Question on Iterated Limits in Relativity

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Abstract

ABSTRACT. Two iterated limits may not commute with each other, in general. Thus, we present an example when the massless limit of the function of $E, \mathbf{p}, m$ does not exist in some calculations within quantum field theory.

RESUMÉ. Deux limites itérées ne sont pas égales en général. Ainsi, nous présentons un exemple oú la limite sans masse de la fonction de $E, \mathbf{p}, m$ n'existe pas dans certains calculs de la théorie quantique des champs.

KEYWORDS: Iterated limits, Relativity, Massless limit.

In the previous paper [1] we found some intrinsic contradictions related to the mathematical foundations of modern physics. It is well known that the partial derivatives commute in the Minkowski space (as well as in the 4-dimensional momentum space). However, if we consider that energy is an implicit function of the 3-momenta and mass (thus, approaching the mass hyperboloid formalism, $E^2 - \mathbf{p}^2 c^2 = m^2 c^4$) then we may be interested in the commutators of the whole-partial derivatives [2] instead. The whole-partial derivatives do not commute, in general. If they are associated with the corresponding physical operators, we would have the uncertainty relations for dynamically-conjugated physical quantities in the latter case. This is an intrinsic contradiction. While we start from the same postulates, on using two different ways of reasoning we arrive at the two different physical conclusions.

In the present note I would like to ask another question related to the mathematical foundations of special relativity. Sometimes, when calculating dynamical invariants (and other physical quantities in quantum field theory), and when studying the corresponding massless limits we need to calculate iterated limits. We may encounter a rare situation when two iterated limits are not equal each other in physics. See, for example, Ref. [3]. We were puzzled calculating the iterated limits of the aggregate $\frac{E^2-\mathbf{p}^2}{m^2}$ (or the inverse one, $\frac{m^2}{E^2-\mathbf{p}^2}$, $\quad c=1$).[1]

$$\lim_{m\rightarrow 0}\lim_{E\rightarrow\pm\sqrt{\mathbf{p}^2+m^2}}(\frac{m^2}{E^2-\mathbf{p}^2}) \quad = \quad 1\,, \tag{1}$$

$$\lim_{E\rightarrow\pm\sqrt{\mathbf{p}^2+m^2}}\lim_{m\rightarrow 0}(\frac{m^2}{E^2-\mathbf{p}^2}) \quad = \quad 0\,. \tag{2}$$

Physics should have well-defined dynamical invariants. Which iterated limit should be applied in the study of massless limits?

The question of the iterated limits was studied in [4, 5]. However, the answers leave room for misunderstandings and contradictions with the experiments.

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Note Added. Some physicists may say: "The two limits are of very different sorts: the limit of $E\rightarrow\pm\sqrt{\mathbf{p}^2+m^2}$ is a limit that subsumes the statement under the theory of Special Relativity. Such limits should be done first, because they define the physical rule book of the rest of the game. The other limit, $m\rightarrow 0$, defines a value of a parameter within the theory defined by the first limit. Limits like $m\rightarrow 0$ should be done only after the theory-setting limits are performed."

However, cases exist when the limit $E\rightarrow\pm\sqrt{\mathbf{p}^2+m^2}$ cannot be applied (or its application leads to the loss of the information). For example, we have for the causal Green's function used in the scalar field theory and in the $m\rightarrow 0$ quantum electrodynamics (QED):

$$D^c(x)=\frac{1}{(2\pi)^4}\int d^4p\frac{e^{-ip\cdot x}}{m^2-p^2-i\epsilon}= \tag{3}$$

$$=\quad \frac{1}{4\pi}\delta(\lambda)-\frac{m}{8\pi\sqrt{\lambda}}\theta(\lambda)[J_1(m\sqrt{\lambda})-iN_1(m\sqrt{\lambda})]+\frac{im}{4\pi^2\sqrt{-\lambda}}\theta(-\lambda)K_1(m\sqrt{-\lambda}),$$

$\lambda=(x^0)^2-\mathbf{x}^2$; J_1, N_1, K_1 are the Bessel functions of the first order. The application of $E\rightarrow\pm\sqrt{\mathbf{p}^2+m^2}-i\delta$ results in non-existence of the integral. Meanwhile, the massless limit is made in the integrand in the Feynman gauge with no problems. Please remember that integrals are also the limits of the Riemann integral sums. The $m\rightarrow 0$ limits are made first sometimes.

The application of the mass shell condition in the Weinberg-Tucker-Hammer $2(2S+1)$-formalism leads to the fact that we would not be able to write the dynamical equation in the covariant form $[\gamma^{\mu\nu}\partial_\mu\partial_\nu-m^2]\Psi_{(6)}(x)=0$. The information about the propagation of the longitudinal modes would be lost (cf. formulas (19,20,27,28) of the first paper [3]). Moreover, the Weinberg equation and the mapping of the Tucker-Hammer equation to the antisymmetric tensor formalism have different physical contents on the interaction level [7, 8].

However, I take this opportunity to note that problems (frequently forgotten) may exist with the direct application of $m\rightarrow 0$ in relativistic quantum equations.

[1] Similar mathematical examples are presented in https://en.wikipedia.org/wiki/Iterated limit.

The case is: when the solutions are constructed on using the relativistic boosts in the momentum space (the mass may appear in the denominator, $\sim 1/m^n$, which cancels the mass terms of the equation giving the non-zero corresponding result).

Next, if we would always apply the mass shell condition first then we come to the derivative paradox of the previous paper [1]. Finally, the condition $E^2 - \mathbf{p}^2 = m^2$ does not always imply the generally-accepted special relativity only. For instance, see the Kapuscik work, Ref. [9], who showed that similar expressions for energy and momentum exist for particles with $V > c$ and $m_\infty \in \Re e$.

Meanwhile, the case $m = 0$ appears to be equivalent to the light cone condition $r = ct$, which can be taken even without the mass shell condition to study the theories extending the special relativity. Not all realizes that it can be used to deduce the Lorentz transformations between two different reference frames. Just take squares and use the lineality: $r_1^2 - c^2 t_1^2 = 0 = r_2^2 - c^2 t_2^2$. Hence, in $d = 1 + 1$ we have

$$x_2 = \gamma(x_1 - vt_1)\,, \quad y_2 = y_1\,, \quad z_2 = z_1\,, \quad t_2 = \alpha(t_1 - \frac{\beta}{c}x_1) \tag{4}$$

with $\alpha = \gamma = 1/\sqrt{1 - \frac{v^2}{c^2}}$, the Lorentz factor; $\beta = v/c$.

Thus, while for physicists everything is obvious in the solutions of the paradoxes, this is not so for mathematicians.

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