard space $(\mathcal{Z}, \mathfrak{F}, \mu)$ and the fiber Hilbert spaces $\mathcal{H}_z$, $z \in \mathcal{Z}$, are unique up to certain natural equivalences. So, one can speak about *the* decomposition of $\mathcal{H}$ with respect to $\mathfrak{N}$. For more details, see [12, Theorem 9.2]. The decomposability of separable Hilbert spaces is pivotal in the sequel and an analog property holds true for von Neumann algebras and representations of C^*-algebras.

6.5 Direct Integrals of Representations of Separable Unital Banach $*$-Algebras

In this section, $\mathcal{X}$ denotes an arbitrary, separable, unital Banach $*$-algebra, also named (separable, unital) involutive Banach algebra. This means that $\mathcal{X}$ is a (complex) Banach algebra with a unit $\mathfrak{1}$ and is endowed with an antilinear involution $A \mapsto A^*$ from $\mathcal{X}$ to itself satisfying $(AB)^* = B^*A^*$ for all $A, B \in \mathcal{X}$. Like Hilbert spaces under consideration, here $\mathcal{X}$ is always assumed to be separable. The main example we have in mind is $\mathcal{X}$ being a separable unital C^*-algebra, i.e., a (separable) Banach $*$-algebra such that $\|A^*A\|_{\mathcal{X}} = \|A\|_{\mathcal{X}}^2$ for $A \in \mathcal{X}$. Recall that a representation[27] π of $\mathcal{X}$ on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism of $\mathcal{X}$ to $\mathcal{B}(\mathcal{H})$.

Definition 6.22 (Field of representations of separable unital Banach $*$-algebras)
(i) *For any set $\mathcal{Z}$, a field $\pi_{\mathcal{Z}}$ of representations of a separable unital Banach $*$-algebra $\mathcal{X}$ is a family $\pi_{\mathcal{Z}} \doteq (\pi_z)_{z\in\mathcal{Z}}$ of representations π_z of $\mathcal{X}$ on a separable (complex) Hilbert space $\mathcal{H}_z$ for all $z \in \mathcal{Z}$.*
(ii) *If $(\mathcal{Z}, \mathfrak{F})$ is a measurable space and $\alpha_{\mathcal{Z}}$ is a coherence for $\mathcal{H}_{\mathcal{Z}} \doteq (\mathcal{H}_z)_{z\in\mathcal{Z}}$, as defined in (i), we say that $\pi_{\mathcal{Z}}$ is $\alpha_{\mathcal{Z}}$-measurable iff $\pi_{\mathcal{Z}}(A) \doteq (\pi_z(A))_{z\in\mathcal{Z}}$ is a $\alpha_{\mathcal{Z}}$-measurable field of operators for all $A \in \mathcal{X}$. See Definition 6.7 (i).*

Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a σ-finite measure space and $\mathcal{H}_{\mathcal{Z}} \doteq (\mathcal{H}_z)_{z\in\mathcal{Z}}$ be the family of Definition 6.22. As representations are norm-contractive [15, Proposition 2.3.1], if $\pi_{\mathcal{Z}}$ is a $\alpha_{\mathcal{Z}}$-measurable field of representations of $\mathcal{X}$ then, for all $A \in \mathcal{X}$, $\pi_{\mathcal{Z}}(A)$ defines a decomposable bounded operator on the direct integral $\mathcal{H}_{\mathcal{Z}}^{\oplus} \equiv \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \mathcal{H}_z \mu(\mathrm{d}z)$ (Definitions 6.9 and 6.15). It is easy to check that the mapping

$$\pi_{\mathcal{Z}}^{\oplus} \equiv \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \pi_z \mu(\mathrm{d}z) : \mathcal{X} \to M_{\mathcal{Z}} \subseteq \mathcal{B}(\mathcal{H}_{\mathcal{Z}}^{\oplus}) \tag{150}$$

defined by

$$\pi_{\mathcal{Z}}^{\oplus}(A) \doteq \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \pi_z(A) \mu(\mathrm{d}z) , \qquad A \in \mathcal{X} , \tag{151}$$

is a representation of $\mathcal{X}$. We term it the *direct integral (representation)* of the representation field $\pi_{\mathcal{Z}}$. Note, additionally, that $N_{\mathcal{Z}} \subseteq [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'$, i.e., the subspace $N_{\mathcal{Z}}$ of diagonalizable operators (Definition 6.16) belongs to the commutant of the space $\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})$. See, e.g., Theorem 6.19 (i).

Similar to Definition 6.21, one can decompose a representation via an abelian von Neumann algebra as follows:

Definition 6.23 (Decomposition of representations via abelian von Neumann algebras)
Let Π be a representation of a separable unital Banach $*$-algebra $\mathcal{X}$ on the separable Hilbert space $\mathcal{H}$, $\mathfrak{N}$ an abelian von Neumann subalgebra of $[\Pi(\mathcal{X})]'$, $(\mathcal{Z}, \mathfrak{F}, \mu)$ a standard measure space and $\pi_{\mathcal{Z}}$ a $\alpha_{\mathcal{Z}}$-measurable field of representations of $\mathcal{X}$, as in Definition 6.22. $\pi_{\mathcal{Z}}^{\oplus} \equiv \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \pi_z \mu(\mathrm{d}z)$ is a direct integral decomposition of Π with respect to $\mathfrak{N}$ if there is a unitary mapping $\mathrm{U} : \mathcal{H} \to \mathcal{H}_{\mathcal{Z}}^{\oplus}$ such that, for all $A \in \mathcal{X}$,

$$\pi_{\mathcal{Z}}^{\oplus}(A) = \mathrm{U}\Pi(A)\mathrm{U}^* \qquad \textit{and} \qquad N_{\mathcal{Z}} = \mathrm{U}\mathfrak{N}\mathrm{U}^* ,$$

i.e., $\mathrm{U}\mathfrak{N}\mathrm{U}^*$ is precisely the algebra of diagonal operators on $\mathcal{H}_{\mathcal{Z}}^{\oplus}$, see Definition 6.16. In this case, we say that Π is decomposable with respect to $\mathfrak{N}$.

[27] A representation of a $*$-algebra $\mathcal{X}$ is also defined to be a pair, in this case $(\mathcal{H}, \pi)$. See [15, Definition 2.3.2].

The following result ensures the existence of decompositions of representations:

Theorem 6.24 (Decompositions of representations – I)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a σ-finite measure space and Π a representation of a separable unital Banach $*$-algebra $\mathcal{X}$ on a direct integral $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ (Definition 6.9) such that $N_{\mathcal{Z}} \subseteq [\Pi(\mathcal{X})]'$ (Definition 6.16). Then, there is a $\alpha_{\mathcal{Z}}$-measurable field $\pi_{\mathcal{Z}}$ of representations of $\mathcal{X}$ such that $\Pi = \pi_{\mathcal{Z}}^{\oplus}$.

Proof. By Theorem 6.19 (i), the assumption $N_{\mathcal{Z}} \subseteq [\Pi(\mathcal{X})]'$ implies that $\Pi(\mathcal{X}) \subseteq [\Pi(\mathcal{X})]'' \subseteq M_{\mathcal{Z}}$, i.e., for any $A \in \mathcal{X}$, there is a μ-essentially bounded, $\alpha_{\mathcal{Z}}$-measurable operator field $(A_z)_{z\in\mathcal{Z}}$ such that

$$\Pi(A) = \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z \mu(\mathrm{d}z) .$$

See Definition 6.15. It means that the field $\pi_{\mathcal{Z}}$ of mappings from $\mathcal{X}$ to $\mathcal{B}(\mathcal{H}_z)$ defined by $\pi_z(A) \doteq A_z$ may be a good candidate for a $\alpha_{\mathcal{Z}}$-measurable field of representations of $\mathcal{X}$. In fact, all $*$-algebraic operations in $M_{\mathcal{Z}}$ (Theorem 6.19 (i)) refers, μ-almost everywhere, to the corresponding operations in each fiber. There is, however, a complication here in converting $*$-algebraic operations in $M_{\mathcal{Z}}$ into fiberwise ones, because these only hold true μ-almost everywhere, on subsets of $\mathcal{Z}$ that possibly depend on the elements of $\mathcal{X}$ that are taken. Therefore, as $\mathcal{X}$ is a *separable* (unital) Banach $*$-algebra, one takes a *countable* dense subset $\mathcal{X}_0$ of $\mathcal{X}$ which is a $*$-algebra over $\mathbb{Q} + i\mathbb{Q}$. Since the countable union of measurable sets of zero measure has zero measure, there is a fixed subset $\mathcal{Z}_0 \subseteq \mathcal{Z}$ satisfying $\mu(\mathcal{Z}\backslash\mathcal{Z}_0) = 0$ such that the definitions $\pi_z(A) \doteq A_z$ for $z \in \mathcal{Z}_0$ and $\pi_z(A) \doteq 0$ for $z \in \mathcal{Z}\backslash\mathcal{Z}_0$ lead to a $\alpha_{\mathcal{Z}}$-measurable field $\pi_{\mathcal{Z}}$ of representations of $\mathcal{X}$ satisfying $\Pi = \pi_{\mathcal{Z}}^{\oplus}$. The desired properties of this family $\pi_{\mathcal{Z}}$ of mappings is a consequence of the density of $\mathcal{X}_0 \subseteq \mathcal{X}$ and Theorem 6.19 (iii), together with [15, Proposition 2.3.1]. For more details, see [12, Theorem 12.3]. ■

Corollary 6.25 (Decompositions of representations – II)
Let Π be any representation of a separable unital Banach $*$-algebra $\mathcal{X}$ on a (separable) Hilbert space $\mathcal{H}$. If $\mathfrak{N}$ is an abelian von Neumann subalgebra of $[\Pi(\mathcal{X})]'$, then it is decomposable with respect to $\mathfrak{N}$.

Proof. The assertion is a consequence of Theorems 6.20 and 6.24. ■
Observe that decompositions of a given representation Π with respect to $\mathfrak{N}$ are unique, up to natural equivalences. In particular, one can speak in this case about *the* direct integral decompositions of Π. For more details, see [12, Theorems 12.1 and 12.4].

Different types of direct integral decompositions can be defined:

Definition 6.26 (Special direct integral representations)
Under Conditions of Definition 6.23 we define the following terminology for the direct integral decomposition $\pi_{\mathcal{Z}}^{\oplus} \equiv \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \pi_z \mu(\mathrm{d}z)$ of Π with respect to $\mathfrak{N}$:
(i.1) *$\pi_{\mathcal{Z}}^{\oplus}$ is a maximal decomposition if $\mathfrak{N}$ is a maximal abelian von Neumann subalgebra of $[\Pi(\mathcal{X})]'$, i.e.,*

$$\mathfrak{N}' \cap [\Pi(\mathcal{X})]' \subseteq \mathfrak{N} .$$

(i.2) *$\pi_{\mathcal{Z}}^{\oplus}$ is an irreducible decomposition whenever π_z is irreducible for μ-almost everywhere $z \in \mathcal{Z}$, i.e., $\{0\}$ and $\mathcal{H}_z$ are the only closed subspaces of $\mathcal{H}_z$ that are invariant under the action of $\pi_z(\mathcal{X})$.*
(ii.1) *$\pi_{\mathcal{Z}}^{\oplus}$ is a subcentral decomposition if $\mathfrak{N}$ is contained in the center of the von Neumann algebra $[\Pi(\mathcal{X})]'$, i.e.,*

$$\mathfrak{N} \subseteq [\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]'' .$$

If the equality holds true, then we speak about the central decomposition.
(ii.2) *$\pi_{\mathcal{Z}}^{\oplus}$ is a factor decomposition whenever π_z is a factor representation of $\mathcal{X}$ for μ-almost everywhere $z \in \mathcal{Z}$, i.e.,*

$$[\pi_z(\mathcal{X})]' \cap [\pi_z(\mathcal{X})]'' = \mathbb{C}\mathbf{1}_{\mathcal{H}_z} \qquad (\mu\text{-almost everywhere}).$$

The maximality of the abelian von Neumann algebra is equivalent to the irreducibility of the representations appearing in direct integral decompositions, i.e., Definitions 6.26 (i.1) and (i.2) are equivalent:

Theorem 6.27 (Maximal versus irreducible decompositions)
Let Π be a representation of a separable unital Banach $*$-algebra $\mathcal{X}$ on a (separable) Hilbert space $\mathcal{H}$ and $\mathfrak{N}$ an abelian von Neumann subalgebra of $[\Pi(\mathcal{X})]'$. Let $\pi_{\mathcal{Z}}^{\oplus}$ be the direct integral decomposition of Π with respect to $\mathfrak{N}$ (Corollary 6.25). Then, $\mathfrak{N}$ is a maximal abelian von Neumann subalgebra of $[\Pi(\mathcal{X})]'$ iff π_z is irreducible for μ-almost everywhere $z \in \mathcal{Z}$.

Proof. Without loss of generality we assume that Π is a representation of $\mathcal{X}$ on a direct integral $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ (Definition 6.9) such that $N_{\mathcal{Z}} = \mathfrak{N} \subseteq [\Pi(\mathcal{X})]'$ (Definition 6.16), where $(\mathcal{Z}, \mathfrak{F}, \mu)$ is a standard measure space.

Assume that π_z is irreducible for μ-almost everywhere $z \in \mathcal{Z}$. $N_{\mathcal{Z}}$ is maximal abelian in $[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'$ iff $N_{\mathcal{Z}}' \cap [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]' \subseteq N_{\mathcal{Z}}$. Therefore, take $A \in N_{\mathcal{Z}}' \cap [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'$. By Theorem 6.19 (i), there is a μ-essentially bounded, $\alpha_{\mathcal{Z}}$-measurable operator field $(A_z)_{z\in\mathcal{Z}}$ such that $A = \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z \mu(\mathrm{d}z)$, see Definition 6.15. Using similar arguments as in the proof of Theorem 6.24, one uses a countable dense subset of $\mathcal{X}$ to prove that $A \in [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'$ yields $A_z \in [\pi_z(\mathcal{X})]'$ for μ-almost everywhere $z \in \mathcal{Z}$. Since π_z is μ-almost everywhere irreducible, $[\pi_z(\mathcal{X})]' = \mathbb{C}\mathbf{1}_{\mathcal{H}_z}$ for μ-almost everywhere $z \in \mathcal{Z}$, by [15, Proposition 2.3.8]. As a consequence, $A_z \in [\pi_z(\mathcal{X})]'$ for μ-almost everywhere $z \in \mathcal{Z}$ implies that $A \in N_{\mathcal{Z}}$.

The converse statement, namely the fact that $N_{\mathcal{Z}}' \cap [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]' \subseteq N_{\mathcal{Z}}$ yields the irreducibility of π_z for μ-almost everywhere $z \in \mathcal{Z}$, requires more arguments. First, an assertion which is similar to Lemmata 6.10 and 6.17 holds true for the direct integral representation $\pi_{\mathcal{Z}}^{\oplus}$, see [12, Eq. (*) in Section 11, p. 46]. So, one can assume without loss of generality that $\mathcal{H}_{\mathcal{Z}}$ is a family of constant separable Hilbert spaces. In this case, one then uses the properties of standard measure spaces (see [12, Theorems 4.1 and 4.3]) to show the existence of two Borel sets $\mathcal{Z}_0, \mathcal{Z}_1 \subseteq \mathcal{Z}$ and a $\alpha_{\mathcal{Z}}$-measurable operator field $(A_z)_{z\in\mathcal{Z}_0}$ over $\mathcal{H}_{\mathcal{Z}_0} \doteq (\mathcal{H}_z)_{z\in\mathcal{Z}_0}$ such that

$$\mu\left(\mathcal{Z}_1 \backslash \mathcal{Z}_0\right) = 0 , \qquad \int_{\mathcal{Z}_0}^{\alpha_{\mathcal{Z}}} A_z \mu(\mathrm{d}z) \in (M_{\mathcal{Z}} \backslash N_{\mathcal{Z}}) \cap [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'$$

and

$$\mathcal{Z}_0 \subseteq S \doteq \{z \in \mathcal{Z} : \pi_z \text{ is not irreducible}\} \subseteq \mathcal{Z}_1 .$$

For the precise arguments leading to these facts, see [12, Theorem 13.1]. By Theorem 6.19 (i) recall that $M_{\mathcal{Z}} = N_{\mathcal{Z}}'$ and since, by assumption, $N_{\mathcal{Z}}' \cap [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]' \subseteq N_{\mathcal{Z}}$, it follows that $\mu(\mathcal{Z}_0) = 0$, which in turn implies that $\mu(\mathcal{Z}_1) = 0$. ∎

By Theorem 6.27, if Π is a representation of $\mathcal{X}$ on $\mathcal{H}$ then *maximal* abelian von Neumann subalgebras of $[\Pi(\mathcal{X})]'$ determine decompositions of Π with respect to *irreducible* representations. Observe that the representation Π is irreducible iff $[\Pi(\mathcal{X})]' = \mathbb{C}\mathbf{1}_{\mathcal{H}}$, by [15, Proposition 2.3.8]. Such a representation is a particular example of a factor representation, for one trivially has in this case that

$$[\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]'' = \mathbb{C}\mathbf{1}_{\mathcal{H}} .$$

The next result establishes a relation between the central decomposition, i.e., the decompositions with respect to the center $[\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]''$ of the von Neumann algebra $[\Pi(\mathcal{X})]'$, and factor decompositions:

Theorem 6.28 (Central versus factor decompositions)
Let Π be a representation of a separable unital Banach $*$-algebra $\mathcal{X}$ on a (separable) Hilbert space $\mathcal{H}$ and $\mathfrak{N}$ an abelian von Neumann subalgebra of $[\Pi(\mathcal{X})]'$. Let $\pi_{\mathcal{Z}}^{\oplus}$ be the direct integral decomposition

of Π *with respect to* $\mathfrak{N}$ *(Corollary 6.25).*
(i) *If* π_z *is a factor representation of* $\mathcal{X}$ *for* μ*-almost everywhere* $z \in \mathcal{Z}$ *then* $[\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]'' \subseteq \mathfrak{N}$.
(ii) *If* $\mathfrak{N} = [\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]''$ *then* π_z *is a factor representation of* $\mathcal{X}$ *for* μ*-almost everywhere* $z \in \mathcal{Z}$.

Proof. Without loss of generality we assume that $\Pi = \pi_{\mathcal{Z}}^{\oplus}$ is a representation of $\mathcal{X}$ on a direct integral $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ (Definition 6.9) such that $\mathfrak{N} = N_{\mathcal{Z}} \subseteq [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'$ (Definition 6.16), where $(\mathcal{Z}, \mathfrak{F}, \mu)$ is a standard measure space.

(i) On the one hand, by Theorem 6.19 (i), $N_{\mathcal{Z}} \subseteq [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'$ yields $[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'' \subseteq M_{\mathcal{Z}}$. Therefore, for any $A \in [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]' \cap [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]''$, there is a μ-essentially bounded, $\alpha_{\mathcal{Z}}$-measurable operator field $(A_z)_{z\in\mathcal{Z}}$ such that $A = \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z \mu(\mathrm{d}z)$ and $A_z \in [\pi_z(\mathcal{X})]'$ for μ-almost everywhere $z \in \mathcal{Z}$, using similar arguments as in the proof of Theorem 6.27. On the other hand, since $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ is separable (Theorem 6.14), one can apply the Kaplanski density theorem [12, Theorem A.2] to approximate, in the strong topology, any decomposable operator

$$A = \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z \mu(\mathrm{d}z) \in [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'' \cap M_{\mathcal{Z}}$$

by a sequence[28] $(\pi_{\mathcal{Z}}^{\oplus}(A_n))_{n\in\mathbb{N}}$ with $A_n \in \mathcal{X}$. One then uses Theorem 6.19 (iv) to show that $A_z \in [\pi_z(\mathcal{X})]''$ for μ-almost everywhere $z \in \mathcal{Z}$. As a consequence, if $A \in [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]' \cap [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]''$ then, for μ-almost everywhere $z \in \mathcal{Z}$, A_z belongs to the center of $[\pi_z(\mathcal{X})]'$, which consists of multiples of the identity, by assumption. Therefore,

$$[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]' \cap [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'' \subseteq N_{\mathcal{Z}} \ .$$

(ii) Since $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ is separable (Theorem 6.14), by the Kaplanski density theorem [12, Theorem A.2], there is a sequence $(C_n)_{n\in\mathbb{N}}$ of operators that is dense in the weak-operator topology within the unit ball of the commutant $[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'$. As $\mathcal{X}$ is separable and since representations are norm-contractive [15, Proposition 2.3.1], we construct the separable unital C^*-algebra $\mathcal{C}$ of operators on the direct integral $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ that is generated by $\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})$ and $(C_n)_{n\in\mathbb{N}} \subseteq [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'$. This C^*-algebra satisfies, by construction, the equality

$$\mathcal{C}' = [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]' \cap [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'' \ ,$$

which equals $N_{\mathcal{Z}}$, by assumption. Using now [15, Proposition 2.3.1] and Theorem 6.19 (iii) together with Corollary 6.25 and Theorem 6.27, where the separable Banach $*$-algebra, the representation and the abelian sublagebra, are respectively $\mathcal{C}$, the identity mapping and $N_{\mathcal{Z}} = \mathcal{C}'$, we deduce the existence of a $\alpha_{\mathcal{Z}}$-measurable field $\varkappa_{\mathcal{Z}}$ of μ-almost everywhere *irreducible* representations of $\mathcal{C}$ on $\mathcal{H}_z$ such that

$$A = \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \varkappa_z (A) \, \mu(\mathrm{d}z) \ , \qquad A \in \mathcal{C} \ ,$$

with $\pi_z = \varkappa_z \circ \pi_{\mathcal{Z}}^{\oplus}$ and $\varkappa_z(C_n) \in [\pi_z(\mathcal{X})]'$ for μ-almost everywhere $z \in \mathcal{Z}$ and $n \in \mathbb{N}$. By the weak-operator density of $(C_n)_{n\in\mathbb{N}}$ in $[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'$ and [15, Proposition 2.3.8], it follows that

$$[\pi_z(\mathcal{X})]' \cap [\pi_z(\mathcal{X})]'' = [\varkappa_z \circ \pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]' \cap [(\varkappa_z(C_n))_{n\in\mathbb{N}}]' = [\varkappa_z(\mathcal{C})]' = \mathbb{C}\mathbf{1}_{\mathcal{H}}$$

for μ-almost everywhere $z \in \mathcal{Z}$. ■
In general, $[\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]'' \subseteq \mathfrak{N}$ does not necessarily imply that π_z is a factor representation of $\mathcal{X}$ for μ-almost everywhere $z \in \mathcal{Z}$. See [12, Example 15.3]. In particular, by Theorem 6.28 (ii), central decompositions (Definition 6.26 (ii.1)) are always factor decompositions (Definition 6.26 (ii.2)), but the converse is false, in general. In other words, the two definitions are not equivalent.

By Theorem 6.28, observe that any representation of $\mathcal{X}$ admits a central decomposition. One remarkable fact about central, or subcentral, decompositions is that the corresponding factor representations are non-redundant in the following sense:

[28] Recall that $\mathcal{X}$ is assumed to be unital.

Theorem 6.29 (Non-redundancy of subcentral decompositions)
Let Π be a representation of a separable unital Banach $*$-algebra $\mathcal{X}$ on a (separable) Hilbert space $\mathcal{H}$ and $\pi_{\mathcal{Z}}^{\oplus}$ be the direct integral decomposition of Π with respect to $\mathfrak{N} \subseteq [\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]''$ (Corollary 6.25). Then, for some measurable subset $\mathcal{Z}_0 \subseteq \mathcal{Z}$, $\mu(\mathcal{Z}_0) = 0$, and all $z_1, z_2 \in \mathcal{Z}\backslash\mathcal{Z}_0$, $z_1 \neq z_2$, there exists no $*$-isomorphism $\kappa : [\pi_{z_1}(\mathcal{X})]'' \mapsto [\pi_{z_2}(\mathcal{X})]''$ such that $\kappa \circ \pi_{z_1} = \pi_{z_2}$.

Proof. As in previous proofs, let $\Pi = \pi_{\mathcal{Z}}^{\oplus}$ be a representation of $\mathcal{X}$ on a direct integral $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ such that $\mathfrak{N} = N_{\mathcal{Z}} \subseteq [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]' \cap [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]''$, where $(\mathcal{Z}, \mathfrak{F}, \mu)$ is a standard measure space. By [12, Proposition B.5], it suffices to prove that π_{z_1} and π_{z_2} are disjoint for all $z_1, z_2 \in \mathcal{Z}\backslash\mathcal{Z}_0$ with $z_1 \neq z_2$. This means that there is no non-zero operator T from $\mathcal{H}_{z_1}$ to $\mathcal{H}_{z_2}$ such that

$$T\pi_{z_1}(A) = \pi_{z_2}(A)\,T\,, \qquad A \in \mathcal{X}\,. \tag{152}$$

To this end, one takes a sequence $(\mathcal{Z}_n)_{n\in\mathbb{N}}$ of Borel subsets of $\mathcal{Z}$ separating the points of $\mathcal{Z}$. By assumption, $N_{\mathcal{Z}} \subseteq [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]''$ and we deduce that

$$\chi_n \doteq \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \mathbf{1}\left[z \in \mathcal{Z}_n\right] \mathbf{1}_{\mathcal{H}_z} \mu(\mathrm{d}z) \in [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]''\,, \qquad n \in \mathbb{N}\,. \tag{153}$$

Assume now that T is an operator from $\mathcal{H}_{z_1}$ to $\mathcal{H}_{z_2}$ satisfying (152) with $z_1 \in \mathcal{Z}_n$ and $z_2 \notin \mathcal{Z}_n$ for some $n \in \mathbb{N}$. Then, by (152)-(153), we *formally* expect that, for μ-almost everywhere $z_1, z_2 \in \mathcal{Z}$, $z_1 \neq z_2$,

$$T\mathbf{1}\left[z_1 \in \mathcal{Z}_n\right] = T\pi_{z_1}(\chi_n) = \pi_{z_2}(\chi_n)\,T = 0\,, \tag{154}$$

implying $T = 0$. To complete the arguments, by making sense of the formal elements $\pi_z(\chi_n)$, one uses a convenient approximation of $(\chi_n)_{n\in\mathbb{N}} \subseteq [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]''$, in the strong topology, via the Kaplanski density theorem [12, Theorem A.2], as in Theorem 6.28 (i). Then, we use Theorem 6.19 (iv) and the fact that a countable union of measurable sets of zero measure has zero measure to obtain a set $\mathcal{Z}_0$ of zero measure such that Equation (154) holds true for any $z_1 \in \mathcal{Z}_n \cap \mathcal{Z}\backslash\mathcal{Z}_0$ and $z_2 \notin \mathcal{Z}_n \cap \mathcal{Z}\backslash\mathcal{Z}_0$. For more details, see [12, Theorem 13.3]. ■

Theorem 6.29 means that π_{z_1} and π_{z_2} are not *quasi-equivalent*[29] representations for any $z_1, z_2 \in \mathcal{Z}\backslash\mathcal{Z}_0$, $z_1 \neq z_2$. When we only have a factor decomposition of a representation Π with respect to $\mathfrak{N}$ (Corollary 6.25) such that

$$[\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]'' \subsetneq \mathfrak{N} \subseteq [\Pi(\mathcal{X})]', \tag{155}$$

we have a redundancy in the following sense: Since two factor representations π_1, π_2 are either quasi-equivalent or disjoint [12, Proposition B.5], when (155) holds true, the direct integral representation $\pi_{\mathcal{Z}}^{\oplus}$ of Π is constructed from a $\alpha_{\mathcal{Z}}$-measurable field $\pi_{\mathcal{Z}}$ of representations of $\mathcal{X}$ having quasi-equivalent representations within a set of non-zero measure. This redundancy disappears when $\mathfrak{N}$ is exactly the center of $[\Pi(\mathcal{X})]'$, or $[\Pi(\mathcal{X})]''$, by Theorem 6.29. If

$$\mathfrak{N} \subsetneq [\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]'' \tag{156}$$

then there is also no redundancy, in the same way. However, by Theorem 6.27, when (156) holds true, the irreducibility of fiber representations cannot be true μ-almost everywhere since $\mathfrak{N}$ is clearly not a maximal abelian von Neumann subalgebra of $[\Pi(\mathcal{X})]'$. Note that (156) implies that Π *cannot* be an irreducible representation of a separable unital Banach $*$-algebra $\mathcal{X}$ on a (separable) Hilbert space $\mathcal{H}$.

Note finally that, even if there exists a $*$-isomorphism $\kappa : [\pi_1(\mathcal{X})]'' \mapsto [\pi_2(\mathcal{X})]''$ such that $\kappa \circ \pi_1 = \pi_2$, two representations π_1, π_2 of the same C^*-algebra $\mathcal{X}$ on two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, respectively, are not necessarily (unitarily) equivalent, which means the existence of a unitary operator U from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $\pi_1(A) = \mathrm{U}^*\pi_2(A)\mathrm{U}$ for any $A \in \mathcal{X}$. This is related with the question

[29] The definition of quasi-equivalent representations in [15, Definition 2.4.25] differs from the one of [12, Appendix B, p. 146], but they are equivalent, by [15, Theorem 2.4.26].

whether isomorphisms between von Neumann algebras can be unitarily implemented. By [15, Theorem 2.4.26], if π_1, π_2 are two quasi-equivalent representations then π_1, π_2 are (unitarily) equivalent up to multiplicity.

6.6 Direct Integrals of von Neumann Algebras

In this section, we study the direct integrals of von Neumann algebras $\mathfrak{M}$ on separable Hilbert spaces $\mathcal{H}$, i.e., a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ so that $\mathfrak{M}'' = \mathfrak{M}$. See [15, Definition 2.4.8. and Theorem 2.4.11].

Definition 6.30 (Fields of von Neumann algebras)
(i) *For any set $\mathcal{Z}$, a field of von Neumann algebras over a measurable family $\mathcal{H}_\mathcal{Z}$ of separable Hilbert spaces is a family $\mathfrak{M}_\mathcal{Z} \doteq (\mathfrak{M}_z)_{z\in\mathcal{Z}}$ of von Neumann algebras $\mathfrak{M}_z$ acting on $\mathcal{H}_z$ for all $z \in \mathcal{Z}$.*
(ii) *If $(\mathcal{Z}, \mathfrak{F})$ is a measurable space and $\alpha_\mathcal{Z}$ is a coherence for the family $\mathcal{H}_\mathcal{Z}$ of (i), we say that $\mathfrak{M}_\mathcal{Z}$ is $\alpha_\mathcal{Z}$-measurable iff there exists a sequence $(A^{(n)})_{n\in\mathbb{N}}$ of $\alpha_\mathcal{Z}$-measurable fields of operators such that $\{A_z^{(n)} : n \in \mathbb{N}\} \subseteq \mathcal{B}(\mathcal{H}_z)$ generates[30] $\mathfrak{M}_z$ for all $z \in \mathcal{Z}$. Such a sequence $(A^{(n)})_{n\in\mathbb{N}}$ is named a $\alpha_\mathcal{Z}$-measurable generating sequence for $\mathfrak{M}_\mathcal{Z}$.*

If $\mathcal{H}_\mathcal{Z}$ is a measurable family of separable Hilbert spaces (see Definition 6.4), then $(\mathbb{C}\mathbf{1}_{\mathcal{H}_z})_{z\in\mathcal{Z}}$ and $(\mathcal{B}(\mathcal{H}_z))_{z\in\mathcal{Z}}$ are trivial examples of $\alpha_\mathcal{Z}$-measurable fields of von Neumann algebras on $\mathcal{H}_\mathcal{Z}$. Another less trivial example is given by the following lemma:

Lemma 6.31 (Fields of bicommutants of representations)
Let $(\mathcal{Z}, \mathfrak{F})$ be a measurable space and $\pi_\mathcal{Z}$ a $\alpha_\mathcal{Z}$-measurable field of representations of a separable unital Banach $*$-algebra $\mathcal{X}$ over a measurable family $\mathcal{H}_\mathcal{Z}$ of separable Hilbert spaces. Then, the field $([\pi_z(\mathcal{X})]'')_{z\in\mathcal{Z}}$ of von Neumann algebras is $\alpha_\mathcal{Z}$-measurable.

Proof. For any $z \in \mathcal{Z}$, $[\pi_z(\mathcal{X})]''$ is generated by the set $\pi_z(\mathcal{X})$, while $\mathcal{X}$ is separable and a representation is norm-contractive [15, Proposition 2.3.1]. Therefore, the lemma is a direct consequence of the $\alpha_\mathcal{Z}$-measurability of $\pi_\mathcal{Z}$. See Definitions 6.22 (ii) and 6.30 (ii). ■
In fact, all $\alpha_\mathcal{Z}$-measurable fields of von Neumann algebras are of this form, by [12, Lemma 18.1].

It turns out that measurable fields of von Neumann algebras are stable with respect to simple point-wise operations on fields of von Neumann algebras:

Lemma 6.32 (Stability of $\alpha_\mathcal{Z}$-measurable fields of von Neumann algebras)
Let $(\mathcal{Z}, \mathfrak{F})$ be a measurable space and $\mathfrak{M}_\mathcal{Z}$, $\mathfrak{M}_\mathcal{Z}^{(n)}$, $n \in \mathbb{N}$, $\alpha_\mathcal{Z}$-measurable fields $\mathfrak{M}_\mathcal{Z}$ of von Neumann algebras over a measurable family $\mathcal{H}_\mathcal{Z}$ of separable Hilbert spaces. Then, the fields $(\mathfrak{M}_z')_{z\in\mathcal{Z}}$, $(\cap_{n\in\mathbb{N}}\mathfrak{M}_z^{(n)})_{z\in\mathcal{Z}}$, $([\cup_{n\in\mathbb{N}}\mathfrak{M}_z^{(n)}]'')_{z\in\mathcal{Z}}$ of von Neumann algebras over $\mathcal{H}_\mathcal{Z}$ are also $\alpha_\mathcal{Z}$-measurable.

Proof. This is a consequence of the properties of the so-called Effros-Borel structure. See [12, Theorem 17.1]. We omit the details. ■

We now define direct integrals of von Neumann algebras via the direct integral of Hilbert spaces (Definition 6.9) and the subalgebra of decomposable operators (Definition 6.15):

Definition 6.33 (Direct integrals of von Neumann algebras)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a σ-finite measure space, $\mathfrak{M}_\mathcal{Z}$ a $\alpha_\mathcal{Z}$-measurable field of von Neumann algebras over a measurable family $\mathcal{H}_\mathcal{Z}$ of separable Hilbert spaces. The direct integral

$$\mathfrak{M}_\mathcal{Z}^\oplus \equiv \int_\mathcal{Z}^{\alpha_\mathcal{Z}} \mathfrak{M}_z \mu(\mathrm{d}z) \subseteq M_\mathcal{Z} \subseteq \mathcal{B}(\mathcal{H}_\mathcal{Z}^\oplus)$$

of $\mathfrak{M}_\mathcal{Z}$ with respect to μ and $\alpha_\mathcal{Z}$ is the $*$-subalgebra of decomposable operators $A = \int_\mathcal{Z}^{\alpha_\mathcal{Z}} A_z \mu(\mathrm{d}z)$ for which $A_z \in \mathfrak{M}_z \subseteq \mathcal{B}(\mathcal{H}_z)$ for all μ-almost everywhere $z \in \mathcal{Z}$.

[30] I.e., $\mathfrak{M}_z$ is the closure, in the strong or weak operator topology, of the $*$-algebra generated by this set.

Using this definition

$$\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} [\mathbb{C}\mathbf{1}_{\mathcal{H}_z}]\mu(\mathrm{d}z) = N_{\mathcal{Z}} \qquad \text{and} \qquad \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \mathcal{B}(\mathcal{H}_z)\mu(\mathrm{d}z) = M_{\mathcal{Z}}$$

are nothing else as the von Neumann algebras $N_{\mathcal{Z}}$ and $M_{\mathcal{Z}}$ of diagonalizable and decomposable operators on $\mathcal{H}_{\mathcal{Z}}^{\oplus}$, respectively. See Definitions 6.15 and 6.16. Observe further that, for any $\alpha_{\mathcal{Z}}$-measurable field $\mathfrak{M}_{\mathcal{Z}}$ of von Neumann algebras over $\mathcal{H}_{\mathcal{Z}}$,

$$\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} [\mathbb{C}\mathbf{1}_{\mathcal{H}_z}]\mu(\mathrm{d}z) \subseteq \mathfrak{M}_{\mathcal{Z}}^{\oplus} \cap \left[\mathfrak{M}_{\mathcal{Z}}^{\oplus}\right]' ,$$

i.e., the algebra $N_{\mathcal{Z}}$ of diagonalizable operator is always a subalgebra of the center of $\mathfrak{M}_{\mathcal{Z}}^{\oplus}$.

The next theorem gives a sufficient condition on direct integrals of von Neumann algebras to be themselves von Neumann algebras:

Theorem 6.34 (Direct integrals of von Neumann algebras as von Neumann algebras)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a standard measure space, $\mathfrak{M}_{\mathcal{Z}}$ a $\alpha_{\mathcal{Z}}$-measurable field of von Neumann algebras over a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. Then, $\mathfrak{M}_{\mathcal{Z}}^{\oplus}$ is the von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_{\mathcal{Z}}^{\oplus})$ generated by $N_{\mathcal{Z}}$ and $\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z^{(n)}\mu(\mathrm{d}z)$, $n \in \mathbb{N}$, where $(A^{(n)})_{n\in\mathbb{N}}$ is any $\alpha_{\mathcal{Z}}$-measurable generating sequence for $\mathfrak{M}_{\mathcal{Z}}$.

Proof. Fix all parameters of the theorem. Pick, in particular, an arbitrary $\alpha_{\mathcal{Z}}$-measurable generating sequence $(A^{(n)})_{n\in\mathbb{N}}$ for $\mathfrak{M}_{\mathcal{Z}}$. Denote by $\mathfrak{M}$ the von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_{\mathcal{Z}}^{\oplus})$ generated by $N_{\mathcal{Z}}$ and $\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z^{(n)}\mu(\mathrm{d}z)$, $n \in \mathbb{N}$. If $B \in \mathfrak{M}'$ then there is a μ-essentially bounded, $\alpha_{\mathcal{Z}}$-measurable operator field $(B_z)_{z\in\mathcal{Z}}$ such that $B = \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} B_z\mu(\mathrm{d}z) \in M_{\mathcal{Z}}$, by Theorem 6.19 (i), and, for μ-almost everywhere $z \in \mathcal{Z}$, $B_z \in \mathfrak{M}_z'$ by similar arguments as in the proof of Theorem 6.24. Therefore, $\mathfrak{M}' \subseteq [\mathfrak{M}_{\mathcal{Z}}^{\oplus}]'$, which in turn implies that $\mathfrak{M}_{\mathcal{Z}}^{\oplus} \subseteq \mathfrak{M}$.

To show the reverse inclusion, use the Kaplanski density theorem [12, Theorem A.2] to approximate, in the strong topology, any $B \in \mathfrak{M}$ by a sequence $(B^{(n)})_{n\in\mathbb{N}} \subseteq \mathfrak{M}_{\mathcal{Z}}^{\oplus}$ of elements of the $*$-algebra generated by $N_{\mathcal{Z}}$ and $\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} A_z^{(n)}\mu(\mathrm{d}z)$, $n \in \mathbb{N}$. Observe that $\mathfrak{M} \subseteq M_{\mathcal{Z}}$, because $N_{\mathcal{Z}} \subseteq \mathfrak{M}'$. See also Theorem 6.19 (i). In other words, $B, (B^{(n)})_{n\in\mathbb{N}} \in \mathfrak{M}$ are all decomposable operators and, using $(B^{(n)})_{n\in\mathbb{N}} \subseteq \mathfrak{M}_{\mathcal{Z}}^{\oplus}$ and Theorem 6.19 (iv), we arrive at $B \in \mathfrak{M}_{\mathcal{Z}}^{\oplus}$, i.e., $\mathfrak{M} \subseteq \mathfrak{M}_{\mathcal{Z}}^{\oplus}$. ■

Corollary 6.35 (Direct integrals of von Neumann algebras and fiber inclusions)
Let $(\mathcal{Z}, \mathfrak{F}, \mu)$ be a standard measure space, $\mathfrak{M}_{\mathcal{Z}}$, $\widetilde{\mathfrak{M}}_{\mathcal{Z}}$ two $\alpha_{\mathcal{Z}}$-measurable field of von Neumann algebras over the same measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. Then, $\mathfrak{M}_{\mathcal{Z}} \subseteq \widetilde{\mathfrak{M}}_{\mathcal{Z}}$ iff $\mathfrak{M}_z \subseteq \widetilde{\mathfrak{M}}_z$ for all μ-almost everywhere $z \in \mathcal{Z}$.

Proof. It is an obvious consequence of Theorem 6.34. ■

Another consequence of Theorem 6.34 concerns the difference between the von Neumann algebra constructed from a direct integral representation and the direct integral of the von Neumann algebras constructed from the fields of von Neumann algebras generated by the corresponding fiber representation, as stated in Lemma 6.31. These von Neumann algebras are in general different. Necessary and sufficient conditions to have equality are given in the following corollary:

Corollary 6.36 (Direct integrals of von Neumann algebras and representations)
Let Π be a representation of a separable unital Banach $*$-algebra $\mathcal{X}$ on a (separable) Hilbert space $\mathcal{H}$ and $\mathfrak{N}$ an abelian von Neumann subalgebra of $[\Pi(\mathcal{X})]'$. Let $\pi_{\mathcal{Z}}^{\oplus}$ be the direct integral decomposition of Π with respect to $\mathfrak{N}$ (Corollary 6.25). Then,

$$\left[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})\right]'' = \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} [\pi_z(\mathcal{X})]''\,\mu(\mathrm{d}z) \qquad \textit{iff} \qquad \mathfrak{N} \subseteq [\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]'' .$$

Proof. Let $\pi_{\mathcal{Z}}^{\oplus}$ be the direct integral decomposition of a representation Π of a separable unital Banach $*$-algebra $\mathcal{X}$ with respect to an abelian von Neumann subalgebra of $[\Pi(\mathcal{X})]'$, as stated in Definition 6.23. See also Corollary 6.25. Then, by Lemma 6.31 and Theorem 6.34, the field $(\mathfrak{M}_z \doteq [\pi_z(\mathcal{X})]'')_{z\in\mathcal{Z}}$ of von Neumann algebras is $\alpha_{\mathcal{Z}}$-measurable and its direct integral $\mathfrak{M}_{\mathcal{Z}}^{\oplus}$ is a von Neumann algebra. As $\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X}) \subseteq \mathfrak{M}_{\mathcal{Z}}^{\oplus}$, we *always* have the natural inclusion

$$[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'' \subseteq \mathfrak{M}_{\mathcal{Z}}^{\oplus} \equiv \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} [\pi_z(\mathcal{X})]''\,\mu(\mathrm{d}z)\ , \tag{157}$$

keeping in mind that $[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]''$ is the smallest von Neumann algebra containing $\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})$. In fact, by Theorem 6.34, $\mathfrak{M}_{\mathcal{Z}}^{\oplus}$ is the von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_{\mathcal{Z}}^{\oplus})$ generated by $N_{\mathcal{Z}}$ and the bicommutant $[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]''$. Since obviously $N_{\mathcal{Z}} \subseteq [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'$, (157) holds true with equality iff $N_{\mathcal{Z}}$ is contained in the center of $[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]'$, i.e., $\mathfrak{N} \subseteq [\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]''$, by Definition 6.23. ■

Note that the bicommutant $[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]''$ of Corollary 6.36 does not necessarily include all diagonal operators and in this case,

$$\left[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})\right]'' \subsetneq \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} [\pi_z(\mathcal{X})]''\,\mu(\mathrm{d}z)\ .$$

See the argument just below (157) that uses Theorem 6.34. In any case, (157) always holds true and, for instance, any separating vector $\Psi = (\Psi_z)_{z\in\mathcal{Z}} \in \tilde{\mathcal{H}}_{\mathcal{Z}}^{\oplus} \equiv \mathcal{H}_{\mathcal{Z}}^{\oplus}$ such that Ψ_z is separating for $[\pi_z(\mathcal{X})]''$ for all μ-almost everywhere $z \in \mathcal{Z}$ yields a separating vector $\Psi \equiv [\Psi]$ for $[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]''$, i.e., $A\Psi = 0$ implies $A = 0$ for all $A \in [\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})]''$.

By [12, Theorem 13.3], or the proof of Theorem 6.29, the condition of Corollary 6.36, that is,

$$\mathfrak{N} \subseteq [\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]'',$$

is directly related with the fact that the fiber representations are mutually disjoint, or equivalently, for some measurable subset $\mathcal{Z}_0 \subseteq \mathcal{Z}$ with $\mu(\mathcal{Z}_0) = 0$ and all $z_1, z_2 \in \mathcal{Z}\backslash\mathcal{Z}_0$, there exists no $*$-isomorphism $\kappa : [\pi_{z_1}(\mathcal{X})]'' \mapsto [\pi_{z_2}(\mathcal{X})]''$ such that $\kappa \circ \pi_{z_1} = \pi_{z_2}$ whenever $z_1 \neq z_2$. In other words, to have the equality

$$\left[\pi_{\mathcal{Z}}^{\oplus}(\mathcal{X})\right]'' = \int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} [\pi_z(\mathcal{X})]''\,\mu(\mathrm{d}z)\ ,$$

it is necessary to have *no* redundancy in the fiber representations, as expected. From Theorem 6.29 and discussions after Equation (156), we see that the natural candidate for an abelian von Neumann subalgebra $\mathfrak{N} \subseteq [\Pi(\mathcal{X})]'$, with respect to which $[\Pi(\mathcal{X})]''$ is decomposed, is precisely the center $[\Pi(\mathcal{X})]' \cap [\Pi(\mathcal{X})]''$.

By Theorem 6.19 (i), note that $\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} \mathcal{B}(\mathcal{H}_z)\mu(\mathrm{d}z)$ is the commutant of $\int_{\mathcal{Z}}^{\alpha_{\mathcal{Z}}} [\mathbb{C}\mathbf{1}_{\mathcal{H}_z}]\mu(\mathrm{d}z)$. Thus, if a von Neumann algebra $\mathfrak{M}$ over $\mathcal{H}_{\mathcal{Z}}^{\oplus}$ is a direct integral of a $\alpha_{\mathcal{Z}}$-measurable field $\mathfrak{M}_{\mathcal{Z}}$ of von Neumann algebras over $\mathcal{H}_{\mathcal{Z}}$, then $N_{\mathcal{Z}} \subseteq \mathfrak{M} \subseteq N_{\mathcal{Z}}'$ and hence, the center of $\mathfrak{M}$ contains all diagonalizable operators. By Corollary 6.36, the converse should clearly be true and this refers to [12, Theorem 19.4] and the existence of a direct integral decomposition of a von Neumann algebra via an abelian von Neumann algebra, similar to Definitions 6.21 and 6.23:

Definition 6.37 (Direct integral decomposition of von Neumann algebras)
Let $\mathfrak{M}$ be a von Neumann algebra on a separable Hilbert space $\mathcal{H}$, $\mathfrak{N}$ an abelian von Neumann subalgebra of $\mathfrak{M}'$, $(\mathcal{Z}, \mathfrak{F}, \mu)$ a standard measure space and $\mathfrak{M}_{\mathcal{Z}}$ a $\alpha_{\mathcal{Z}}$-measurable field of von Neumann algebras over a measurable family $\mathcal{H}_{\mathcal{Z}}$ of separable Hilbert spaces. $\mathfrak{M}_{\mathcal{Z}}^{\oplus}$ is a direct integral decomposition of $\mathfrak{M}$ with respect to $\mathfrak{N}$ if there is a $*$-isomorphism $\phi : L^\infty(\mathcal{Z}, \mu) \to \mathfrak{N}$ and a unitary mapping $\mathrm{U} : \mathcal{H} \to \mathcal{H}_{\mathcal{Z}}^{\oplus}$ such that,

$$\mathfrak{M}_{\mathcal{Z}}^{\oplus} = \mathrm{U}\mathfrak{M}\mathrm{U}^*\ , \quad N_{\mathcal{Z}} = \mathrm{U}\mathfrak{N}\mathrm{U}^*\ , \quad \mathrm{U}\phi(f)\mathrm{U}^* = \int_{\mathcal{Z}_{\mathfrak{N}}}^{\alpha_{\mathcal{Z}_{\mathfrak{N}}}} f(z)\mathbf{1}_{\mathcal{H}_z}\mu(\mathrm{d}z)\ , \quad f \in L^\infty(\mathcal{Z}, \mu)\ .$$

In this case, we say that $\mathfrak{M}$ is decomposable with respect to $\mathfrak{N}$.

Recall that (i) any separable Hilbert has a decomposition with respect to any abelian von Neumann algebra on it and (ii) such a decomposition is related to a standard measure space, by Theorem 6.20. The following result, which is similar to Theorem 6.20 and Corollary 6.25, ensures the existence of direct integral decompositions of von Neumann algebras:

Theorem 6.38 (Direct integral decomposition of von Neumann algebras)
Let $\mathfrak{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ and $\mathfrak{N}$ an abelian von Neumann subalgebra of $\mathfrak{M}'$.Then, $\mathfrak{M}$ is decomposable with respect to $\mathfrak{N}$ iff $\mathfrak{N} \subseteq \mathfrak{M}' \cap \mathfrak{M}''$.

Proof. If $\mathfrak{M}$ is decomposable with respect to $\mathfrak{N}$ then we must have $\mathfrak{N} \subseteq \mathfrak{M}' \cap \mathfrak{M}''$. See discussion before Definition 6.37. Conversely, any von Neumann algebra $\mathfrak{M}$ on a separable Hilbert space $\mathcal{H}$ is the strong closure of a norm-separable C^*-algebra $\mathcal{X}$: To see this, note that the unit closed ball of any von Neumann algebra $\mathfrak{M}$ on $\mathcal{H}$ is compact with respect to the weak-operator topology. Therefore, if the Hilbert space $\mathcal{H}$ is separable, the weak-operator topology is metrizable on any ball of $\mathfrak{M}$, which is thus separable in this topology. In particular, a von Neumann algebra $\mathfrak{M}$ on a separable Hilbert space is separable with respect to the weak-operator topology. Thus take any weak-operator-dense countable subset $\mathcal{X}_0 \subseteq \mathfrak{M}$ and let $\mathcal{X}$ be the separable C^*-algebra generated by $\mathcal{X}_0$. Clearly, $\mathcal{X}'' = \mathfrak{M}$. Define the representation Π on $\mathcal{X}$ to be the identity mapping. Let $\pi_{\mathcal{Z}}^{\oplus}$ be the direct integral decomposition of Π with respect to $\mathfrak{N}$ (Corollary 6.25). The assertion then follows from Corollary 6.36 as $[\Pi(\mathcal{X})]'' = \mathfrak{M}$. See also Theorem 6.20 for the existence of the $*$-isomorphism $\phi : L^\infty(\mathcal{Z}, \mu) \rightarrow \mathfrak{N}$. ■

Hence, in the context of standard spaces, the theory of direct integrals of fields of von Neumann algebras corresponds to the study of von Neumann subalgebras of the algebra of decomposable operators whose center contains the diagonalizable ones.

When $\mathfrak{N} = \mathfrak{M}' \cap \mathfrak{M}''$, one talks about the *central* decomposition of $\mathfrak{M}$, similar to Definition 6.26 (ii.1). One can also talk about *factor* decompositions of von Neumann algebras, similar to Definition 6.26 (ii.2).

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