Classical Dynamics Generated by Long-Range Interactions for Lattice Fermions and Quantum Spins

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Abstract

We study the macroscopic dynamical properties of fermion and quantum-spin systems with long-range, or mean-field, interactions. The results obtained are far beyond previous ones and require the development of a mathematical framework to accommodate the macroscopic long-range dynamics, which corresponds to an intricate combination of classical and short-range quantum dynamics. In this paper we focus on the classical part of the long-range, or mean-field, macroscopic dynamics, but we already introduce the full framework. The quantum part of the macroscopic dynamics is studied in a subsequent paper. We show that the classical part of the macroscopic dynamics results from self-consistency equations within the (quantum) state space. As is usual, the classical dynamics is driven by Liouville's equation.

Dedicated to V.A. Zagrebnov for his important contributions to the mathematics of quantum many-body theory.

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Contents

1 Introduction

More than seventy years ago, Bogoliubov proposes an ansatz, widely known as the Bogoliubov approximation, which corresponds to replace, in many-boson Hamiltonians, the annihilation and creation operators of zero-impulsion particles with complex numbers to be determined *self-consistently*. See [1, Section 1.1] for more details. His motivation comes from the observation that these (unbounded) operators almost commute in the thermodynamic limit, leading to some *classical* field in the macroscopic Bose system. In 1968, Ginibre [2] shows the exactness of the approximation in which concerns the thermodynamic pressure of superstable Bose gases. However, even nowadays, the mathematical validity of this approximation with respect to the primordial dynamics of (stable) many-boson Hamiltonians with usual two-body interactions is an open problem.

In the context of many-fermion systems, ten years after Bogoliubov's ansatz, a similar approximation is used in the BCS theory of (conventional) superconductivity, as explained by Bogliubov in 1958 [3] and Haag in 1962 [4]. In 1966, this approximation is shown [8] to be exact at the level of the thermodynamic pressure for fermion systems that are similar to the BCS model. See also the so-called approximating Hamiltonian method used on the level of the pressure of fermionic systems in [8–11].

The validity of the approximation with respect to the primordial dynamics was an open question that Thirring and Wehrl [5, 6] solve in 1967 for an exactly solvable permutation-invariant fermion model. An attempt to generalize Thirring and Wehrl's results to a general class of fermionic models, including the BCS theory, has been done in 1978 [7], but at the cost of technical assumptions that are difficult to verify in practice.

In 1973, Hepp and Lieb [12] made explicit, for the first time, the existence of Poisson brackets in some (commutative) algebra of functions, related to the classical effective dynamics. This is done for a permutation-invariant quantum-spin system with mean-field interactions. This research direction has been strongly developed by many authors until 1992, see [13–32]. All these papers study dynamical properties of *permutation-invariant* quantum-spin systems with mean-field interactions. Even if we are rather interested in mean-field dynamics, note meanwhile that equilibrium properties of such quantum systems are also extensively studied in the same period, see for instance [33–35] and references therein.

Thereafter, the mathematical research activity on this subject considerably decreases until the early 2000s when emerges, within the mathematical physics community, a new interest in such quantum systems, partially because of new experiments like those on ultracold atoms (via laser and evaporative coolings). See, for instance, the paper [36] on mean-field dynamics, published in the year 2000. In 2005, Ginibre's result on Bose gases at thermodynamical equilibrium is generalized [37]. There is also an important research activity on the mathematical foundation of the Gross-Pitaevskii[1] (GP) or Hartree theories, starting after 1998. For more details on the GP theory and mean-field dynamics for indistinguishable particles (bosons), see [38–42] and references therein. In which concern lattice-fermion or quantum-spin systems with long-range, or mean-field, interactions at equilibrium, see, e.g., [43–46]. Concerning the dynamics of fermion systems in the continuum with mean-field interactions, see [47–56], as well as [39, Sections 6 and 7]. Such mean-field problems are even related to other academic disciplines, like mathematical economics, via the so-called mean-field game theory [57] developed from 2006 by Lasry and Lions. Mean-field theory in its extended sense is, in fact, a major research field of mathematics, even nowadays, and is still studied in physics, see, e.g., [58] and references therein. The current paper belongs to this research field, since we hereby study the dynamical properties of fermion and quantum-spin systems with long-range, or mean-field, interactions.

All mathematical studies [39, 47–56] of the 2000s on mean-field fermionic dynamics are on the continuum and use approximating *quasi-free*[2] states (or a mixture of them) as initial states to derive the Hartree-Fock equation, which is originally based on the assumption that the many-fermion wavefunction is a Slater determinant. Quoting [39, p. 79]: "*Slater determinants are relevant at zero temperature because they provide (or at least they are expected to provide) a good approximation to the fermionic ground state of Hamilton operators like (6.1) in the mean-field limit. At positive temperature, equilibrium states are mixed; in the mean-field regime, they are expected to be approximately quasi-free mixed states, like the Gibbs state of a non-interacting gas.*"

These arguments are probably true in the mean-field regimes considered in these studies because the non-mean-field part of the model is always associated with a one-particle Hamiltonian, but one cannot expect this property to hold true for general fermion models with long-range, or mean-field, interactions. Equilibrium states of the usual BCS theory is a mixture of quasi-free states, even at zero temperature, in presence of a superconducting phase, because of gauge invariance. Adding a Hubbard interaction to such BCS-type models directly destroys the quasi-free property of equilibrium states, even in the sense of a mixture. Indeed, a general many-fermion wavefunction cannot be expressed as a single determinant, even at zero temperature, and, consequently, we cannot expect the Hartree-Fock equation to be generally correct. For instance, this method usually overestimates the full (ground state) energy. To solve that problem in computational chemistry and condensed matter physics one can use either extensions of the Hartree-Fock method, or the Density Functional Theory[3], which is based on some energy functional for the one-fermion density only, via the Hohenberg-Kohn theorems. See for instance [59].

In contrast with results [39, 47–56] of the 2000s, we consider fermions on the lattice, not in the continuum, and we meanwhile study quantum-spin systems. Our analysis of the corresponding mean-field dynamics does *not* require to have approximating quasi-free states (or a mixture of them) as initial states.

[1]The so-called GP limit is not really a mean-field limit, but it looks similar.

[2]In some papers, only (approximating) Slater determinants are considered.

[3]A not necessarly better, but computationally more efficient approximation.

The recent studies [39, 47–56] of the 2000s never extract effective classical dynamics within the quantum one. The reason, for most of them [39, 49, 51–54, 56], is that their mean-field scales reveal a *semi-classical structure*, similar to what is first done in 1981 to derive the Vlasov hierarchy in [60, 61] from quantum dynamics. In order to see the intricate combination of a classical dynamics and a quantum one, one has to go back to previous results [12–32], initiated by Hepp and Lieb. In our opinion, the most elaborated and interesting results in that context are obtained by Bóna [17, 19–21] in 1987-1990, who studies in detail the dynamics of a large class of permutation-invariant quantum-spin systems with mean-field interactions. Bóna formalizes a new view point on quantum mechanics in 1991 [25] and later in a mature textbook published in 2000 and revised in 2012 [62], describing what he names "extended quantum mechanics": It is an intricate combination of classical and quantum dynamics. Bóna's approach does not seem to be incorporated by the physics community, until now.

In contrast with the results [12–32] of the 70s-90s (1973-1992), our study does *not* require the permutation invariance of quantum-spin systems. In fact, we are able to study dynamical properties of a very general class of lattice-fermion, or quantum-spin, systems with long-range, or mean-field, *translation-invariant*[4] interactions. In this context, the initial state is only assumed to be periodic with respect to space translations. In fact, by Proposition 2.3, the set of all initial states allowed in our study is weak*-dense within the set of even states, the physically relevant ones. Even for permutation-invariant systems, our results go beyond previous ones, since the class of long-range, or mean-field, interactions we are able to handle is much larger.

Our study reveals an entanglement of classical and quantum dynamics, as explained in [63], which revisits Bóna's approach [62] on "extended quantum mechanics". In [63], we do not really follow Bóna's path but highlight the importance of self-consistency equations, instead. This study leads to a new theory which strongly differs from the so-called quantum-classical hybrid theories of theoretical physics. In this paper and in the subsequent one [64], we show that this mathematical framework is imperative to describe the dynamical properties of lattice-fermion or quantum-spin systems with long-range, or mean-field, interactions, because of the necessity of coupled classical and quantum evolution equations.

Here, we focus on the classical part of the long-range, or mean-field, dynamics of the quantum systems under consideration, even if we have to consider the full algebraic framework, as described in [63]. The quantum part of the dynamics, shortly discussed in Section 6.6, is studied in a companion paper [64] because it additionally involves the theory of direct integrals of measurable families of Hilbert spaces, operators, von Neumann algebras and C^*-algebra representations, as presented, for instance, in the monograph [65].

The classical part of the dynamics of lattice-fermion, or quantum-spin, systems with long-range, or mean-field, interactions is shown to result from the solution of a self-consistency equation within the (quantum) state space. In general, it is a *non-linear* dynamics generated by a Poisson bracket. In other words, it is driven by some Liouville's equation, similar to what has been recently observed [42] by Ammari, Liard, and Rouffort for Bose gases in the mean-field limit. As soon as the classical part of this dynamics is concerned, there is no need to assume any particular property, neither on initial states, nor on local Hamiltonians. Translation invariance, or periodicity, is only required for the analysis of the quantum part of the dynamics. So, these particular spatial features are only pivotal in [64].

It would have been interesting to add applications of our general theory, but we still have to present (in [64]) the quantum part of such dynamics before doing that. In fact, several applications on quasi-free models as well as permutation-invariant systems will be presented in separated papers. See, e.g., [66]. We will also discuss in a forthcoming paper such mean-field dynamics within the representation of an arbitrary (generalized) equilibrium state[5] of the quantum system under consideration.

[4]This can be easily extended to periodically invariant systems, by redefining the finite spin set.

[5]This class of states is studied in detail in [46].

Our main results are Theorems 6.5, 6.10, Proposition 6.7, and Corollaries 6.9, 6.11. Note that Theorem 6.10 is a highly non-trivial mathematical statement in the theory of non-autonomous evolution equations [67–71]. It results from Lieb-Robinson bounds on multi-commutators of high orders derived in 2017 [72].

The paper is organized as follows: Section 3 explains the well-established dynamical properties of lattice-fermion systems with *short-range* interactions. In this context, the celebrated Lieb-Robinson bounds are pivotal and are also an important tool to obtain the full dynamics of systems with long-range, or mean-field, interactions. Section 4 defines what we name "long-range interactions" and we make explicit the problem of the thermodynamic limit of their associated dynamics. Like in [46] we prefer to use, from now on, the term "long-range" instead of "mean-field", because the latter can refer to different scalings, in particular in the recent works [39, 47–56]. In Section 6, we present the self-consistency equations as well as the classical part of long-range dynamics. This requires the mathematical framework of [63], which has thus to be presented in the first subsections of Section 6. Note that we shortly explain what the quantum part of the long-range dynamics is in Section 6.6, even if this will be done in detail in [64]. Finally, Section 7 gives the proofs of Theorems 6.5 and 6.10. We add in Section 8 an appendix relating our current notion of long-range interactions to the one of [46].

In this paper, we only focus on lattice-fermion systems which are, from a technical point of view, slightly more difficult than quantum-spin systems, because of a non-commutativity issue at different lattice sites. However, all the results presented here and in the subsequent papers hold true for quantum-spin systems, via obvious modifications.

Notation 1.1
(i) *A norm on a generic vector space $\mathcal{X}$ is denoted by $\|\cdot\|_{\mathcal{X}}$ and the identity mapping of $\mathcal{X}$ by $\mathbf{1}_{\mathcal{X}}$. The space of all bounded linear operators on $(\mathcal{X},\|\cdot\|_{\mathcal{X}})$ is denoted by $\mathcal{B}(\mathcal{X})$. The unit element of any algebra $\mathcal{X}$ is denoted by $\mathfrak{1}$, provided it exists.*
(ii) *For any topological space $\mathcal{X}$ and normed space $(\mathcal{Y},\|\cdot\|_{\mathcal{Y}})$, $C(\mathcal{X};\mathcal{Y})$ denotes the space of continuous mappings from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{X}$ is a locally compact topological space, then $C_b(\mathcal{X};\mathcal{Y})$ denotes the Banach space of bounded continuous mappings from $\mathcal{X}$ to $\mathcal{Y}$ along with the topology of uniform convergence.*
(iii) *The notion of an automorphism depends on the structure of the corresponding domain. In algebra, a ($\ast$-) automorphism acting on a $\ast$-algebra is a bijective $\ast$-homomorphism from this algebra to itself. In topology, an automorphism acting on a topological space is a self-homeomorphism, that is, a homeomorphism of the space to itself.*

2 Algebraic Formulation of Lattice Fermion Systems

2.1 Background Lattice

Fix once and for all $d\in\mathbb{N}$, the dimension of the lattice. Choose also a subset $\mathfrak{L}\subseteq\mathbb{Z}^d$ (which will be omitted in our notation, unless it is important to specify it). The main example we have in mind is $\mathfrak{L}=\mathbb{Z}^d$, as is done in [46], but it is convenient to keep this set as being any arbitrary subset of the d-dimensional lattice. Let $\mathcal{P}_f\equiv\mathcal{P}_f(\mathfrak{L})\subseteq 2^{\mathfrak{L}}$ be the set of all non-empty finite subsets of $\mathfrak{L}$. In order to define the thermodynamic limit, for simplicity, we use the cubic boxes

$$\Lambda_L\doteq\{(x_1,\ldots,x_d)\in\mathbb{Z}^d:|x_1|,\ldots,|x_d|\leq L\}\cap\mathfrak{L}\,,\qquad L\in\mathbb{N}\,,\tag{1}$$

as a so-called van Hove net.

2.2 The CAR Algebra

For any $\Lambda \in \mathcal{P}_f$, $\mathcal{U}_\Lambda$ is the finite dimensional unital C^*-algebra generated by elements $\{a_{x,\mathrm{s}}\}_{x\in\Lambda,\mathrm{s}\in\mathrm{S}}$ satisfying the canonical anti-commutation relations (CAR), S being some finite set of spins:

$$a_{x,\mathrm{s}}a_{y,\mathrm{t}} + a_{y,\mathrm{t}}a_{x,\mathrm{s}} = 0\ , \qquad a_{x,\mathrm{s}}a_{y,\mathrm{t}}^* + a_{y,\mathrm{t}}^*a_{x,\mathrm{s}} = \delta_{\mathrm{s},\mathrm{t}}\delta_{x,y}\mathbf{1} \tag{2}$$

for all $x, y \in \mathfrak{L}$ and $\mathrm{s}, \mathrm{t} \in \mathrm{S}$. Observe that $(\mathcal{U}_{\Lambda_L})_{L\in\mathbb{N}}$ is an increasing sequence of C^*-algebras. Hence, the set

$$\mathcal{U}_0 \doteq \bigcup_{L\in\mathbb{N}} \mathcal{U}_{\Lambda_L} \tag{3}$$

of so-called local elements is a normed $*$-algebra with $\|A\|_{\mathcal{U}_0} = \|A\|_{\mathcal{U}_{\Lambda_L}}$ for all $A \in \mathcal{U}_{\Lambda_L}$ and $L \in \mathbb{N}$. The CAR C^*-algebra $\mathcal{U} \equiv \mathcal{U}_{\mathfrak{L}}$ of the full system, whose norm is denoted by $\|\cdot\|_{\mathcal{U}}$, is by definition the completion of the normed $*$-algebra $\mathcal{U}_0$. It is separable, by finite dimensionality of $\mathcal{U}_\Lambda$ for $\Lambda \in \mathcal{P}_f$. Equivalently, the C^*-algebra $\mathcal{U}$ is the universal C^*-algebra [73, Section II.8.3] associated with the relations (2) for all $x, y \in \mathfrak{L}$ and $\mathrm{s}, \mathrm{t} \in \mathrm{S}$. The (real) Banach subspace of all self-adjoint elements of $\mathcal{U}$ is denoted by $\mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U}$.

2.3 Important $*$-Automorphisms of the CAR Algebra

Parity: Observe that the condition

$$\sigma(a_{x,\mathrm{s}}) = -a_{x,\mathrm{s}}, \qquad x \in \Lambda,\ \mathrm{s} \in \mathrm{S}\ , \tag{4}$$

defines a unique $*$-automorphism σ of the C^*-algebra $\mathcal{U}$. Elements $A_1, A_2 \in \mathcal{U}$ satisfying $\sigma(A_1) = A_1$ and $\sigma(A_2) = -A_2$ are respectively called *even* and *odd*. Note that the set

$$\mathcal{U}^+ \equiv \mathcal{U}^+_{\mathfrak{L}} \doteq \{A \in \mathcal{U} : A = \sigma(A)\} \subseteq \mathcal{U} \tag{5}$$

of all even elements is a $*$-algebra. By continuity of σ, $\mathcal{U}^+$ is additionally norm-closed and, hence, a C^*-subalgebra of $\mathcal{U}$. Even elements are crucial here since, for any finite subsets $\Lambda^{(1)}, \Lambda^{(2)} \in \mathcal{P}_f$ with $\Lambda^{(1)} \cap \Lambda^{(2)} = \emptyset$,

$$[A_1, A_2] \doteq A_1A_2 - A_2A_1 = 0\ , \qquad A_1 \in \mathcal{U}_{\Lambda^{(1)}} \cap \mathcal{U}^+,\ A_2 \in \mathcal{U}_{\Lambda^{(2)}}\ . \tag{6}$$

However, this relation is wrong in general when A_1 is not an even element. For instance, the CAR (2) trivially yields $[a_{x,\mathrm{s}}, a_{y,\mathrm{t}}] = 2a_{x,\mathrm{s}}a_{y,\mathrm{t}} \neq 0$ for any $x, y \in \mathfrak{L}$ and $\mathrm{s}, \mathrm{t} \in \mathrm{S}$, $(x, \mathrm{s}) \neq (y, \mathrm{t})$.

The condition (6) is the expression of the local causality in quantum field theory. Using well-known constructions[6], the C^*-algebra $\mathcal{U}$, generated by anticommuting elements, can be recovered from $\mathcal{U}^+$. As a consequence, the C^*-algebra $\mathcal{U}^+$ should thus be seen as more fundamental than $\mathcal{U}$ in Physics. In fact, $\mathcal{U}$ corresponds in this context to the so-called local field algebra. See, e.g., [74, Sections 4.8 and 6].

Note that any element $A \in \mathcal{U}$ can be written as a sum of even and odd elements, respectively denoted by A^+ and A^-:

$$A = A^+ + A^- \qquad \text{with} \qquad A^\pm \doteq \frac{1}{2}\left(A \pm \sigma\left(A\right)\right)\ . \tag{7}$$

This directly follows from the fact that σ is an involution (i.e., $\sigma \circ \sigma = \mathbf{1}_{\mathcal{U}}$).

[6] More precisely, the so-called sector theory of quantum field theory.

Translations: $\mathfrak{L} = \mathbb{Z}^d$ is an important case here, because invariant states under the action of the group $(\mathbb{Z}^d, +)$ of lattice translations on $\mathcal{U} \equiv \mathcal{U}_{\mathbb{Z}^d}$ are pivotal in the full description of macroscopic long-range dynamics. The translations in $\mathcal{U}$ refer to the group homomorphism $x \mapsto \alpha_x$ from $\mathbb{Z}^d$ to the group of $*$-automorphisms of $\mathcal{U}$, which is uniquely defined by the condition

$$\alpha_x(a_{y,\mathrm{s}}) = a_{y+x,\mathrm{s}} , \quad y \in \mathbb{Z}^d, \ \mathrm{s} \in \mathrm{S} . \tag{8}$$

This group homomorphism is used below to define the notion of (space) periodicity of states of lattice-fermion systems (Section 2.5), as well as the notion of translation-invariance of interactions (Section 3.1).

Permutations: Let Π be the set of all bijective mappings from $\mathfrak{L}$ to itself which leave all but finitely many elements invariant. It is a group with respect to the composition of mappings. The condition

$$\mathfrak{p}_\pi : a_{x,\mathrm{s}} \mapsto a_{\pi(x),\mathrm{s}}, \quad x \in \mathfrak{L}, \ \mathrm{s} \in \mathrm{S} , \tag{9}$$

defines a group homomorphism $\pi \mapsto \mathfrak{p}_\pi$ from Π to the group of $*$-automorphisms of $\mathcal{U}$.

2.4 State Space

For any $\Lambda \in \mathcal{P}_f$, we denote by

$$E_\Lambda \doteq \{\rho_\Lambda \in \mathcal{U}_\Lambda^* : \rho_\Lambda \geq 0, \ \rho_\Lambda(\mathfrak{1}) = 1\} = \{\rho_\Lambda \in \mathcal{U}_\Lambda^* : \|\rho_\Lambda\|_{\mathcal{U}_\Lambda^*} = \rho_\Lambda(\mathfrak{1}) = 1\} \tag{10}$$

the set of all states on $\mathcal{U}_\Lambda$. By finite dimensionality of $\mathcal{U}_\Lambda$ for $\Lambda \in \mathcal{P}_f$, the set E_Λ is a norm-compact convex subset of the dual space $\mathcal{U}_\Lambda^*$. It is not a simplex, being affinely equivalent to the set of states on a matrix algebra. It does not have of course a dense set of extreme points. In comparison, the structure of the set of states for infinite systems is more subtle.

The state space associated with $\mathcal{U}$ is defined by

$$E \doteq \{\rho \in \mathcal{U}^* : \rho \geq 0, \ \rho(\mathfrak{1}) = 1\} = \{\rho \in \mathcal{U}^* : \|\rho\|_{\mathcal{U}^*} = \rho(\mathfrak{1}) = 1\} . \tag{11}$$

In particular, if $\mathfrak{L} \in \mathcal{P}_f$ then $E = E_{\mathfrak{L}}$. In any case, for $\mathcal{U}$ is a separable Banach space, E is a metrizable and weak*-compact convex subset of the dual space $\mathcal{U}^*$, by [87, Theorems 3.15 and 3.16]. It is also the state space of the classical dynamics [63, Definition 2.1] introduced later on.

By the Krein-Milman theorem [87, Theorem 3.23], E is the weak*-closure of the convex hull of the (non-empty) set $\mathcal{E}(E)$ of its extreme points:

$$E = \overline{\mathrm{co}\mathcal{E}\,(E)} . \tag{12}$$

When $\mathfrak{L}$ is an infinite set, $\mathcal{U}$ is antiliminal and simple, because it is a UHF (uniformly hyperfinite) algebra. See, e.g., [63, Section 8]. Therefore, by [63, Lemma 8.5], the state space E has a weak*-dense subset of extreme points:

$$E = \overline{\mathcal{E}(E)} . \tag{13}$$

This fact is well-known and already discussed in [77, p. 226]. As a matter of fact, the property of a convex weak*-compact set having a weak*-dense set of extreme points is not accidental, but *generic*[7] in infinite dimension, by [63, Theorem 2.4].

[7] More precisely, by [63, Theorem 2.4], the set of all convex weak*-compact subsets of the dual $\mathcal{X}^*$ of an infinite-dimensional separable Banach space $\mathcal{X}$ is a weak*-Hausdorff-dense G_δ subset of the space of convex weak*-compact subsets of $\mathcal{X}^*$.

For any $\Lambda \in \mathcal{P}_f$, the restriction of a state $\rho \in E$ to $\mathcal{U}_\Lambda$ yields a state in E_Λ, always denoted by $\rho_\Lambda \in E_\Lambda$. Conversely, for any $\Lambda \in \mathcal{P}_f$, a state $\rho_\Lambda \in E_\Lambda$ can be seen as a state $\tilde{\rho}$ on $\mathcal{U}$ by setting[8]

$$\tilde{\rho}(AB) = \rho_\Lambda(A)\mathrm{tr}(B) \ , \qquad A \in \mathcal{U}_\Lambda, \ B \in \mathcal{U}_\mathcal{Z}, \ \mathcal{Z} \in \mathcal{P}_f, \ \mathcal{Z} \cap \Lambda = \emptyset \ , \tag{14}$$

where tr is the unique tracial state on $\mathcal{U}$, also named normalized trace, see [78, Section 4.2]. In particular, similar to (3), the set

$$E_0 \doteq \bigcup_{L \in \mathbb{N}} E_{\Lambda_L}$$

can be seen as a weak*-dense subset of E, by density of $\mathcal{U}_0 \subseteq \mathcal{U}$.

2.5 Even States

As explained below Equation (5), recall that the C^*-algebra $\mathcal{U}^+$ should thus be seen as more fundamental than $\mathcal{U}$ in Physics, because of the local causality in quantum field theory. As a consequence, states on the C^*-algebra $\mathcal{U}^+$ should be seen as being the physically relevant ones. As it is explicitly shown in the proof of Proposition 2.1, the set of states on $\mathcal{U}^+$ is canonically identified with the set of *even* states on $\mathcal{U}$, defined by

$$E^+ \doteq \{\rho \in E : \rho \circ \sigma = \rho\} \ , \tag{15}$$

σ being the unique $*$-automorphism of $\mathcal{U}$ satisfying (4). E^+ is again a metrizable and weak*-compact convex set and has a (non-empty) set $\mathcal{E}(E^+)$ of extreme points such that

$$E^+ = \overline{\mathrm{co}\mathcal{E}\left(E^+\right)} \ ,$$

by the Krein-Milman theorem [87, Theorem 3.23]. Like for the space E of *all* states (cf. (13)), if $\mathfrak{L}$ is an infinite set then $\mathcal{E}(E^+)$ is a *dense* subset of the set of all even states:

Proposition 2.1 (Weak*-density of extreme even states)
Let $\mathfrak{L} \subseteq \mathbb{Z}^d$ be an infinite set. The set $\mathcal{E}(E^+)$ of extreme points of E^+ is a weak*-dense subset of E^+, i.e., $E^+ = \overline{\mathcal{E}(E^+)}$.

Proof. Let $\tilde{E}^+$ be the set of all states on $\mathcal{U}^+$. Obviously, $E^+ \subseteq \tilde{E}^+$ by seeing any state on E as a state on $\mathcal{U}^+$, by restriction. Conversely, for any state $\tilde{\rho} \in \tilde{E}^+$, we define the linear functional

$$\rho \doteq \tilde{\rho} \circ \left(\frac{\sigma + \mathbf{1}_\mathcal{U}}{2}\right) \in \mathcal{U}^* \ .$$

Note that $\|\sigma\|_{\mathcal{B}(\mathcal{U}^*)} = 1$ because σ is a $*$-automorphism of a C^*-algebra. Since $\rho(\mathfrak{1}) = 1$ and $\|\rho\|_{\mathcal{U}^*} \leq 1$, we deduce from [75, Proposition 2.3.11] that $\rho \in E$. By construction, the restriction of ρ to $\mathcal{U}^+$ is $\tilde{\rho}$ while $\rho(A) = 0$ for all odd elements $A \in \mathcal{U}$. In other words, $\rho \in E^+$ and one can identify $\tilde{E}^+$ with E^+. It easy to check that the mapping $\tilde{\rho} \mapsto \rho$ is an affine weak* homeomorphism from $\tilde{E}^+$ to E^+. By [76], if $\mathfrak{L}$ is an infinite set then $\mathcal{U}^+$ is $*$-isomorphic to $\mathcal{U}$. Thus, by (13), the assertion follows. ■

Remark 2.2 (States of quantum-spin systems)
As already mentioned, all results presented here can be extended to quantum-spin systems on a lattice, the corresponding C^*-algebra being an infinite tensor product of finite-dimensional matrix algebras attached to each lattice site $x \in \mathfrak{L} = \mathbb{Z}^d$. In this case, there is no even property to consider, which corresponds to take $E^+ = E$ in all the discussions.

[8](14) is well defined because of [78, Theorem 11.2], the tracial state tr being even.

Important examples of even states are the so-called periodic states: Let $\mathfrak{L} = \mathbb{Z}^d$. Consider the sub-groups $(\mathbb{Z}^d_{\vec{\ell}}, +) \subseteq (\mathbb{Z}^d, +)$, $\vec{\ell} \in \mathbb{N}^d$, where

$$\mathbb{Z}^d_{\vec{\ell}} \doteq \ell_1\mathbb{Z} \times \cdots \times \ell_d\mathbb{Z} \ .$$

Any state $\rho \in E$ satisfying $\rho \circ \alpha_x = \rho$ for all $x \in \mathbb{Z}^d_{\vec{\ell}}$ is called $\mathbb{Z}^d_{\vec{\ell}}$*-invariant* on $\mathcal{U}$ or $\vec{\ell}$*-periodic*, α_x being the unique $*$-automorphism of $\mathcal{U}$ satisfying (8). Translation-invariant states refer to $(1, \cdots, 1)$-periodic states. The set of all periodic states is denoted by

$$E_{\mathrm{p}} \doteq \bigcup_{\vec{\ell} \in \mathbb{N}^d} E_{\vec{\ell}} \,, \tag{16}$$

where, for any $\vec{\ell} \in \mathbb{N}^d$,

$$E_{\vec{\ell}} \doteq \left\{ \rho \in E : \rho \circ \alpha_x = \rho \quad \text{for all } x \in \mathbb{Z}^d_{\vec{\ell}} \right\} \ . \tag{17}$$

By [46, Lemma 1.8], periodic states are even and form a weak*-dense subset of even states:

Proposition 2.3 (Weak*-density of periodic states)
Let $\mathfrak{L} = \mathbb{Z}^d$. The set E_{p} of periodic states is a weak*-dense set of E^+, i.e., $E^+ = \overline{E_{\mathrm{p}}}$.

Proof. For any $\rho \in E^+$ and $n \in \mathbb{N}$, we define the state $\tilde{\rho}_n$ to be some $(2n+1, \ldots, 2n+1)$-periodic state for which $(\tilde{\rho}_n)_{\Lambda_n} = \rho_{\Lambda_n}$ in the cubic box Λ_n (1). Such a state always exists because of [78, Theorem 11.2], since ρ is, by definition, even. Clearly, $\{\tilde{\rho}_n\}_{n\in\mathbb{N}}$ converges, as $n \to \infty$, towards $\rho \in E^+$ with respect to the weak*-topology, by density of $\mathcal{U}_0 \subseteq \mathcal{U}$. ■

The sets $E_{\vec{\ell}}$, $\vec{\ell} \in \mathbb{N}^d$, of $\vec{\ell}$-periodic states all share the *same* peculiar geometrical structure: Like the set E of all states for $\mathfrak{L} = \mathbb{Z}^d$, they are metrizable and weak*-compact convex sets with a weak*-dense set $\mathcal{E}(E_{\vec{\ell}})$ of extreme points:

$$E_{\vec{\ell}} = \overline{\mathrm{co}\mathcal{E}(E_{\vec{\ell}})} = \overline{\mathcal{E}(E_{\vec{\ell}})} \,, \qquad \vec{\ell} \in \mathbb{N}^d \ . \tag{18}$$

Compare with Equations (12)-(13). In fact, up to an affine homeomorphism, for any $\vec{\ell} \in \mathbb{N}^d$, $E_{\vec{\ell}}$ is the so-called *Poulsen simplex* [46, Theorem 1.12]. This property is well-known and also holds true for quantum-spin systems [75, p. 405-406, 464]. The fact that all $E_{\vec{\ell}}$, $\vec{\ell} \in \mathbb{N}^d$, have the same topological structure is not so surprising since, for any fixed $\vec{\ell} \in \mathbb{N}^d$, we can redefine the spin set $\mathrm{S} = \mathrm{S}_{\vec{\ell}}$ and, as a consequence, the CAR algebra $\mathcal{U} = \mathcal{U}_{\vec{\ell}}$ to see any $\vec{\ell}$-periodic state $\rho \in E_{\vec{\ell}}$ as a translation-invariant state on the new CAR algebra $\mathcal{U}_{\vec{\ell}}$. By [63, Proposition 6.14], E^+ can also be seen as the *weak*-Hausdorff limit* of the increasing sequence $\{E_{\vec{\ell}}\}_{\vec{\ell}\in\mathbb{N}^d}$ of weak*-compact *simplices* with weak*-dense set of extreme points.

By [46, Theorem 1.16] note that the set of all ($\vec{\ell}$-) *ergodic* states, as defined by [46, Definition 1.15] for any $\vec{\ell} \in \mathbb{N}^d$, is equal to

$$\mathcal{E}_{\mathrm{p}} \doteq \bigcup_{\vec{\ell} \in \mathbb{N}^d} \mathcal{E}\left(E_{\vec{\ell}}\right) \subseteq E_{\mathrm{p}} \subseteq E^+ \ . \tag{19}$$

By Proposition 2.3, $\mathcal{E}_{\mathrm{p}}$ is also a weak*-dense set of E^+.

The set E_{p} is important because, in all cyclic representations of $\mathcal{U}$ associated with a state $\rho \in E_{\mathrm{p}}$, the infinite-volume dynamics of interacting lattice fermions with long-range interactions exists. The ergodicity of states of $\mathcal{E}_{\mathrm{p}}$ plays a crucial rôle in this context.

Instead of the set E_{p} of periodic states, previous studies extracting classical dynamics for lattice-fermion and quantum-spin systems with long-range interactions (cf. [5, 6, 12–32]) use classical flows within the subset

$$E_{\Pi} \doteq \{\rho \in E : \rho \circ \mathfrak{p}_\pi = \rho \quad \text{for all } \pi \in \Pi\} \tag{20}$$

of *permutation-invariant* states, $\mathfrak{p}_\pi$ being the unique $*$-automorphism of $\mathcal{U}$ satisfying (9). This set has a much simpler structure than the set E_{p} of periodic states:

$$E_\Pi \subseteq \bigcap_{\vec{\ell}\in\mathbb{N}^d} E_{\vec{\ell}} \subseteq E_{\mathrm{p}} \ .$$

Compare this assertion with (16). E_Π is a closed metrizable face of $E_{\vec{\ell}}$ for all $\vec{\ell} \in \mathbb{N}^d$ and a Bauer simplex, i.e., a (Choquet) simplex whose set of extreme points is closed. Indeed, by Størmer's theorem [46, Theorem 5.2], extreme points of E_Π are so-called *product states*. See [46, Section 5.1] for more details on permutation-invariant states.

3 Lattice Fermions with Short-Range Interactions

3.1 Banach Spaces of Short-Range Interactions

A (complex) *interaction* is a mapping $\Phi : \mathcal{P}_f \to \mathcal{U}^+$ such that $\Phi_\Lambda \in \mathcal{U}_\Lambda$ for any $\Lambda \in \mathcal{P}_f$. The set of all interactions can be naturally endowed with the structure of a complex vector space as follows:

$$(\Phi + \tilde{\Phi})_\Lambda \doteq \Phi_\Lambda + \tilde{\Phi}_\Lambda \qquad \text{and} \qquad (\lambda\Phi)_\Lambda \doteq \lambda\Phi_\Lambda \tag{21}$$

for all interactions $\Phi, \tilde{\Phi}$ and $\lambda \in \mathbb{C}$. The mapping

$$\Phi \mapsto \Phi^* \doteq (\Phi_\Lambda^*)_{\Lambda\in\mathcal{P}_f} \tag{22}$$

is a natural involution on the vector space of interactions. Self-adjoint interactions are interactions Φ satisfying $\Phi = \Phi^*$. The set of all self-adjoint interactions forms a real subspace of the space of all interactions. Any interaction Φ can be decomposed into its real and imaginary parts, which are self-adjoint interactions respectively defined by

$$\mathrm{Re}\,\{\Phi\} \doteq \frac{1}{2}\left(\Phi^* + \Phi\right) \qquad \text{and} \qquad \mathrm{Im}\,\{\Phi\} \doteq \frac{1}{2i}\left(\Phi - \Phi^*\right) \ . \tag{23}$$

We now define a Banach space $\mathcal{W}$ of short-range interactions by introducing a norm for interactions that take into account their spacial decay. To this end, we use a positive-valued symmetric function $\mathbf{F} : \mathfrak{L}^2 \to (0,1]$ with maximum value $\mathbf{F}(x,x) = 1$ for all $x \in \mathfrak{L}$. Like for instance in [72], we impose the following conditions on $\mathbf{F}$:

- *Summability on $\mathfrak{L}$.*
$$\|\mathbf{F}\|_{1,\mathfrak{L}} \doteq \sup_{y\in\mathfrak{L}} \sum_{x\in\mathfrak{L}} \mathbf{F}(x,y) \in [1,\infty) \ . \tag{24}$$

- *Bounded convolution constant.*
$$\mathbf{D} \doteq \sup_{x,y\in\mathfrak{L}} \sum_{z\in\mathfrak{L}} \frac{\mathbf{F}(x,z)\,\mathbf{F}(z,y)}{\mathbf{F}(x,y)} < \infty \ . \tag{25}$$

Examples of functions $\mathbf{F} : \mathfrak{L}^2 \to (0,1]$ satisfying (24)-(25) for any lattice $\mathfrak{L} \subseteq \mathbb{Z}^d$ ($d \in \mathbb{N}$) are given by

$$\mathbf{F}(x,y) = (1 + |x-y|)^{-(d+\epsilon)} \qquad \text{or} \qquad \mathbf{F}(x,y) = \mathrm{e}^{-\varsigma|x-y|}(1 + |x-y|)^{-(d+\epsilon)} \tag{26}$$

for every $\varsigma, \epsilon \in \mathbb{R}^+$. In all the paper, (24)-(25) are assumed to be satisfied.

Then, a norm for interactions Φ is defined by

$$\|\Phi\|_{\mathcal{W}} \doteq \sup_{x,y\in\mathfrak{L}} \sum_{\Lambda\in\mathcal{P}_f,\ \Lambda\supseteq\{x,y\}} \frac{\|\Phi_\Lambda\|_{\mathcal{U}}}{\mathbf{F}(x,y)} \tag{27}$$

and

$$\mathcal{W} \equiv \mathcal{W}^{(\mathfrak{L})} \equiv (\mathcal{W}, \|\cdot\|_{\mathcal{W}})$$

denotes the separable Banach space of interactions Φ satisfying $\|\Phi\|_{\mathcal{W}} < \infty$. Elements $\Phi \in \mathcal{W}$ are named *short-range* interactions on $\mathfrak{L} \subseteq \mathbb{Z}^d$. The (real) Banach subspace of all self-adjoint interactions is denoted by $\mathcal{W}^{\mathbb{R}} \subsetneq \mathcal{W}$, similar to $\mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U}$.

By definition, an interaction Φ on $\mathfrak{L} = \mathbb{Z}^d$ is translation-invariant if, for all $x \in \mathbb{Z}^d$ and $\Lambda \in \mathcal{P}_f$, $\Phi_{\Lambda+x} = \alpha_x(\Phi_\Lambda)$, where

$$\Lambda + x \doteq \left\{y + x \in \mathbb{Z}^d : y \in \Lambda\right\} \ . \tag{28}$$

Recall that $\{\alpha_x\}_{x\in\mathbb{Z}^d}$ is the family of (translation) $*$-automorphisms of $\mathcal{U}$ defined by (8). We denote by $\mathcal{W}_1 \subsetneq \mathcal{W}$ the (separable) Banach subspace of translation-invariant, short-range interactions on $\mathfrak{L} = \mathbb{Z}^d$.

For any $\Phi \in \mathcal{W}$ and $\vec{\ell} \in \mathbb{N}^d$, we define the even observable

$$\mathfrak{e}_{\Phi,\vec{\ell}} \doteq \frac{1}{\ell_1 \cdots \ell_d} \sum_{x=(x_1,\ldots,x_d),\ x_i\in\{0,\ldots,\ell_i-1\}} \ \sum_{\mathcal{Z}\in\mathcal{P}_f,\ \mathcal{Z}\ni x} \frac{\Phi_{\mathcal{Z}}}{|\mathcal{Z}|} \ . \tag{29}$$

From (24) and (27), note that

$$\|\mathfrak{e}_{\Phi,\vec{\ell}}\|_{\mathcal{U}} \leq \|\mathbf{F}\|_{1,\mathfrak{L}} \|\Phi\|_{\mathcal{W}} \ , \qquad \Phi \in \mathcal{W},\ \vec{\ell} \in \mathbb{N}^d. \tag{30}$$

For any self-adjoint, translation-invariant interaction Φ, they refer to the energy density observables of [46, Eq. (1.16)]. See also Proposition 3.2 below. Additionally, for any $\Lambda \in \mathcal{P}_f$, we define the closed subspaces[9]

$$\mathcal{W}_\Lambda \doteq \{\Phi \in \mathcal{W}_1 : \Phi_{\mathcal{Z}} = 0 \text{ whenever } \mathcal{Z} \nsubseteq \Lambda,\ \mathcal{Z} \ni 0\} \tag{31}$$

of finite-range translation-invariant interactions. Note that, for any $\Lambda \in \mathcal{P}_f$ and $\vec{\ell} \in \mathbb{N}^d$,

$$\mathcal{W}_\Lambda \subseteq \left\{\Phi \in \mathcal{W}_1 : \mathfrak{e}_{\Phi,\vec{\ell}} \in \mathcal{U}_{\Lambda^{(\vec{\ell})}}\right\} \subseteq \mathcal{W}_1 \ , \tag{32}$$

where

$$\Lambda^{(\vec{\ell})} \doteq \cup\left\{\Lambda + x : x = (x_1, \ldots, x_d),\ x_i \in \{0, \ldots, \ell_i - 1\}\right\} \in \mathcal{P}_f \ . \tag{33}$$

Recall that, for $\Lambda \in \mathcal{P}_f$, $\mathcal{U}_\Lambda \subseteq \mathcal{U}$ is the finite-dimensional unital C^*-algebra generated by elements $\{a_{x,\mathrm{s}}\}_{x\in\Lambda,\mathrm{s}\in\mathrm{S}}$ satisfying the CAR (2). Similar to (3),

$$\mathcal{W}_0 \doteq \bigcup_{L\in\mathbb{N}} \mathcal{W}_{\Lambda_L} \subseteq \mathcal{W}_1 \tag{34}$$

is a dense subspace of $\mathcal{W}_1$. Elements of $\mathcal{W}_0$ are *finite-range*, translation-invariant interactions.

[9]This follows from the continuity and linearity of the mappings $\Phi \mapsto \Phi_{\mathcal{Z}}$ for all $\mathcal{Z} \in \mathcal{P}_f$.

3.2 Local Energy Elements

We define a sequence of local elements associated with any complex interaction $\Phi \in \mathcal{W}$ as follows:

Definition 3.1 (Local energy)
The local energy elements of a complex interaction $\Phi \in \mathcal{W}$ are

$$U_L^\Phi \doteq \sum_{\Lambda \subseteq \Lambda_L} \Phi_\Lambda \in \mathcal{U}_{\Lambda_L} \cap \mathcal{U}^+ , \qquad L \in \mathbb{N} .$$

If $\Phi \in \mathcal{W}^{\mathbb{R}}$, then $(U_L^\Phi)_{L\in\mathbb{N}} \in \mathcal{U}^{\mathbb{R}}$ and so, Definition 3.1 yields a sequence of local Hamiltonians, which are used to generate finite-volume dynamics.

By straightforward estimates using (24) and (27), as in (30), note that, for any complex interactions $\Phi, \Psi \in \mathcal{W}$,

$$\left\| U_L^\Phi - U_L^\Psi \right\|_{\mathcal{U}} = \left\| U_L^{\Phi-\Psi} \right\|_{\mathcal{U}} \leq |\Lambda_L| \, \|\mathbf{F}\|_{1,\mathfrak{L}} \, \|\Phi - \Psi\|_{\mathcal{W}} , \qquad L \in \mathbb{N} . \tag{35}$$

In addition, similar to [46, Lemma 1.32], local energy elements yield energy densities for translation-invariant short-range interactions:

Proposition 3.2 (Energy density of periodic states)
For any $\vec{\ell} \in \mathbb{N}^d$, $\vec{\ell}$-periodic state $\rho \in E_{\vec{\ell}}$ (17) and translation-invariant complex interaction $\Phi \in \mathcal{W}_1$,

$$\lim_{L\to\infty} \frac{\rho\left(U_L^\Phi\right)}{|\Lambda_L|} = \rho(\mathfrak{e}_{\Phi,\vec{\ell}})$$

with $\mathfrak{e}_{\Phi,\vec{\ell}}$ being the even observable defined by (29).

Proof. All arguments of the proof can be found in [46, Lemma 4.17]. ■

By Proposition 3.2, the energy density

$$e_\Phi(\rho) \doteq \lim_{L\to\infty} \frac{\rho\left(U_L^\Phi\right)}{|\Lambda_L|} , \qquad \Phi \in \mathcal{W}_1 ,$$

exists for all periodic states, which form a weak*-dense subset E_{p} of the set E^+ of even states, by Proposition 2.3. Note that there are infinitely many (uncountable) other states with this property. Examples of non-periodic states for which $e_\Phi(\rho)$ exists can be constructed by using KMS states associated with *random* interactions together with the Akcoglu-Krengel ergodic theorem [80]. See, e.g., [81]. For all $\rho \in E$ such that $e_\Phi(\rho)$ exists, observe from Inequality (35) that

$$|e_\Phi(\rho) - e_\Psi(\rho)| \leq \|\mathbf{F}\|_{1,\mathfrak{L}} \, \|\Phi - \Psi\|_{\mathcal{W}} , \qquad \Phi, \Psi \in \mathcal{W}_1 . \tag{36}$$

3.3 Derivations on the CAR Algebra

For any short-range interaction $\Phi \in \mathcal{W}$, the elements of Definition 3.1 determine a sequence of bounded operators on $\mathcal{U}$:

Definition 3.3 (Derivations on the CAR algebra for short-range interactions)
The derivations $\{\delta_L^\Phi\}_{L\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{U})$ associated with any interaction $\Phi \in \mathcal{W}$ are defined by

$$\delta_L^\Phi(A) \doteq i\left[U_L^\Phi, A\right] \doteq i\left(U_L^\Phi A - A U_L^\Phi\right) , \qquad A \in \mathcal{U}, \ L \in \mathbb{N} .$$

They are (bounded) derivations on $\mathcal{U}$ since, for any $\Phi \in \mathcal{W}$ and $L \in \mathbb{N}$,

$$\delta_L^\Phi (AB) = \delta_L^\Phi (A) B + A\delta_L^\Phi (B) \ , \qquad A, B \in \mathcal{U} \ . \tag{37}$$

They are symmetric (or $*$-derivations) when $\Phi \in \mathcal{W}^{\mathbb{R}}$:

$$\delta_L^\Phi (A)^* = \delta_L^\Phi(A^*) \ , \qquad A \in \mathcal{U}, \ L \in \mathbb{N} \ . \tag{38}$$

In the thermodynamic limit $L \to \infty$, this sequence of derivations leads to a limit derivation defined on the (dense) subset $\mathcal{U}_0$ (3) of local elements:

Proposition 3.4 (Strong convergence of finite-volume derivations)
For any $\Phi \in \mathcal{W}$, $\Lambda \in \mathcal{P}_f$, $A \in \mathcal{U}_\Lambda$ and $L_2 \geq L_1 \geq 0$,

$$\left\| \delta_{L_2}^\Phi (A) - \delta_{L_1}^\Phi (A) \right\|_{\mathcal{U}} \leq 2 \left|\Lambda\right| \left\|A\right\|_{\mathcal{U}} \left\|\Phi\right\|_{\mathcal{W}} \sup_{y\in\Lambda} \sum_{x \in \Lambda_{L_1}^c} \mathbf{F}(x,y)$$

and

$$\sup_{L\in\mathbb{N}} \left\| \delta_L^\Phi (A) \right\|_{\mathcal{U}} \leq 2 \left|\Lambda\right| \left\|A\right\|_{\mathcal{U}} \left\|\Phi\right\|_{\mathcal{W}} \left\|\mathbf{F}\right\|_{1,\mathfrak{L}} \ .$$

Proof. Fixing all parameters of the proposition, we straightforwardly get the estimate

$$\begin{aligned} \left\| \delta_{L_2}^\Phi (A) - \delta_{L_1}^\Phi (A) \right\|_{\mathcal{U}} &\leq \sum_{y\in\Lambda} \sum_{x\in\Lambda_{L_1}^c} \sum_{\mathcal{Z}\in\mathcal{P}_f, \ \mathcal{Z}\supseteq\{x,y\}} \left\| [\Phi_{\mathcal{Z}}, A] \right\|_{\mathcal{U}} \\ &\leq 2 \left\|A\right\|_{\mathcal{U}} \left\|\Phi\right\|_{\mathcal{W}} \sum_{y\in\Lambda} \sum_{x\in\Lambda_{L_1}^c} \mathbf{F}(x,y) \ , \end{aligned}$$

which implies the first assertion. The second assertion is even simpler to prove. We omit the details.
■

Corollary 3.5 (Generators of infinite-volume short-range dynamics)
For any $\Phi \in \mathcal{W}$ and $A \in \mathcal{U}_0$, the limit

$$\delta^\Phi (A) = \lim_{L\to\infty} \delta_L^\Phi (A)$$

exists and defines a (densely defined) derivation δ^Φ from $\mathcal{U}_0 \subseteq \mathcal{U}$ to $\mathcal{U}$ satisfying the following bound:

$$\left\| \delta^\Phi (A) \right\|_{\mathcal{U}} \leq 2 \left|\Lambda\right| \left\|A\right\|_{\mathcal{U}} \left\|\Phi\right\|_{\mathcal{W}} \left\|\mathbf{F}\right\|_{1,\mathfrak{L}} \ , \qquad A \in \mathcal{U}_\Lambda, \ \Lambda \in \mathcal{P}_f \ .$$

Additionally, δ^Φ is symmetric (or a $*$-derivation) whenever $\Phi \in \mathcal{W}^{\mathbb{R}}$.

Proof. Combine Proposition 3.4 with Equations (37)-(38) and the completeness of $\mathcal{U}$. ■

Remark 3.6 (Closure of limit derivations)
If $\Phi \in \mathcal{W}^{\mathbb{R}}$ then the symmetric derivation δ^Φ is (norm-) closable [72, Lemma 4.6]. It is proven from its dissipativity [82, Definition 1.4.6, Proposition 1.4.7], which is, in turn, deduced from [82, Theorem 1.4.9] because $A \in \mathcal{U}_0$ and $A \geq 0$ implies $A^{1/2} \in \mathcal{U}_0$. Moreover, its closure generates a strongly continuous group of $*$-automorphisms of $\mathcal{U}$, by [72, Theorem 4.8]. See also Proposition 3.7 below.

3.4 Dynamics Generated by Short-Range Interactions

We now consider time-dependent interactions. Let $\Psi \in C(\mathbb{R}; \mathcal{W})$ be a continuous function from $\mathbb{R}$ to the Banach space $\mathcal{W}$ of interactions on $\mathfrak{L} \subseteq \mathbb{Z}^d$. Then, for any $L \in \mathbb{N}$, there is a unique (fundamental) solution $(\tau_{t,s}^{(L,\Psi)})_{s,t\in\mathbb{R}}$ in $\mathcal{B}(\mathcal{U})$ to the (finite-volume) non-autonomous evolution equations

$$\forall s,t \in \mathbb{R}: \qquad \partial_s \tau_{t,s}^{(L,\Psi)} = -\delta_L^{\Psi(s)} \circ \tau_{t,s}^{(L,\Psi)}, \qquad \tau_{t,t}^{(L,\Psi)} = \mathbf{1}_{\mathcal{U}}, \tag{39}$$

and

$$\forall s,t \in \mathbb{R}: \qquad \partial_t \tau_{t,s}^{(L,\Psi)} = \tau_{t,s}^{(L,\Psi)} \circ \delta_L^{\Psi(t)}, \qquad \tau_{s,s}^{(L,\Psi)} = \mathbf{1}_{\mathcal{U}}. \tag{40}$$

In these two equations, $\mathbf{1}_{\mathcal{U}}$ refers to the identity mapping of $\mathcal{U}$. Note also that, for any $L \in \mathbb{N}$ and $\Psi \in C(\mathbb{R}; \mathcal{W})$, $(\tau_{t,s}^{(L,\Psi)})_{s,t\in\mathbb{R}}$ is a continuous two-parameter family of bounded operators that satisfies the (reverse) cocycle property

$$\forall s,r,t \in \mathbb{R}: \qquad \tau_{t,s}^{(L,\Psi)} = \tau_{r,s}^{(L,\Psi)} \tau_{t,r}^{(L,\Psi)}.$$

If $\Psi \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$ then $\delta_L^{\Psi(t)}$ is always a symmetric derivation (or $*$-derivation) and thus, in this case, $\tau_{t,s}^{(L,\Psi)}$ is a $*$-automorphism of $\mathcal{U}$ for all lengths $L \in \mathbb{N}$ and times $s,t \in \mathbb{R}$. Moreover, in the thermodynamic limit $L \to \infty$, the family $(\tau_{t,s}^{(L,\Psi)})_{s,t\in\mathbb{R}}$ strongly converges to a strongly continuous two-parameter family of $*$-automorphisms of $\mathcal{U}$, associated with the family $\{\delta^{\Psi(t)}\}_{t\in\mathbb{R}}$ of limit symmetric derivations of Corollary 3.5:

Proposition 3.7 (Infinite-volume short-range dynamics)
For any $\Psi \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$, as $L \to \infty$, $(\tau_{t,s}^{(L,\Psi)})_{s,t\in\mathbb{R}}$ converges strongly, uniformly for s,t on compacta, to a strongly continuous two-parameter family $(\tau_{t,s}^{\Psi})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{U}$, which is the unique solution in $\mathcal{B}(\mathcal{U})$ to the non-autonomous evolutions equation

$$\forall s,t \in \mathbb{R}: \qquad \partial_t \tau_{t,s}^{\Psi} = \tau_{t,s}^{\Psi} \circ \delta^{\Psi(t)}, \qquad \tau_{s,s}^{\Psi} = \mathbf{1}_{\mathcal{U}}, \tag{41}$$

in the strong sense on the dense subspace $\mathcal{U}_0 \subseteq \mathcal{U}$. In particular, it satisfies the reverse cocycle property:

$$\forall s,r,t \in \mathbb{R}: \qquad \tau_{t,s}^{\Psi} = \tau_{r,s}^{\Psi} \tau_{t,r}^{\Psi}. \tag{42}$$

Proof. See [72, Corollary 5.2]. ■

It is convenient to introduce at this point the notation

$$\partial_\Psi \Lambda \doteq \{x \in \Lambda : \exists \mathcal{Z} \in \mathcal{P}_f \text{ with } x \in \mathcal{Z},\ \Psi_{\mathcal{Z}} \neq 0 \text{ and } \mathcal{Z} \cap \Lambda \neq 0,\ \mathcal{Z} \cap \Lambda^c \neq 0\}$$

for any interaction Ψ and any finite subset $\Lambda \in \mathcal{P}_f$ with complement $\Lambda^c \doteq \mathfrak{L} \backslash \Lambda$. For any $s,t \in \mathbb{R}$, $t \wedge s$ and $t \vee s$ stand, respectively, for the minimum and maximum of the set $\{s,t\}$. We are now in a position to give additional estimates on the limit dynamics, like the celebrated Lieb-Robinson bounds:

Proposition 3.8 (Estimates on short-range dynamics)
For any $\Psi \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$, $(\tau_{t,s}^{\Psi})_{s,t\in\mathbb{R}}$ satisfies the following bounds:

(i) *Lieb-Robinson bounds (cf. (6)). For any $s,t \in \mathbb{R}$, sets $\Lambda^{(1)}, \Lambda^{(2)} \in \mathcal{P}_f$ with $\Lambda^{(1)} \cap \Lambda^{(2)} = \emptyset$, every even element $A_1 \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda^{(1)}}$ and all $A_2 \in \mathcal{U}_{\Lambda^{(2)}}$,*

$$\left\| \left[\tau_{t,s}^{\Psi}(A_1), A_2 \right] \right\|_{\mathcal{U}} \leq 2\mathbf{D}^{-1} \|A_1\|_{\mathcal{U}} \|A_2\|_{\mathcal{U}} \left(\mathrm{e}^{2\mathbf{D} \int_{t\wedge s}^{t\vee s} \|\Psi(\alpha)\|_{\mathcal{W}} \mathrm{d}\alpha} - 1 \right) \sum_{x \in \partial_\Psi \Lambda^{(1)}} \sum_{y \in \Lambda^{(2)}} \mathbf{F}(x,y).$$

(ii) *Rate of convergence. For any* $s,t\in\mathbb{R}$, $\Lambda\in\mathcal{P}_f$, $A\in\mathcal{U}_\Lambda$ *and* $L\in\mathbb{N}$ *such that* $\Lambda\subseteq\Lambda_L$,

$$\begin{aligned}&\left\|\tau_{t,s}^{\Psi}(A)-\tau_{t,s}^{(L,\Psi)}(A)\right\|_{\mathcal{U}}\\ \leq\ & 2\|A\|_{\mathcal{U}}\int_{t\wedge s}^{t\vee s}\left(\|\Psi(\alpha_1)\|_{\mathcal{W}}\,\mathrm{e}^{2\mathbf{D}\int_{\alpha_1\wedge s}^{\alpha_1\vee s}\|\Psi(\alpha_2)\|_{\mathcal{W}}\mathrm{d}\alpha_2}\right)\mathrm{d}\alpha_1\sum_{y\in\mathfrak{L}\backslash\Lambda_L}\sum_{x\in\Lambda}\mathbf{F}(x,y)\ .\end{aligned}$$

(iii) *Lipschitz continuity with respect to* Ψ. *For any* $s,t\in\mathbb{R}$, $\Psi,\tilde{\Psi}\in C\left(\mathbb{R};\mathcal{W}^{\mathbb{R}}\right)$, $\Lambda\in\mathcal{P}_f$ *and* $A\in\mathcal{U}_\Lambda$,

$$\left\|\tau_{t,s}^{\Psi}(A)-\tau_{t,s}^{\tilde{\Psi}}(A)\right\|_{\mathcal{U}}\leq 2|\Lambda|\,\|A\|_{\mathcal{U}}\,\|\mathbf{F}\|_{1,\mathfrak{L}}\int_{t\wedge s}^{t\vee s}\mathrm{e}^{2\mathbf{D}\int_{t\wedge\alpha}^{t\vee\alpha}\|\Psi(\alpha_1)\|_{\mathcal{W}}\mathrm{d}\alpha_1}\left\|\tilde{\Psi}(\alpha)-\Psi(\alpha)\right\|_{\mathcal{W}}\mathrm{d}\alpha\ .$$

(iv) *Uniform continuity with respect to times. For any* $s_1,s_2,t_1,t_2\in\mathbb{R}$, $\Psi\in C(\mathbb{R};\mathcal{W}^{\mathbb{R}})$, $\Lambda\in\mathcal{P}_f$ *and* $A\in\mathcal{U}_\Lambda$,

$$\begin{aligned}&\left\|\tau_{t_1,s_1}^{\Psi}(A)-\tau_{t_2,s_2}^{\Psi}(A)\right\|_{\mathcal{U}}\\ \leq\ & 2|\Lambda|\,\|A\|_{\mathcal{U}}\,\|\mathbf{F}\|_{1,\mathfrak{L}}\left(\int_{t_1\wedge t_2}^{t_1\vee t_2}\|\Psi(\alpha)\|_{\mathcal{W}}\mathrm{d}\alpha+\int_{s_1\wedge s_2}^{s_1\vee s_2}\mathrm{e}^{2\mathbf{D}\int_{t_2\wedge\alpha}^{t_2\vee\alpha}\|\Psi(\alpha_1)\|_{\mathcal{W}}\mathrm{d}\alpha_1}\|\Psi(\alpha)\|_{\mathcal{W}}\mathrm{d}\alpha\right)\ .\end{aligned}$$

Proof. The proof of Assertion (i) is almost done in [72, Theorem 5.1, Corollary 5.2 (ii)]. However, the bound there refers to the supremum with respect to α of the norm $\|\Psi(\alpha)\|_{\mathcal{W}}$. Here, we need a slightly more accurate estimate (a point-wise estimate). In fact, by [72, equation after Eq. (5.4)] and similar arguments as in [72, Eqs. (4.16)-(4.18)], we get Assertion (i). Then, Assertion (ii) is proven exactly like in the proof of [72, Theorem 5.1 (ii)], by replacing [72, Theorem 5.1 (i)] with Assertion (i). It remains to prove Assertions (iii) and (iv). We start with (iii):

For any $s,t\in\mathbb{R}$, $\Psi,\tilde{\Psi}\in C(\mathbb{R};\mathcal{W}^{\mathbb{R}})$, $\Lambda\in\mathcal{P}_f$, $A\in\mathcal{U}_\Lambda$ and any sufficiently large $L\in\mathbb{N}$ such that $\mathcal{U}_\Lambda\subseteq\mathcal{U}_{\Lambda_L}$, by (39) and Proposition 3.7, observe that

$$\begin{aligned}\tau_{t,s}^{\tilde{\Psi}}(A)-\tau_{t,s}^{(L,\Psi)}(A) &= \int_s^t\tau_{\alpha,s}^{\tilde{\Psi}}\circ\left(\delta^{\tilde{\Psi}(\alpha)}-\delta^{\Psi(\alpha)}\right)\circ\tau_{t,\alpha}^{(L,\Psi)}(A)\,\mathrm{d}\alpha\\ &\quad+\int_s^t\tau_{\alpha,s}^{\tilde{\Psi}}\circ\left(\delta^{\Psi(\alpha)}-\delta^{(L,\Psi(\alpha))}\right)\circ\tau_{t,\alpha}^{(L,\Psi)}(A)\,\mathrm{d}\alpha\ .\end{aligned}\tag{43}$$

By Definitions 3.1, 3.3 and Corollary 3.5, it follows that

$$\begin{aligned}\left\|\tau_{t,s}^{\tilde{\Psi}}(A)-\tau_{t,s}^{(L,\Psi)}(A)\right\|_{\mathcal{U}} &\leq \int_{t\wedge s}^{t\vee s}\sum_{\mathcal{Z}\in\mathcal{P}_f}\left\|\left[\left(\tilde{\Psi}(\alpha)-\Psi(\alpha)\right)_{\mathcal{Z}},\tau_{t,\alpha}^{(L,\Psi)}(A)\right]\right\|_{\mathcal{U}}\mathrm{d}\alpha\\ &\quad+\int_{t\wedge s}^{t\vee s}\sum_{\mathcal{Z}\in\mathcal{P}_f,\ \mathcal{Z}\cap\Lambda_L^c\neq\emptyset}\left\|\left[\Psi(\alpha)_{\mathcal{Z}},\tau_{t,\alpha}^{(L,\Psi)}(A)\right]\right\|_{\mathcal{U}}\mathrm{d}\alpha\ ,\end{aligned}\tag{44}$$

where $\Lambda_L^c\doteq\mathfrak{L}\backslash\Lambda_L$ is the complement of the cubic box Λ_L (1). Now, by using Assertion (i) for $\Psi_L\in C(\mathbb{R};\mathcal{W}^{\mathbb{R}})$ defined[10], for $L\in\mathbb{N}$, by

$$\Psi_L(t)_{\mathcal{Z}}=\Psi(t)_{\mathcal{Z}}\,\mathbf{1}\left[\mathcal{Z}\subseteq\Lambda_L\right]\ ,\qquad\mathcal{Z}\in\mathcal{P}_f,\ t\in\mathbb{R}\ ,$$

together with $\|\Psi_L(t)\|_{\mathcal{W}}\leq\|\Psi(t)\|_{\mathcal{W}}$, (24)-(25) and Equation (27), we get that

$$\begin{aligned}&\sum_{\mathcal{Z}\in\mathcal{P}_f,\ \mathcal{Z}\cap\Lambda_L^c\neq\emptyset}\left\|\left[\Psi(\alpha)_{\mathcal{Z}},\tau_{t,\alpha}^{(L,\Psi)}(A)\right]\right\|_{\mathcal{U}}\\ \leq\ & 2\|A\|_{\mathcal{U}}\left(\mathrm{e}^{2\mathbf{D}\int_{t\wedge\alpha}^{t\vee\alpha}\|\Psi(\alpha_1)\|_{\mathcal{W}}\mathrm{d}\alpha_1}\right)\|\Psi(\alpha)\|_{\mathcal{W}}\sum_{x\in\Lambda}\sum_{y\in\Lambda_L^c}\mathbf{F}(x,y)\end{aligned}\tag{45}$$

[10] $\mathbf{1}[p]=1$ when the proposition p is true and 0, else.

and

$$\sum_{\mathcal{Z}\in\mathcal{P}_f}\left\|\left[\left(\tilde{\Psi}\left(\alpha\right)-\Psi\left(\alpha\right)\right)_{\mathcal{Z}},\tau_{t,\alpha}^{(L,\Psi)}\left(A\right)\right]\right\|_{\mathcal{U}} \tag{46}$$

$$\leq\quad 2\left|\Lambda\right|\left\|A\right\|_{\mathcal{U}}\left\|\mathbf{F}\right\|_{1,\mathfrak{L}}\mathrm{e}^{2\mathbf{D}\int_{t\wedge\alpha}^{t\vee\alpha}\left\|\Psi(\alpha_1)\right\|_{\mathcal{W}}\mathrm{d}\alpha_1}\left\|\tilde{\Psi}\left(\alpha\right)-\Psi\left(\alpha\right)\right\|_{\mathcal{W}}.$$

To prove these two inequalities, see [72, Eqs. (4.25)-(4.25)]. Since

$$\lim_{L\to\infty}\sum_{x\in\Lambda}\sum_{y\in\Lambda_L^c}\mathbf{F}\left(x,y\right)=0\,, \tag{47}$$

because of (24), Assertion (iii) follows by combining (44)-(47) with Assertion (ii).

Finally, to get Assertion (iv), note first that Corollary 3.5 directly implies that

$$\left\|\tau_{t_1,s}^{\Psi}(A)-\tau_{t_2,s}^{\Psi}(A)\right\|_{\mathcal{U}}\leq 2\left|\Lambda\right|\left\|A\right\|_{\mathcal{U}}\left\|\mathbf{F}\right\|_{1,\mathfrak{L}}\int_{t_1\wedge t_2}^{t_1\vee t_2}\left\|\Psi\left(\alpha\right)\right\|_{\mathcal{W}}\mathrm{d}\alpha \tag{48}$$

for any $s,t_1,t_2\in\mathbb{R}$, $\Psi\in C(\mathbb{R};\mathcal{W}^{\mathbb{R}})$, $\Lambda\in\mathcal{P}_f$ and $A\in\mathcal{U}_\Lambda$. Meanwhile, fix $s_1,s_2,t\in\mathbb{R}$, $\Psi\in C(\mathbb{R};\mathcal{W}^{\mathbb{R}})$, $\Lambda\in\mathcal{P}_f$ and $A\in\mathcal{U}_\Lambda$. By Assertion (ii), for any $\varepsilon\in\mathbb{R}^+$ there is $L\in\mathbb{R}$ such that

$$\left\|\tau_{t,s_1}^{\Psi}(A)-\tau_{t,s_2}^{\Psi}(A)\right\|_{\mathcal{U}}\leq\left\|\tau_{t,s_1}^{(L,\Psi)}(A)-\tau_{t,s_2}^{(L,\Psi)}(A)\right\|_{\mathcal{U}}+\varepsilon\,,$$

which, by Equation (39), implies that

$$\left\|\tau_{t,s_1}^{\Psi}(A)-\tau_{t,s_2}^{\Psi}(A)\right\|_{\mathcal{U}}\leq\int_{s_1\wedge s_2}^{s_1\vee s_2}\sum_{\mathcal{Z}\in\mathcal{P}_f}\left\|\left[\Psi\left(\alpha\right)_{\mathcal{Z}},\tau_{t,\alpha}^{(L,\Psi)}\left(A\right)\right]\right\|_{\mathcal{U}}\mathrm{d}\alpha+\varepsilon\,. \tag{49}$$

Similar to (46), it follows that

$$\left\|\tau_{t,s_1}^{\Psi}(A)-\tau_{t,s_2}^{\Psi}(A)\right\|_{\mathcal{U}}\leq 2\left|\Lambda\right|\left\|A\right\|_{\mathcal{U}}\left\|\mathbf{F}\right\|_{1,\mathfrak{L}}\int_{s_1\wedge s_2}^{s_1\vee s_2}\mathrm{e}^{2\mathbf{D}\int_{t\wedge\alpha}^{t\vee\alpha}\left\|\Psi(\alpha_1)\right\|_{\mathcal{W}}\mathrm{d}\alpha_1}\left\|\Psi\left(\alpha\right)\right\|_{\mathcal{W}}\mathrm{d}\alpha\,. \tag{50}$$

Assertion (iv) is a combination of (48) and (50). ■

4 Lattice Fermions with Long-Range Interactions

4.1 Banach Space of Long-Range Models

Fix now $\mathfrak{L}=\mathbb{Z}^d$, $d\in\mathbb{N}$. Let $\mathbb{S}$ be the unit sphere of the Banach space $\mathcal{W}_1$ of translation-invariant (complex) interactions. Observe that any finite signed Borel measure $\mathfrak{a}$ on $\mathbb{S}$ defines an interaction

$$\int_{\mathbb{S}}\Psi\;\mathfrak{a}\left(\mathrm{d}\Psi\right)\in\mathcal{W}_1 \tag{51}$$

by

$$\left(\int_{\mathbb{S}}\Psi\;\mathfrak{a}\left(\mathrm{d}\Psi\right)\right)_\Lambda\doteq\int_{\mathbb{S}}\Psi_\Lambda\mathfrak{a}\left(\mathrm{d}\Psi\right)\;,\qquad\Lambda\in\mathcal{P}_f\,. \tag{52}$$

This last integral is well-defined because, for each $\Lambda\in\mathcal{P}_f$, the integrand is an absolutely integrable function taking values in a finite-dimensional normed space, which is $\mathcal{U}_\Lambda$. Below, we extend this observation to define long-range, or mean-field, models. Note that (51) can also be seen as a Bochner integral because the measure $\mathfrak{a}$ is finite and $\mathcal{W}$ is a separable Banach space. See, e.g., [89, Theorems 1.1 and 1.2].

For any $n \in \mathbb{N}$ and any finite signed Borel measure $\mathfrak{a}$ on the Cartesian product $\mathbb{S}^n$ (endowed with its product topology), we define the finite signed Borel measure $\mathfrak{a}^*$ to be the pushforward of $\mathfrak{a}$ through the automorphism

$$\left(\Psi^{(1)}, \ldots, \Psi^{(n)}\right) \mapsto ((\Psi^{(n)})^*, \ldots, (\Psi^{(1)})^*) \in \mathbb{S}^n \tag{53}$$

of $\mathbb{S}^n$ as a topological space. A finite signed Borel measure $\mathfrak{a}$ on $\mathbb{S}^n$ is, by definition, *self-adjoint* whenever $\mathfrak{a}^* = \mathfrak{a}$.

For any $n \in \mathbb{N}$, we denote the space of *self-adjoint*, finite, signed Borel measures on $\mathbb{S}^n$ by $\mathcal{S}(\mathbb{S}^n)$, which is a real Banach space with the norm of the total variation

$$\|\mathfrak{a}\|_{\mathcal{S}(\mathbb{S}^n)} \doteq |\mathfrak{a}|(\mathbb{S}^n) , \qquad n \in \mathbb{N} . \tag{54}$$

The set of all sequences $\mathfrak{a} \equiv (\mathfrak{a}_n)_{n\in\mathbb{N}}$ of self-adjoint, finite, signed Borel measures $\mathfrak{a}_n \in \mathcal{S}(\mathbb{S}^n)$ is a real vector space, where

$$(\mathfrak{a} + \tilde{\mathfrak{a}})_n \doteq \mathfrak{a}_n + \tilde{\mathfrak{a}}_n \qquad \text{and} \qquad (\lambda\mathfrak{a})_n \doteq \lambda\mathfrak{a}_n , \qquad n \in \mathbb{N} ,$$

for any sequence $\mathfrak{a} \equiv (\mathfrak{a}_n)_{n\in\mathbb{N}}, \tilde{\mathfrak{a}} \equiv (\tilde{\mathfrak{a}}_n)_{n\in\mathbb{N}}$ and all $\lambda \in \mathbb{R}$. We define the (real) space $\mathcal{S}$ to be the set of all sequences $\mathfrak{a} \equiv (\mathfrak{a}_n)_{n\in\mathbb{N}}$ of self-adjoint, finite signed Borel measures $\mathfrak{a}_n \in \mathcal{S}(\mathbb{S}^n)$ with

$$\|\mathfrak{a}\|_{\mathcal{S}} \doteq \sum_{n\in\mathbb{N}} n^2 \, \|\mathbf{F}\|_{1,\mathbb{Z}^d}^{n-1} \, \|\mathfrak{a}_n\|_{\mathcal{S}(\mathbb{S}^n)} < \infty , \tag{55}$$

where we recall that $\mathbf{F} : \mathbb{Z}^d \times \mathbb{Z}^d \to (0, 1]$ is the decay function with maximum value $\mathbf{F}(x, x) = 1$ for $x \in \mathbb{Z}^d$ and satisfying Conditions (24)-(25). See also (27). Observe that $(\mathcal{S}, \| \cdot \|_{\mathcal{S}})$ is a real Banach space. We are now in a position to define *long-range* models:

Definition 4.1 (Long-range models)
The (real) Banach space of long-range models is the space $\mathcal{M} \doteq \mathcal{W}^{\mathbb{R}} \times \mathcal{S}$ *along with the norm*

$$\|\mathfrak{m}\|_{\mathcal{M}} \doteq \|\Phi\|_{\mathcal{W}} + \|\mathfrak{a}\|_{\mathcal{S}} , \qquad \mathfrak{m} \doteq (\Phi, \mathfrak{a}) \in \mathcal{M} .$$

Note that $\mathcal{W}^{\mathbb{R}}$ and $\mathcal{S}$ are canonically seen as subspaces of $\mathcal{M}$, i.e.,

$$\mathcal{W}^{\mathbb{R}} \subseteq \mathcal{M} \qquad \text{and} \qquad \mathcal{S} \subseteq \mathcal{M} .$$

We emphasize that long-range models are *not necessarily* translation-invariant since, obviously,

$$\mathcal{M}_1 \doteq \left(\mathcal{W}_1 \cap \mathcal{W}^{\mathbb{R}}\right) \times \mathcal{S} \subsetneq \mathcal{M} . \tag{56}$$

Similar to (29)-(31), we define the subsets

$$\mathcal{M}_\Lambda \doteq \mathcal{W}^{\mathbb{R}} \times \mathcal{S}_\Lambda \subseteq \mathcal{M} , \qquad \Lambda \in \mathcal{P}_f , \tag{57}$$

where, for any $\Lambda \in \mathcal{P}_f$,

$$\mathcal{S}_\Lambda \doteq \{(\mathfrak{a}_n)_{n\in\mathbb{N}} \in \mathcal{S} : \forall n \in \mathbb{N}, \ |\mathfrak{a}_n|(\mathbb{S}^n) = |\mathfrak{a}_n|((\mathbb{S} \cap \mathcal{W}_\Lambda)^n)\} . \tag{58}$$

Note that the short-range part of models in $\mathcal{M}_\Lambda$ is *not necessarily* finite-range, but their long-range interactions are built from finite-range interactions. Similar to (34) we can define a dense subspace

$$\mathcal{M}_0 \doteq \bigcup_{L\in\mathbb{N}} \mathcal{M}_{\Lambda_L} \subseteq \mathcal{M} . \tag{59}$$

4.2 Local Hamiltonians and Derivations on the CAR Algebra

Similar to Definition 3.1, we define a sequence of local Hamiltonians for any model $\mathfrak{m} \in \mathcal{M}$: At any fixed $n \in \mathbb{N}$ and $L \in \mathbb{N}$, the mapping

$$\left(\Psi^{(1)}, \ldots, \Psi^{(n)}\right) \mapsto U_L^{\Psi^{(1)}} \cdots U_L^{\Psi^{(n)}}$$

from $\mathbb{S}^n$ to $\mathcal{U}$ is continuous (see (35)), and so, for any long-range model $\mathfrak{m} \in \mathcal{M}$, we can define the following self-adjoint element of $\mathcal{U}$:

Definition 4.2 (Hamiltonians)
The local Hamiltonians of any model $\mathfrak{m} \in \mathcal{M}$ *are*

$$U_L^{\mathfrak{m}} \doteq U_L^{\Phi} + \sum_{n \in \mathbb{N}} \frac{1}{|\Lambda_L|^{n-1}} \int_{\mathbb{S}^n} U_L^{\Psi^{(1)}} \cdots U_L^{\Psi^{(n)}} \mathfrak{a}_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right) , \qquad L \in \mathbb{N} .$$

Note that

$$U_L^{\mathfrak{m}} \in \mathcal{U}_{\Lambda_L} \cap \mathcal{U}^{\mathbb{R}} \cap \mathcal{U}^{+} , \qquad L \in \mathbb{N} ,$$

and straightforward estimates using Equations (35), (54)-(55) and Definition 4.1 yield the bound

$$\|U_L^{\mathfrak{m}}\|_{\mathcal{U}} \leq |\Lambda_L| \, \|\mathbf{F}\|_{1,\mathfrak{L}} \, \|\mathfrak{m}\|_{\mathcal{M}} , \qquad L \in \mathbb{N} . \tag{60}$$

(This upper bound is relatively coarse, in general.)

For any translation-invariant long-range model $\mathfrak{m} \doteq (\Phi, \mathfrak{a}) \in \mathcal{M}_1$ (cf. (56)), observe that

$$U_L^{\mathfrak{m}} \doteq U_L^{\tilde{\Phi}} + \sum_{n=2}^{\infty} \frac{1}{|\Lambda_L|^{n-1}} \int_{\mathbb{S}^n} U_L^{\Psi^{(1)}} \cdots U_L^{\Psi^{(n)}} \mathfrak{a}_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right) , \qquad L \in \mathbb{N} ,$$

where

$$\tilde{\Phi} \doteq \Phi + \int_{\mathbb{S}} \Psi \, \mathfrak{a}_1 \left(\mathrm{d}\Psi\right) \in \mathcal{W} ,$$

the last integral being defined by (52). If the model is finite-range and translation-invariant, i.e., $\Phi, \tilde{\Phi} \in \mathcal{W}_1$, then the interaction $\tilde{\Phi}$ can be reproduced by some self-adjoint, finite, signed Borel measure $\tilde{\mathfrak{a}}_1$, leading to the definition of a new model $\tilde{\mathfrak{m}} \doteq (0, \tilde{\mathfrak{a}}) \in \mathcal{M}_1$ such that $U_L^{\mathfrak{m}} = U_L^{\tilde{\mathfrak{m}}}$. In other words, if one is only interested in finite-range translation-invariant long-range models, then one can directly use the Banach space $\mathcal{S}$. Finally, for any translation-invariant long-range model $\mathfrak{m} = (\Phi, (0, \mathfrak{a}_2, 0, \ldots)) \in \mathcal{M}_1$, remark that

$$U_L^{\mathfrak{m}} \doteq U_L^{\Phi} + \frac{1}{|\Lambda_L|} \int_{\mathbb{S}^2} U_L^{\Psi^{(1)}} U_L^{\Psi^{(2)}} \mathfrak{a}_2 \left(\mathrm{d}\Psi^{(1)}, \mathrm{d}\Psi^{(2)}\right) , \tag{61}$$

which can be seen as a local Hamiltonian of a long-range model in the sense of [46], as explained in Section 8.

Like in Definition 3.3, any model $\mathfrak{m} \in \mathcal{M}$ yields a sequence of bounded derivations:

Definition 4.3 (Derivations on the CAR algebra for long-range interactions)
The (symmetric) derivations $\{\delta_L^{\mathfrak{m}}\}_{L \in \mathbb{N}} \subseteq \mathcal{B}(\mathcal{U})$ *associated with any model* $\mathfrak{m} \in \mathcal{M}$ *are defined by*

$$\delta_L^{\mathfrak{m}}(A) \doteq i \left[U_L^{\mathfrak{m}}, A\right] \doteq i \left(U_L^{\mathfrak{m}} A - A U_L^{\mathfrak{m}}\right) , \qquad A \in \mathcal{U}, \ L \in \mathbb{N} .$$

4.3 Dynamical Problem Associated with Long-Range Interactions

For any long-range model $\mathfrak{m} \in \mathcal{M}$, the finite-volume dynamics are always well-defined: For all $L \in \mathbb{N}$ there is a strongly continuous one-parameter group $(\tau_t^{(L,\mathfrak{m})})_{t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{U}$ generated by $\delta_L^{\mathfrak{m}} \in \mathcal{B}(\mathcal{U})$:

$$\tau_t^{(L,\mathfrak{m})}(A) \doteq \mathrm{e}^{itU_L^{\mathfrak{m}}} A \mathrm{e}^{-itU_L^{\mathfrak{m}}} , \qquad A \in \mathcal{U} . \tag{62}$$

Compare with (39)-(40) for short-range interactions in the autonomous situation.

Nevertheless, in contrast with short-range interactions (cf. Corollary 3.5 and Proposition 3.7), in the thermodynamic limit $L \to \infty$, the finite-volume dynamics does not generally converge within the C^*-algebra $\mathcal{U}$. To see this, consider the following elementary example: Choose a model $\mathfrak{m} = (0,(0,\mathfrak{a}_2,0,\ldots)) \in \mathcal{M}_1$ such that

$$U_L^{\mathfrak{m}} = \frac{1}{2|\Lambda_L|} N_L^2 \qquad \text{with} \qquad N_L \doteq \sum_{x\in\Lambda_L,\mathrm{s}\in\mathrm{S}} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}} .$$

Take $A = a_{0,\mathrm{s}} \in \mathcal{U}_0$ for some fixed spin $\mathrm{s} \in \mathrm{S}$. Observe that

$$\tau_t^{(L,\mathfrak{m})}(a_{0,\mathrm{s}}) = \mathrm{e}^{it(2|\Lambda_L|)^{-1}} \mathrm{e}^{it|\Lambda_L|^{-1} a_{0,\mathrm{s}}^* a_{0,\mathrm{s}} N_L} a_{0,\mathrm{s}} \mathrm{e}^{-it|\Lambda_L|^{-1} a_{0,\mathrm{s}}^* a_{0,\mathrm{s}} N_L} .$$

Therefore, for any $t \in \mathbb{R}$,

$$\mathrm{e}^{-it(2|\Lambda_L|)^{-1}} \tau_t^{(L,\mathfrak{m})}(a_{0,\mathrm{s}}) = a_{0,\mathrm{s}} + it|\Lambda_L|^{-1} \left[a_{0,\mathrm{s}}^* a_{0,\mathrm{s}} N_L, a_{0,\mathrm{s}} \right] + R_L(t) \tag{63}$$

with $\|R_L(t)\|_{\mathcal{U}} \leq 2t^2$. Note that

$$|\Lambda_L|^{-1} \left[a_{0,\mathrm{s}}^* a_{0,\mathrm{s}} N_L, a_{0,\mathrm{s}} \right] = -|\Lambda_L|^{-1} a_{0,\mathrm{s}} N_L$$

and it is straightforward to check that this last element does not converge in $\mathcal{U}$, as $L \to \infty$. By Equation (63), at least at small times $|t| > 0$, $(\tau_t^{(L,\mathfrak{m})}(A))_{L\in\mathbb{N}} \subseteq \mathcal{U}$ does *not* converge, as $L \to \infty$.

The non-convergence property is generic: For any integer $n \geq 2$, $\Psi^{(1)},\ldots,\Psi^{(n)} \in \mathcal{W}$, $A \in \mathcal{U}$ and all $L \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{|\Lambda_L|^{n-1}} \left[U_L^{\Psi^{(1)}} \cdots U_L^{\Psi^{(n)}}, A \right] \quad &= \quad \left[U_L^{\Psi^{(1)}}, A \right] \frac{U_L^{\Psi^{(2)}}}{|\Lambda_L|} \cdots \frac{U_L^{\Psi^{(n)}}}{|\Lambda_L|} + \frac{U_L^{\Psi^{(1)}}}{|\Lambda_L|} \cdots \frac{U_L^{\Psi^{(n-1)}}}{|\Lambda_L|} \left[U_L^{\Psi^{(n)}}, A \right] \\ &\quad + \sum_{m=2}^{n-1} \frac{U_L^{\Psi^{(1)}}}{|\Lambda_L|} \cdots \frac{U_L^{\Psi^{(m-1)}}}{|\Lambda_L|} \left[U_L^{\Psi^{(m)}}, A \right] \frac{U_L^{\Psi^{(m+1)}}}{|\Lambda_L|} \cdots \frac{U_L^{\Psi^{(n)}}}{|\Lambda_L|} .\end{aligned} \tag{64}$$

(Compare with Definitions 4.2 and 4.3.) Note that the element (64) of $\mathcal{U}$ is uniformly bounded with respect to $L \in \mathbb{N}$, since, by Corollary 3.5, the commutators in the right-hand side of this last equation have a limit in $\mathcal{U}$ for any $A \in \mathcal{U}_0$, as $L \to \infty$. However, $|\Lambda_L|^{-1} U_L^{\Psi}$ does not generally converge in the norm sense of $\mathcal{U}$: Since

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \left[U_L^{\Psi}, A \right] = 0 , \qquad A \in \mathcal{U} ,$$

if the sequence $(|\Lambda_L|^{-1} U_L^{\Psi})_{L\in\mathbb{N}}$ would converge in $\mathcal{U}$, as $L \to \infty$, then its limit would be an element of the center of $\mathcal{U}$. For $\mathcal{U}$ is a simple algebra (cf. [75, Corollary 2.6.19]), its center is trivial. Therefore, $(|\Lambda_L|^{-1} U_L^{\Psi})_{L\in\mathbb{N}}$ would converge to $c\mathfrak{1}$ for some $c \in \mathbb{C}$. In particular, by Proposition 3.2, for $\Psi \in \mathcal{W}_1$,

$$\left(\mathfrak{e}_{\Psi,\vec{\ell}} - c\mathfrak{1} \right) \in \bigcap_{\rho\in E_{\vec{\ell}}} \ker \rho ,$$

which is clearly wrong, in general. This observation is well-known. See, e.g., [19, p. 2225].

By contrast, taking $\Psi \in \mathcal{W}_1$, $\vec{\ell} \in \mathbb{N}^d$ and any cyclic representation $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ of an *extreme* (or ergodic) $\vec{\ell}$-periodic state $\rho \in E_{\vec{\ell}}$ (17), one has that the sequence $(|\Lambda_L|^{-1}\pi_\rho(U_L^\Psi))_{L\in\mathbb{N}}$ does strongly converge to $\pi_\rho(c)$ for some $c = c_\rho \in \mathbb{C}$. This is basically Haag's argument [4] proposed in 1962 in order to give a mathematical meaning to the dynamics of the BCS model. In the more general case of a not necessarily extreme state $\rho \in E_{\vec{\ell}}$, the strong operator limit of the sequence is an element of the (possibly non-trivial) center of the von Neumann algebra $\pi_\rho(\mathcal{U})''$. These facts compel us to consider a more general setting, in particular the notion of *state-dependent* observables and interactions.

5 C^*-Algebra of Continuous Functions on States

In this section, we define the extended quantum framework introduced in [63], starting with the classical C^*-algebra. Note that we consider here the C^*-algebra $\mathcal{U}$, with state space E, as the primordial C^*-algebra $\mathcal{X}$ of [63]. Nevertheless, from the point of view of physics, only the subalgebra $\mathcal{U}^+ \subseteq \mathcal{U}$ of even elements is relevant, as already explained in Section 2.3. In this case, the physical state space is the set $E^+ \subseteq E$ of all even states.

5.1 The Classical C^*-Algebra of Continuous Functions on States

Recall that E stands for the metrizable, weak*-compact and convex set of states on $\mathcal{U}$, as defined by (11). It is the state space of the classical dynamics defined on the space $C(E;\mathbb{C})$ of complex-valued weak*-continuous functions on E:

Classical algebra: Endowed with the point-wise operations and complex conjugation, $C(E;\mathbb{C})$ becomes a unital commutative C^*-algebra denoted by

$$\mathfrak{C} \doteq \left(C\left(E;\mathbb{C}\right), +, \cdot_{\mathbb{C}}, \times, \overline{(\cdot)}, \|\cdot\|_{\mathfrak{C}}\right) , \tag{65}$$

where the corresponding C^*-norm is

$$\|f\|_{\mathfrak{C}} \doteq \max_{\rho\in E} |f(\rho)| , \qquad f \in \mathfrak{C} . \tag{66}$$

Note that the "$\max$" in the definition of the norm is well-defined because of the continuity of f together with the compactness of E. $\mathfrak{C}$ is the classical C^*-algebra of weak*-continuous complex-valued functions on states. The (real) Banach subspace of all real-valued functions is denoted by $\mathfrak{C}^{\mathbb{R}} \subsetneq \mathfrak{C}$. The C^*-algebra $\mathfrak{C}$ is separable, E being metrizable and compact.

Gelfand transform: Elements of the (separable and unital) C^*-algebra $\mathcal{U}$ naturally define continuous and affine functions $\hat{A} \in \mathfrak{C}$ by

$$\hat{A}(\rho) \doteq \rho(A) , \qquad \rho \in E,\ A \in \mathcal{U} . \tag{67}$$

This is the well-known Gelfand transform. Note that $A \neq B$ implies $\hat{A} \neq \hat{B}$, as states separates elements of $\mathcal{U}$. Since

$$\|A\|_{\mathcal{U}} = \max_{\rho\in E} |\rho(A)| , \qquad A \in \mathcal{U}^{\mathbb{R}} , \tag{68}$$

the mapping $A \mapsto \hat{A}$ defines a linear isometry from the Banach space $\mathcal{U}^{\mathbb{R}}$ of all self-adjoint elements to the space $\mathfrak{C}^{\mathbb{R}}$ of all real-valued functions on E.

Dense classical subalgebra: Recall that $\mathcal{U}_0$ is the normed $*$-algebra of local elements of $\mathcal{U}$ defined by (3). We denote respectively by

$$\mathfrak{C}_{\mathcal{U}_0} \doteq \mathbb{C}[\{\hat{A} : A \in \mathcal{U}_0\}] \qquad \text{and} \qquad \mathfrak{C}_{\mathcal{U}} \doteq \mathbb{C}[\{\hat{A} : A \in \mathcal{U}\}] \tag{69}$$

the subalgebras of polynomials in the elements of $\{\hat{A} : A \in \mathcal{U}_0\}$ and $\{\hat{A} : A \in \mathcal{U}\}$, with complex coefficients. The unit $\hat{\mathfrak{1}} \in \mathfrak{C}$ mapping any state to 1 belongs to $\mathfrak{C}_{\mathcal{U}_0}$, by definition. Since $\mathcal{U}_0$ is, by construction, dense in $\mathcal{U}$, the subalgebra $\mathfrak{C}_{\mathcal{U}_0}$ separates states. Therefore, by the Stone-Weierstrass theorem, $\mathfrak{C}_{\mathcal{U}_0} \subseteq \mathfrak{C}_{\mathcal{U}}$ is dense in $\mathfrak{C}$, i.e., $\mathfrak{C} = \overline{\mathfrak{C}_{\mathcal{U}_0}}$.

5.2 Poisson Structure Associated with the State Space

We define $\mathcal{A}(E;\mathbb{C}) \subseteq \mathfrak{C}$ to be the closed subspace of all *affine*, weak*-continuous complex-valued functions over E. By [63, Definition 3.7], the convex Gâteaux derivative of $f \in \mathfrak{C}$ at a fixed state $\rho \in E$ is an affine weak*-continuous complex-valued function over E, defined as follows:

Definition 5.1 (Convex weak*-continuous Gâteaux derivative)
For any $f \in \mathfrak{C}$ and $\rho \in E$, we say that $\mathrm{d}f(\rho) : E \to \mathbb{C}$ is the (unique) convex weak*-continuous Gâteaux derivative of f at $\rho \in E$ if $\mathrm{d}f(\rho) \in \mathcal{A}(E;\mathbb{C})$ and

$$\lim_{\lambda \to 0^+} \lambda^{-1} \left(f\left((1-\lambda)\rho + \lambda \upsilon\right) - f(\rho) \right) = \left[\mathrm{d}f(\rho)\right](\upsilon) \ , \qquad \rho, \upsilon \in E \ .$$

A function $f \in \mathfrak{C}$ such that $\mathrm{d}f(\rho)$ exists for all $\rho \in E$ is called (convex-)*differentiable* and we use the notation

$$\mathrm{d}f \equiv (\mathrm{d}f(\rho))_{\rho \in E} : E \to \mathcal{A}(E;\mathbb{C}) \ .$$

If $f \in \mathcal{A}(E;\mathbb{C})$ then

$$\mathrm{d}f(\rho) = f - f(\rho) \ , \qquad \rho \in E \ ,$$

which means that affine functions of $\mathfrak{C}$ are continuously (convex-)differentiable, as expected.

We define the (non-empty) subspace of continuously differentiable complex-valued functions over the convex and weak*-compact set E by

$$\mathfrak{Y} \equiv \mathfrak{Y}(E) \doteq \{ f \in \mathfrak{C} : \mathrm{d}f \in C(E;\mathfrak{C}) \} \ . \tag{70}$$

We endow this vector space with the norm

$$\|f\|_{\mathfrak{Y}} \doteq \|f\|_{\mathfrak{C}} + \max_{\rho \in E} \|\mathrm{d}f(\rho)\|_{\mathfrak{C}} < \infty \ , \qquad f \in \mathfrak{Y} \ , \tag{71}$$

in order to obtain a Banach space, again denoted by $\mathfrak{Y}$. The "$\max$" in the definition of the norm is well-defined because of the continuity of f and $\mathrm{d}f$ together with the compactness of E. The normed vector space $\mathfrak{Y}$ is complete, see [63, Section 3.4].

The (real) Banach subspace of all continuously differentiable real-valued functions is denoted by $\mathfrak{Y}^{\mathbb{R}} \subsetneq \mathfrak{Y}$. By [63, Proposition 3.8] together with Equations (66)-(68) and (70), for any $f \in \mathfrak{Y}^{\mathbb{R}} \subsetneq \mathfrak{C}^{\mathbb{R}}$, there is a unique $\mathrm{D}f \in C(E;\mathcal{U}^{\mathbb{R}})$ such that

$$\mathrm{d}f(\rho) = \widehat{\mathrm{D}f(\rho)} \ , \qquad \rho \in E \ . \tag{72}$$

By (66) and (68), note that

$$\|\mathrm{D}f(\rho)\|_{\mathcal{U}} = \|\mathrm{d}f(\rho)\|_{\mathfrak{C}} \ , \qquad \rho \in E \ . \tag{73}$$

Since the convex weak*-continuous Gâteaux derivative is linear and because any function $f \in \mathfrak{Y}$ can be decomposed into real and imaginary parts, it follows that, for any $f \in \mathfrak{Y} \subsetneq \mathfrak{C}$, there is a unique $\mathrm{D}f \in C(E;\mathcal{U})$ satisfying (72). For instance,

$$\mathrm{D}\hat{A}\,(\rho) = A - \rho\,(A)\,\mathfrak{1}\;, \qquad A \in \mathcal{U}\;.$$

Therefore, using [63, Definition 3.9] and the standard notation $[A, B] \doteq AB - BA$, $A, B \in \mathcal{U}$, for the commutator, we can define a Poisson bracket for continuously differentiable complex-valued functions on the state space:

Definition 5.2 (Poisson bracket)
We define the mapping $\{\cdot,\cdot\} : \mathfrak{Y} \times \mathfrak{Y} \to \mathfrak{C}$ *by*

$$\{f, g\}\,(\rho) \doteq \rho\,(i\,[\mathrm{D}f\,(\rho)\,,\mathrm{D}g\,(\rho)])\;, \qquad f, g \in \mathfrak{Y}\;.$$

By [63, Proposition 3.10 and discussions afterwards], this mapping is a Poisson bracket on $\mathfrak{C}_{\mathcal{U}}$, that is, for complex-valued polynomials. In other words, it is a skew-symmetric biderivation on $\mathfrak{C}_{\mathcal{U}}$ satisfying the Jacobi identity.

In fact, by generalizing the well-known construction of a Poisson bracket for the polynomial functions on the dual space of finite dimensional Lie groups [79, Section 7.1], we define in [63, Section 3.2] a Poisson bracket for the polynomial functions on the hermitian continuous functional (like the states) on any C^*-algebra. Then, in [63, Section 3.3], the Poisson bracket is localized on the state and phase[11] spaces associated with this algebra, by taking quotients with respect to conveniently chosen Poisson ideals. In particular, this leads, in an elegant way, to a Poisson bracket for polynomial functions of the classical C^*-algebra $\mathfrak{C}$. Definitions 5.1-5.2 just yield a very convenient explicit expression of this Poisson bracket for functions on the state space.

5.3 The Quantum C^*-Algebra of Continuous Functions on States

The long-range dynamics takes place in the space $C(E;\mathcal{U})$ of weak*-continuous $\mathcal{U}$-valued functions on the metrizable compact space E, which can be endowed with a C^*-algebra structure:

Quantum algebra: Endowed with the point-wise $*$-algebra operations inherited from $\mathcal{U}$, $C(E;\mathcal{U})$ is a unital *non-commutative* C^*-algebra denoted by

$$\mathfrak{U} \equiv \mathfrak{U}_{\mathfrak{L}} \doteq (C\,(E;\mathcal{U})\,, +, \cdot_{\mathbb{C}}, \times, ^*, \|\cdot\|_{\mathfrak{U}})\;. \tag{74}$$

The unique C^*-norm $\|\cdot\|_{\mathfrak{U}}$ is the supremum norm for functions on E taking values in the normed space $\mathcal{U}$, i.e.,

$$\|f\|_{\mathfrak{U}} \doteq \max_{\rho \in E} \|f\,(\rho)\|_{\mathcal{U}}\;, \qquad f \in \mathfrak{U}\;. \tag{75}$$

Recall that $\mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U}$ is the (real) Banach subspace of all self-adjoint elements of $\mathcal{U}$. The (real) Banach subspace of all $\mathcal{U}^{\mathbb{R}}$-valued functions of $\mathfrak{U}$ is similarly denoted by $\mathfrak{U}^{\mathbb{R}} \subsetneq \mathfrak{U}$.

We identify the primordial C^*-algebra $\mathcal{U}$, on which the quantum dynamics is usually defined, with the subalgebra of constant functions of $\mathfrak{U}$. Meanwhile, the classical dynamics appears in the algebra $\mathfrak{C}$ of complex-valued weak*-continuous functions on E. This unital commutative C^*-algebra is identified with the subalgebra of functions of $\mathfrak{U}$ whose values are multiples of the unit $\mathfrak{1} \in \mathcal{U}$. In other words, we have the canonical inclusions

$$\mathcal{U} \subseteq \mathfrak{U} \qquad \text{and} \qquad \mathfrak{C} \subseteq \mathfrak{U}\;. \tag{76}$$

[11] The phase space in [63, Definition 2.2] is the weak* closure of the subset $\mathcal{E}(E)$ of extreme points of the state space E. For general C^*-algebras, E and the weak* closure of $\mathcal{E}(E)$ are not necessary the same. But here, if $\mathfrak{L}$ is infinite then $E = \overline{\mathcal{E}(E)}$, i.e., phase and state spaces coincide.

See [63, Eq. (67)].

Dense subalgebras: Similar to (3), we define the $*$-subalgebras

$$\mathfrak{U}_{\Lambda} \doteq \{f \in \mathfrak{U} : f(E) \subseteq \mathcal{U}_{\Lambda}\} \ , \qquad \Lambda \in \mathcal{P}_f \ , \tag{77}$$

and

$$\mathfrak{U}_0 \doteq \bigcup_{L \in \mathbb{N}} \mathfrak{U}_{\Lambda_L} \subseteq \mathfrak{U} \ . \tag{78}$$

The union $\mathfrak{C}_{\mathcal{U}_0} \cup \mathcal{U}_0$ generates the C^*-algebra $\mathfrak{U}$ and

$$\mathfrak{U}_0 = \mathrm{span}\,\{\mathfrak{C}\mathcal{U}_0\} \supseteq \mathrm{span}\,\{\mathfrak{C}_{\mathcal{U}_0}\mathcal{U}_0\} \tag{79}$$

are dense $*$-subalgebras of $\mathfrak{U}$. To see this, use the density of $\mathcal{U}_0 \subseteq \mathcal{U}$ and $\mathfrak{C}_{\mathcal{U}_0}$, as well as the compactness of E together with the existence of partitions of unity subordinated to any open cover of the metrizable (weak*-compact) space E (paracompactness and metrizability of E).

Positivity of elements of $\mathfrak{U}$: The positivity of elements of $\mathfrak{U}$ is equivalent to their point-wise positivity. This is a direct consequence of the following lemma:

Lemma 5.3 (Spectrum)
For any $f \in \mathfrak{U}$, its spectrum equals

$$\mathrm{spec}\,(f) = \bigcup_{\rho \in E} \mathrm{spec}\,(f(\rho)) \ .$$

Proof. Fix $f \in \mathfrak{U}$ and take any $z \in \mathbb{C}$ in the resolvent set of f. Then, clearly, z also belongs to the resolvent set of $f(\rho) \in \mathcal{U}$ for all $\rho \in E$. It follows that

$$\bigcup_{\rho \in E} \mathrm{spec}\,(f(\rho)) \subseteq \mathrm{spec}\,(f) \ . \tag{80}$$

Take now $z \in \mathbb{C}$ in the resolvent set of $f(\rho) \in \mathcal{U}$ for all $\rho \in E$. Using the Neumann series

$$(A - B - z\mathbf{1}_{\mathcal{U}})^{-1} = (z\mathbf{1}_{\mathcal{U}} - A)^{-1} \sum_{n=1}^{\infty} \left(B\,(z\mathbf{1}_{\mathcal{U}} - A)^{-1}\right)^n$$

for all $A, B \in \mathcal{U}$ with $\|B\,(z\mathbf{1}_{\mathcal{U}} - A)^{-1}\|_{\mathcal{U}} < 1$, one sees that the mapping

$$\rho \mapsto (z\mathbf{1}_{\mathcal{U}} - f(\rho))^{-1}$$

from E to $\mathcal{U}$ is weak*-continuous. In particular, this mapping is an element of $\mathfrak{U}$ which, by construction, is the resolvent of f at $z \in \mathbb{C}$. It follows that

$$\mathrm{spec}\,(f) \subseteq \bigcup_{\rho \in E} \mathrm{spec}\,(f(\rho)) \ . \tag{81}$$

By (80)-(81), the assertion follows. ■

Corollary 5.4 (Positivity)
Any element $f \in \mathfrak{U}^{\mathbb{R}}$ is positive iff $f(\rho) \in \mathcal{U}^{\mathbb{R}}$ is positive for all $\rho \in E$.

Functional calculus for elements of $\mathfrak{U}$: It turns out that the continuous functional calculus in $\mathfrak{U}$ coincides with the point-wise continuous functional calculus in $\mathcal{U}$:

Lemma 5.5 (Continuous functional calculus)
For any self-adjoint $f \in \mathfrak{U}^{\mathbb{R}}$ *with spectrum* $\mathrm{spec}(f)$ *and any continuous function* $\varphi \in C(\mathrm{spec}(f);\mathbb{C})$*,* $\varphi(f) = \varphi \circ f$*, where*

$$\varphi \circ f\,(\rho) \doteq \varphi\,(f\,(\rho))\ , \qquad \rho \in E\ .$$

Note that $\varphi\,(f\,(\rho))$ *is well-defined because* $\mathrm{spec}\,(f\,(\rho)) \subseteq \mathrm{spec}(f)$ *for all* $\rho \in E$*, by Lemma 5.3.*

Proof. Note first that, for all self-adjoint $f \in \mathfrak{U}^{\mathbb{R}}$ and any polynomial function $\varphi \in C(\mathrm{spec}(f);\mathbb{C})$, $\varphi\,(f) = \varphi \circ f \in \mathfrak{U}$. Observe next that, for any $\varphi_1, \varphi_2 \in C(\mathrm{spec}(f);\mathbb{C})$ and $\rho \in E$,

$$\|\varphi_1 \circ f\,(\rho) - \varphi_2 \circ f\,(\rho)\|_{\mathcal{U}} \leq \sup_{\rho \in \mathrm{spec}(f(\rho))} |\varphi_1\,(\rho) - \varphi_2\,(\rho)| \leq \|\varphi_1 - \varphi_2\|_{\infty}\ , \tag{82}$$

by Lemma 5.3. If $\varphi \in C(\mathrm{spec}(f);\mathbb{C})$ is a general continuous function then take any sequence of polynomial functions $\varphi_n \in C(\mathrm{spec}(f);\mathbb{C})$ converging uniformly to φ. Such a sequence always exists, by the Stone-Weierstrass theorem and the compactness of $\mathrm{spec}(f)$. Therefore, we infer from (82) that $\varphi \circ f$ is the uniform limit of the sequence of weak*-continuous function $(\varphi_n \circ f)_{n\in\mathbb{N}} \subseteq \mathfrak{U}$. In particular, $\varphi \circ f$ is weak*-continuous, i.e., $\varphi \circ f \in \mathfrak{U}$. It is easy to check that the mapping $\varphi \mapsto \varphi \circ f$ is a $*$-homomorphism from $C(\mathrm{spec}(f);\mathbb{C})$ to $\mathfrak{U}$ with $1 \circ f = \mathfrak{1}$ being the unit of $\mathfrak{U}$ and $\mathrm{id}_{\mathrm{spec}(f)} \circ f = f$. By the uniqueness of the continuous functional calculus, $\varphi \circ f = \varphi(f)$ for all $f \in \mathfrak{U}^{\mathbb{R}}$ and $\varphi \in C(\mathrm{spec}(f);\mathbb{C})$. ■

5.4 Important $*$-Automorphisms of the Quantum C^*-Algebra

Parity: The $*$-automorphism σ of $\mathcal{U}$ uniquely defined by the condition (4) naturally induces a $*$-automorphism Ξ of $\mathfrak{U}$ defined by

$$[\Xi\,(f)]\,(\rho) \doteq \sigma\,(f\,(\rho))\ , \qquad \rho \in E,\ f \in \mathfrak{U}\ . \tag{83}$$

Elements $f_1, f_2 \in \mathcal{U}$ satisfying $\Xi(f_1) = f_1$ and $\Xi(f_2) = -f_2$ are respectively called *even* and *odd*. The set

$$\mathfrak{U}^+ \equiv \mathfrak{U}^+_{\mathfrak{L}} \doteq \{f \in \mathfrak{U} : f = \Xi(f)\} = \left\{f \in \mathfrak{U} : f\,(E) \subseteq \mathcal{U}^+\right\} \subseteq \mathfrak{U} \tag{84}$$

of all even weak*-continuous $\mathcal{U}$-valued functions on states is a C^*-subalgebra. Compare with (5).

Translations: Let $\mathfrak{L} = \mathbb{Z}^d$. The $*$-automorphisms α_x, $x \in \mathbb{Z}^d$, of $\mathcal{U}$ uniquely defined by the condition (8) naturally induce a group homomorphism $x \mapsto \mathrm{A}_x$ from $\mathbb{Z}^d$ to the group of $*$-automorphisms of $\mathfrak{U}$, defined by

$$[\mathrm{A}_x\,(f)]\,(\rho) \doteq \alpha_x\,(f\,(\rho))\ , \qquad \rho \in E,\ f \in \mathfrak{U},\ x \in \mathbb{Z}^d\ . \tag{85}$$

These $*$-automorphisms represent the translation group in $\mathfrak{U}$.

Permutations: In the same way, we can define a group homomorphism $\pi \mapsto \mathfrak{P}_\pi$ from the set Π of all bijective mappings from $\mathfrak{L}$ into itself, which leave all but finitely many elements invariant, to the group of $*$-automorphisms of $\mathfrak{U}$ by using (9) and the definition

$$[\mathfrak{P}_\pi\,(f)]\,(\rho) \doteq \mathfrak{p}_\pi\,(f\,(\rho))\ , \qquad \rho \in E,\ f \in \mathfrak{U},\ \pi \in \Pi\ .$$

The $*$-automorphisms representing the permutation group in $\mathfrak{U}$ are not used here and are only given for completeness.

6 Limit Long-Range Dynamics – Classical Part

6.1 Banach Spaces of State-Dependent Short-Range Interactions

Similar to what is done in Section 3.1, a state-dependent (complex) interaction is defined to be a mapping $\mathbf{\Phi} : \mathcal{P}_f \to \mathfrak{U}^+$ such that $\Phi_\Lambda \in \mathfrak{U}_\Lambda$ for any $\Lambda \in \mathcal{P}_f$. See Equations (77) and (84). Similar to (21) and (22), the set of all state-dependent interactions is naturally endowed with the structure of a complex vector space on which the natural involution

$$\mathbf{\Phi} \mapsto \mathbf{\Phi}^* \doteq (\mathbf{\Phi}_\Lambda^*)_{\Lambda \in \mathcal{P}_f} \tag{86}$$

is defined. Self-adjoint state-dependent interactions $\mathbf{\Phi}$ are, by definition, those satisfying $\mathbf{\Phi} = \mathbf{\Phi}^*$. A state-dependent interaction $\mathbf{\Phi}$ can be identified with a mapping $\rho \mapsto \mathbf{\Phi}(\rho)$ from E to the vector space of usual interactions (of Section 3.1) via the definition

$$\mathbf{\Phi}(\rho)_\Lambda \doteq \mathbf{\Phi}_\Lambda(\rho) \ , \qquad \Lambda \in \mathcal{P}_f \ .$$

In this paper, we only consider a particular space of state-dependent short-range interactions: Using the Banach space $\mathcal{W}$ of (usual) short-range interactions, we define

$$\mathfrak{W} \doteq \left(C(E;\mathcal{W}), +, \cdot_{\mathbb{C}}, ^*, \|\cdot\|_{\mathfrak{W}}\right)$$

to be the Banach space of weak*-continuous, state-dependent short-range interactions, along with the supremum norm

$$\|\mathbf{\Phi}\|_{\mathfrak{W}} \doteq \max_{\rho \in E} \|\mathbf{\Phi}(\rho)\|_{\mathcal{W}} \ , \qquad \mathbf{\Phi} \in \mathfrak{W} \ , \tag{87}$$

where $\|\cdot\|_{\mathcal{W}}$ is defined by (27). Note that (86) is an isometric antilinear involution on $\mathfrak{W}$. Recall that $\mathcal{W}^{\mathbb{R}} \subsetneq \mathcal{W}$ is the (real) Banach subspace of all self-adjoint interactions and the (real) Banach subspace of all self-adjoint state-dependent interactions is similarly denoted by

$$\mathfrak{W}^{\mathbb{R}} \equiv \left(C\left(E;\mathcal{W}^{\mathbb{R}}\right), +, \cdot_{\mathbb{R}}, \|\cdot\|_{\mathfrak{W}}\right) \subsetneq \mathfrak{W} \ .$$

To simplify notation, for any $\mathbf{\Psi} \in C(\mathbb{R};\mathfrak{W})$ and $\rho \in E$, $\mathbf{\Psi}(\rho) \in C(\mathbb{R};\mathcal{W})$ stands for the time-dependent interaction defined by

$$\mathbf{\Psi}(\rho)(t) \doteq \mathbf{\Psi}(t;\rho) \ , \qquad \rho \in E, \ t \in \mathbb{R} \ . \tag{88}$$

When $\mathfrak{L} = \mathbb{Z}^d$, $\mathbf{\Phi} \in \mathfrak{W}$ is, by definition, translation-invariant if, for all $x \in \mathbb{Z}^d$ and $\Lambda \in \mathcal{P}_f$, $\mathbf{\Phi}_{\Lambda+x} = \mathrm{A}_x(\mathbf{\Phi}_\Lambda)$, see (28). Recall that $\{\mathrm{A}_x\}_{x\in\mathbb{Z}^d}$ is the family of (translation) $*$-automorphisms on $\mathfrak{U}$ defined by (85). Similar to the (separable) Banach subspace $\mathcal{W}_1 \subsetneq \mathcal{W}$ of translation-invariant, short-range interactions on $\mathfrak{L} = \mathbb{Z}^d$, we denote by $\mathfrak{W}_1 \subsetneq \mathfrak{W}$ the Banach subspace of translation-invariant, state-dependent, short-range interactions on $\mathfrak{L} = \mathbb{Z}^d$.

6.2 Derivations on the Quantum C^*-Algebra of Functions and Dynamics

For any state-dependent (short-range) interaction $\mathbf{\Phi} \in \mathfrak{W}$, we can naturally define limit derivations in the quantum C^*-algebra $\mathfrak{U}$:

Definition 6.1 (Derivations for state-dependant interactions)
The symmetric derivations $\boldsymbol{\delta}^{\mathbf{\Phi}}$ associated with any $\mathbf{\Phi} \in \mathfrak{W}$ is defined on the dense subset $\mathfrak{U}_0 = \mathrm{span}\{\mathfrak{C}\mathcal{U}_0\}$ (see (79)) by

$$\left[\boldsymbol{\delta}^{\mathbf{\Phi}}(fA)\right](\rho) \doteq f(\rho)\,\delta^{\mathbf{\Phi}(\rho)}(A) \ , \qquad \rho \in E, \ f \in \mathfrak{C}, \ A \in \mathcal{U}_0 \ .$$

The right-hand side of the last equation defines an element of $\mathfrak{U}$, by Corollary 3.5. Similar to Remark 3.6, observe also that, if $\Phi \in \mathfrak{W}^{\mathbb{R}}$, then the symmetric derivation $\boldsymbol{\delta}^{\Phi}$ is (norm-) closable: By Corollary 5.4 and Lemma 5.5, for all $f \in \mathfrak{U}_0$, $f \geq 0$ implies $f^{1/2} \in \mathfrak{U}_0$. It follows from [82, Theorem 1.4.9] that $\boldsymbol{\delta}^{\Phi}$ is dissipative [82, Definition 1.4.6], and, by [82, Proposition 1.4.7], it thus norm-closable and its closure is also dissipative.

Any $\boldsymbol{\Psi} \in C(\mathbb{R}; \mathfrak{W}^{\mathbb{R}})$ determines a two-parameter family $\mathfrak{T}^{\boldsymbol{\Psi}} \equiv (\mathfrak{T}^{\boldsymbol{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of the quantum C^*-algebra $\mathfrak{U}$ defined by (74):

$$\left[\mathfrak{T}^{\boldsymbol{\Psi}}_{t,s}(f)\right](\rho) \doteq \tau^{\boldsymbol{\Psi}(\rho)}_{t,s}(f(\rho)) \ , \qquad \rho \in E,\ f \in \mathfrak{U},\ s,t \in \mathbb{R} \ . \tag{89}$$

For all $f \in \mathfrak{U}_0$, the right -hand side of (89) defines an element of $\mathfrak{U}$, by Proposition 3.8 (iii) and Lebesgue's dominated convergence theorem. By density of $\mathfrak{U}_0 \subseteq \mathfrak{U}$ and the fact that $\tau^{\boldsymbol{\Psi}(\rho)}_{t,s}$ is a contraction, it follows that the right-hand side of (89) defines an element of $\mathfrak{U}$ for all $f \in \mathfrak{U} \supsetneq \mathfrak{U}_0$. Similar to Proposition 3.7, this family satisfies a non-autonomous evolution equation with infinitesimal generator $\boldsymbol{\delta}^{\boldsymbol{\Psi}(t)}$ for $t \in \mathbb{R}$:

Proposition 6.2 (Infinite-volume state-dependent short-range dynamics)
For any $\boldsymbol{\Psi} \in C(\mathbb{R}; \mathfrak{W}^{\mathbb{R}})$, $\mathfrak{T}^{\boldsymbol{\Psi}} \equiv (\mathfrak{T}^{\boldsymbol{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$ is a strongly continuous two-parameter family of $*$-automorphisms of $\mathfrak{U}$, which is the unique solution in $\mathcal{B}(\mathfrak{U})$ to the non-autonomous evolution equation

$$\forall s,t \in \mathbb{R}: \qquad \partial_t \mathfrak{T}^{\boldsymbol{\Psi}}_{t,s} = \mathfrak{T}^{\boldsymbol{\Psi}}_{t,s} \circ \boldsymbol{\delta}^{\boldsymbol{\Psi}(t)} \ , \qquad \mathfrak{T}^{\boldsymbol{\Psi}}_{s,s} = \mathbf{1}_{\mathfrak{U}} \ , \tag{90}$$

in the strong sense on the dense subspace $\mathfrak{U}_0 \subseteq \mathfrak{U}$, $\mathbf{1}_{\mathfrak{U}}$ being the identity mapping of $\mathfrak{U}$. In particular, it satisfies the reverse cocycle property:

$$\forall s,r,t \in \mathbb{R}: \qquad \mathfrak{T}^{\boldsymbol{\Psi}}_{t,s} = \mathfrak{T}^{\boldsymbol{\Psi}}_{r,s}\mathfrak{T}^{\boldsymbol{\Psi}}_{t,r} \ . \tag{91}$$

Proof. Fix $\boldsymbol{\Psi} \in C(\mathbb{R}; \mathfrak{W}^{\mathbb{R}})$. The fact that $(\mathfrak{T}^{\boldsymbol{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$ is a family of $*$-automorphisms of $\mathfrak{U}$ satisfying (91) is a direct consequence of (89), since $(\tau^{\boldsymbol{\Psi}(\rho)}_{t,s})_{s,t\in\mathbb{R}}$ is a family of $*$-automorphisms of $\mathcal{U}$ satisfying (42) for any $\rho \in E$.

We now prove that the family $(\mathfrak{T}^{\boldsymbol{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$ is strongly continuous: By (89) and [63, Lemma 5.1 (iii)], it suffice to prove that $(\tau^{\boldsymbol{\Psi}(\rho)}_{t,s})_{(\rho,s,t)\in E\times\mathbb{R}^2}$ is a strongly continuous family. To this end, take three sequences $(s_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and $(\rho_n)_{n\in\mathbb{N}} \subseteq E$ converging respectively to $s,t \in \mathbb{R}$ and $\rho \in E$. For any $A \in \mathcal{U}$,

$$\left\|\tau^{\boldsymbol{\Psi}(\rho_n)}_{t_n,s_n}(A) - \tau^{\boldsymbol{\Psi}(\rho)}_{t,s}(A)\right\|_{\mathcal{U}} \leq \left\|\tau^{\boldsymbol{\Psi}(\rho_n)}_{t_n,s_n}(A) - \tau^{\boldsymbol{\Psi}(\rho)}_{t_n,s_n}(A)\right\|_{\mathcal{U}} + \left\|\tau^{\boldsymbol{\Psi}(\rho)}_{t_n,s_n}(A) - \tau^{\boldsymbol{\Psi}(\rho)}_{t,s}(A)\right\|_{\mathcal{U}} \ . \tag{92}$$

By Proposition 3.7,

$$\lim_{n\to\infty}\left\|\tau^{\boldsymbol{\Psi}(\rho)}_{t_n,s_n}(A) - \tau^{\boldsymbol{\Psi}(\rho)}_{t,s}(A)\right\|_{\mathcal{U}} = 0 \ , \qquad A \in \mathcal{U} \ . \tag{93}$$

Because $\boldsymbol{\Psi} \in C\left(\mathbb{R}; \mathfrak{W}^{\mathbb{R}}\right)$, observe from (87) that, for $T \in \mathbb{R}^+$,

$$\sup_{\alpha\in[-T,T]}\sup_{\rho\in E}\|\boldsymbol{\Psi}(\alpha;\rho)\|_{\mathcal{W}} = \sup_{\alpha\in[-T,T]}\|\boldsymbol{\Psi}(\alpha)\|_{\mathfrak{W}} < \infty \ . \tag{94}$$

Therefore, we infer from Proposition 3.8 (iii) and Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty}\left\|\tau^{\boldsymbol{\Psi}(\rho_n)}_{t_n,s_n}(A) - \tau^{\boldsymbol{\Psi}(\rho)}_{t_n,s_n}(A)\right\|_{\mathcal{U}} = 0 \ , \qquad A \in \mathcal{U}_0 \ .$$

By Combining (92)-(93) with (89), [63, Lemma 5.1 (iii)], the density of $\mathcal{U}_0$ in $\mathcal{U}$ and the fact that $\tau^{\boldsymbol{\Psi}(\rho)}_{t,s}$ is a contraction for any $s,t \in \mathbb{R}$, we deduce that $(\mathfrak{T}^{\boldsymbol{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$ is a strongly continuous two-parameter family.

We next prove the non-autonomous evolution equation for $(\mathfrak{T}_{t,s}^{\mathbf{\Psi}})_{s,t\in\mathbb{R}}$: By Equation (89),

$$\forall s,t\in\mathbb{R},\ f\in\mathfrak{C}\subseteq\mathfrak{U}:\qquad \partial_t\mathfrak{T}_{t,s}^{\mathbf{\Psi}}(f)=0\ ,\qquad \mathfrak{T}_{s,s}^{\mathbf{\Psi}}=\mathbf{1}_{\mathfrak{U}}\ . \tag{95}$$

For any $s,t\in\mathbb{R}$, $A\in\mathcal{U}_0\subseteq\mathfrak{U}_0$ and $h\in\mathbb{R}\backslash\{0\}$, observe additionally that

$$\begin{aligned}&\left\|h^{-1}\left(\mathfrak{T}_{t+h,s}^{\mathbf{\Psi}}(A)-\mathfrak{T}_{t,s}^{\mathbf{\Psi}}(A)\right)-\mathfrak{T}_{t,s}^{\mathbf{\Psi}}\circ\boldsymbol{\delta}^{\mathbf{\Psi}(t)}(A)\right\|_{\mathfrak{U}}\\ =\ &\sup_{\rho\in E}\left\|h^{-1}\left(\tau_{t+h,t}^{\mathbf{\Psi}(\rho)}(A)-A\right)-\delta^{\mathbf{\Psi}(t;\rho)}(A)\right\|_{\mathcal{U}}\ ,\end{aligned} \tag{96}$$

using Proposition 3.7. Now, by contradiction, assume the existence of a zero sequence $(h_n)_{n\in\mathbb{N}}$, a sequence $(\rho_n)_{n\in\mathbb{N}}\subseteq E$ and a positive constant $D>0$ such that

$$\inf_{n\in\mathbb{N}}\left\|h_n^{-1}\left(\tau_{t+h_n,t}^{\mathbf{\Psi}(\rho_n)}(A)-A\right)-\delta^{\mathbf{\Psi}(t;\rho_n)}(A)\right\|_{\mathcal{U}}\geq D>0.$$

By weak*-compactness of E, we can assume without loss of generality that $(\rho_n)_{n\in\mathbb{N}}$ converges in the weak* topology to some $\rho\in E$, as $n\rightarrow\infty$. From Corollary 3.5, it follows that

$$\liminf_{n\to\infty}\left\|h_n^{-1}\left(\tau_{t+h_n,t}^{\mathbf{\Psi}(\rho_n)}(A)-A\right)-\delta^{\mathbf{\Psi}(t;\rho)}(A)\right\|_{\mathcal{U}}\geq D>0\ . \tag{97}$$

Meanwhile, using Proposition 3.8 (iii), for any $\varepsilon\in\mathbb{R}^+$, there is $n_0\in\mathbb{N}$ such that, for all $n\geq n_0$,

$$\begin{aligned}h_n^{-1}\left(\tau_{t+h_n,t}^{\mathbf{\Psi}(\rho_n)}-\tau_{t+h_n,t}^{\mathbf{\Psi}(\rho)}\right)(A)\ &\leq\ 2\left|\Lambda\right|\left\|A\right\|_{\mathcal{U}}\left\|\mathbf{F}\right\|_{1,\mathfrak{L}}\mathrm{e}^{2\mathbf{D}\int_{-\varepsilon}^{\varepsilon}\|\Psi(\rho;t+\alpha_1)\|_{\mathcal{W}}\mathrm{d}\alpha_1}\\ &\qquad\times\max_{\alpha\in[-\varepsilon,\varepsilon]}\left\|\mathbf{\Psi}(t+\alpha;\rho_n)-\mathbf{\Psi}(t+\alpha;\rho)\right\|_{\mathcal{W}}\ .\end{aligned} \tag{98}$$

Note that the mapping $(t,\rho)\mapsto\mathbf{\Psi}(t;\rho)$ is (jointly) continuous on $\mathbb{R}\times E$, by definition of $\mathfrak{W}$, and we infer from Inequalities (97)-(98) that

$$\liminf_{n\to\infty}\left\|h_n^{-1}\left(\tau_{t+h_n,t}^{\mathbf{\Psi}(\rho)}(A)-A\right)-\delta^{\mathbf{\Psi}(t;\rho)}(A)\right\|_{\mathcal{U}}\geq D>0\ ,$$

which contradicts (41) for $\Psi=\mathbf{\Psi}(\rho)$. By Equation (96), it follows that

$$\forall s,t\in\mathbb{R},\ A\in\mathcal{U}_0\subseteq\mathfrak{U}_0:\qquad \partial_t\mathfrak{T}_{t,s}^{\mathbf{\Psi}}(A)=\mathfrak{T}_{t,s}^{\mathbf{\Psi}}\circ\boldsymbol{\delta}^{\mathbf{\Psi}(t)}(A)\ . \tag{99}$$

By using that $\boldsymbol{\delta}^{\mathbf{\Psi}(t)}$, $t\in\mathbb{R}$, are derivations and $\mathfrak{T}_{t,s}^{\mathbf{\Psi}}$, $s,t\in\mathbb{R}$, are $*$-automorphisms of $\mathfrak{U}$, we deduce (90) on $\mathfrak{U}_0$, from (95) and (99). Recall that $\mathfrak{U}_0=\mathrm{span}\{\mathfrak{C}\mathcal{U}_0\}$, by (79).

Finally, in order to prove the uniqueness of the solution to (90), assume that $(\tilde{\mathfrak{T}}_{t,s})_{s,t\in\mathbb{R}}\subseteq\mathcal{B}(\mathfrak{U})$ is a two-parameter family satisfying (90) on $\mathfrak{U}_0$. Since $\boldsymbol{\delta}^{\mathbf{\Psi}(t)}(\mathfrak{C})=\{0\}$, $\mathfrak{C}$ is a subspace of the fixed point algebra of $(\tilde{\mathfrak{T}}_{t,s})_{s,t\in\mathbb{R}}$. In particular, by [63, Lemma 5.2], it comes from a strongly continuous family $(\tilde{\tau}_{t,s}^{\rho})_{(\rho,s,t)\in E\times\mathbb{R}^2}$ defined by

$$\tilde{\tau}_{t,s}^{\rho}(A)\doteq[\tilde{\mathfrak{T}}_{t,s}(A)](\rho)\ ,\qquad \rho\in E,\ A\in\mathcal{U}\subseteq\mathfrak{U},\ s,t\in\mathbb{R}\ .$$

Through (90) (cf. (96)), for each $\rho\in E$, $(\tau_{t,s}^{\mathbf{\Psi}(\rho)})_{s,t\in\mathbb{R}}$ and $(\tilde{\tau}_{t,s}^{\rho})_{s,t\in\mathbb{R}}$ are both solution in $\mathcal{B}(\mathcal{U})$ to the non-autonomous evolution equation (41) on $\mathcal{U}_0$ for $\Psi=\mathbf{\Psi}(\rho)$. Therefore, by Proposition 3.7, the solution to the non-autonomous evolution equation (90) on $\mathfrak{U}_0$ is also unique. ■

6.3 From Quantum Dynamics to Classical Flows

Proposition 6.2 means that, for any $\mathbf{\Psi} \in C(\mathbb{R}; \mathfrak{W}^{\mathbb{R}})$, $(\mathfrak{U}, \mathfrak{T}^{\mathbf{\Psi}})$ is a state-dependent C^*-dynamical system, as defined in [63, Definition 5.3]. Therefore, as explained in [63, Section 5.2], for any $\mathbf{\Psi} \in C(\mathbb{R}; \mathfrak{W}^{\mathbb{R}})$, $(\mathfrak{U}, \mathfrak{T}^{\mathbf{\Psi}})$ induces a Feller dynamics within the classical C^*-algebra $\mathfrak{C}$ defined by (65)-(66):

State-space trajectories: Let $C(E; E)$ be the set of weak*-continuous functions from the state space E to itself endowed with the topology of uniform convergence. In other words, any net $(f_j)_{j\in J} \subseteq C(E; E)$ converges to $f \in C(E; E)$ whenever

$$\lim_{j\in J} \max_{\rho\in E} |f_j(\rho)(A) - f(\rho)(A)| = 0 , \qquad \text{for all } A \in \mathcal{U} . \tag{100}$$

We denote by $\mathrm{Aut}(E) \subsetneq C(E; E)$ the subspace of all automorphisms of E, i.e., element of $C(E; E)$ with weak*-continuous inverse. Equivalently, $\mathrm{Aut}(E)$ is the set of all bijective mappings in $C(E; E)$, because E is a compact Hausdorff space. From the family $\mathfrak{T}^{\mathbf{\Psi}} \equiv (\mathfrak{T}^{\mathbf{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$, we define a continuous family $(\phi^{\mathbf{\Psi}}_{t,s})_{s,t\in\mathbb{R}} \subseteq \mathrm{Aut}(E)$ by

$$\phi^{\mathbf{\Psi}}_{t,s}(\rho) \doteq \rho \circ \tau^{\mathbf{\Psi}(\rho)}_{t,s} , \qquad \rho \in E,\ s,t \in \mathbb{R} , \tag{101}$$

where $(\tau^{\mathbf{\Psi}(\rho)}_{t,s})_{(\rho,s,t)\in E\times\mathbb{R}^2}$ is the unique strongly continuous family of $*$-automorphisms of $\mathcal{U}$ satisfying (89), see also [63, Lemma 5.2].

Classical flows as Feller evolution systems: The state-space trajectories, in turn, yield a strongly continuous two-parameter family $(V^{\mathbf{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of the classical C^*-algebra $\mathfrak{C}$, defined by

$$V^{\mathbf{\Psi}}_{t,s} f \doteq f \circ \phi^{\mathbf{\Psi}}_{t,s} , \qquad f \in \mathfrak{C},\ s,t \in \mathbb{R} . \tag{102}$$

This classical dynamics is a *Feller evolution system* in the following sense: As a $*$-automorphism, $V^{\mathbf{\Psi}}_{t,s}$ is self-adjointness- and positivity-preserving while $\|V^{\mathbf{\Psi}}_{t,s}\|_{\mathcal{B}(\mathfrak{C}^{\mathbb{R}})} = 1$; $(V^{\mathbf{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$ is a strongly continuous two-parameter family satisfying

$$\forall s,r,t \in \mathbb{R} : \qquad V^{\mathbf{\Psi}}_{t,s} = V^{\mathbf{\Psi}}_{r,s} \circ V^{\mathbf{\Psi}}_{t,r} , \tag{103}$$

by (91). Therefore, as explained in [63, Section 4.4], the classical dynamics defined as the restriction of $(V^{\mathbf{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$ to the real space $\mathfrak{C}^{\mathbb{R}}$ can be associated in this case with Feller processes[12] in probability theory: By the Riesz-Markov representation theorem and the monotone convergence theorem, there is a unique two-parameter group $(p^{\mathbf{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$ of Markov transition kernels $p^{\mathbf{\Psi}}_{t,s}(\cdot,\cdot)$ on E such that

$$V^{\mathbf{\Psi}}_{t,s} f(\rho) = \int_E f(\hat{\rho})\, p^{\mathbf{\Psi}}_{t,s}(\rho, \mathrm{d}\hat{\rho}) , \qquad f \in \mathfrak{C}^{\mathbb{R}} .$$

The right-hand side of the above identity makes sense for bounded measurable functions from E to $\mathbb{R}$. In fact, one can naturally extend $(V^{\mathbf{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$ to this more general class of functions on E.

The notion of Feller evolution system, which is only an extension of Feller semigroups to non-autonomous two-parameter families, has been introduced (at least) in 2014 [83].

Parity: For any $\mathbf{\Psi} \in C(\mathbb{R}; \mathfrak{W}^{\mathbb{R}})$, the family $(\mathfrak{T}^{\mathbf{\Psi}}_{t,s})_{s,t\in\mathbb{R}}$ is parity-preserving, i.e.,

$$\Xi \circ \mathfrak{T}^{\mathbf{\Psi}}_{t,s} = \mathfrak{T}^{\mathbf{\Psi}}_{t,s} \circ \Xi , \qquad s,t \in \mathbb{R} ,$$

where Ξ is the $*$-automorphism of $\mathfrak{U}$ defined by (83). It follows that

$$\phi^{\mathbf{\Psi}}_{t,s}(E^+) \subseteq E^+ , \qquad \phi^{\mathbf{\Psi}}_{t,s}(E \backslash E^+) \subseteq E \backslash E^+ , \qquad s,t \in \mathbb{R} , \tag{104}$$

[12] The positivity and norm-preserving property are reminiscent of Markov semigroups.

which in turn implies that $V_{t,s}^{\mathbf{\Psi}}$ can be seen as a mapping on either $C(E^+,\mathbb{C})$ or $C(E\backslash E^+,\mathbb{C})$:

$$V_{t,s}^{\mathbf{\Psi}}\left(f|_{E^+}\right) \doteq \left(V_{t,s}^{\mathbf{\Psi}}f\right)|_{E^+}\ , \qquad V_{t,s}^{\mathbf{\Psi}}\left(f|_{E\backslash E^+}\right) \doteq \left(V_{t,s}^{\mathbf{\Psi}}f\right)|_{E\backslash E^+}\ , \qquad f\in\mathfrak{C},\ s,t\in\mathbb{R}\ . \tag{105}$$

Recall that E^+ is the weak*-compact convex set of even states defined by (15), which is the physical state space. Similar to [63, Corollary 4.3], the set $\mathcal{E}(E^+)$ of extreme points of E^+ is also conserved by the flow.

<u>Translations:</u> Let $\mathfrak{L}=\mathbb{Z}^d$. For any continuous mapping $\mathbf{\Psi}\in C(\mathbb{R};\mathfrak{W}_1\cap\mathfrak{W}^{\mathbb{R}})$ from $\mathbb{R}$ to the space $\mathfrak{W}_1\cap\mathfrak{W}^{\mathbb{R}}$ of translation-invariant, self-adjoint and state-dependent interactions, the mapping $x\mapsto \mathrm{A}_x$ from $\mathbb{Z}^d$ to the group of $*$-automorphisms of $\mathfrak{U}$, defined by (85), is a symmetry group of the state-dependent C^*-dynamical system $(\mathfrak{U},\mathfrak{T}^{\mathbf{\Psi}})$, i.e.,

$$\mathrm{A}_x\circ\mathfrak{T}_{t,s}^{\mathbf{\Psi}}=\mathfrak{T}_{t,s}^{\mathbf{\Psi}}\circ\mathrm{A}_x\ , \qquad s,t\in\mathbb{R},\ x\in\mathbb{Z}^d\ .$$

As a consequence, for any $\vec{\ell}\in\mathbb{N}^d$,

$$\phi_{t,s}^{\mathbf{\Psi}}\left(E_{\vec{\ell}}\right)\subseteq E_{\vec{\ell}}\ , \qquad \phi_{t,s}^{\mathbf{\Psi}}\left(E\backslash E_{\vec{\ell}}\right)\subseteq E\backslash E_{\vec{\ell}}\ , \qquad s,t\in\mathbb{R}\ , \tag{106}$$

which in turn implies that $V_{t,s}^{\mathbf{\Psi}}$ can be seen as a mapping on either $C(E_{\vec{\ell}},\mathbb{C})$ or $C(E\backslash E_{\vec{\ell}},\mathbb{C})$:

$$V_{t,s}^{\mathbf{\Psi}}(f|_{E_{\vec{\ell}}})\doteq(V_{t,s}^{\mathbf{\Psi}}f)|_{E_{\vec{\ell}}}\ , \qquad V_{t,s}^{\mathbf{\Psi}}(f|_{E\backslash E_{\vec{\ell}}})\doteq(V_{t,s}^{\mathbf{\Psi}}f)|_{E\backslash E_{\vec{\ell}}}\ , \qquad f\in\mathfrak{C},\ s,t\in\mathbb{R}\ . \tag{107}$$

Recall that $E_{\vec{\ell}}\subseteq E^+$ is the weak*-compact convex set of $\vec{\ell}$-periodic states defined by (17) for any $\vec{\ell}\in\mathbb{N}^d$. Similar to [63, Corollary 4.3], for every $\vec{\ell}\in\mathbb{N}^d$, the set $\mathcal{E}(E_{\vec{\ell}})$ of extreme points of $E_{\vec{\ell}}$ is, in this case, also conserved by the flow.

6.4 Self-Consistency Equations

By using Equation (64), Proposition 3.2 and Corollary 3.5 together with the linearity of the mapping $\Phi\mapsto\delta^{\Phi}$, for every integer $n\geq 2$, any translation-invariant interactions $\Psi^{(1)},\ldots,\Psi^{(n)}\in\mathcal{W}_1$, each local element $A\in\mathcal{U}_0$, $\vec{\ell}\in\mathbb{N}^d$ and every extreme $\vec{\ell}$-periodic state $\rho\in\mathcal{E}(E_{\vec{\ell}})\subseteq E_{\vec{\ell}}$, one can prove that

$$\lim_{L\to\infty}\frac{1}{|\Lambda_L|^{n-1}}\rho\left(i\left[U_L^{\Psi^{(1)}}\cdots U_L^{\Psi^{(n)}},A\right]\right)=\rho\circ\delta^{\mathbf{\Psi}(\rho)}(A)\ , \tag{108}$$

where, in this case, the state-dependent interaction $\mathbf{\Psi}\in\mathfrak{W}$ equals

$$\mathbf{\Psi}(\rho)=\left\lfloor\rho;\Psi^{(1)},\ldots,\Psi^{(n)}\right\rfloor_{\vec{\ell}}\doteq\sum_{m=1}^{n}\Psi^{(m)}\prod_{j\in\{1,\ldots,n\},j\neq m}\rho(\mathfrak{e}_{\Psi^{(j)},\vec{\ell}})\in\mathcal{W}_1\ , \qquad \rho\in E\ . \tag{109}$$

Compare (108) with Definitions 4.2 and 4.3.

The proof of (108) uses the *ergodicity* of extreme periodic states in a crucial way. It is non-trivial and will be performed in detail in [64]. One central argument in this proof, like in Haag's approach [4] to mean-field theories, is that, for any translation-invariant interaction $\Phi\in\mathcal{W}_1$ and any cyclic representation $(\mathcal{H}_\rho,\pi_\rho,\Omega_\rho)$ of an *extreme* state $\rho\in\mathcal{E}(E_{\vec{\ell}})$ at fixed $\vec{\ell}\in\mathbb{N}^d$, the uniformly bounded family

$$\left\{\frac{\pi_\rho\left(U_L^{\Phi}\right)}{|\Lambda_L|}\right\}_{L\in\mathbb{N}}\subseteq\mathcal{B}(\mathcal{H}_\rho)$$

converges to the operator $\rho(\mathfrak{e}_{\Phi,\vec{\ell}})\mathbf{1}_{\mathcal{H}_\rho}$, in the sense of the strong operator topology. See [64, Section 5.3] for more details. Compare with Proposition 3.2.

Having in mind the discussions of Section 4.3 and the linearity of the mapping $\Phi \mapsto \delta^{\Phi}$, we give here the limit (108) in order to convince the reader that appropriate approximating (state-dependent, short-range) interactions naturally appears in the description of the infinite-volume dynamics of lattice-fermion (or quantum-spin) systems with long-range interactions. To define them, recall that the integral of interactions is defined by (51)-(52).

Definition 6.3 (Non-autonomous approximating interactions)
For $\vec{\ell} \in \mathbb{N}^d$ and any continuous functions $\mathfrak{m} = (\Phi(t), \mathfrak{a}(t))_{t\in\mathbb{R}} \in C(\mathbb{R}; \mathcal{M})$, $\xi \in C(\mathbb{R}; E)$, we define the mapping $\Phi^{(\mathfrak{m},\xi)}$ from $\mathbb{R}$ to $\mathcal{W}^{\mathbb{R}}$ by

$$\Phi^{(\mathfrak{m},\xi)}(t) \doteq \Phi(t) + \sum_{n\in\mathbb{N}} \int_{\mathbb{S}^n} \left\lfloor \xi(t); \Psi^{(1)}, \ldots, \Psi^{(n)} \right\rfloor_{\vec{\ell}} \mathfrak{a}(t)_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right), \qquad t \in \mathbb{R},$$

with $\lfloor \rho; \Psi \rfloor_{\vec{\ell}} \doteq \Psi$. If $\boldsymbol{\xi} \in C(\mathbb{R}; \mathrm{Aut}(E))$, then a mapping $\boldsymbol{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}$ from $\mathbb{R}$ to $\mathfrak{W}^{\mathbb{R}}$ is defined, for any $\rho \in E$ and $t \in \mathbb{R}$, by

$$\boldsymbol{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}(t;\rho) \doteq \Phi(t) + \sum_{n\in\mathbb{N}} \int_{\mathbb{S}^n} \left\lfloor \boldsymbol{\xi}(t;\rho); \Psi^{(1)}, \ldots, \Psi^{(n)} \right\rfloor_{\vec{\ell}} \mathfrak{a}(t)_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right).$$

Such approximating interactions can be used to define, via Proposition 3.7 or Proposition 6.2, $*$-automorphisms of $\mathcal{U}$ or $\mathfrak{U}$, because they are always bounded continuous functions:

Lemma 6.4 (Continuity of approximating interactions)
Let $\vec{\ell} \in \mathbb{N}^d$ and $\mathfrak{m} \in C(\mathbb{R}; \mathcal{M})$. For any $\xi \in C(\mathbb{R}; E)$, $\Phi^{(\mathfrak{m},\xi)} \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$ and, for any $\boldsymbol{\xi} \in C(\mathbb{R}; \mathrm{Aut}(E))$, $\boldsymbol{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})} \in C(\mathbb{R}; \mathfrak{W}^{\mathbb{R}})$ with

$$\left\| \Phi^{(\mathfrak{m},\xi)}(t) \right\|_{\mathcal{W}} \leq \|\mathfrak{m}(t)\|_{\mathcal{M}} \qquad \textit{and} \qquad \left\| \boldsymbol{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}(t) \right\|_{\mathfrak{W}} \leq \|\mathfrak{m}(t)\|_{\mathcal{M}}, \qquad t \in \mathbb{R}. \tag{110}$$

Proof. Inequalities (110) are direct consequences of Equations (30), (54)-(55), Definition 4.1 and the triangle inequality, recalling that $\mathbb{S}^n$ is the n-fold Cartesian product of the unit sphere $\mathbb{S}$ of the Banach space $\mathcal{W}_1$. Using the same arguments, note also that, for any $\mathfrak{m} \in C(\mathbb{R}; \mathcal{M})$, $\xi \in C(\mathbb{R}; E)$ and $t_1, t_2 \in \mathbb{R}$,

$$\begin{aligned} &\left\| \Phi^{(\mathfrak{m},\xi)}(t_1) - \Phi^{(\mathfrak{m},\xi)}(t_2) \right\|_{\mathcal{W}} \\ &\leq \|\Phi(t_1) - \Phi(t_2)\|_{\mathcal{W}} + \sum_{n\in\mathbb{N}} n \|\mathbf{F}\|_{1,\mathfrak{L}}^{n-1} \|\mathfrak{a}(t_1)_n - \mathfrak{a}(t_2)_n\|_{\mathcal{S}(\mathbb{S}^n)} \\ &\quad + \sum_{n\in\mathbb{N}} \|\mathbf{F}\|_{1,\mathfrak{L}}^{n-1} \int_{\mathbb{S}^n} \sum_{j=1}^{n} \left| (\xi(t_1) - \xi(t_2)) (\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}) \right| \, |\mathfrak{a}(t_1)_n| \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right). \end{aligned} \tag{111}$$

Since $\mathfrak{m} \in C(\mathbb{R}; \mathcal{M})$ and $\xi \in C(\mathbb{R}; E)$, we invoke Lebesgue's dominated convergence theorem to deduce from the last inequality that $\Phi^{(\mathfrak{m},\xi)} \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$. The difference of $\boldsymbol{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}$ for two different times at a fixed state ρ satisfies the same inequality as (111), $\xi(t_1), \xi(t_2)$ being replaced with $\boldsymbol{\xi}(t_1;\rho), \boldsymbol{\xi}(t_2;\rho)$, respectively. Since $\mathrm{Aut}(E) \subsetneq C(E; E)$ is the subspace of all automorphisms of E endowed with the topology of uniform convergence, as stated in Equation (100), one also infers from Lebesgue's dominated convergence theorem that $\boldsymbol{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})} \in C(\mathbb{R}; \mathfrak{W}^{\mathbb{R}})$, provided $\mathfrak{m} \in C(\mathbb{R}; \mathcal{M})$ and $\boldsymbol{\xi} \in C(\mathbb{R}; \mathrm{Aut}(E))$. ■

As is usual, we do the following identifications for the subspaces of constant functions:

$$E \subseteq C(\mathbb{R}; E) \qquad \text{and} \qquad \mathrm{Aut}(E) \subseteq C(\mathbb{R}; \mathrm{Aut}(E)). \tag{112}$$

When $\xi \in E$ and $\mathfrak{m} = (\Phi, (0, \mathfrak{a}_2, 0, \ldots)) \in \mathcal{M}_1$, the first part of Definition 6.3 corresponds to the (autonomous) approximating interactions first introduced in [46, Definition 2.31], see also Section 8. They are used there to characterize (generalized equilibrium) states minimizing the free-energy density through *self-consistency equations* (gap equations), whose solutions are related to non-cooperative equilibria of a two-person zero-sum game (thermodynamic game).

More generally, by Proposition 6.2 and Lemma 6.4, for any $\boldsymbol{\xi} \in C(\mathbb{R}; \mathrm{Aut}(E))$, there is a strongly continuous two-parameter family $(\mathfrak{T}_{t,s}^{\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{U}$ satisfying the reverse cocycle property. This family satisfies a non-autonomous evolution equation, similar to Equation (41). It is used to construct the infinite-volume limit of the non-autonomous dynamics of time-dependent long-range models of $\mathcal{M}$ within a cyclic representation associated with an arbitrary periodic state. We show that, generically, a long-range (or mean-field) dynamics is equivalent to an intricate combination of a classical *and* short-range quantum dynamics. Both dynamics will be (non-trivial) consequences of the well-posedness of *self-consistency equations*, which are reminiscent of [63, Theorem 4.1].

To present these equations, recall that, for any $\Lambda \in \mathcal{P}_f$, $\mathcal{M}_\Lambda$ belongs to the dense subset $\mathcal{M}_0 \subseteq \mathcal{M}$ of models with an arbitrary short-range part, while the long-range interactions are polynomials of finite-range translation-invariant interactions. See (57)-(59) and Definition 4.2.

Theorem 6.5 (Self-consistency equations)
Fix $\Lambda \in \mathcal{P}_f$ and $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda)$. There is a unique $\boldsymbol{\varpi}^{\mathfrak{m}} \in C(\mathbb{R}^2; \mathrm{Aut}(E))$ such that

$$\boldsymbol{\varpi}^{\mathfrak{m}}(s,t) = \phi_{t,s}^{\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(\alpha,\cdot))}}|_{\alpha=s} \ , \qquad s,t \in \mathbb{R} \ ,$$

where $(\phi_{t,s}^{\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}})_{s,t\in\mathbb{R}}$ is the continuous family of automorphisms of E defined by (101) with the state-dependent interaction $\mathbf{\Psi} = \mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}$ of Definition 6.3 for $\boldsymbol{\xi} \in C(\mathbb{R}; \mathrm{Aut}(E))$.

Proof. The theorem is a consequence of Lemmata 7.2 and 7.8. ∎

Remark 6.6
Section 7 proves stronger results than Theorem 6.5. See, in particular, Lemma 7.4.

At fixed $\Lambda \in \mathcal{P}_f$ and long-range model $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda)$, Theorem 6.5 means that, for any $s, t \in \mathbb{R}$, $\rho \in E$ and $A \in \mathcal{U}$,

$$\rho_{s,t}(A) = \rho \circ \tau_{t,s}^{\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(\alpha,\cdot))}(\rho)}(A)|_{\alpha=s} \qquad \text{with} \qquad \rho_{s,t} \doteq \boldsymbol{\varpi}^{\mathfrak{m}}(s,t;\rho) \doteq \boldsymbol{\varpi}^{\mathfrak{m}}(s,t)(\rho) \in E \ . \quad (113)$$

See Equations (88), (89) and (101). Let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be an arbitrary family of closed sets

$$\mathcal{C}_n \subseteq (\mathbb{S} \cap \mathcal{W}_\Lambda)^n \ , \qquad n \in \mathbb{N} \ ,$$

such that, for $t \in \mathbb{R}$,

$$|\mathfrak{a}(t)_n|(\mathcal{C}_n) = |\mathfrak{a}(t)_n|(\mathbb{S}^n) \quad \text{with} \quad \mathfrak{m} = (\Phi(t), \mathfrak{a}(t))_{t\in\mathbb{R}} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda)$$

(cf. (31) and (57)-(58)). Then, (113) for the time-dependent expectation

$$\rho_{s,t}(A) \in \mathbb{C} \ , \qquad s,t \in \mathbb{R} \ ,$$

of elements

$$A \in \mathcal{V}_{\mathfrak{m}} \doteq \left\{ \mathfrak{e}_{\Psi,\vec{\ell}} : \Psi \in \mathcal{C}_n, \ n \in \mathbb{N} \right\}$$

leads to a systems of non-autonomous, coupled and non-linear equations, in general. These self-consistency equations are strongly related to the self-consistency equations (gap equations) explained in [46, Section 2.8] for the special case of (generalized) equilibrium states. By contrast, Equations (113) for elements $A \in \mathcal{U}\backslash\mathcal{V}_{\mathfrak{m}}$ are not coupled to each other.

Last but not least, for any $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda)$, observe from (104) that

$$\varpi^{\mathfrak{m}} \left(s, t; E^+\right) \subseteq E^+ , \qquad \varpi^{\mathfrak{m}} \left(s, t; E\backslash E^+\right) \subseteq E\backslash E^+ , \qquad s, t \in \mathbb{R} .$$

Recall that E^+ is the weak*-compact convex set of even states defined by (15). When long-range models are translation-invariant, that is, if $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_1 \cap \mathcal{M}_\Lambda)$ (see (56)), the non-autonomous approximating interactions of Definition 6.3 are also translation-invariant. By (106), it follows, in this case, that $\varpi^{\mathfrak{m}}(s,t)$ maps periodic states to periodic states: for any $\vec{\ell} \in \mathbb{N}^d$,

$$\varpi^{\mathfrak{m}} \left(s, t; E_{\vec{\ell}}\right) \subseteq E_{\vec{\ell}} , \qquad \varpi^{\mathfrak{m}} \left(s, t; E\backslash E_{\vec{\ell}}\right) \subseteq E\backslash E_{\vec{\ell}} , \qquad s, t \in \mathbb{R} . \tag{114}$$

Recall that $E_{\vec{\ell}} \subseteq E$ is the weak*-compact convex subset of $\vec{\ell}$-periodic states defined by (17). Like E, it has a dense set of extreme points ($\vec{\ell}$-ergodic states). Additionally, in this case, similar to [63, Corollary 4.3], for every $\vec{\ell} \in \mathbb{N}^d$, the set $\mathcal{E}(E_{\vec{\ell}})$ of extreme points of $E_{\vec{\ell}}$ is conserved by the flow.

6.5 Classical Part of Long-Range Dynamics

Similar to (102), the continuous family $\varpi^{\mathfrak{m}}$ of Theorem 6.5 yields a family $(V^{\mathfrak{m}}_{t,s})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{C}$ defined by

$$V^{\mathfrak{m}}_{t,s}(f) \doteq f \circ \varpi^{\mathfrak{m}}(s,t) , \qquad f \in \mathfrak{C},\ s, t \in \mathbb{R} . \tag{115}$$

It is a strongly continuous two-parameter family defining a classical dynamics on the commutative C^*-algebra $\mathfrak{C}$ of continuous complex-valued functions on states defined by (65)-(66):

Proposition 6.7 (Classical dynamics as Feller evolution system)
Fix $\Lambda \in \mathcal{P}_f$ and $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda)$. Then, $(V^{\mathfrak{m}}_{t,s})_{s,t\in\mathbb{R}}$ is a strongly continuous two-parameter family of $*$-automorphisms of $\mathfrak{C}$ satisfying the reverse cocycle property:

$$\forall s, r, t \in \mathbb{R} : \qquad V^{\mathfrak{m}}_{t,s} = V^{\mathfrak{m}}_{r,s} \circ V^{\mathfrak{m}}_{t,r} .$$

If $\mathfrak{m} \in \mathcal{M}_\Lambda \subsetneq C_b(\mathbb{R}; \mathcal{M}_\Lambda)$, i.e., $\mathfrak{m}$ is constant in time, then $V^{\mathfrak{m}}_{t,s} = V^{\mathfrak{m}}_{t-s,0}$ for any $s, t \in \mathbb{R}$ and $(V^{\mathfrak{m}}_{t,0})_{t\in\mathbb{R}}$ is a C_0-group of $*$-automorphisms of $\mathfrak{C}$.

Proof. In order to prove these assertions, one simply adapts the argument used to prove [63, Proposition 3.4], having in mind results of Section 7. We omit the details. ■

Like (102), $(V^{\mathfrak{m}}_{t,s})_{s,t\in\mathbb{R}}$ can be associated with a Feller process in probability theory. If $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda \cap \mathcal{M}_1)$, note that the classical flow conserves the Poulsen simplex $E_{\vec{\ell}}$, $\vec{\ell} \in \mathbb{N}^d$, and $V^{\mathfrak{m}}_{t,s}$ can be seen as either a mapping from $C(E_{\vec{\ell}}; \mathbb{C})$ to itself or from $C(E\backslash E_{\vec{\ell}}; \mathbb{C})$ to itself:

$$V^{\mathfrak{m}}_{t,s}(f|_{E_{\vec{\ell}}}) \doteq (V^{\mathfrak{m}}_{t,s} f)|_{E_{\vec{\ell}}} , \qquad V^{\mathfrak{m}}_{t,s}(f|_{E\backslash E_{\vec{\ell}}}) \doteq (V^{\mathfrak{m}}_{t,s} f)|_{E\backslash E_{\vec{\ell}}} , \qquad f \in \mathfrak{C},\ s, t \in \mathbb{R} , \tag{116}$$

using (114). Compare with Equation (107). For all $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda)$, the same holds true for the weak*-compact convex set E^+ of even states.

For any constant function $\mathfrak{m} \in \mathcal{M}_\Lambda \subsetneq C_b(\mathbb{R}; \mathcal{M}_\Lambda)$, $(V^{\mathfrak{m}}_{t,0})_{t\in\mathbb{R}}$ is a C_0-group of $*$-automorphisms of $\mathfrak{C}$ and we denote by $\daleth^{\mathfrak{m}}$ its (well-defined) generator. By [84, Chap. II, Sect. 3.11], it is a closed (linear) operator densely defined in $\mathfrak{C}$. Since $V^{\mathfrak{m}}_{t,0}$, $t \in \mathbb{R}$, are $*$-automorphisms, we infer from the Nelson theorem [82, Theorem 1.5.4], or the Lumer-Phillips theorem [75, Theorem 3.1.16], that $\pm\daleth^{\mathfrak{m}}$ are dissipative operators, i.e., $\daleth^{\mathfrak{m}}$ is conservative. The $*$-homomorphism property of $V^{\mathfrak{m}}_{t,0}$, $t \in \mathbb{R}$, is reflected by the fact that $\urcorner^{\mathfrak{m}}$ has to be a symmetric derivation of $\mathfrak{C}$. In fact, similar to [63, Theorem 4.5], $\urcorner^{\mathfrak{m}}$ is directly related to a Poissonian symmetric derivation.

In order to understand this fact, we need to figure out the appropriate *classical energy functions*, which correspond in [63, Theorem 4.5] to the function h. Recall that $\hat{A} \in \mathfrak{C}$ is the continuous and affine function defined by (67) for any $A \in \mathcal{U}$, while $\{\cdot,\cdot\}$ is the Poisson bracket of Definition 5.2. Now, having in mind Definition 4.2, it is natural to define the following family of classical energy functions of $\mathfrak{C}$:

Definition 6.8 (Classical energy functions of long-range dynamics)
For any $\mathfrak{m} \in \mathcal{M}$, we define the functions

$$\mathrm{h}_L^{\mathfrak{m}} \doteq \widehat{U_L^{\Phi}} + \sum_{n\in\mathbb{N}} \frac{1}{|\Lambda_L|^{n-1}} \int_{\mathbb{S}^n} \widehat{U_L^{\Psi^{(1)}}} \cdots \widehat{U_L^{\Psi^{(n)}}}\ \mathfrak{a}(t)_n \left(\mathrm{d}\Psi^{(1)},\ldots,\mathrm{d}\Psi^{(n)}\right) \in \mathfrak{C}^{\mathbb{R}}, \qquad L\in\mathbb{N},$$

which we name the local classical energy functions associated with $\mathfrak{m}$.

The integral in Definition 6.8 is well-defined by the same reasons than in Definition 4.2. It is important to stress that, although these two definitions look similar, $\mathrm{h}_L^{\mathfrak{m}} \neq \widehat{U_L^{\mathfrak{m}}}$, in general. Local classical energy functions are continuously differentiable real-valued functions over the convex and weak*-compact set E, i.e., $\{\mathrm{h}_L^{\mathfrak{m}}\}_{L\in\mathbb{N}} \subseteq \mathfrak{Y}^{\mathbb{R}}$, the subspace $\mathfrak{Y}^{\mathbb{R}} \subseteq \mathfrak{C}^{\mathbb{R}}$ being defined by (70): For any $\mathfrak{m} \in \mathcal{M}$, by straightforward estimates using Equations (35), (54)-(55) and Definition 4.1 together with Lebesgue's dominated convergence theorem, one checks that, for any $L \in \mathbb{N}$, $\mathrm{h}_L^{\mathfrak{m}}$ is continuously differentiable and

$$\mathrm{Dh}_L^{\mathfrak{m}} = U_L^{\Phi} - \widehat{U_L^{\Phi}} + \sum_{n\in\mathbb{N}}\sum_{m=1}^{n} \int_{\mathbb{S}^n} \left(U_L^{\Psi^{(m)}} - \widehat{U_L^{\Psi^{(m)}}}\right) \prod_{j\in\{1,\ldots,n\},j\neq m} \frac{\widehat{U_L^{\Psi^{(j)}}}}{|\Lambda_L|} \mathfrak{a}(t)_n \left(\mathrm{d}\Psi^{(1)},\ldots,\mathrm{d}\Psi^{(n)}\right), \tag{117}$$

where $\mathrm{Dh}_L^{\mathfrak{m}} \in C(E;\mathcal{U}^{\mathbb{R}}) = \mathfrak{U}^{\mathbb{R}}$ is the function defined by (72) for $f = \mathrm{h}_L^{\mathfrak{m}}$. Moreover, by (35), (68), (71) and (73), for any $\mathfrak{m} \in \mathcal{M}$,

$$\|\mathrm{h}_L^{\mathfrak{m}}\|_{\mathfrak{Y}} = \|\mathrm{h}_L^{\mathfrak{m}}(\rho)\|_{\mathfrak{C}} + \max_{\rho\in E} \|\mathrm{Dh}_L^{\mathfrak{m}}(\rho)\|_{\mathcal{U}} \leq 3\,|\Lambda_L|\,\|\mathbf{F}\|_{1,\mathfrak{L}}\,\|\mathfrak{m}\|_{\mathcal{M}}, \qquad L\in\mathbb{N},$$

which is similar to Inequality (60) that bounds the norm of the Hamiltonians $U_L^{\mathfrak{m}}$, $L \in \mathbb{N}$.

Last but not least, it is very instructive to compare (117) with the local Hamiltonians associated with approximating interactions of Definition 6.3, in the light of Proposition 3.2: All terms of the form

$$\frac{\widehat{U_L^{\Psi}}(\rho)}{|\Lambda_L|} = \frac{\rho\left(U_L^{\Psi}\right)}{|\Lambda_L|}, \qquad \Psi \in \mathbb{S},\ \rho\in E,$$

in (117) should converge, as $L \to \infty$. By Proposition 3.2, we know this holds true whenever ρ is a periodic state, the limit being $\rho(\mathfrak{e}_{\Phi,\vec{\ell}})$ for some $\vec{\ell} \in \mathbb{N}^d$. The periodicity of states is therefore a very useful property in this context. Recall that periodic states form a weak*-dense set E_{p} (16) of the physically relevant set E^+ (15) of even states, by Proposition 2.3.

So, restricting our study to periodic states, we are now in a position to link the generator $\urcorner^{\mathfrak{m}}$ to a Poissonian symmetric derivation:

Corollary 6.9 (Classical evolutions via Poisson brackets)
For any $\Lambda \in \mathcal{P}_f$ and $\mathfrak{m} \in \mathcal{M}_\Lambda$, the dense $*$-subalgebra $\mathfrak{C}_{\mathcal{U}_0}$ (69) belongs to the domain of $\urcorner^{\mathfrak{m}}$ and

$$\urcorner^{\mathfrak{m}}(f)\,|_{E_{\mathrm{p}}} = \lim_{L\to\infty} \{\mathrm{h}_L^{\mathfrak{m}}, f\}\,|_{E_{\mathrm{p}}}, \qquad f\in\mathfrak{C}_{\mathcal{U}_0},$$

where the limit has to be understood point-wise on the weak*-dense subspace $E_{\mathrm{p}} \subseteq E^+$ of all periodic states.

Proof. Fix $\Lambda \in \mathcal{P}_f$ and $\mathfrak{m} \in \mathcal{M}_\Lambda$. Comparing Definitions 3.3 and 6.3 with Definition 5.2 and (117) and using Proposition 3.2 and Corollary 3.5 together with Lebesgue's dominated convergence theorem, one computes that

$$\lim_{L\to\infty} \{\mathrm{h}_L^{\mathfrak{m}}, f\}(\rho) = \rho \circ \delta^{\Phi^{(\mathfrak{m},\rho)}}(\mathrm{D}f(\rho)) , \qquad f \in \mathfrak{C}_{\mathcal{U}_0},\ \rho \in E_{\mathrm{p}} . \tag{118}$$

Note that one can interchange the local quantum derivation (which is a commutator) in the left-hand side of (118) with every integral over $(\mathbb{S}\cap\mathcal{W}_\Lambda)^n$, $n \in \mathbb{N}$, by finite dimensionality of the space $\mathcal{U}_{\Lambda_L}$ for all $L \in \mathbb{N}$. Here, we use (112) to define the approximating interaction $\Phi^{(\mathfrak{m},\rho)}$, as well as the fact that $\mathrm{D}f(\rho) \in \mathcal{U}_0$ whenever $f \in \mathfrak{C}_{\mathcal{U}_0}$. Now, the rest of the proof is very similar to the one of [63, Theorem 4.5]: By Lemma 7.4, one verifies that, for any $A \in \mathcal{U}_0$ and $t \in \mathbb{R}$,

$$\partial_t V_{t,0}^{\mathfrak{m}}(\hat{A})(\rho) = V_{t,0}^{\mathfrak{m}} \circ \urcorner^{\mathfrak{m}}(\hat{A})(\rho) = \boldsymbol{\varpi}^{\mathfrak{m}}(0,t;\rho) \circ \delta^{\Phi^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(0,t;\rho))}}(A) . \tag{119}$$

Since $\urcorner^{\mathfrak{m}}$ and $\mathfrak{d}^{\mathrm{h}_L^{\mathfrak{m}}}$, $L \in \mathbb{N}$, are symmetric derivations, the assertion follows by combining (118) and (119). ∎

Equation (119) also holds true for any time-dependent model $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda)$ and $\Lambda \in \mathcal{P}_f$, with $\urcorner^{\mathfrak{m}}$ being replaced with $\urcorner^{\mathfrak{m}(t)}$ in (119). Therefore, in the non-autonomous situation, for any $s,t \in \mathbb{R}$, $\rho \in E$ and polynomial function $f \in \mathfrak{C}_{\mathcal{U}_0}$,

$$\partial_t V_{t,s}^{\mathfrak{m}}(f)(\rho) = V_{t,s}^{\mathfrak{m}} \circ \urcorner^{\mathfrak{m}(t)}(f)(\rho) = \boldsymbol{\varpi}^{\mathfrak{m}}(s,t;\rho) \circ \delta^{\Phi^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(s,t;\rho))}}(f(\rho)) . \tag{120}$$

Similar (point-wise) identities for $\partial_s V_{t,s}^{\mathfrak{m}}$ like

$$\partial_s V_{t,s}^{\mathfrak{m}}(f) = -\urcorner^{\mathfrak{m}(s)} \circ V_{t,s}^{\mathfrak{m}}(f) \tag{121}$$

are not at all obvious. In fact, no unified theory of non-autonomous evolution equations that gives a complete characterization of the existence of fundamental solutions in terms of properties of generators, analogously to the Hille-Yosida generation theorems for the autonomous case, is available. See, e.g., [67–71] and references therein.

An important, *highly non-trivial*, result in that direction is proven in the next theorem, which depends in a crucial way on Lieb-Robinson bounds for multi-commutators of [72, Theorems 4.11, 5.4]:

Theorem 6.10 (Non–autonomous classical dynamics)
Fix $\Lambda \in \mathcal{P}_f$, $\varsigma,\epsilon \in \mathbb{R}^+$ and take

$$\mathbf{F}(x,y) = \mathrm{e}^{-2\varsigma|x-y|}(1+|x-y|)^{-(d+\epsilon)} , \qquad x,y \in \mathfrak{L} ,$$

as the decay function, see (26). Then, $(V_{t,s}^{\mathfrak{m}})_{s,t\in\mathbb{R}}$ is a strongly continuous two-parameter family of $*$-automorphisms of $\mathfrak{C}$ satisfying, on the dense $*$-subalgebra $\mathfrak{C}_{\mathcal{U}_0}$ (69),

$$\forall s,t \in \mathbb{R}: \qquad \partial_t V_{t,s}^{\mathfrak{m}}(f)|_{E_{\mathrm{p}}} = \lim_{L\to\infty} V_{t,s}^{\mathfrak{m}}\left(\{\mathrm{h}_L^{\mathfrak{m}(t)}, f\}\right)|_{E_{\mathrm{p}}} , \qquad V_{s,s}^{\mathfrak{m}} = \mathbf{1}_{\mathfrak{C}} , \tag{122}$$

for any $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda \cap \mathcal{M}_1)$ while, for any $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda)$, $s,t \in \mathbb{R}$ and $f \in \mathfrak{C}_{\mathcal{U}_0}$, $V_{t,s}^{\mathfrak{m}}(f) \in \mathfrak{Y}$ with

$$\forall s,t \in \mathbb{R}: \qquad \partial_s V_{t,s}^{\mathfrak{m}}(f)|_{E_{\mathrm{p}}} = -\lim_{L\to\infty} \{\mathrm{h}_L^{\mathfrak{m}(s)}, V_{t,s}^{\mathfrak{m}}(f)\}|_{E_{\mathrm{p}}} , \qquad V_{t,t}^{\mathfrak{m}} = \mathbf{1}_{\mathfrak{C}} . \tag{123}$$

All limits have to be understood point-wise on the weak*-dense subspace $E_{\mathrm{p}} \subseteq E^+$ of all periodic states.

Proof. Fix all parameters of the theorem. Equation (122) results from (114), (118), (120), Lemma 3.2 and the fact that $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda \cap \mathcal{M}_1)$ (cf. (116)). In order to prove $V_{t,s}^{\mathfrak{m}}(f) \in \mathfrak{Y}$ and (123), it suffices to invoke Lemma 7.13, which says that

$$\partial_s V_{t,s}^{\mathfrak{m}}(\hat{A})|_{E_{\mathrm{p}}} = - \lim_{L\to\infty} \{\mathrm{h}_L^{\mathfrak{m}(s)}, V_{t,s}^{\mathfrak{m}}(\hat{A})\}|_{E_{\mathrm{p}}}$$

for any $s, t \in \mathbb{R}$ and $A \in \mathcal{U}_0$. Since $(V_{t,s}^{\mathfrak{m}})_{s,t\in\mathbb{R}}$ is a family of $*$-automorphisms of $\mathfrak{C}$, by using the (bi)linearity and Leibniz's rule satisfied by the derivatives and the bracket $\{\cdot, \cdot\}$, we deduce that $V_{t,s}^{\mathfrak{m}}(f) \in \mathfrak{Y}$ and (123) for all polynomial functions $f \in \mathfrak{C}_{\mathcal{U}_0}$ and times $s, t \in \mathbb{R}$. ■

By Corollary 6.9 and Theorem 6.10, Equation (121) seems to hold true, but it cannot be directly deduced from Theorem 6.10. We refrain from doing such a study in this paper, since Theorem 6.10 already shows that we have a non-autonomous classical dynamics in the usual sense. Note that, in the case $\mathfrak{L} = \mathbb{Z}^d$, we have to consider the limit $L \to \infty$. This is the classical counterpart of the thermodynamic limit $L \to \infty$ in the derivations of Corollary 3.5.

In the autonomous situation, as already suggested by Corollary 6.9, we obtain from Theorem 6.10 the usual (autonomous) dynamics of classical mechanics written in terms of Poisson brackets (see, e.g., [85, Proposition 10.2.3]), i.e., *Liouville's equation*:

Corollary 6.11 (Liouville's equation)
Under conditions of Theorem 6.10, for any $t \in \mathbb{R}$ and $f \in \mathfrak{C}_{\mathcal{U}_0}$,

$$\partial_t V_{t,0}^{\mathfrak{m}}(f) = V_{t,0}^{\mathfrak{m}} \circ \daleth^{\mathfrak{m}}(f) = V_{t,0}^{\mathfrak{m}}\left(\lim_{L\to\infty} \{\mathrm{h}_L^{\mathfrak{m}}, f\}\right) = \lim_{L\to\infty} \left\{\mathrm{h}_L^{\mathfrak{m}}, V_{t,0}^{\mathfrak{m}}(f)\right\} = \daleth^{\mathfrak{m}} \circ V_{t,0}^{\mathfrak{m}}(f) \ ,$$

where all limits have to be understood point-wise on the weak*-dense subspace $E_{\mathrm{p}} \subseteq E^+$ of all periodic states.

Proof. Combine Corollary 6.9 with Theorem 6.10. ■

Writing the classical dynamics in terms of Liouville's equation, as in Corollary 6.11, is conceptually illuminating and also very useful from a purely mathematical point of view. For instance, the Gross-Pitaevskii and Hartree hierarchies mathematically derived from Bose gases with *mean-field* interactions are *infinite* systems of coupled PDEs and therefore, a direct proof of the uniqueness of its solutions is technically quite demanding, usually involving Feynman graphs, multilinear estimates, etc. In [42], the authors show that a solution to these hierarchies is basically a family of time-dependent correlation functions associated with a certain positive measure on the unit ball of a L^2-space, whose dynamical evolution is driven by Liouville's equation, similarly to Corollary 6.11. Uniqueness of a solution to such Liouville's equation can be proven in a general setting, as shown in 2018 [42, 86], implying the uniqueness of a solution to the Gross-Pitaevskii and Hartree hierarchies without highly technical issues depending on the particularities of the hierarchies under consideration.

6.6 Quantum Part of Long-Range Dynamics

The classical part of the dynamics of lattice-fermion systems with long-range interactions, which is defined within the classical C^*-algebras $\mathfrak{C}$ of continuous complex-valued functions on states, is shown to result from self-consistency equations, as explained in Theorem 6.5. Since $\mathfrak{C}$ can be seen as a subalgebra (76) of the quantum C^*-algebras $\mathfrak{U}$ of continuous $\mathcal{U}$-valued functions on states, defined by (74)-(75), there is a natural extension of the classical dynamics on $\mathfrak{C}$: The continuous family $\boldsymbol{\varpi}^{\mathfrak{m}}$ of Theorem 6.5 yields a family $(\mathfrak{V}_{t,s}^{\mathfrak{m}})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{U}$ defined by

$$\mathfrak{V}_{t,s}^{\mathfrak{m}}(f) \doteq f \circ \boldsymbol{\varpi}^{\mathfrak{m}}(s,t) \ , \qquad f \in \mathfrak{U}, \ s, t \in \mathbb{R} \ .$$

In particular, by (115), $\mathfrak{V}_{t,s}^{\mathfrak{m}}|_{\mathfrak{C}} = V_{t,s}^{\mathfrak{m}}$ for any $s, t \in \mathbb{R}$.

However, it is not what we have in mind here: Emphasizing rather the inclusion $\mathcal{U} \subseteq \mathfrak{U}$, in the long-range dynamics, the classical algebra $\mathfrak{C}$ becomes a subalgebra of the fixed-point algebra of the state-dependent long-range dynamics on $\mathfrak{U}$. In [64] we describe in detail the quantum part of the long-range dynamics, which is defined in a representation of the C^*-algebra $\mathfrak{U}$. We give below a few key points of this study:

<u>(i):</u> As soon as the classical part of the long-range dynamics is concerned, there is no need to impose any additional property on initial states to define it. By contrast, for the quantum part, periodicity of initial states is needed. Note, however, that the set E_{p} of all periodic states is still a weak*-dense subset of the physically relevant set E^+ of all even states, by Proposition 2.3.

<u>(ii):</u> From now on, fix $\vec{\ell} \in \mathbb{N}^d$ and consider the set $E_{\vec{\ell}}$ of $\vec{\ell}$-periodic states defined by (17). Note that any $\vec{\ell}$-periodic state $\rho \in E_{\vec{\ell}} \subseteq \mathcal{U}^*$ naturally extends to a state on $\mathfrak{U}$: There is a natural conditional expectation Ξ from $\mathfrak{U}$ to $\mathfrak{C} \subseteq \mathfrak{U}$ defined by

$$\Xi\left(f\right)\left(\rho\right) = \rho\left(f\left(\rho\right)\right) \ , \qquad \rho \in E \ .$$

Then, since $E_{\vec{\ell}}$ is a Choquet simplex, any state $\rho \in E_{\vec{\ell}}$ can be uniquely identified with its Choquet measure μ_ρ which, in turn, is a state of $\mathfrak{C}$, canonically viewed as a measure on E. The state of $\mathfrak{U}$ extending $\rho \in E_{\vec{\ell}}$ is the state $\mu_\rho \circ \Xi$. Moreover, if $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ is a cyclic representation of $\rho \in E_{\vec{\ell}}$, then, by [64, Proposition 4.2], there is a representation

$$\Pi_\rho : \mathfrak{U} \to \mathcal{B}\left(\mathcal{H}_\rho\right)$$

such that

$$\left[\Pi_\rho\left(\mathfrak{U}\right)\right]'' = \left[\Pi_\rho\left(\mathcal{U}\right)\right]'' \qquad \text{and} \qquad \Pi_\rho\left(A\right) = \pi_\rho\left(A\right)$$

for any $A \in \mathcal{U} \subseteq \mathfrak{U}$. In particular,

$$\left[\Pi_\rho\left(\mathfrak{C}\right)\right]'' \subseteq \left[\pi_\rho\left(\mathcal{U}\right)\right]' \cap \left[\pi_\rho\left(\mathcal{U}\right)\right]'' \ .$$

In fact, for $\rho \in E_{\vec{\ell}}$, $(\mathcal{H}_\rho, \Pi_\rho, \Omega_\rho)$ is a cyclic representation of the state $\mu_\rho \circ \Xi \in \mathfrak{U}^*$. The existence of such an extension of π_ρ strongly depends on the orthogonality of the ($\vec{\ell}$-) ergodic decomposition of $\vec{\ell}$-periodic states, as it is explained in [64]. The ergodicity property of extreme states of $E_{\vec{\ell}}$ is pivotal in order to get the limit dynamics stated in the third point.

<u>(iii):</u> Let $\mathfrak{m} \in C_b(\mathbb{R}; \mathcal{M}_\Lambda \cap \mathcal{M}_1)$ for some $\Lambda \in \mathcal{P}_f$. Assume that the state at initial time $s \in \mathbb{R}$ is $\rho \in E_{\vec{\ell}}$. Then, by taking the above cyclic representation $(\mathcal{H}_\rho, \Pi_\rho, \Omega_\rho)$ of ρ, seen as the state $\mu_\rho \circ \Xi$ on $\mathfrak{U}$, we show in [64, Theorem 4.3] that, for any $t \in \mathbb{R}$ and $A \in \mathcal{U} \subseteq \mathfrak{U}$, in the thermodynamic limit $L \to \infty$,

$$\Pi_\rho\left(\mathfrak{T}_{t,s}^{\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(\alpha,\cdot))}}\left(A\right)|_{\alpha=s} - \tau_{t,s}^{(L,\mathfrak{m})}\left(A\right)\right) \tag{124}$$

converges to 0 in the σ-weak topology within $\mathcal{B}(\mathcal{H}_\rho)$. In particular, the restriction to $\mathcal{U}$ of the state

$$\rho \circ \mathfrak{T}_{t,s}^{\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(\alpha,\cdot))}}|_{\alpha=s} \tag{125}$$

can be seen as the state of the system at any time $t \in \mathbb{R}$ when ρ is the (initial) state at time $t = s$. Here,

$$\boldsymbol{\varpi}^{\mathfrak{m}} \in C\left(\mathbb{R}^2; \mathrm{Aut}\left(E\right)\right)$$

results from Theorem 6.5, $\mathbf{\Phi}^{(\mathfrak{m},\boldsymbol{\xi})}$ is the state-dependent interaction of Definition 6.3 for

$$\boldsymbol{\xi} \in C\left(\mathbb{R}; \mathrm{Aut}\left(E\right)\right) \ ,$$

$(\mathfrak{T}_{t,s}^{\mathbf{\Psi}})_{s,t\in\mathbb{R}}$ is the strongly continuous two-parameter family of $*$-automorphisms of $\mathfrak{U}$ of Proposition 6.2 for $\mathbf{\Psi} \in C(\mathbb{R};\mathfrak{W}^{\mathbb{R}})$, and $(\tau_t^{(L,\mathfrak{m})})_{t\in\mathbb{R}}$ is the strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{U}$ defined by (62). Note that, even if the local dynamics is autonomous, the limit dynamics can still be *non-autonomous*.

A similar result holds true for non-autonomous long-range dynamics, i.e., for all time-dependent models

$$\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda \cap \mathcal{M}_1) \ , \qquad \Lambda \in \mathcal{P}_f \ .$$

In this case, one replaces the autonomous local dynamics in (124) with the non-autonomous local one, similar to (39)-(40).

7 Technical Proofs

The aim of this section is to prove Theorems 6.5 and 6.10. In fact, we prove here stronger results than these theorems. The proof of Theorem 6.5 is done in five lemmata and two corollaries. The proof of Theorem 6.10 is a direct consequence of Lemma 7.13. Note that those proofs are a much more involved version of the ones performed in [63, Section 7] to prove [63, Theorems 4.1 and 4.6].

We start with preliminary definitions: In all the present section, fix once and for all $\vec{\ell} \in \mathbb{N}^d$, $\Lambda \in \mathcal{P}_f$ and a time-dependent model

$$\mathfrak{m} = ((\Phi(t), \mathfrak{a}(t)))_{t\in\mathbb{R}} \in C_b(\mathbb{R};\mathcal{M}_\Lambda) \ , \qquad \text{with} \qquad \mathcal{M}_\Lambda \doteq \mathcal{W}^{\mathbb{R}} \times \mathcal{S}_\Lambda \subseteq \mathcal{M} \ ,$$

(see (57)), $\mathcal{S}_\Lambda \subseteq \mathcal{S}$ being defined by (58). By (32)-(33), we can assume without loss of generality that $\mathfrak{e}_{\Phi,\vec{\ell}} \in \mathcal{U}_\Lambda$. In order to simplify mathematical expressions, we use the standard notation

$$\|\mathfrak{m}\|_\infty \equiv \|\mathfrak{m}\|_{C_b(\mathbb{R};\mathcal{M})} \doteq \sup_{t\in\mathbb{R}} \|\mathfrak{m}(t)\|_{\mathcal{M}} \ , \qquad \mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda) \ . \tag{126}$$

Recall that $E_\Lambda \subseteq \mathcal{U}_\Lambda^*$, which is defined by (10), is the norm-compact set of states on the finite-dimensional C^*-algebra $\mathcal{U}_\Lambda$. For every continuous function $\zeta \in C(\mathbb{R}; E_\Lambda)$, let

$$\Psi^\zeta(t) \doteq \Phi(t) + \sum_{n\in\mathbb{N}} \int_{(\mathbb{S}\cap\mathcal{W}_\Lambda)^n} \left\lfloor \zeta(t); \Psi^{(1)}, \ldots, \Psi^{(n)} \right\rfloor_{\vec{\ell}} \mathfrak{a}(t)_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right) \ , \qquad t \in \mathbb{R} \ , \tag{127}$$

with $\mathcal{W}_\Lambda \subseteq \mathcal{W}_1$ being defined by (31). In particular, by Definition 6.3 and Equation (58), for all continuous functions $\xi \in C(\mathbb{R}; E)$,

$$\Psi^{\xi|_{\mathcal{U}_\Lambda}}(t) = \Phi^{(\mathfrak{m},\xi)}(t) \ , \qquad t \in \mathbb{R} \ . \tag{128}$$

Note that such approximating interactions can be used to define a strongly continuous two-parameter family $(\tau_{t,s}^{\Psi^\zeta})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{U}$ for any $\zeta \in C(\mathbb{R}; E_\Lambda)$, by Proposition 3.7 and Lemma 6.4. Such approximating dynamics satisfy the following estimate:

Lemma 7.1 (Estimates on approximating dynamics)
For any $s,t \in \mathbb{R}$, $\tilde{\Lambda} \in \mathcal{P}_f$, $A \in \mathcal{U}_{\tilde{\Lambda}}$ and $\zeta_1, \zeta_2 \in C(\mathbb{R}; E_\Lambda) \subseteq C(\mathbb{R};\mathcal{U}_\Lambda^*)$,

$$\left\| \left(\tau_{t,s}^{\Psi^{\zeta_1}} - \tau_{t,s}^{\Psi^{\zeta_2}}\right)(A) \right\|_{\mathcal{U}} \leq 2|\tilde{\Lambda}| \, \|A\|_{\mathcal{U}} \, \|\mathbf{F}\|_{1,\mathfrak{L}} \, \|\mathfrak{m}\|_\infty \, \mathrm{e}^{2\mathbf{D}\|\mathfrak{m}\|_\infty |t-s|} \int_{t\wedge s}^{t\vee s} \|\zeta_1(\alpha) - \zeta_2(\alpha)\|_{\mathcal{U}_\Lambda^*} \mathrm{d}\alpha \ .$$

Here, $\mathcal{U}_\Lambda^*$ is endowed with the usual norm for continuous linear functionals.

Proof. Similar to Inequality (111), by Equations (30), (54)-(55) and (128), note that, for any $\zeta_1, \zeta_2 \in C(\mathbb{R}; E_\Lambda)$,

$$\left\| \Psi^{\zeta_1}(\alpha) - \Psi^{\zeta_2}(\alpha) \right\|_{\mathcal{W}} \leq \| \mathfrak{m}(\alpha) \|_{\mathcal{M}} \| \zeta_1(\alpha) - \zeta_2(\alpha) \|_{\mathcal{U}_\Lambda^*} , \qquad \alpha \in \mathbb{R} .$$

Combining this inequality with Proposition 3.8 (iii) and Lemma 6.4, we obtain the assertion. Note that $C(\mathbb{R}; E_\Lambda) \subseteq C(\mathbb{R}; \mathcal{U}_\Lambda^*)$, because the norm and weak* topologies of $\mathcal{U}_\Lambda^*$ are the same, by finite dimensionality of $\mathcal{U}_\Lambda$ for $\Lambda \in \mathcal{P}_f$. ■

We now show the existence and uniqueness of the solution to the self-consistency equation:

Lemma 7.2 (Self-consistency equations)
For any $s \in \mathbb{R}$ and $\rho \in E$, there is a unique solution $\varpi_{\rho,s}$ to the following equation in $\xi \in C(\mathbb{R}; E)$:

$$\forall t \in \mathbb{R} : \qquad \xi(t) = \rho \circ \tau_{t,s}^{\Psi^{\xi|_{\mathcal{U}_\Lambda}}} . \tag{129}$$

Moreover, $\varpi_{\rho,s}(t) = \varpi_{\varpi_{\rho,s}(r),r}(t)$ for any $r, s, t \in \mathbb{R}$.

Proof. The proof is similar to the one of [63, Lemma 7.3]: Fix the initial time $s \in \mathbb{R}$ and state $\rho \in E$. The existence and uniqueness of a solution $\varpi_{\rho,s}$ to (129) is proven via the Banach fixed point theorem:

Step 1: Fix $T \in \mathbb{R}^+$ and observe that $C([s-T, s+T]; E_\Lambda)$ is a closed bounded subset of the Banach space $C([s-T, s+T]; \mathcal{U}_\Lambda^*)$, with $\mathcal{U}_\Lambda^*$ being endowed with the usual norm for continuous linear functionals and

$$\| \zeta \|_{C([s-T,s+T];\mathcal{U}_\Lambda^*)} \doteq \sup_{t \in [s-T,s+T]} \| \zeta(t) \|_{\mathcal{U}_\Lambda^*} , \qquad \zeta \in C([s-T, s+T]; \mathcal{U}_\Lambda^*) .$$

Define the mapping $\mathfrak{F}$ from $C([s-T, s+T]; E_\Lambda)$ to itself by

$$\mathfrak{F}(\zeta)(t) \doteq \rho \circ \tau_{t,s}^{\Psi^\zeta} \Big|_{\mathcal{U}_\Lambda} , \qquad t \in [s-T, s+T] . \tag{130}$$

The existence of such a $*$-automorphism $\tau_{t,s}^{\Psi^\zeta}$ for any $\zeta \in C([s-T, s+T]; E_\Lambda)$ and $t \in [s-T, s+T]$ follows from Proposition 3.7 and Lemma 6.4. By Proposition 3.7 (or Proposition 3.8 (iv)), (130) defines a mapping from $C([s-T, s+T]; E_\Lambda)$ to itself. Moreover, by (127) and (130), we infer from Lemma 7.1 that, for any $\zeta_1, \zeta_2 \in C([s-T, s+T]; E_\Lambda)$,

$$\begin{aligned} & \| \mathfrak{F}(\zeta_1)(t) - \mathfrak{F}(\zeta_2)(t) \|_{C([s-T,s+T];\mathcal{U}_\Lambda^*)} \\ \leq \; & 4T |\Lambda| \| \mathbf{F} \|_{1,\mathfrak{L}} \| \mathfrak{m} \|_\infty \mathrm{e}^{4\mathbf{D}T\|\mathfrak{m}\|_\infty} \| \zeta_1 - \zeta_2 \|_{C([s-T,s+T];\mathcal{U}_\Lambda^*)} . \end{aligned} \tag{131}$$

Therefore, by fixing the time parameter $T \in \mathbb{R}^+$ such that

$$T \leq \frac{\mathrm{e}^{-4\mathbf{D}T\|\mathfrak{m}\|_\infty}}{8 |\Lambda| \| \mathbf{F} \|_{1,\mathfrak{L}} \| \mathfrak{m} \|_\infty} , \tag{132}$$

the function $\mathfrak{F}$ is a contraction. Hence, we obtain a unique solution $\daleth_{\rho,s} \in C([s-T, s+T]; E_\Lambda)$ to Equation (129) with $\xi|_{\mathcal{U}_\Lambda} = \daleth_{\rho,s}$ at fixed $s \in \mathbb{R}$ and $\rho \in E$.

Step 2: By (128), the restriction of any solution $\varpi_{\rho,s} \in C([s-T, s+T]; E)$ to (129) to the subspace $\mathcal{U}_\Lambda \subseteq \mathcal{U}$ must equal $\daleth_{\rho,s} \in C([s-T, s+T]; E_\Lambda)$ and $\Phi^{(\mathfrak{m},\varpi_{\rho,s}(t))} = \Psi^{\daleth_{\rho,s}}(t)$. With this observation, we see that

$$\varpi_{\rho,s}(t) \doteq \rho \circ \tau_{t,s}^{\Psi^{\daleth_{\rho,s}}} , \qquad t \in [s-T, s+T] ,$$

is the unique solution in $C([s-T,s+T];E)$ to (129) at fixed initial time $s\in\mathbb{R}$ and state $\rho\in E$.

Step 3: In the same way we prove the existence and uniqueness of a solution to (129) at fixed $s\in\mathbb{R}$ and $\rho\in E$, one shows that, for each $r\in[s-T,s+T]$, the self-consistency equation

$$\forall t\in[r-\tilde{T},r+\tilde{T}]:\qquad \xi(t)=\varpi_{\rho,s}(r)\circ\tau_{t,r}^{\Psi^{\xi|\mathcal{U}_\Lambda}}\ , \tag{133}$$

has also a unique solution $\varpi_{\varpi_{\rho,s}(r),r}$ in $C([r-\tilde{T},r+\tilde{T}];E)$ for any $\tilde{T}\in(0,T]$. By the reverse cocycle property (42), at fixed $s\in\mathbb{R}$ and $\rho\in E$, $\varpi_{\rho,s}$ solves the self-consistency equation (133) for any $r\in(s-T,s+T)$ and $t\in[s-\tilde{T},s+\tilde{T}]$ with $\tilde{T}=T-|s-r|\in\mathbb{R}^+$, whence

$$\varpi_{\rho,s}(t)=\varpi_{\varpi_{\rho,s}(r),r}(t)$$

for any $r\in(s-T,s+T)$ and $t\in[s-\tilde{T},s+\tilde{T}]$.

Step 4: Assume the existence and uniqueness of a solution $\varpi_{\rho,s}$ in $C([s-T_0,s+T_0];E)$ to Equation (129) for some parameter $T_0\in\mathbb{R}^+$. Take

$$r\in(s-T_0,s-T_0+T)\cup(s+T_0-T,s+T_0)\ .$$

By combining the existence and uniqueness of a solution $\varpi_{\varpi_{\rho,s}(r),r}$ to (133) in $C([r-\tilde{T},r+\tilde{T}];E)$ for any $\tilde{T}\in(0,T]$ together with the reverse cocycle property (42), we deduce that

$$\varpi_{\rho,s}(t)=\varpi_{\varpi_{\rho,s}(r),r}(t)\ ,\qquad t\in(s-T_0,s+T_0)\ ,$$

as well as the existence of a unique solution $\varpi_{\rho,s}$ to (129) in $C([s-T_0-T,s+T_0+T];E)$. As a consequence, one can infer from a contradiction argument the existence and uniqueness of a solution in $\xi\in C(\mathbb{R};E)$ to the self-consistency equation (129). Moreover, this solution must satisfy the equality $\varpi_{\rho,s}(t)=\varpi_{\varpi_{\rho,s}(r),r}(t)$ for any $r,s,t\in\mathbb{R}$. ■

Corollary 7.3 (Bijectivity of the solution to the self-consistency equation)
For any $s,t\in\mathbb{R}$, $\varpi_s(t)\equiv(\varpi_{\rho,s}(t))_{\rho\in E}$ is a bijective mapping from E to itself.

Proof. The assertion is a direct consequence of Lemma 7.2, in particular the equality $\varpi_{\rho,s}(t)=\varpi_{\varpi_{\rho,s}(r),r}(t)$ for $r,s,t\in\mathbb{R}$. ■

By combining Lemma 7.2 with Proposition 3.7, note that, for any $s\in\mathbb{R}$, $\rho\in E$ and $A\in\mathcal{U}_0$,

$$\partial_t\{\varpi_{\rho,s}(t)(A)\}=\rho\circ\tau_{t,s}^{\Psi^{\daleth}}\circ\delta^{\Psi^{\daleth(t)}}(A)\ ,\qquad t\in\mathbb{R}\ ,$$

with $\daleth\doteq\varpi_{\rho,s}|_{\mathcal{U}_\Lambda}$. This property can be strengthened within the local C^*-algebra $\mathcal{U}_\Lambda$:

Lemma 7.4 (Differentiability of the solution – I)
For any $s\in\mathbb{R}$ and $\rho\in E$, $\daleth\doteq\varpi_{\rho,s}|_{\mathcal{U}_\Lambda}\in C^1(\mathbb{R};\mathcal{U}_\Lambda^*)$ with derivative given by

$$\partial_t\{\varpi_{\rho,s}(t)|_{\mathcal{U}_\Lambda}\}=\rho\circ\tau_{t,s}^{\Psi^{\daleth}}\circ\delta^{\Psi^{\daleth(t)}}|_{\mathcal{U}_\Lambda}\ ,\qquad t\in\mathbb{R}\ .$$

Here, $\mathcal{U}_\Lambda^*$ is endowed with the usual norm for continuous linear functionals.

Proof. To prove that $\daleth\doteq\varpi_{\rho,s}|_{\mathcal{U}_\Lambda}\in C^1(\mathbb{R};\mathcal{U}_\Lambda^*)$ at fixed $s\in\mathbb{R}$ and $\rho\in E$, we first remark that, for any $A\in\mathcal{U}_\Lambda$ and $h\in\mathbb{R}\backslash\{0\}$,

$$\begin{aligned}&\left|h^{-1}\left(\rho\circ\tau_{t+h,s}^{\Psi^{\daleth}}(A)-\rho\circ\tau_{t,s}^{\Psi^{\daleth}}(A)\right)-\rho\circ\tau_{t,s}^{\Psi^{\daleth}}\circ\delta^{\Psi^{\daleth(t)}}(A)\right| \\ \leq\ & \max_{\alpha\in[t-h,t+h]}\left\|\delta^{\Psi^{\daleth(\alpha)}-\Psi^{\daleth(t)}}(A)\right\|_{\mathcal{U}}+\max_{\alpha\in[t-h,t+h]}\left\|\left(\tau_{\alpha,s}^{\Psi^{\daleth}}-\tau_{t,s}^{\Psi^{\daleth}}\right)\circ\delta^{\Psi^{\daleth(t)}}(A)\right\|_{\mathcal{U}}\ .\end{aligned} \tag{134}$$

Since $\daleth \in C(\mathbb{R}; E_\Lambda)$, by Corollary 3.5, for any $A \in \mathcal{U}_\Lambda$ with $\|A\|_\mathcal{U} = 1$,

$$\max_{\alpha\in[t-h,t+h]} \left\| \delta^{\Psi^{\daleth(\alpha)}-\Psi^{\daleth(t)}}(A) \right\|_\mathcal{U} \leq 2|\Lambda| \|\mathbf{F}\|_{1,\mathfrak{L}} \max_{\alpha\in[t-h,t+h]} \left\| \Psi^{\daleth(\alpha)} - \Psi^{\daleth(t)} \right\|_\mathcal{W}$$

and hence, by using Lemma 6.4, we arrive at

$$\lim_{h\to 0} \sup_{A\in\mathcal{U}_\Lambda, \|A\|_\mathcal{U}=1} \max_{\alpha\in[t-h,t+h]} \left\| \delta^{\Psi^{\daleth(\alpha)}-\Psi^{\daleth(t)}}(A) \right\|_\mathcal{U} = 0 . \tag{135}$$

Meanwhile, by Proposition 3.4, for any $A \in \mathcal{U}_\Lambda$ satisfying $\|A\|_\mathcal{U} = 1$,

$$\begin{aligned}\left\| \left(\tau_{\alpha,s}^{\Psi^\daleth} - \tau_{t,s}^{\Psi^\daleth}\right) \circ \delta^{\Psi^{\daleth(t)}}(A) \right\|_\mathcal{U} \leq \left\| \left(\tau_{\alpha,s}^{\Psi^\daleth} - \tau_{t,s}^{\Psi^\daleth}\right) \circ \delta_L^{\Psi^{\daleth(t)}}(A) \right\|_\mathcal{U} & \\ + 2|\Lambda| \left\| \Psi^{\daleth(t)} \right\|_\mathcal{W} \sup_{y\in\Lambda} \sum_{x\in\Lambda_L^c} \mathbf{F}(x,y) & .\end{aligned}$$

Thus, by (24), for any fixed $\varepsilon \in \mathbb{R}^+$, there is $L \in \mathbb{N}$ such that, for any $A \in \mathcal{U}_\Lambda$ with $\|A\|_\mathcal{U} = 1$,

$$\max_{\alpha\in[t-h,t+h]} \left\| \left(\tau_{\alpha,s}^{\Psi^\daleth} - \tau_{t,s}^{\Psi^\daleth}\right) \circ \delta^{\Psi^{\daleth(t)}}(A) \right\|_\mathcal{U} \leq \max_{\alpha\in[t-h,t+h]} \left\| \left(\tau_{\alpha,s}^{\Psi^\daleth} - \tau_{t,s}^{\Psi^\daleth}\right) \circ \delta_L^{\Psi^{\daleth(t)}}(A) \right\|_\mathcal{U} + \varepsilon ,$$

while we obtain from Proposition 3.8 (iv) that

$$\lim_{h\to 0} \sup_{A\in\mathcal{U}_\Lambda, \|A\|_\mathcal{U}=1} \max_{\alpha\in[t-h,t+h]} \left\| \left(\tau_{\alpha,s}^{\Psi^\daleth} - \tau_{t,s}^{\Psi^\daleth}\right) \circ \delta_L^{\Psi^{\daleth(t)}}(A) \right\|_\mathcal{U} = 0 .$$

It follows that

$$\lim_{h\to 0} \sup_{A\in\mathcal{U}_\Lambda, \|A\|_\mathcal{U}=1} \max_{\alpha\in[t-h,t+h]} \left\| \left(\tau_{\alpha,s}^{\Psi^\daleth} - \tau_{t,s}^{\Psi^\daleth}\right) \circ \delta^{\Psi^{\daleth(t)}}(A) \right\|_\mathcal{U} = 0 .$$

We finally combine the last limit with (134)-(135) to deduce that $\daleth \in C^1(\mathbb{R}; \mathcal{U}_\Lambda^*)$ with derivative given by

$$\partial_t \left\{ \varpi_{\rho,s}(t)|_{\mathcal{U}_\Lambda} \right\} = \partial_t \daleth(t) = \rho \circ \tau_{t,s}^{\Psi^\daleth} \circ \delta^{\Psi^{\daleth(t)}}|_{\mathcal{U}_\Lambda} , \qquad t \in \mathbb{R} ,$$

at any fixed $s \in \mathbb{R}$ and $\rho \in E$. ■

Lemma 7.5 (Continuity with respect to the initial condition)
For any $s, t \in \mathbb{R}$, $\varpi_s(t) \equiv (\varpi_{\rho,s}(t))_{\rho\in E} \in C(E; E)$.

Proof. Take $s \in \mathbb{R}$ and two states $\rho_1, \rho_2 \in E$. Then, define the quantity

$$\mathbf{X}(T) \doteq \max_{t\in[s-T,s+T]} \left\| \left(\varpi_{\rho_1,s}(t) - \varpi_{\rho_2,s}(t)\right)|_{\mathcal{U}_\Lambda} \right\|_{\mathcal{U}_\Lambda^*} , \qquad T \in \mathbb{R}^+ . \tag{136}$$

By Proposition 3.8 (ii) and Lemma 6.4 together with Equations (24) and (128), for any $T, \varepsilon \in \mathbb{R}^+$, there is $L \in \mathbb{N}$ such that, for any $A \in \mathcal{U}_\Lambda$ with $\|A\|_\mathcal{U} = 1$,

$$\sup_{t\in[s-T,s+T]} \sup_{\zeta\in C(\mathbb{R};E_\Lambda)} \left\| \tau_{t,s}^{\Psi^\zeta}(A) - \tau_{t,s}^{(L,\Psi^\zeta)}(A) \right\|_\mathcal{U} \leq \varepsilon . \tag{137}$$

By combining Lemmata 7.1 (for $\mathfrak{L} = \Lambda_L$) and 7.2 with (137), we thus obtain the bound

$$\mathbf{X}(T) \leq 2\varepsilon + \left\| (\rho_1 - \rho_2)|_{\mathcal{U}_{\Lambda_L}} \right\|_{\mathcal{U}_{\Lambda_L}^*} + 2|\Lambda| \|\mathbf{F}\|_{1,\mathfrak{L}} \|\mathfrak{m}\|_\infty \mathrm{e}^{2\mathbf{D}T\|\mathfrak{m}\|_\infty} \int_0^T \mathbf{X}(\alpha)\,\mathrm{d}\alpha ,$$

where $\|\mathfrak{m}\|_{\infty}$ is defined by (126). By Grönwall's inequality,

$$\mathbf{X}(T) \leq \left(2\varepsilon + \left\|(\rho_1 - \rho_2)|_{\mathcal{U}_{\Lambda_L}}\right\|_{\mathcal{U}^*_{\Lambda_L}}\right)\left(1 + 2|\Lambda| \|\mathbf{F}\|_{1,\mathfrak{L}} \|\mathfrak{m}\|_{\infty} \mathrm{e}^{2\mathbf{D}T\|\mathfrak{m}\|_{\infty}} \mathbf{Y}(T)\right) \tag{138}$$

with

$$\mathbf{Y}(T) \doteq \int_0^T \exp\left\{2|\Lambda| \|\mathbf{F}\|_{1,\mathfrak{L}} \|\mathfrak{m}\|_{\infty} \mathrm{e}^{2\mathbf{D}T\|\mathfrak{m}\|_{\infty}} \alpha\right\} \mathrm{d}\alpha \,. \tag{139}$$

By finite dimensionality of $\mathcal{U}_{\Lambda_L}$, the norm and weak* topologies of $\mathcal{U}^*_{\Lambda_L}$ are the same and, by the weak* continuity property of $\varpi_s(t)$, we infer from (136) and (138)-(139) that

$$(\varpi_{\rho,s}(t)|_{\mathcal{U}_\Lambda})_{\rho\in E} \in C(E;\mathcal{U}^*_\Lambda) \ , \qquad s,t\in\mathbb{R} \,. \tag{140}$$

The continuity is even uniform for times t in compact sets. Now, by Lemma 7.2, for any $s,t \in \mathbb{R}$, $\rho_1,\rho_2 \in E$ and $A \in \mathcal{U}$, one obviously gets from the triangle inequality that

$$\begin{aligned} &\left|\left(\varpi_{\rho_1,s}(t) - \varpi_{\rho_2,s}(t)\right)(A)\right| \\ \leq\ &\left|(\rho_1-\rho_2)\circ\tau_{t,s}^{\Psi^{\varpi_{\rho_1,s}|\mathcal{U}_\Lambda}}(A)\right| + \left\|\left(\tau_{t,s}^{\Psi^{\varpi_{\rho_1,s}|\mathcal{U}_\Lambda}} - \tau_{t,s}^{\Psi^{\varpi_{\rho_2,s}|\mathcal{U}_\Lambda}}\right)(A)\right\|_{\mathcal{U}} \,. \end{aligned} \tag{141}$$

Combined with Lemma 7.1, Lebesgue's dominated convergence theorem and Equation (140), this last inequality leads to

$$(\varpi_{\rho,s}(t)(A))_{\rho\in E} \in C(E;\mathbb{C}) \ , \qquad A\in\mathcal{U}_0 \,.$$

By density of $\mathcal{U}_0 \subseteq \mathcal{U}$ and the fact that any state $\rho \in E$ satisfies $\|\rho\|_{\mathcal{U}^*} = 1$, the assertion follows. ∎

Corollary 7.6 (Solution to the self-consistency equation as self-homeomorphisms)
At fixed $s,t\in\mathbb{R}$, $\varpi_s(t) \equiv (\varpi_{\rho,s}(t))_{\rho\in E} \in \mathrm{Aut}(E)$, i.e., $\varpi_s(t)$ is an automorphism of the state space E. Moreover, it satisfies a cocycle property:

$$\forall s,r,t\in\mathbb{R}: \qquad \varpi_s(t) = \varpi_r(t)\circ\varpi_s(r) \,. \tag{142}$$

Proof. The proof is the same as the one of [63, Corollary 7.7]: By Corollary 7.3, for any $s,t \in \mathbb{R}$, $\varpi_s(t)$ is a weak*-continuous bijective mapping from E to itself. Since E is weak*-compact, its inverse is also weak*-continuous. The cocycle property is a rewriting of the equality $\varpi_{\rho,s}(t) = \varpi_{\varpi_{\rho,s}(r),r}(t)$ of Lemma 7.2. ∎

Before stating the next lemma, recall that the topology used in the subspace $\mathrm{Aut}(E) \subsetneq C(E;E)$ of all automorphisms of E is the one of uniform convergence of weak*-continuous functions, as stated in Equation (100).

Lemma 7.7 (Well-posedness of the self-consistency equation)
For any $s\in\mathbb{R}$,

$$\boldsymbol{\varpi}_s^{\mathfrak{m}} \equiv (\varpi_s(t))_{t\in\mathbb{R}} \equiv ((\varpi_{\rho,s}(t))_{\rho\in E})_{t\in\mathbb{R}} \in C(\mathbb{R};\mathrm{Aut}(E)) \,.$$

At fixed $s\in\mathbb{R}$, the mapping $t\mapsto\varpi_s(t)$ is uniformly continuous for times t in compact sets, i.e., for any $T,\varepsilon\in\mathbb{R}^+$ and $A\in\mathcal{U}$ there is $\eta\in\mathbb{R}^+$ such that, for all $t_1,t_2\in[-T,T]$ with $|t_1-t_2|\leq\eta$,

$$\max_{\rho\in E}|\varpi_s(t_1)(A) - \varpi_s(t_2)(A)| \leq \varepsilon \,.$$

Proof. The proof is not exactly the same as the one of [63, Lemma 7.8], but it is similar: Take any sequence $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converging to $t\in\mathbb{R}$. Assume that $\varpi_s(t_n)$ does not converge to $\varpi_s(t)$, uniformly. So, by density of $\mathcal{U}_0$, there are $(\rho_n)_{n\in\mathbb{N}} \subseteq E$, $k \in \mathbb{N}$, $L \in \mathbb{N}$, $\varepsilon_1, \ldots, \varepsilon_k \in \mathbb{R}^+$ and $A_1, \ldots, A_k \in \mathcal{U}_{\Lambda_L}$ such that

$$\liminf_{n\to\infty} \left| \left[\varpi_{\rho_n,s}(t_n) - \varpi_{\rho_n,s}(t) \right](A_j) \right| \geq \varepsilon_j > 0 \ , \qquad j \in \{1, \ldots, k\} \ . \tag{143}$$

By weak*-compactness and metrizability of E, we can assume without loss of generality that the sequence $(\rho_n)_{n\in\mathbb{N}}$ weak*-converges to some $\rho \in E$. By Lemma 7.2 and Equation (143), this in turn implies that

$$\liminf_{n\to\infty} \left| \left[\rho_n \circ \tau_{t_n,s}^{\Psi^{\varpi_{\rho_n,s}|\mathcal{U}_\Lambda}} - \varpi_{\rho,s}(t) \right](A_j) \right| \geq \varepsilon_j > 0 \ , \qquad j \in \{1, \ldots, k\} \ , \tag{144}$$

since $\varpi_s(t) \in C(E; E)$. Using Lemma 7.1, Lebesgue's dominated convergence theorem and Equation (140), we obtain from (144) that

$$\liminf_{n\to\infty} \left| \left[\rho_n \circ \tau_{t_n,s}^{\Psi^{\varpi_{\rho,s}|\mathcal{U}_\Lambda}} - \varpi_{\rho,s}(t) \right](A_j) \right| \geq \varepsilon_j > 0 \ , \qquad j \in \{1, \ldots, k\} \ . \tag{145}$$

But this is a contradiction because $(\tau_{t,s}^{\Psi})_{s,t\in\mathbb{R}}$ is a strongly continuous two-parameter family (Proposition 3.7) and hence,

$$\lim_{n\to\infty} \rho_n \circ \tau_{t_n,s}^{\Psi^{\varpi_{\rho,s}|\mathcal{U}_\Lambda}}(A_j) = \rho \circ \tau_{t,s}^{\Psi^{\varpi_{\rho,s}|\mathcal{U}_\Lambda}}(A_j) = [\varpi_{\rho,s}(t)](A_j)$$

for all $j \in \{1, \ldots, k\}$. By (136) and (138)-(139) taken for some arbitrarily large (but finite) cubic box $\Lambda_L \supseteq \Lambda$, note that $(\varpi_{\rho,s}(t)|_{\mathcal{U}_\Lambda})_{\rho\in E} \in C\left(E; \mathcal{U}_{\Lambda_L}^*\right)$ is uniformly continuous for times t in compact sets and, as $n \to \infty$,

$$\left(\tau_{t,s}^{\Psi^{\varpi_{\rho_n,s}|\mathcal{U}_\Lambda}} - \tau_{t,s}^{\Psi^{\varpi_{\rho,s}|\mathcal{U}_\Lambda}}\right)(A) \ , \qquad A \in \mathcal{U}_0 \ ,$$

converges to $0 \in \mathcal{U}$ uniformly with respect to times t in compact sets, by Lemma 7.1. It means that the mapping $t \mapsto \varpi_s(t)$ is in fact uniformly continuous for times t in compact sets, at fixed $s \in \mathbb{R}$. ■

Lemma 7.8 (Joint continuity with respect to initial and final times)
The solution to the self-consistency equation is jointly continuous with respect to initial and final times:

$$\boldsymbol{\varpi}^{\mathfrak{m}} \equiv (\boldsymbol{\varpi}_s^{\mathfrak{m}})_{s\in\mathbb{R}} \equiv (\varpi_s(t))_{s,t\in\mathbb{R}} \equiv ((\varpi_{\rho,s}(t))_{\rho\in E})_{s,t\in\mathbb{R}} \in C\left(\mathbb{R}^2; \mathrm{Aut}(E)\right) \ .$$

Proof. The proof is not the same as the one of [63, Lemma 7.9], but it is similar: Fix $\rho \in E$, $s \in \mathbb{R}$ and $T \in \mathbb{R}^+$ and remark that $C([s-T, s+T]^2; E_\Lambda)$ is a closed bounded subset of the Banach space $C([s-T, s+T]^2; \mathcal{U}_\Lambda^*)$, where

$$\|\zeta\|_{C\left([s-T,s+T]^2;\mathcal{U}_\Lambda^*\right)} \doteq \sup_{\alpha,t\in[s-T,s+T]} \|\zeta(\alpha,t)\|_{\mathcal{U}_\Lambda^*} \ , \qquad \zeta \in C\left([s-T, s+T]^2; \mathcal{U}_\Lambda^*\right) \ .$$

Similar to (130), we define the mapping $\mathfrak{F}$ from $C([s-T, s+T]^2; E_\Lambda)$ to itself by

$$\mathfrak{F}(\zeta)(\alpha, t) \doteq \rho \circ \tau_{t,\alpha}^{\Psi^\zeta}|_{\mathcal{U}_\Lambda} \ , \qquad \alpha, t \in [s-T, s+T] \ .$$

See Proposition 3.7 and Lemma 6.4. By Inequality (131), $\mathfrak{F}$ is also a contraction when the time $T \in \mathbb{R}^+$ satisfies (132). Hence, in this case, for any $\rho \in E$ and $s \in \mathbb{R}$, there is a unique

$$\tilde{\mathfrak{I}} \in C^2\left([s-T, s+T]^2; E_\Lambda\right) \tag{146}$$

such that

$$\forall \alpha, t \in [s-T, s+T]: \qquad \tilde{\mathfrak{I}}(\alpha, t) = \rho \circ \tau_{t,\alpha}^{\Psi^{\tilde{\mathfrak{I}}(\alpha,\cdot)}}|_{\mathcal{U}_\Lambda} \ .$$

Now, by uniqueness of the solution to (129),

$$\varpi_{\rho,\alpha}(t)|_{\mathcal{U}_\Lambda} = \tilde{\mathfrak{J}}(\alpha,t) = \rho\circ\tau_{t,\alpha}^{\Psi^{\tilde{\mathfrak{J}}(\alpha,\cdot)}}|_{\mathcal{U}_\Lambda} \tag{147}$$

for any $\alpha,t\in[s-T,s+T]$. Similar to (141), for any $A\in\mathcal{U}$ and $\alpha_1,\alpha_2,t_1,t_2\in[s-T,s+T]^2$, note that

$$\begin{aligned}|(\varpi_{\rho,\alpha_1}(t_1)-\varpi_{\rho,\alpha_2}(t_2))(A)| \leq{}& \left\|\left(\tau_{t_1,\alpha_1}^{\Psi^{\tilde{\mathfrak{J}}(\alpha_1,\cdot)}}-\tau_{t_2,\alpha_1}^{\Psi^{\tilde{\mathfrak{J}}(\alpha_1,\cdot)}}\right)(A)\right\|_{\mathcal{U}}\\ &+\left\|\left(\tau_{t_2,\alpha_1}^{\Psi^{\tilde{\mathfrak{J}}(\alpha_1,\cdot)}}-\tau_{t_2,\alpha_2}^{\Psi^{\tilde{\mathfrak{J}}(\alpha_1,\cdot)}}\right)(A)\right\|_{\mathcal{U}}+\left\|\left(\tau_{t_2,\alpha_2}^{\Psi^{\tilde{\mathfrak{J}}(\alpha_1,\cdot)}}-\tau_{t_2,\alpha_2}^{\Psi^{\tilde{\mathfrak{J}}(\alpha_2,\cdot)}}\right)(A)\right\|_{\mathcal{U}}.\end{aligned}$$

Using this elementary inequality together with Proposition 3.8 (iv), Lemmata 6.4, 7.1 and Equations (146)-(147), one gets that, for any local element $A\in\mathcal{U}_0$, the mapping

$$(\alpha,t)\mapsto\varpi_{\rho,\alpha}(t)(A)$$

from $[s-T,s+T]^2$ to $\mathbb{C}$ is continuous. By density of $\mathcal{U}_0\subsetneq\mathcal{U}$, we deduce that

$$(\varpi_{\rho,\alpha}(t))_{\alpha,t\in\mathbb{R}}\in C\left([s-T,s+T]^2;E\right), \qquad \rho\in E,\ s\in\mathbb{R},$$

the parameter $T\in\mathbb{R}^+$ satisfying (132). Via Corollary 7.6 and Lemma 7.7, we then deduce that

$$(\varpi_{\rho,s}(t))_{s,t\in\mathbb{R}}\in C\left(\mathbb{R}^2;E\right), \qquad \rho\in E. \tag{148}$$

To get the assertion, it only remains to reproduce the compactness argument performed in the proof of Lemma 7.7: Take two sequences $(s_n)_{n\in\mathbb{N}},(t_n)_{n\in\mathbb{N}}\subseteq\mathbb{R}$ converging to $s,t\in\mathbb{R}$, respectively. Assume that $\varpi_{s_n}(t_n)$ does not converge to $\varpi_s(t)$, uniformly, which corresponds to have Equation (144), the term $\tau_{t_n,s}^{\Psi^{\varpi_{\rho_n,s}|\mathcal{U}_\Lambda}}$ being replaced with $\tau_{t_n,s_n}^{\Psi^{\varpi_{\rho_n,s_n}|\mathcal{U}_\Lambda}}$. Thanks to the triangle inequality and Lemma 7.1, for any $L\in\mathbb{N}$, $A\in\mathcal{U}_{\Lambda_L}$, sufficiently large $T\in\mathbb{R}^+$ and $n\in\mathbb{N}$ such that $s,t\in(-T,T)$ and $s_n,t_n\in[-T,T]$,

$$\begin{aligned}\left\|\left(\tau_{t_n,s_n}^{\Psi^{\varpi_{\rho_n,s_n}|\mathcal{U}_\Lambda}}-\tau_{t_n,s_n}^{\Psi^{\varpi_{\rho,s}|\mathcal{U}_\Lambda}}\right)(A)\right\|_{\mathcal{U}} \leq{}& 2|\Lambda_L|\,\|A\|_{\mathcal{U}}\,\|\mathbf{F}\|_{1,\mathfrak{L}}\,\|\mathfrak{m}\|_\infty\,\mathrm{e}^{4\mathbf{D}T\|\mathfrak{m}\|_\infty}\\ &\times\int_{-T}^{T}\left(\left\|(\varpi_{\rho_n,s_n}(\alpha)-\varpi_{\rho,s_n}(\alpha))|_{\mathcal{U}_\Lambda}\right\|_{\mathcal{U}_\Lambda^*}+\|(\varpi_{\rho,s_n}(\alpha)-\varpi_{\rho,s}(\alpha))|_{\mathcal{U}_\Lambda}\|_{\mathcal{U}_\Lambda^*}\right)\mathrm{d}\alpha.\end{aligned}$$

As a consequence, by using (138)-(139) and (148) together with Lebesgue's dominated convergence theorem, one arrives from (144) at Equation (145), $\tau_{t_n,s}^{\Psi^{\varpi_{\rho,s}|\mathcal{U}_\Lambda}}$ being replaced with $\tau_{t_n,s_n}^{\Psi^{\varpi_{\rho,s}|\mathcal{U}_\Lambda}}$. This is not possible because $(\tau_{t,s}^{\Psi})_{s,t\in\mathbb{R}}$ is a strongly continuous two-parameter family, by Proposition 3.7. ■

The proof of Theorem 6.5, being a consequence of Lemmata 7.2 and 7.8, is finished. In order to prove Theorem 6.10, we give now several technical assertions. Concerning the first one, recall Definition 5.1: For any $f\in\mathfrak{C}$ and $\rho\in E$, $\mathrm{d}f(\rho):E\to\mathbb{C}$ is the (unique) convex weak*-continuous Gâteaux derivative of f at $\rho\in E$ if $\mathrm{d}f(\rho)\in\mathcal{A}(E;\mathbb{C})$ and

$$\lim_{\lambda\to0^+}\lambda^{-1}(f((1-\lambda)\rho+\lambda\upsilon)-f(\rho))=[\mathrm{d}f(\rho)](\upsilon), \qquad \upsilon\in E.$$

By Equation (72) extended to the complex case, for any $f\in\mathfrak{Y}$, there is a unique $\mathrm{D}f\in C(E;\mathcal{U})$ such that

$$\mathrm{d}f(\rho)(\upsilon)=\widehat{\mathrm{D}f(\rho)}(\upsilon)=\upsilon(\mathrm{D}f(\rho)), \qquad \rho,\upsilon\in E.$$

In the next lemma, we compute the convex weak*-continuous Gâteaux derivative of the classical evolution (115) of elementary functions defined by (67) for local elements of $\mathcal{U}$:

Lemma 7.9 (Differentiability of the solution – II)
For any $s,t\in\mathbb{R}$ and $A\in\mathcal{U}_0$,

$$\left(\varpi_{\rho,s}\left(t\right)\left(A\right)\right)_{\rho\in E}\equiv\left(\varpi_{\rho,s}\left(t,A\right)\right)_{\rho\in E}\in C^{1}\left(E;\mathbb{C}\right) \tag{149}$$

and, for any $\upsilon\in E$,

$$\left[\mathrm{d}\varpi_{\rho,s}\left(t,A\right)\right]\left(\upsilon\right)=\upsilon\left(\mathrm{D}\varpi_{\rho,s}\left(t,A\right)\right)=\left(\upsilon-\rho\right)\circ\tau_{t,s}^{\Psi^{\gimel}}\left(A\right)+\maltese_{A}\left[\mathrm{d}\varpi_{\rho,s}\left(\cdot,\cdot\right)\left(\upsilon\right)\right]\ ,$$

where $\gimel\equiv\gimel_{\rho,s}\doteq\varpi_{\rho,s}|_{\mathcal{U}_{\Lambda}}$ and, for any continuous function $\xi:\mathbb{R}\times\mathcal{U}_{\Lambda}\rightarrow\mathbb{C}$,

$$\begin{aligned}\maltese_{A}\left[\xi\right]\doteq&\sum_{n\in\mathbb{N}}\sum_{\mathcal{Z}\in\mathcal{P}_{f}}\int_{s}^{t}\mathrm{d}\alpha\int_{\left(\mathbb{S}\cap\mathcal{W}_{\Lambda}\right)^{n}}\mathfrak{a}\left(\alpha\right)_{n}\left(\mathrm{d}\Psi^{(1)},\ldots,\mathrm{d}\Psi^{(n)}\right)\\&\sum_{m_{1},m_{2}=1,m_{2}\neq m_{1}}^{n}\xi(\alpha,\mathfrak{e}_{\Psi^{(m_{2})},\vec{\ell}})\rho\left(i\left[\tau_{\alpha,s}^{\Psi^{\gimel}}\left(\Psi_{\mathcal{Z}}^{(m_{1})}\right),\tau_{t,s}^{\Psi^{\gimel}}\left(A\right)\right]\right)\\&\prod_{j\in\{1,\ldots,n\}\backslash\{m_{1},m_{2}\}}\varpi_{\rho,s}\left(\alpha,\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}\right)\ .\end{aligned} \tag{150}$$

The above series is absolutely summable. Moreover, for any $s\in\mathbb{R}$, $\rho\in E$ and $\tilde{\Lambda}\in\mathcal{P}_{f}$, the mapping $(t,A)\mapsto\mathrm{D}\varpi_{\rho,s}\left(t,A\right)$ from $\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}}$ to $\mathcal{U}$ is continuous.

Proof. We start with a preliminary observation: By Proposition 3.7 and (39) together with Lieb-Robinson bounds for multi-commutators (cf. Proposition 3.8 (i) and [72, Theorems 4.11, 5.4]) and Lebesgue's dominated convergence theorem, observe that

$$\left(\tau_{t,s}^{\Psi_{1}}-\tau_{t,s}^{\Psi_{2}}\right)\left(A\right)=\sum_{\mathcal{Z}\in\mathcal{P}_{f}}\int_{s}^{t}\mathrm{d}\alpha\tau_{\alpha,s}^{\Psi_{1}}\left(i\left[\left(\Psi_{1}\left(\alpha\right)-\Psi_{2}\left(\alpha\right)\right)_{\mathcal{Z}},\tau_{t,\alpha}^{\Psi_{2}}\left(A\right)\right]\right) \tag{151}$$

for any $s,t\in\mathbb{R}$, $A\in\mathcal{U}_0$ and $\Psi_1,\Psi_2\in C(\mathbb{R};\mathcal{W}^{\mathbb{R}}\cap\mathcal{W}_{\Lambda})$. (Note that, in this case, Ψ_1,Ψ_2 are finite-range interactions.) The above series is absolutely summable, because of Lieb-Robinson bounds, as stated in Proposition 3.8 (i). In order to arrive at (151), we also use the equality

$$\left(\tau_{t,s}^{\Psi_{1}}-\tau_{t,s}^{\Psi_{2}}\right)|_{\mathcal{U}_{\tilde{\Lambda}}}=\lim_{L\rightarrow\infty}\int_{s}^{t}\tau_{\alpha,s}^{\Psi_{1}}\circ\left(\delta^{\Psi_{1}(\alpha)}-\delta_{L}^{\Psi_{2}(\alpha)}\right)\circ\tau_{t,\alpha}^{(\Psi_{2},L)}|_{\mathcal{U}_{\tilde{\Lambda}}}\mathrm{d}\alpha$$

for any $\tilde{\Lambda}\in\mathcal{P}_f$, $s,t\in\mathbb{R}$ and $\Psi_1,\Psi_2\in C(\mathbb{R};\mathcal{W}^{\mathbb{R}})$, deduced from (39) and Proposition 3.7. See also Definitions 3.1, 3.3 and Corollary 3.5. In particular, to get (151), in the light of Equations (43)-(44), we have to estimate multi-commutators of order *three*. This is done by the extension to multicommutators of the Lieb-Robinson bounds, contributed in [72, Theorems 4.11, 5.4]. Lieb-Robinson bounds for multi-commutators of order three stated in [72, Theorems 4.11, 5.4] require sufficient polynomial decays of interactions. An obvious sufficient condition for such decays is to take $\Psi_1\left(t\right),\Psi_2\left(t\right)\in\mathcal{W}_{\Lambda}$ for all $t\in\mathbb{R}$, meaning that they are all finite-range. See (31).

Fix now all parameters of the lemma. For any $s,t\in\mathbb{R}$, $\rho,\upsilon\in E$, $h\in(0,1]$ and $A\in\mathcal{U}_0$,

$$\begin{aligned}\daleth\left(h,t,A;\upsilon\right)\ \doteq&\ h^{-1}\left(\varpi_{(1-h)\rho+h\upsilon,s}\left(t,A\right)-\varpi_{\rho,s}\left(t,A\right)\right)\\=&\ \left(\upsilon-\rho\right)\circ\tau_{t,s}^{\Psi^{\gimel_{\rho,s}}}\left(A\right)+h^{-1}\left(\left(1-h\right)\rho+h\upsilon\right)\circ\left(\tau_{t,s}^{\Psi^{\gimel_{(1-h)\rho+h\upsilon,s}}}-\tau_{t,s}^{\Psi^{\gimel_{\rho,s}}}\right)\left(A\right)\ .\end{aligned}$$

Using now (109), (127) and (151) we deduce that, for any $\upsilon \in E$ and $A \in \mathcal{U}_0$,

$$\begin{aligned}
\beth(h,t,A;\upsilon) &= (\upsilon-\rho)\circ\tau_{t,s}^{\Psi^{\beth_{\rho,s}}}(A) \qquad (152)\\
&\quad +\sum_{n\in\mathbb{N}}\sum_{\mathcal{Z}\in\mathcal{P}_f}\int_s^t \mathrm{d}\alpha \int_{(\mathbb{S}\cap\mathcal{W}_\Lambda)^n}\mathfrak{a}(\alpha)_n\left(\mathrm{d}\Psi^{(1)},\ldots,\mathrm{d}\Psi^{(n)}\right)\\
&\qquad \sum_{m_1,m_2=1,m_2\neq m_1}^{n}\beth\left(h,\alpha,\mathfrak{e}_{\Psi^{(m_2)},\vec{\ell}};\upsilon\right)\\
&\qquad\quad \times((1-h)\rho+h\upsilon)\circ\tau_{\alpha,s}^{\Psi^{\beth_{(1-h)\rho+h\upsilon,s}}}\left(i\left[\Psi_{\mathcal{Z}}^{(m_1)},\tau_{t,\alpha}^{\Psi^{\beth_{\rho,s}}}(A)\right]\right)\\
&\qquad\quad \times\prod_{j\in\{1,\ldots,m_2-1\}\backslash\{m_1\}}\varpi_{\rho,s}\left(\alpha,\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}\right)\\
&\qquad\quad \times\prod_{j\in\{m_2+1,\ldots,n\}\backslash\{m_1\}}\varpi_{(1-h)\rho+h\upsilon,s}\left(\alpha,\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}\right) ,
\end{aligned}$$

where the two products over j are, by definition, equal to 1 when j ranges over the empty set[13]. Note that (151) can be used here because $\mathfrak{m}\in C_b(\mathbb{R};\mathcal{M}_\Lambda)$, implying that $\Psi^{\beth}\in C_b(\mathbb{R};\mathcal{W}^{\mathbb{R}}\cap\mathcal{W}_\Lambda)$. See Lemma 6.4 and Equation (128).

From Equation (152), one sees that $\beth(\lambda,t,A;\upsilon)$ is given by a Dyson-type series which is absolutely summable, uniformly with respect to $h\in(0,1]$. To show that, use $\mathfrak{m}\in C_b(\mathbb{R};\mathcal{M}_\Lambda)$ (see (55) and (57)-(58)), Lemma 6.4 together with Equation (128), the fact that $(\tau_{t,s}^{\Psi^{\xi}})_{s,t\in\mathbb{R}}$ is a family of $*$-automorphisms of $\mathcal{U}$ for any $\xi\in C(\mathbb{R};E)$, Inequality (30) and the (usual) Lieb-Robinson bounds (Proposition 3.8 (i)). Note that, for any $\tilde{\Lambda}\in\mathcal{P}_f$, the mapping $\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}}$ to $\mathbb{C}$ defined by

$$(t,A)\mapsto(\upsilon-\rho)\circ\tau_{t,s}^{\Psi^{\beth_{\rho,s}}}(A)$$

is continuous. By Lemma 7.1 and Lebesgue's dominated convergence theorem,

$$\lim_{h\to0^+}((1-h)\rho+h\upsilon)\circ\tau_{\alpha,s}^{\Psi^{\beth_{(1-h)\rho+h\upsilon,s}}}\left(i\left[\Psi_{\mathcal{Z}}^{(m_1)},\tau_{t,\alpha}^{\Psi^{\beth_{\rho,s}}}(A)\right]\right)=\rho\circ\tau_{\alpha,s}^{\Psi^{\beth_{\rho,s}}}\left(i\left[\Psi_{\mathcal{Z}}^{(m_1)},\tau_{t,\alpha}^{\Psi^{\beth_{\rho,s}}}(A)\right]\right) ,$$

uniformly for α in a compact set and, by Equations (136) and (138)-(139),

$$\lim_{h\to0^+}\max_{\alpha\in[s-T,s+T]}\left\|\left(\varpi_{(1-h)\rho+h\upsilon,s,s}(\alpha)-\varpi_{\rho,s}(\alpha)\right)|_{\mathcal{U}_\Lambda}\right\|_{\mathcal{U}_\Lambda^*}=0 , \qquad T\in\mathbb{R}^+ .$$

Hence, using again Lebesgue's dominated convergence theorem, we deduce that

$$\beth(0,t,A;\upsilon)\doteq\lim_{h\to0^+}\beth(h,t,A;\upsilon)=\lim_{h\to0^+}h^{-1}\left(\varpi_{(1-h)\rho+h\upsilon,s}(t,A)-\varpi_{\rho,s}(t,A)\right) \qquad (153)$$

exists for all $s,t\in\mathbb{R}$, $\rho,\upsilon\in E$ and $A\in\mathcal{U}_0$, as given by a Dyson-type series. In particular, for any $\upsilon\in E$ and $\tilde{\Lambda}\in\mathcal{P}_f$, the complex-valued function $(t,A)\mapsto\beth(0,t,A,\upsilon)$ on $\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}}$ is the unique solution in $\xi\in C(\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}};\mathbb{C})$ to the equation

$$\xi(t,A)=(\upsilon-\rho)\circ\tau_{t,s}^{\Psi^{\beth}}(A)+\maltese_A[\xi] \qquad (154)$$

with $\maltese_A$ defined by (150). Compare with (152) taken at $h=0$.

The uniqueness of the solution in $\xi\in C(\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}};\mathbb{C})$ to (154) is a consequence of the fact that one can iterate Equation (154) in order to prove that ξ is the Dyson-type series $\beth(0,t,A,\upsilon)$: First, it suffices to show this fact for $\tilde{\Lambda}=\Lambda$ and $A\in\mathcal{U}_\Lambda$ with $\|A\|_{\mathcal{U}}\leq\|\mathbf{F}\|_{1,\mathfrak{L}}$ because this case fixes

[13]This happens when $m_2\in\{1,n\}$.

all $\xi(t,\mathfrak{e}_{\Psi,\vec{\ell}})$ for all $\Psi \in \mathbb{S}\cap\mathcal{W}_\Lambda$ and $t\in\mathbb{R}$ (cf. (30)). Secondly, observe that any function $\xi \in C(\mathbb{R}\times\mathcal{U}_\Lambda;\mathbb{C})$ is bounded on compact sets of $\mathbb{R}\times\mathcal{U}_\Lambda$ and any norm-closed ball of $\mathcal{U}_\Lambda$ is compact, by finite dimensionality of $\mathcal{U}_\Lambda$. Using these observations together with (24)-(25), (30), (55), Lieb-Robinson bounds (Proposition 3.8 (i)), Lemma 6.4 and tedious computations, one checks that, for any $A\in\mathcal{U}_\Lambda$ satisfying $\|A\|_{\mathcal{U}}\leq\|\mathbf{F}\|_{1,\mathfrak{L}}$, arbitrary time $T\in\mathbb{R}^+$ and $t\in[s-T,s+T]$,

$$|\maltese_A[\xi]| \leq 2\|\mathbf{F}\|_{1,\mathfrak{L}}\,\mathrm{e}^{4\mathbf{D}T\|\mathfrak{m}\|_\infty}\,|\Lambda|\,\|\mathfrak{m}\|_\infty \sup_{t\in[s-T,s+T]}\ \sup_{B\in\mathcal{U}_\Lambda:\|B\|_{\mathcal{U}}\leq\|\mathbf{F}\|_{1,\mathfrak{L}}} |\xi(t,B)|\left(\int_s^t \mathrm{d}\alpha\right). \quad (155)$$

More generally, we can reconstruct from (154) the kth first terms of the Dyson-type series $\beth(0,t,A,\upsilon)$ and, in the same way one obtains (155) (i.e., the $(0+1)$th remaining term), the $(k+1)$th remaining term is bounded by

$$\sup_{t\in[s-T,s+T]}\ \sup_{B\in\mathcal{U}_\Lambda:\|B\|_{\mathcal{U}}\leq\|\mathbf{F}\|_{1,\mathfrak{L}}} |\xi(t,B)|\left(2\|\mathbf{F}\|_{1,\mathfrak{L}}\,\mathrm{e}^{4\mathbf{D}T\|\mathfrak{m}\|_\infty}\,|\Lambda|\,\|\mathfrak{m}\|_\infty\right)^{k+1}\int_s^t \mathrm{d}\alpha_1\cdots\int_s^{\alpha_{k-1}}\mathrm{d}\alpha_k\,,$$

and thus vanishes in the limit $k\to\infty$. So, by Lebesgue's dominated convergence theorem, any solution $\xi\in C(\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}};\mathbb{C})$ to (154) is equal to $\beth(0,t,A,\upsilon)$.

Similar to (152), using in particular the Lieb-Robinson bounds (Proposition 3.8 (i)), note that the integral equation

$$\begin{aligned}\mathfrak{D}(t,A) &= \tau_{t,s}^{\Psi^{\daleth}}(A)-\rho\circ\tau_{t,s}^{\Psi^{\daleth}}(A)\mathbf{1} \\ &\quad+\sum_{n\in\mathbb{N}}\sum_{\mathcal{Z}\in\mathcal{P}_f}\int_s^t\mathrm{d}\alpha\int_{(\mathbb{S}\cap\mathcal{W}_\Lambda)^n}\mathfrak{a}(\alpha)_n\left(\mathrm{d}\Psi^{(1)},\ldots,\mathrm{d}\Psi^{(n)}\right) \\ &\quad\quad\sum_{m_1,m_2=1,m_2\neq m_1}^{n}\mathfrak{D}\left(\alpha,\mathfrak{e}_{\Psi^{(m_2)},\vec{\ell}}\right) \\ &\quad\quad\quad\times\rho\circ\tau_{\alpha,s}^{\Psi^{\daleth}}\left(i\left[\Psi_{\mathcal{Z}}^{(m_1)},\tau_{t,\alpha}^{\Psi^{\daleth}}(A)\right]\right) \\ &\quad\quad\quad\quad\times\prod_{j\in\{1,\ldots,n\}\backslash\{m_1,m_2\}}\varpi_{\rho,s}\left(\alpha,\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}\right).\end{aligned} \quad (156)$$

uniquely determines, by absolutely summable (in $\mathfrak{U}$) Dyson-type series, a continuous mapping $(t,A)\mapsto\mathfrak{D}(t,A)$ from $\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}}$ to $\mathcal{U}$ for any $\tilde{\Lambda}\in\mathcal{P}_f$, which, via (154), satisfies

$$\upsilon(\mathfrak{D}(t,A))=\beth(0,t,A;\upsilon)\doteq\lim_{h\to0^+}h^{-1}\left(\varpi_{(1-h)\rho+h\upsilon,s}(t,A)-\varpi_{\rho,s}(t,A)\right) \quad (157)$$

for all $s,t\in\mathbb{R}$, $\rho,\upsilon\in E$ and $A\in\mathcal{U}_0$. Observe that the integrals in the corresponding Dyson-type series are well-defined as Bochner integrals: For any $\tilde{\Lambda}\in\mathcal{P}_f$, the mapping from $\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}}$ to $\mathcal{U}$ defined by

$$(t,A)\mapsto\tau_{t,s}^{\Psi^{\daleth}}(A)-\rho\circ\tau_{t,s}^{\Psi^{\daleth}}(A)\mathbf{1} \quad (158)$$

is continuous. Since the measures $\mathfrak{a}(\alpha)_n$, $n\in\mathbb{N}$, are finite and $\mathcal{U}$ is a separable Banach space, by [89, Theorems 1.1 and 1.2], all terms appearing in the arguments of the integrals are Bochner-integrable. By Definition 5.1, the assertion follows. ■

For the last assertions, we assume that the decay function equals

$$\mathbf{F}(x,y)=\mathrm{e}^{-2\varsigma|x-y|}(1+|x-y|)^{-(d+\epsilon)}\,,\qquad x,y\in\mathfrak{L}\,, \quad (159)$$

for some fixed $\varsigma,\epsilon\in\mathbb{R}^+$. See (26).

Lemma 7.10 (Graph norm continuity of dynamics on local elements)
Assume (159). For any $\Psi \in C(\mathbb{R};\mathcal{W}^{\mathbb{R}})$, $\Phi \in \mathcal{W}$, $s,t \in \mathbb{R}$ and $A \in \mathcal{U}_0$, $\tau_{t,s}^{\Psi}(A) \in \mathrm{dom}(\delta^{\Phi})$ and

$$\delta^{\Phi} \circ \tau_{t,s}^{\Psi}(A) = \lim_{L\to\infty} \delta^{\Phi} \circ \tau_{t,s}^{(L,\Psi)}(A) = \lim_{L\to\infty} \delta_L^{\Phi} \circ \tau_{t,s}^{\Psi}(A) \ ,$$

uniformly for s,t on compacta. Additionally, for any $s \in \mathbb{R}$ and $\tilde{\Lambda} \in \mathcal{P}_f$, the mapping $(t,A) \mapsto \delta^{\Phi} \circ \tau_{t,s}^{\Psi}(A)$ from $\mathbb{R} \times \mathcal{U}_{\tilde{\Lambda}}$ to $\mathcal{U}$ is continuous.

Proof. For any $\Psi \in C(\mathbb{R};\mathcal{W}^{\mathbb{R}})$, $\Phi \in \mathcal{W}$, $s,t \in \mathbb{R}$ and $A \in \mathcal{U}_0$, a simple adaptation of [72, Equation (5.47), $m=1$] (using Lieb-Robinson bounds for multi-commutators of order three [72, Theorems 4.11, 5.4]) implies that

$$\lim_{L_0\to\infty} \sup_{L\in\mathbb{N}} \left\| \delta_L^{\Phi} \circ \left(\tau_{t,s}^{\Psi} - \tau_{t,s}^{(L_0,\Psi)} \right)(A) \right\|_{\mathcal{U}} = 0 \, . \tag{160}$$

Using again a similar argument together with Corollary 3.5 and the closedness of δ^{Φ}, we also deduce that

$$\delta^{\Phi} \circ \tau_{t,s}^{\Psi}(A) = \lim_{L\to\infty} \delta^{\Phi} \circ \tau_{t,s}^{(L,\Psi)}(A) \qquad \text{and} \qquad \tau_{t,s}^{\Psi}(A) \in \mathrm{dom}(\delta^{\Phi}) \, . \tag{161}$$

Note that the limits in Equations (160)-(161) are uniform for s,t on compacta (cf. [72, Theorem 5.6 (i)]). For any $\Psi \in C(\mathbb{R};\mathcal{W}^{\mathbb{R}})$, $\Phi \in \mathcal{W}$, $L, L_0 \in \mathbb{N}$, $s,t \in \mathbb{R}$ and $A \in \mathcal{U}_0$, observe meanwhile that

$$\begin{aligned} \left\| \delta^{\Phi} \circ \tau_{t,s}^{\Psi}(A) - \delta_L^{\Phi} \circ \tau_{t,s}^{\Psi}(A) \right\|_{\mathcal{U}} \ \leq \ & \left\| \delta^{\Phi} \circ \left(\tau_{t,s}^{\Psi} - \tau_{t,s}^{(L_0,\Psi)} \right)(A) \right\|_{\mathcal{U}} \\ & + \left\| \delta_L^{\Phi} \circ \left(\tau_{t,s}^{\Psi} - \tau_{t,s}^{(L_0,\Psi)} \right)(A) \right\|_{\mathcal{U}} \\ & + \left\| \left(\delta^{\Phi} - \delta_L^{\Phi} \right) \circ \tau_{t,s}^{(L_0,\Psi)}(A) \right\|_{\mathcal{U}} \, . \end{aligned}$$

By (160)-(161) and Corollary 3.5, it follows that, for any $\Phi \in \mathcal{W}$, $s,t \in \mathbb{R}$ and $A \in \mathcal{U}_0$,

$$\delta^{\Phi} \circ \tau_{t,s}^{\Psi}(A) = \lim_{L\to\infty} \delta_L^{\Phi} \circ \tau_{t,s}^{\Psi}(A) \ ,$$

uniformly for s,t on compacta.

For any $L \in \mathbb{R}$, the mapping from $\mathbb{R} \times \mathcal{U}_0$ to $\mathcal{U}$ defined by

$$(t,A) \mapsto \delta^{\Phi} \circ \tau_{t,s}^{(L,\Psi)}(A) \tag{162}$$

is continuous, by Corollary 3.5. By (161), as $L \to \infty$, this mapping on any fixed $A \in \mathcal{U}_0$ converges uniformly for s,t on compacta to the mapping from $\mathbb{R} \times \mathcal{U}_0$ to $\mathcal{U}$ defined by

$$(t,A) \mapsto \delta^{\Phi} \circ \tau_{t,s}^{\Psi}(A) \ , \tag{163}$$

which is thus continuous with respect to $t \in \mathbb{R}$. By finite dimensionality of $\mathcal{U}_{\tilde{\Lambda}}$ for $\tilde{\Lambda} \in \mathcal{P}_f$ and linearity of the mapping (163) with respect to $A \in \mathcal{U}_0$, the function (163) is continuous on $\mathbb{R} \times \mathcal{U}_{\tilde{\Lambda}}$. ∎

Proposition 7.11 (Graph norm continuity of convex derivatives)
Assume (159). For any $\Phi \in \mathcal{W}$, $s,t \in \mathbb{R}$, $\rho \in E$ and $A \in \mathcal{U}_0$, $\mathrm{D}\varpi_{\rho,s}(t,A) \in \mathrm{dom}(\delta^{\Phi})$ and

$$\delta^{\Phi}\left(\mathrm{D}\varpi_{\rho,s}(t,A)\right) = \lim_{L\to\infty} \delta_L^{\Phi}\left(\mathrm{D}\varpi_{\rho,s}(t,A)\right) \ .$$

Additionally, for any $s \in \mathbb{R}$, $\rho \in E$ and $\tilde{\Lambda} \in \mathcal{P}_f$, the mapping $(t,A) \mapsto \delta^{\Phi}(\mathrm{D}\varpi_{\rho,s}(t,A))$ from $\mathbb{R} \times \mathcal{U}_{\tilde{\Lambda}}$ to $\mathcal{U}$ is continuous.

Proof. Fix all parameters of the proposition. Let

$$\mathcal{G} \doteq \left(\mathrm{dom}(\delta^{\Phi}), \|\cdot\|_{\mathcal{G}}\right)$$

be the Banach space obtained by endowing the domain of δ^{Φ} with its graph norm. By [72, Theorem 4.8 (ii)], $\mathcal{U}_0$ is a core of the derivation δ^{Φ} and hence, $\mathcal{G}$ is a separable Banach space.

By using Lemma 7.10, the closedness of δ^{Φ} and the fact that all terms appearing in the arguments of the integrals in the Dyson-type series of $\mathfrak{D}(t,A)$ are Bochner-integrable, one checks that the Dyson-type series deduced from (156) yields an element $\mathfrak{D}(t,A) \in \mathcal{G}$ with $(t,A) \mapsto \delta^{\Phi}(\mathfrak{D}(t,A))$ on $\mathbb{R} \times \mathcal{U}_{\tilde{\Lambda}}$ ($\tilde{\Lambda} \in \mathcal{P}_f$) being the unique solution in $\mathfrak{E} \in C(\mathbb{R} \times \mathcal{U}_{\tilde{\Lambda}}; \mathcal{U})$ to the equation

$$\begin{aligned}\mathfrak{E}(t,A) \;=\;& \delta^{\Phi} \circ \tau_{t,s}^{\Psi^{\daleth}}(A) \\ &+ \sum_{n\in\mathbb{N}} \sum_{\mathcal{Z}\in\mathcal{P}_f} \int_s^t \mathrm{d}\alpha \int_{(\mathbb{S}\cap\mathcal{W}_\Lambda)^n} \mathfrak{a}(\alpha)_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right) \\ &\qquad \sum_{m_1,m_2=1, m_2\neq m_1}^{n} \mathfrak{E}\left(\alpha, \mathfrak{e}_{\Psi^{(m_2)},\vec{\ell}}\right) \\ &\qquad\qquad \times \rho \circ \tau_{\alpha,s}^{\Psi^{\daleth}}\left(i\left[\Psi_{\mathcal{Z}}^{(m_1)}, \tau_{t,\alpha}^{\Psi^{\daleth}}(A)\right]\right) \\ &\qquad\qquad\qquad \times \prod_{j\in\{1,\ldots,n\}\backslash\{m_1,m_2\}} \varpi_{\rho,s}\left(\alpha, \mathfrak{e}_{\Psi^{(j)},\vec{\ell}}\right)\end{aligned} \tag{164}$$

for $t\in\mathbb{R}$, $A \in \mathcal{U}_{\tilde{\Lambda}}$ and $\tilde{\Lambda} \in \mathcal{P}_f$. Compare this equation with (156). The integrals in (164) are again well-defined as Bochner integrals: Combine Lemma 7.10 and [89, Theorems 1.1 and 1.2] with the fact that the measure $\mathfrak{a}(\alpha)_n$, $n\in\mathbb{N}$, are finite and $\mathcal{U}$ is separable.

Additionally, for any $s\in\mathbb{R}$, $\rho \in E$ and $A \in \mathcal{U}_0$, the mapping $t \mapsto \delta^{\Phi}(\mathrm{D}\varpi_{\rho,s}(t,A))$ from $\mathbb{R}$ to $\mathcal{U}$ is continuous, again by Lemma 7.10, the last Dyson-type series and the identity

$$\mathrm{D}\varpi_{\rho,s}(t,A) = \mathfrak{D}(t,A) \ . \tag{165}$$

By linearity with respect to $A \in \mathcal{U}_0$ and finite dimensionality of $\mathcal{U}_{\tilde{\Lambda}}$ for $\tilde{\Lambda} \in \mathcal{P}_f$, it follows that the mapping $(t,A) \mapsto \delta^{\Phi}(\mathrm{D}\varpi_{\rho,s}(t,A))$ from $\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}}$ to $\mathcal{U}$ is continuous.

By combining Lemma 7.10 with Lebesgue's dominated convergence theorem and the fact that the unique solution $\mathfrak{E} = \delta^{\Phi} \circ \mathfrak{D}$ to (164) is given by a Dyson-type series, we arrive at

$$\delta^{\Phi}(\mathfrak{D}(t,A)) = \lim_{L\to\infty} \delta_L^{\Phi}(\mathfrak{D}(t,A)) \tag{166}$$

in $\mathcal{U}$. Note that one can interchange the local derivation δ_L^{Φ} (which is a commutator) in the right-hand side of (166) with every integral over $(\mathbb{S}\cap\mathcal{W}_\Lambda)^n$ for any $n\in\mathbb{N}$, because it is a bounded operator on $\mathcal{U}$. By (165), this concludes the proof of the proposition. ■

Lemma 7.12 (Differentiability of the solution – III)
Assume (159). Then, for any $t\in\mathbb{R}$, $\rho\in E$ and $A \in \mathcal{U}_0$,

$$(\varpi_{\rho,s}(t)(A))_{s\in\mathbb{R}} \equiv (\varpi_{\rho,s}(t,A))_{s\in\mathbb{R}} \in C^1(\mathbb{R};\mathbb{C})$$

with derivative satisfying, for any $A\in\mathcal{U}_0$,

$$\partial_s \varpi_{\rho,s}(t,A) = -\rho \circ \delta^{\Psi^{\varpi_{\rho,s}|\mathcal{U}_\Lambda}(s)} \circ \tau_{t,s}^{\Psi^{\varpi_{\rho,s}|\mathcal{U}_\Lambda}}(A) + \maltese_A\left[\partial_s \varpi_{\rho,s}\right] \ . \tag{167}$$

Here, $\maltese_A$ is defined by (150). Additionally, for any $\tilde{\Lambda}\in\mathcal{P}_f$, $(t,A) \mapsto \partial_s\varpi_{\rho,s}(t,A)$ is a continuous function on $\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}}$.

Proof. Fix all parameters of the Lemma. By Lemma 7.2, for any $\rho \in E$, $s,t \in \mathbb{R}$, $A \in \mathcal{U}_0$ and $\varepsilon \in \mathbb{R}\backslash\{0\}$,

$$\begin{aligned}\tilde{\beth}(\varepsilon,t,A) &\doteq \varepsilon^{-1}\left(\varpi_{\rho,s+\varepsilon}(t,A)-\varpi_{\rho,s}(t,A)\right)\\ &= \varepsilon^{-1}\rho\circ(\tau_{t,s+\varepsilon}^{\Psi^{\daleth_{\rho,s}}}-\tau_{t,s}^{\Psi^{\daleth_{\rho,s}}})(A)+\varepsilon^{-1}\rho\circ(\tau_{t,s+\varepsilon}^{\Psi^{\daleth_{\rho,s+\varepsilon}}}-\tau_{t,s+\varepsilon}^{\Psi^{\daleth_{\rho,s}}})(A)\end{aligned}$$

with $\daleth \equiv \daleth_{\rho,s} \doteq \varpi_{\rho,s}|_{\mathcal{U}_\Lambda}$. Similar to (152), via Equations (109), (127) and (151) (keeping in mind that $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda)$), we deduce that

$$\begin{aligned}\tilde{\beth}(\varepsilon,t,A) &= \varepsilon^{-1}\rho\circ(\tau_{t,s+\varepsilon}^{\Psi^{\daleth_{\rho,s}}}-\tau_{t,s}^{\Psi^{\daleth_{\rho,s}}})(A) \qquad (168)\\ &\quad+\sum_{n\in\mathbb{N}}\sum_{\mathcal{Z}\in\mathcal{P}_f}\int_{s+\varepsilon}^{t}\mathrm{d}\alpha\int_{(\mathbb{S}\cap\mathcal{W}_\Lambda)^n}\mathfrak{a}(\alpha)_n\left(\mathrm{d}\Psi^{(1)},\ldots,\mathrm{d}\Psi^{(n)}\right)\\ &\qquad\sum_{m_1,m_2=1,m_2\neq m_1}^{n}\tilde{\beth}\left(\varepsilon,\alpha,\mathfrak{e}_{\Psi^{(m_2)},\vec{\ell}}\right)\\ &\qquad\quad\times\rho\circ\tau_{\alpha,s}^{\Psi^{\daleth_{\rho,s+\varepsilon}}}\left(i\left[\Psi_{\mathcal{Z}}^{(m_1)},\tau_{t,\alpha}^{\Psi^{\daleth_{\rho,s}}}(A)\right]\right)\\ &\qquad\quad\times\prod_{j\in\{1,\ldots,m_2-1\}\backslash\{m_1\}}\varpi_{\rho,s}\left(\alpha,\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}\right)\\ &\qquad\quad\times\prod_{j\in\{m_2+1,\ldots,n\}\backslash\{m_1\}}\varpi_{\rho,s+\varepsilon}\left(\alpha,\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}\right),\end{aligned}$$

where the two products over j are, by definition, equal to 1 when j ranges over the empty set[14]. Again, using the same arguments as for $\beth(h,t,A;\upsilon)$ in (152), one sees from Equation (168) that $\tilde{\beth}(\varepsilon,t,A)$ is given by a Dyson-type series which is absolutely summable, uniformly with respect to ε in a bounded set when $\mathfrak{m} \in C_b(\mathbb{R};\mathcal{M}_\Lambda)$.

Now, assuming (159), we can apply [72, Theorem 5.5] to the interaction $\Psi^{\daleth_{\rho,s}}$:

$$\forall s,r,t\in\mathbb{R}: \qquad \partial_r\tau_{t,r}^{\Psi^{\daleth_{\rho,s}}}=-\delta^{\Psi^{\daleth_{\rho,s}}(r)}\circ\tau_{t,r}^{\Psi^{\daleth_{\rho,s}}}, \tag{169}$$

in the strong sense on the dense set $\mathcal{U}_0$. It is a highly non-trivial outcome resulting again from Lieb-Robinson bounds for multi-commutators of order three given in [72, Theorems 4.11, 5.4]. Similar to (153), by Lemmata 7.1 and 7.8 together with Lebesgue's dominated convergence theorem, we deduce from the Dyson-type series coming from (168) that

$$\partial_s\varpi_{\rho,s}(t,A)\doteq\lim_{\varepsilon\to0}\tilde{\beth}(\varepsilon,t,A)=\lim_{\varepsilon\to0}\varepsilon^{-1}\left(\varpi_{\rho,s+\varepsilon}(t,A)-\varpi_{\rho,s}(t,A)\right)$$

exists, for all $s,t \in \mathbb{R}$, $\rho \in E$ and $A \in \mathcal{U}_0$, and is also given by a Dyson-type series. Note that the Dyson-type series are well-defined because, for any $\tilde{\Lambda} \in \mathcal{P}_f$, the mapping from $\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}}$ to $\mathbb{C}$ defined by

$$(t,A)\mapsto\rho\circ\delta^{\Psi^{\daleth_{\rho,s}}(s)}\circ\tau_{t,s}^{\Psi^{\daleth_{\rho,s}}}(A) \tag{170}$$

is continuous, by Lemma 7.10.

By Proposition 3.8 (i), for any $\tilde{\Lambda} \in \mathcal{P}_f$, the complex-valued function $(t,A)\mapsto\partial_s\varpi_{\rho,s}(t,A)$ on $\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}}$ is the unique solution in $\xi \in C(\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}};\mathbb{C})$ to the equation

$$\xi(t,A)=-\rho\circ\delta^{\Psi^{\daleth_{\rho,s}}(s)}\circ\tau_{t,s}^{\Psi^{\daleth_{\rho,s}}}(A)+\maltese_A[\xi] \tag{171}$$

with $\maltese_A$ defined by (150). Compare with (168) taken at $\varepsilon = 0$. To prove the uniqueness of a solution in $\xi \in C(\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}};\mathbb{C})$ to (171), we use the same arguments than for (154), keeping in mind Lemma

[14] This happens when $m_2 \in \{1,n\}$.

7.10. This is again a consequence of Lieb-Robinson bounds for multi-commutators of order three given in [72, Theorems 4.11, 5.4]. ■

We conclude this section with the derivation of Liouville's equation for (elementary) continuous and affine functions defined by (67), from which Theorem 6.10 is deduced.

Lemma 7.13 (Liouville's equation for affine functions)
Assume (159). Then,

$$\partial_s V_{t,s}^{\mathfrak{m}}(\hat{A})\,(\rho) = -\lim_{L\to\infty}\{\mathrm{h}_L^{\mathfrak{m}(s)}, V_{t,s}^{\mathfrak{m}}(\hat{A})\}\,(\rho)\ , \qquad s,t\in\mathbb{R},\ A\in\mathcal{U}_0,\ \vec{\ell}\in\mathbb{N}^d,\ \rho\in E_{\vec{\ell}}\ ,$$

with $\hat{A}\in\mathfrak{C}$ being defined by (67).

Proof. Fix $s\in\mathbb{R}$ and $\rho\in E$. By (67) and (115), note that

$$V_{t,s}^{\mathfrak{m}}(\hat{A}) = \varpi_{\rho,s}\,(t)\,(A) \equiv \varpi_{\rho,s}\,(t,A)\ , \qquad t\in\mathbb{R},\ A\in\mathcal{U}\ .$$

Therefore, by Lemma 7.9 and (156),

$$\mathrm{D}V_{t,s}^{\mathfrak{m}}(\hat{A})\,(\rho) = \mathrm{D}\varpi_{\rho,s}\,(t;A) = \mathfrak{D}\,(t,A)\ , \qquad t\in\mathbb{R},\ A\in\mathcal{U}_0\ .$$

See also Definition 5.1 and Equation (72). By Proposition 7.11 and (164), the continuous complex-valued function

$$(t,A)\mapsto -\rho\circ\delta^{\Psi^{\varpi_{\rho,s}|\mathcal{U}_\Lambda}(s)}\left(\mathrm{D}V_{t,s}^{\mathfrak{m}}(\hat{A})\,(\rho)\right)$$

on $\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}}$ solves Equation (171), like the well-defined continuous mapping

$$(t,A)\mapsto \partial_s V_{t,s}^{\mathfrak{m}}(\hat{A})\,(\rho) = \partial_s\varpi_{\rho,s}\,(t)\,(A) \equiv \partial_s\varpi_{\rho,s}\,(t,A)$$

from $\mathbb{R}\times\mathcal{U}_{\tilde{\Lambda}}$ to $\mathbb{C}$ (Lemma 7.12), at any fixed $\tilde{\Lambda}\in\mathcal{P}_f$. By uniqueness of the solution to (171),

$$\partial_s V_{t,s}^{h}(\hat{A})\,(\rho) = -\rho\circ\delta^{\Psi^{\varpi_{\rho,s}|\mathcal{U}_\Lambda}(s)}\left(\mathrm{D}V_{t,s}^{\mathfrak{m}}(\hat{A})\,(\rho)\right)\ , \qquad s,t\in\mathbb{R},\ A\in\mathcal{U}_0\ .$$

By tedious computations using Definitions 3.3, 5.2, Corollary 3.5, Lemma 3.2, Proposition 7.11, Lebesgue's dominated convergence theorem and Equations (109), (117) and (127), one meanwhile checks that, for any $\vec{\ell}\in\mathbb{N}^d$ and $\rho\in E_{\vec{\ell}}$,

$$\rho\circ\delta^{\Psi^{\varpi_{\rho,s}|\mathcal{U}_\Lambda}(s)}\left(\mathrm{D}V_{t,s}^{\mathfrak{m}}(\hat{A})\,(\rho)\right) = \lim_{L\to\infty}\{\mathrm{h}_L^{\mathfrak{m}(s)}, V_{t,s}^{\mathfrak{m}}(\hat{A})\}\,(\rho)\ , \qquad s,t\in\mathbb{R},\ A\in\mathcal{U}_0\ .$$

■

8 Equivalent Definition of Translation-Invariant Long-Range Models

Recall that $\mathfrak{L}=\mathbb{Z}^d$, see Section 4.1. In [46, Definition 2.1], we give a definition of translation-invariant long-range models that differs from Equation (56). It turns out that any model of

$$\mathcal{M}_1^{(2)} \doteq \left\{(\Phi,\mathfrak{a})\equiv(\Phi,(\mathfrak{a}_n)_{n\in\mathbb{N}})\in\mathcal{W}_1^{\mathbb{R}}\times\mathcal{S} : \forall n\in\mathbb{N}\backslash\{2\}, \mathfrak{a}_n=0\right\}\ ,$$

where $\mathcal{W}_1^{\mathbb{R}} \doteq \mathcal{W}_1\cap\mathcal{W}^{\mathbb{R}}$, can be identified with a long-range model in the sense of [46, Definition 2.1], and vice-versa. (The notation $\mathcal{W}_1$ in [46] corresponds here to $\mathcal{W}_1^{\mathbb{R}}$.) This identification can be done in a such a way that the sequences of local Hamiltonians associated with each long-range model are the same in both cases:

(i): We start with a preliminary observation which simplifies the arguments. At $L \in \mathbb{N}$, the local Hamiltonian of any model $\mathfrak{m} = (\Phi, (0, \mathfrak{a}_2, 0, \ldots)) \in \mathcal{M}_1^{(2)}$ of Definition 4.2 is equal to

$$U_L^{\mathfrak{m}} = U_L^{\Phi} + \frac{1}{2\,|\Lambda_L|} \int_{\mathbb{S}^2} \left(U_L^{\Psi^{(1)}} U_L^{\Psi^{(2)}} + (U_L^{\Psi^{(2)}})^* (U_L^{\Psi^{(1)}})^* \right) \mathfrak{a}_2 \left(\mathrm{d}\Psi^{(1)}, \mathrm{d}\Psi^{(2)} \right) \ ,$$

because $\mathfrak{a}_2$ is, by definition, self-adjoint, meaning that it equals its pushforward through the homeomorphism (53) for $n = 2$. Since, for any $A, B \in \mathcal{U}$,

$$AB + B^* A^* = \frac{1}{2} \left((A^* + B)^* (A^* + B) - (A^* - B)^* (A^* - B) \right) \ ,$$

observe that

$$U_L^{\mathfrak{m}} = U_L^{\Phi} + \frac{1}{|\Lambda_L|} \int_{\mathbb{S}^2} \left(|U_L^{\Psi^{(1)*} + \Psi^{(2)}}|^2 - |U_L^{\Psi^{(1)*} - \Psi^{(2)}}|^2 \right) \mathfrak{a}_2 \left(\mathrm{d}\Psi^{(1)}, \mathrm{d}\Psi^{(2)} \right)$$

with $|C|^2 \doteq C^* C$ for $C \in \mathcal{U}$. Let $\mathbb{B} \supseteq \mathbb{S}$ be the unit closed ball of the Banach space $\mathcal{W}_1$ of translation-invariant (complex) interactions and define the continuous functions $F^{\pm} : \mathbb{S}^2 \to \mathbb{B}$ by

$$F^{\pm} \left(\Psi^{(1)}, \Psi^{(2)} \right) \doteq \frac{1}{2} \left(\Psi^{(1)*} \pm \Psi^{(2)} \right) \ , \qquad \Psi^{(1)}, \Psi^{(2)} \in \mathbb{S} \ .$$

Denoting by $F_*^{\pm}(\mathfrak{a}_2)$ the two pushforwards of the measure $\mathfrak{a}_2$ through the continuous functions $F^{\pm}$, we arrive at the equality

$$U_L^{\mathfrak{m}} = U_L^{\Phi} + \frac{1}{|\Lambda_L|} \int_{\mathbb{B}} |U_L^{\Psi}|^2 \mathfrak{a}(\mathrm{d}\Psi) \ , \qquad \mathfrak{a} \doteq F_*^{+}(\mathfrak{a}_2) - F_*^{-}(\mathfrak{a}_2) \ . \tag{172}$$

Then, any model of $\mathcal{M}_1^{(2)}$ can be identified with a long-range model in the sense of [46, Definition 2.1]:

- The measure space of [46, Definition 2.1] is $(\mathbb{B}, \Sigma, |\mathfrak{a}|)$, with $\Sigma \equiv \Sigma_{\mathbb{B}}$ being the Borel σ-algebra associated with $\mathbb{B}$. Σ is countably generated, by separability of $\mathcal{W} \supseteq \mathbb{B}$. Ergo, by [88, Proposition 3.4.5], the space $L^2(\mathbb{B}; \mathbb{C}) \doteq L^2(\mathbb{B}, |\mathfrak{a}|; \mathbb{C})$ of square-integrable complex-valued functions on $\mathbb{B}$ is a separable Hilbert space, i.e., $(\mathbb{B}, \Sigma, |\mathfrak{a}|)$ is a separable measure space.

- The corresponding $\mathcal{L}^2$-functions $(\Phi_\Psi)_{\Psi \in \mathbb{B}}, (\Phi'_\Psi)_{\Psi \in \mathbb{B}}$ of [46, Definition 2.1] are defined by

  $$\Phi_\Psi \doteq \mathrm{Re}\{\Psi\} \qquad \text{and} \qquad \Phi'_\Psi \doteq \mathrm{Im}\{\Psi\} \qquad \text{with} \qquad \Psi \in \mathbb{B} \ .$$

  See (23) for the definition of real and imaginary parts of interactions. These two functions are Bochner measurable, by [89, Theorems 1.1 and 1.2], for they are continuous functions and $\mathcal{W}$ is a separable Banach space. Additionally, they are $\mathcal{L}^2$-functions because there are bounded on a space of finite measure.

- The corresponding measurable function $\gamma_\Psi \in \{-1, 1\}$ of [46, Definition 2.3] is obtained from any Hahn decomposition $P_{\mathfrak{a}}, N_{\mathfrak{a}} \in \Sigma$ of the signed measure $\mathfrak{a}$ (172) by

  $$\gamma_\Psi = \mathbf{1}[\Psi \in P_{\mathfrak{a}}] - \mathbf{1}[\Psi \in N_{\mathfrak{a}}] \ , \qquad \Psi \in \mathbb{B} \ ,$$

  where $P_{\mathfrak{a}}$ and $N_{\mathfrak{a}}$ are respectively positive and negative sets for $\mathfrak{a}$.

(ii): Conversely, let $(\mathcal{A}, \mathfrak{A}, \mathfrak{a}_0)$ be a separable measure space with $\mathfrak{A}$ and $\mathfrak{a}_0 : \mathfrak{A} \to \mathbb{R}_0^+$ being respectively some σ-algebra on $\mathcal{A}$ and some measure on $\mathfrak{A}$. Fix a measurable function $a \mapsto \gamma_a \in \{-1, 1\}$ on $\mathcal{A}$ (see [46, Definition 2.3]) and a model

$$(\Phi, (\Phi_a)_{a\in\mathcal{A}}, (\Phi'_a)_{a\in\mathcal{A}}) \in \mathcal{W}_1^{\mathbb{R}} \times \mathcal{L}^2\left(\mathcal{A}, \mathcal{W}_1^{\mathbb{R}}\right) \times \mathcal{L}^2\left(\mathcal{A}, \mathcal{W}_1^{\mathbb{R}}\right) \tag{173}$$

in the sense of [46, Definition 2.1]. ($\mathcal{W}_1$ in [46] refers here to $\mathcal{W}_1^{\mathbb{R}}$.)

Define the set

$$\mathcal{A}_{\Phi,\Phi'} \doteq \{a \in \mathcal{A} : \Phi_a + i\Phi'_a \neq 0\} \ .$$

The mapping

$$a \mapsto \Phi_a + i\Phi'_a$$

from $\mathcal{A}_{\Phi,\Phi'}$ to $\mathbb{R}^+$ is measurable, since the vector space operations in $\mathcal{W}$ are jointly continuous. Hence, $\mathcal{A}_{\Phi,\Phi'_a}$, the preimage of $\mathcal{W}_1\backslash\{0\}$ by this mapping, is an element of the σ-algebra $\mathfrak{A}$. As a consequence, by [46, Definition 2.3], at $L \in \mathbb{N}$, the local Hamiltonian associated with a model of [46, Definition 2.1] equals

$$U_L = U_L^{\Phi} + \frac{1}{|\Lambda_L|} \int_{\mathcal{A}_{\Phi,\Phi'}} \gamma_a |U_{\Lambda_L}^{\Phi_a + i\Phi'_a}|^2 \mathfrak{a}_0\left(\mathrm{d}a\right) \ , \tag{174}$$

recalling that $|C|^2 \doteq C^*C$ for $C \in \mathcal{U}$.

Define the functions $G : \mathcal{A}_{\Phi,\Phi'} \to \mathbb{S}$ by

$$G\left(a\right) \doteq g\left(a\right)\left(\Phi_a + i\Phi'_a\right) \ , \qquad a \in \mathcal{A}_{\Phi,\Phi'} \ ,$$

where

$$g\left(a\right) \doteq \|\Phi_a + i\Phi'_a\|_{\mathcal{W}}^{-1} \ , \qquad a \in \mathcal{A}_{\Phi,\Phi'} \ .$$

The mapping g from $\mathcal{A}_{\Phi,\Phi'}$ to $\mathbb{R}^+$ is measurable, as a composition of a measurable function from $\mathcal{A}_{\Phi,\Phi'}$ to $\mathcal{W}_1\backslash\{0\}$ with a continuous one from $\mathcal{W}_1\backslash\{0\}$ to $\mathbb{R}^+$. Again by the joint continuity of the vector space operations in $\mathcal{W}$, G is a measurable function. As a consequence, at $L \in \mathbb{N}$, by (174), the local Hamiltonian associated with a model of [46, Definition 2.1] equals

$$U_L = U_L^{\Phi} + \frac{1}{|\Lambda_L|} \int_{\mathbb{S}} |U_{\Lambda_L}^{\Psi}|^2 \mathfrak{a}\left(\mathrm{d}\Psi\right) \ , \qquad \mathfrak{a} \doteq G_*(\gamma g^{-1} \mathfrak{a}_0) \ ,$$

where $G_*\left(\gamma g^{-2}\mathfrak{a}_0\right)$ is the pushforward of the signed measure $\gamma g^{-2}\mathfrak{a}_0$ through the measurable function G. Note that $\mathfrak{a}$ is a finite measure on $\mathbb{S}$ because

$$\int_{\mathbb{S}} |\mathfrak{a}|\left(\mathrm{d}\Psi\right) = \int_{\mathbb{S}} \|\Phi_a + i\Phi'_a\|_{\mathcal{W}}^2 \,\mathfrak{a}_0\left(\mathrm{d}a\right) \leq 2\int_{\mathbb{S}} \|\Phi_a\|_{\mathcal{W}}^2 \,\mathfrak{a}_0\left(\mathrm{d}a\right) + 2\int_{\mathbb{S}} \|\Phi'_a\|_{\mathcal{W}}^2 \,\mathfrak{a}_0\left(\mathrm{d}a\right) < \infty \ .$$

Now, using the continuous mapping K from $\mathbb{S}$ to $\mathbb{S} \times \mathbb{S}$ defined by

$$K\left(\Psi\right) \doteq \left(\Psi^*, \Psi\right) \ , \qquad \Psi \in \mathbb{S} \, ,$$

we define

$$\mathfrak{a}_2 \doteq \frac{1}{2}\left(K_*(\mathfrak{a}) + K_*(\mathfrak{a})^*\right)$$

to be the real part of the pushforward of the signed measure $\mathfrak{a}$ through the measurable function K. Then, by construction, $\mathfrak{a}_2 = \mathfrak{a}_2^*$ is self-adjoint and

$$U_L = U_L^{\Phi} + \frac{1}{2\left|\Lambda_L\right|} \int_{\mathbb{S}^2} \left(U_L^{\Psi^{(1)}} U_L^{\Psi^{(2)}} + (U_L^{\Psi^{(2)}})^*(U_L^{\Psi^{(1)}})^*\right) \mathfrak{a}_2\left(\mathrm{d}\Psi^{(1)}, \mathrm{d}\Psi^{(2)}\right)$$

for any $L \in \mathbb{N}$.

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