On $q-$asymptotics for $q-$difference-differential equations with Fuchsian and irregular singularities

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January 9, 2012

Abstract

This work is devoted to the study of a Cauchy problem for a certain family of $q-$difference-differential equations having Fuchsian and irregular singularities. For given formal initial conditions, we first prove the existence of a unique formal power series $\hat{X}(t,z)$ solving the problem. Under appropriate conditions, $q-$Borel and $q-$Laplace techniques (firstly developed by J.-P. Ramis and C. Zhang) help us in order to construct actual holomorphic solutions of the Cauchy problem whose $q-$asymptotic expansion in t, uniformly for z in the compact sets of $\mathbb{C}$, is $\hat{X}(t,z)$. The small divisors phenomenon owing to the Fuchsian singularity causes an increase in the order of $q-$exponential growth and the appearance of a subexponential Gevrey growth in the asymptotics.

Key words: $q-$difference-differential equations, $q-$Laplace transform, formal power series solutions, $q-$Gevrey asymptotic expansions, small divisors, Fuchsian and irregular singularities.
2010 MSC: 34K25, 34M25, 34M30, 33E30.

1 Introduction

Partial differential equations of the form

$$t^{2r_2}\partial_t^{r_2}(z\partial_z)^{r_1}\partial_z^{S}u(t,z) = F(t,z,\partial_t,\partial_z)u(t,z), \tag{1}$$

$S, r_1, r_2 \in \mathbb{N} := \{0,1,...\}$ and where F is some differential operator with polynomial coefficients, have been studied by the second author in [15, 18]. These equations belong to a class of partial differential equations with both irregular singularity at $t=0$ in the sense of Mandai [21] (see [6, 21, 22]) and Fuchsian singularity at $z=0$ (see, for example, [1, 4, 9, 11, 20, 30]).

Departing from $1-$Borel summable formal initial data in some direction $d \in \mathbb{R}$

$$(\partial_z^j \hat{u})(t,0) = \hat{u}_j(t) \in \mathbb{C}[[t]], \quad 0 \leq j \leq S-1 \tag{2}$$

one can construct the formal solution $\hat{u}(t,z) = \sum_{m\geq 0}\hat{u}_m(t)z^m/m! \in (\mathbb{C}[[t]])[[z]]$ of (1), (2).

If $r_1 = 0$, $\hat{u}(t,z)$ is 1-Borel summable with respect to t in the direction d, if this is well chosen, as a series with coefficients in the Banach space of holomorphic functions near the origin (in z) with the supremum norm, see [18]. Whereas if $r_1 \neq 0$, the Gevrey order with respect to t suffers increasement, caused by the presence of small divisors introduced by the Fuchsian operator $(z\partial_z)^{r_1}$, see [15].

As a $q-$analog of the problem (1),(2) where ∂_t is replaced by the operator $(f(qt)-f(t))/(qt-t)$ for $q\in\mathbb{C}$ (which formally tends to ∂_t as $|q|$ tends to 1), we consider the $q-$difference-differential equation

$$((z\partial_z+1)^{r_1}(t\sigma_q)^{r_2}+1)\partial_z^S\hat{X}(t,z)=\sum_{k=0}^{S-1}b_k(z)(t\sigma_q)^{m_{0,k}}(\partial_z^k\hat{X})(t,zq^{-m_{1,k}}) \tag{3}$$

with given initial conditions

$$(\partial_z^j\hat{X})(t,0)=\hat{X}_j(t)\in\mathbb{C}[[t]],\quad 0\le j\le S-1, \tag{4}$$

where S,$m_{0,k}$,$m_{1,k}$ are nonnegative integers for $0\le k\le S-1$ and $q\in\mathbb{C}$ with $|q|>1$. σ_q stands for the dilation operator $(\sigma_q\hat{X})(t,z)=\hat{X}(qt,z)$, and $b_k(z)$ are polynomials in z. As in previous works [16], [17], the map $(t,z)\mapsto(q^{m_{0,k}}t,zq^{-m_{1,k}})$ is assumed to be volume shrinking, meaning that the modulus of the Jacobian determinant $|q|^{m_{0,k}-m_{1,k}}<1$. We will always assume that $r_1\ge 0$, while $r_2\ge 1$.

Advanced/delayed partial differential equations have also been widely studied, see for example [12, 13, 14, 23, 29, 32]. Some authors have considered the use of special functions transforms for the study of the asymptotic properties of the solutions of $q-$difference-differential equations [10, 24]. Our present work is a contribution to this area.

This Cauchy problem (3), (4) has a unique formal solution $\hat{X}(t,z)=\sum_{h\ge 0}\hat{X}_h(t)\frac{z^h}{h!}$, where $\hat{X}_h(t)=\sum_{m\ge 0}f_{m,h}t^m\in\mathbb{C}[[t]]$, $h\ge 0$ (see Lemma 5). Our main result (Theorem 5) states the construction of an actual solution $X(t,z)$ which is asymptotically represented by $\hat{X}(t,z)$ in some sense to precise. For this purpose, we study the auxiliary Cauchy problem

$$((z\partial_z+1)^{r_1}\tau^{r_2}+1)\partial_z^S\hat{W}(\tau,z)=\sum_{k=0}^{S-1}b_k(z)\tau^{m_{0,k}}(\partial_z^k\hat{W})(\tau,zq^{-m_{1,k}}) \tag{5}$$

with initial conditions

$$(\partial_z^j\hat{W})(\tau,0)=\hat{W}_j(\tau)\in\mathbb{C}[[t]],\quad 0\le j\le S-1. \tag{6}$$

The $q-$Laplace transform is the key when reducing the study of (3), (4) to this auxiliary problem (see Lemma 6). The $q-$Laplace transform we consider was introduced by J.-P. Ramis and C. Zhang in [28], and in recent years it has been used with great success in the study of the asymptotic properties of solutions of $q-$difference equations, see [8], in much the same way as the classical Laplace-Borel transform has been applied to the asymptotic study of formal solutions to differential equations and singular perturbation problems in the complex domain (see the works of W. Balser [2, 3], B. Malgrange [19], J.-P. Ramis [26] or O. Costin [7]).
This new Cauchy problem (5), (6) is studied in two respects.

Firstly, we study the behabiour of the solution when departing from initial data W_j being holomorphic functions defined in a $q-$spiral $Vq^{\mathbb{Z}}=\{vq^h : v\in V,\ h\in\mathbb{Z}\}$, with $q-$exponential growth (of order 2). Here, $V\subseteq\mathbb{C}\setminus\{0\}$ is a well chosen bounded open set and q is also well chosen. In Theorem 2 we prove there exists a unique solution of (5), (6),

$$W(\tau,z)=\sum_{h\ge 0}W_h(\tau)\frac{z^h}{h!}, \tag{7}$$

holomorphic on $Vq^{\mathbb{Z}}\times\mathbb{C}$ and of q-exponential growth (of order 1) in τ, in the terminology of [28], uniformly for z in any compact set of $\mathbb{C}$. The increase in the order may be seen as an effect of the small divisors appearing in the problem.

Secondly, if one departs from functions W_j, $0 \leq j \leq S-1$, which are holomorphic near the origin, the coefficients in (7) turn out to be holomorphic functions in discs D_h with radii tending to 0 as h tends to infinity (see Theorem 4). Indeed,

$$\sup_{\tau \in \overline{D}_h} |\partial^n W_h(\tau)| \leq C_1 \Big(\frac{1}{T}\Big)^n \Big(\frac{1}{X_1}\Big)^h n!h!(h+1)^{\frac{r_1 n}{r_2}} |q|^{-h^2/2}, \quad n, h \geq 0.$$

Departing from initial conditions under both assumptions, one can apply the $q-$Laplace transform on W_h (see Proposition 3), obtaining holomorphic functions which are defined in a common domain $\mathcal{T}_{\lambda,q,\delta,r_0}$ (see (9) for its definition) for all $h \geq 0$.

The main result (Theorem 5) states that, if one departs from well chosen formal initial conditions $\hat{X}_j$, $0 \leq j \leq S-1$, one can find a solution of (3), (4)

$$X(t,z) = \sum_{h \geq 0} \mathcal{L}_q^{\lambda}(W_h)(t) \frac{z^h}{h!},$$

which is holomorphic in $\mathcal{T}_{\lambda,q,\delta,r_0} \times \mathbb{C}$, and such that given $R > 0$, there exist constants $\tilde{C}, \tilde{D} > 0$ such that for every $n \in \mathbb{N}$, $n \geq 1$, one has

$$\Big| X(t,z) - \sum_{h \geq 0} \sum_{m=0}^{n-1} f_{m,h} t^m \frac{z^h}{h!} \Big| \leq \tilde{C} \tilde{D}^n \Gamma(\frac{r_1}{r_2}(n+1)) |q|^{n(n-1)/2} |t|^n$$

for every $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$, $z \in D(0,R)$. Again one may note that the small divisors phenomenon has caused the appearance of the term $\Gamma(\frac{r_1}{r_2}(n+1))$.

The paper is organized as follows. Section 2 provides information concerning the $q-$Laplace transform. Section 3 is devoted to the study of a first auxiliary Cauchy problem in suitable weighted Banach spaces of formal Laurent series. This is needed in the following Section. More precisely, in the proof of Theorem 2. A second Cauchy problem in weighted Banach spaces of formal Taylor series is stated Section 5, leading to Theorem 4. Finally, in Section 7, the solution of the main problem is constructed, giving asymptotic properties (see Theorem 5). Some final remarks on the nature of the solution in the special case that $r_1 = 0$, in which no small divisors appear, are remarked.

We fix some conventions. $\mathbb{C}^*$ stands for $\mathbb{C} \setminus \{0\}$, and $\mathbb{N}$ for the set $\{0, 1, 2, \cdots\}$. $D(0,r)$ denotes the open disc with center 0 and radius $r > 0$. Given a set $V \subset \mathbb{C}$ and $q \in \mathbb{C}$, we define

$$Vq^{\mathbb{Z}} = \{vq^h : v \in V,\ h \in \mathbb{Z}\}, \quad Vq^{\mathbb{N}} = \{vq^h : v \in V,\ h \in \mathbb{N}\}.$$

2 A $q-$analogue of the Laplace transform and $q-$asymptotic expansion

In [28] and [31], the authors introduce the concept of a $q-$analog of the Laplace transform. In this section, we recall this concept and some of its main properties. The proof of the next Proposition is in the spirit of the one corresponding to the Proposition 3, so we only briefly sketch the steps taken.

Proposition 1 *Let $q \in \mathbb{C}$ such that $|q| > 1$. Let V be an open and bounded set in $\mathbb{C}^*$ and $D(0,\rho_0)$ a disc such that $V \cap D(0,\rho_0) \neq \emptyset$. Let $(\mathbb{F}, ||.||_{\mathbb{F}})$ be a complex Banach space. We also* *fix a holomorphic function $\phi : Vq^{\mathbb{N}} \cup D(0,\rho_0) \to \mathbb{F}$ which satisfies the following estimates : there exist $C, M > 0$ such that*

$$||\phi(xq^m)||_{\mathbb{F}} \leq M|q|^{m^2/2}C^m \tag{8}$$

for all $m \geq 0$, all $x \in V$. Let Θ be the Theta Jacobi function defined in $\mathbb{C}^*$ by

$$\Theta(x) = \sum_{n\in\mathbb{Z}} q^{-n(n-1)/2}x^n.$$

Let $\delta > 0$ and $\lambda \in V \cap D(0,\rho_0)$. We denote by

$$\mathcal{R}_{\lambda,q,\delta} = \{t \in \mathbb{C}^*/|1 + \frac{\lambda}{tq^k}| > \delta, \forall k \in \mathbb{Z}\}, \quad \mathcal{T}_{\lambda,q,\delta,r_1} = \mathcal{R}_{\lambda,q,\delta} \cap D(0,r_1). \tag{9}$$

The $q-$Laplace transform of ϕ in the direction $\lambda q^{\mathbb{Z}}$ is defined by

$$\mathcal{L}_q^{\lambda}(\phi)(t) := \sum_{m\in\mathbb{Z}} \phi(q^m\lambda)/\Theta(\frac{q^m\lambda}{t})$$

for all $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$, if $r_1 < |\lambda q^{1/2}|/C$. Moreover, $\mathcal{L}_q^{\lambda}(\phi)(t)$ defines a bounded holomorphic function on $\mathcal{T}_{\lambda,q,\delta,r_1}$ with values in $\mathbb{F}$ when $r_1 < |\lambda q^{1/2}|/C$. Assume that the function ϕ has the following Taylor expansion

$$\phi(\tau) = \sum_{n\geq 0} \frac{f_n}{q^{n(n-1)/2}}\tau^n \tag{10}$$

on $D(0,\rho_0)$, where $f_n \in \mathbb{F}$, $n \geq 0$. Then, there exist two constants $D, B > 0$ such that

$$||\mathcal{L}_q^{\lambda}(\phi)(t) - \sum_{m=0}^{n-1} f_m t^m||_{\mathbb{F}} \leq DB^n|q|^{n(n-1)/2}|t|^n \tag{11}$$

for all $n \geq 1$, for all $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$.

Remark: In the situation described by (11) it is said that $\mathcal{L}_q^{\lambda}(\phi)$ admits the series $\sum_{m=0}^{\infty} f_m t^m$ as $q-$Gevrey asymptotic expansion of order 1 (whenever the exponent of $|q|$ in the bounds is $n(n-1)/(2r)$ the order is said to be r). Analogously, a function that satisfies estimates such as (8) is said to have $q-$exponential growth of order 1 in $Vq^{\mathbb{N}}$.
If $\phi(z) = \sum_{n\geq 0} a_n z^n$ is an entire function such that there exists $C > 0$ such that

$$|a_n| \leq C\exp(-(n-\alpha)^2/2)$$

for all $n \geq 0$ and some $\alpha \geq 0$, then ϕ satisfies the estimates (8). For a reference, see [25].

Proof Since the Theta function $\Theta(x)$ satisfies the $q-$difference equation $\Theta(qx) = qx\Theta(x)$ for all $x \in \mathbb{C}^*$, we get that

$$\Theta(\frac{q^m\lambda}{t}) = q^{m(m+1)/2}(\frac{\lambda}{t})^m\Theta(\frac{\lambda}{t}) \tag{12}$$

for all $t \in \mathbb{C}^*$. Moreover, from Lemma 4.6 of [27], there exists $K_1 > 0$ such that

$$|\Theta(q^m\lambda/t)| \geq K_1\delta\sum_{n\in\mathbb{Z}} |q|^{-n(n-1)/2}|\frac{q^m\lambda}{t}|^n \tag{13}$$

for all $t \in \mathcal{R}_{\lambda,q,\delta}$, all $m \in \mathbb{Z}$. From this, we have that

$$|\Theta(\frac{\lambda}{t})| \geq K_1\delta|q|^{-K(K-1)/2}|\frac{\lambda}{t}|^K \tag{14}$$

for all $t \in \mathcal{R}_{\lambda,q,\delta}$.

Let $K \geq 0$ be an integer. The estimates in (11) are achieved in two steps. As a first step, one derives

$$\sum_{m>0} ||\frac{\phi(q^m\lambda)}{\Theta(q^m\lambda/t)}||_{\mathbb{F}} \leq D_1(B_1)^K|q|^{K(K-1)/2}|t|^K, \quad t \in \mathcal{T}_{\lambda,q,\delta,r_1} \tag{15}$$

for some fixed $0 < r_1 < |\lambda||q|^{\frac{1}{2}}/C$, and positive constants B_1, D_1, not depending on K. This part of the proof rests on the estimates satisfied by ϕ (see (8)) and (14). In the second step, one determines

$$||\sum_{m\leq 0}\phi(q^m\lambda)/\Theta(q^m\lambda/t) - \sum_{n=0}^{K} f_nt^n||_{\mathbb{F}} \leq D_2(B_2)^K|q|^{K(K-1)/2}|t|^K, \quad t \in \mathcal{T}_{\lambda,q,\delta,r_1}, \tag{16}$$

for some constants $B_2, D_2 > 0$. This is made by means of a rearrangement of the elements in the series above, and then by splitting the resulting expression in two parts, and taking into account the Taylor expansion of ϕ, (10), near the origin.

One can conclude from the estimates in (15), (16) and

$$||\sum_{m\in\mathbb{Z}}\phi(q^m\lambda)/\Theta(q^m\lambda/t) - \sum_{n=0}^{K-1} f_nt^n||_{\mathbb{F}} \leq ||\sum_{m\in\mathbb{Z}}\phi(q^m\lambda)/\Theta(q^m\lambda/t) - \sum_{n=0}^{K} f_nt^n||_{\mathbb{F}} + ||f_Kt^K||_{\mathbb{F}}.$$

□

It is straightforward to check the following

Proposition 2 *Let V be an open and bounded set in $\mathbb{C}^*$ and $D(0,\rho_0)$ be a disc such that $V \cap D(0,\rho_0) \neq \emptyset$. Let ϕ be a holomorphic function on $Vq^{\mathbb{N}} \cup D(0,\rho_0)$ with values in $(\mathbb{F}, ||.||_{\mathbb{F}})$ which satisfies the estimates : There exist $C, K > 0$, such that $||\phi(xq^m)||_{\mathbb{F}} \leq K|q|^{m^2/2}C^m$ for all $m \geq 0$, all $x \in V$. Then, the function $m\phi(\tau) = \tau\phi(\tau)$ is holomorphic on $Vq^{\mathbb{N}} \cup D(0,\rho_0)$ and satisfies estimates of the form (8). Let $\lambda \in V \cap D(0,\rho_0)$. We have the following equality*

$$\mathcal{L}_q^\lambda(m\phi)(t) = t\mathcal{L}_q^\lambda(\phi)(qt)$$

for all $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$, if $r_1 < |\lambda q^{1/2}|/(C|q|)$.

For convenience, we recall the following concepts.

Definition 1 *A series $\hat{f}(t) = \sum_{n\geq 0} f_nt^n \in \mathbb{C}[[t]]$ is said to be $q-$Gevrey of order 1 if its so-called formal $q-$Borel transform of order 1,*

$$\hat{\mathcal{B}}_q\hat{f}(\tau) = \sum_{n\geq 0}\frac{f_n}{q^{n(n-1)/2}}\tau^n,$$

converges (i.e. it has positive radius of convergence).
The formal $q-$Laplace transform of order 1 of a series $\hat{g}(\tau) = \sum_{n\geq 0} g_n\tau^n \in \mathbb{C}[[\tau]]$ is defined as

$$\hat{\mathcal{L}}_q\hat{g}(t) = \sum_{n\geq 0} q^{n(n-1)/2}g_nt^n,$$

so that these formal transforms are inverse of each other.

It is immediate to check that, in agreement with Proposition 2, that for every $\hat{g} \in \mathbb{C}[[\tau]]$, we have

$$\hat{\mathcal{L}}_q(\tau\hat{g})(t) = t\hat{\mathcal{L}}_q\hat{g}(qt). \tag{17}$$

3 A Cauchy problem in a weighted Banach space of formal Laurent series

With the help of the $q-$Laplace transform we will change our initial problem (3), (4) into an equivalent one (5), (6) whose study will require the consideration of two auxiliary Cauchy problems. The first of them, which emerges in this Section, will be crucial in the study of the $q-$exponential growth of the coefficients of a solution of (5), (6). Although our equation involves a complex number q with $|q| > 1$, in this Section and in Section 5 we will be only concerned with the value $|q|$, so we directly work with a real value $q > 1$.

Definition 2 *We consider the vector space $\mathbb{E}_{q,(T,X)}$ of formal Laurent power series*

$$V(\xi, x) = \sum_{l\in\mathbb{Z}, h\geq 0} v_{l,h}\xi^l \frac{x^h}{h!} \in \mathbb{C}[[\xi, \xi^{-1}, x]] \tag{18}$$

such that

$$||V(\xi, x)||_{(T,X)} := \sum_{l\in\mathbb{Z}, h\geq 0} \frac{|v_{l,h}|}{q^{P(l,h)}} T^l \frac{X^h}{h!} < \infty,$$

where $T, X > 0$, $q > 1$ are positive real numbers and where

$$P(l, h) = \begin{cases} \frac{1}{4}l^2 + \frac{1}{2}lh - \frac{1}{2}h^2 & \text{if } l \geq 0, h \geq 0, \\ -(1/2)h^2 & \text{if } l \leq 0, h \geq 0. \end{cases}$$

The space $(\mathbb{E}_{q,(T,X)}, ||.||_{(T,X)})$ is a Banach space.

Remark: Notice that we have a continuous inclusion $(\mathbb{E}_{q,(T,X')}, ||.||_{(T,X')}) \hookrightarrow (\mathbb{E}_{q,(T,X)}, ||.||_{(T,X)})$ when $0 < X \leq X'$.

We consider the integration operator ∂_x^{-1} defined on $\mathbb{C}[[\xi, \xi^{-1}, x]]$ by

$$\partial_x^{-1}(V(\xi, x)) := \sum_{l\in\mathbb{Z}, h\geq 1} v_{l,h-1}\xi^l \frac{x^h}{h!} \in \mathbb{C}[[\xi, \xi^{-1}, x]].$$

The main result in this Section, Theorem 1, rests on the following technical lemmas whose proofs are omitted for simplicity.

Lemma 1 *Let $m_1, s, h_1, h_2 \geq 0$ be nonnegative integers. Let $T, X > 0$. Assume that the inequalities hold $s + h_2 \geq 2h_1$, $m_1 \geq s + h_2$. Then, there exist $C > 0$ (depending q, s, h_1, h_2, m_1) such that $||x^s(\partial_x^{-h_2}V)(q^{h_1}\xi, \frac{x}{q^{m_1}})||_{(T,X)} \leq CX^{(s+h_2)}||V(\xi, x)||_{(T,X)}$ for all $V(\xi, x) \in \mathbb{E}_{q,(T,X)}$.*

Lemma 2 *Let $s, h_1 \geq 0$ and $T_0, X_0 > 0$. Then, there exists $0 < X_1 < X_0$ and for all $T_1 > 0$ satisfying $q^{-h_1}T_0 \leq T_1 \leq T_0 q^{\frac{s}{2}-h_1}$, there exists a constant $C_1 > 0$ (depending on q, s, h_1, T_0, X_0) such that $||x^sV(q^{h_1}\xi, x)||_{(T_1,X_1)} \leq C_1||V(\xi, x)||_{(T_0,X_0)}$ for all $V(\xi, x) \in \mathbb{E}_{q,(T_0,X_0)}$.*

Lemma 3 *Let $h_2 \geq 0$ and $T_0, X_0 > 0$. Then, there exists $0 < X_1 < X_0$ and for all $T_1 > 0$ satisfying $T_0 \leq T_1 \leq T_0 q^{h_2/2}$, there exists a constant $C_2 > 0$ (depending on q, h_2, T_0, X_0) such that $||\partial_x^{-h_2} V(\xi, x)||_{(T_1, X_1)} \leq C_2 ||V(\xi, x)||_{(T_0, X_0)}$ for all $V(\xi, x) \in \mathbb{E}_{q,(T_0, X_0)}$.*

Let $S, m_{0,k}, m_{1,k}$, $0 \leq k \leq S-1$ be positive integers. Let $\mathcal{D}$ the linear operator from $\mathbb{C}[[\xi, \xi^{-1}, x]]$ into $\mathbb{C}[[\xi, \xi^{-1}, x]]$ defined by

$$\mathcal{D}(V(\xi, x)) := \partial_x^S V(\xi, x) - \sum_{k=0}^{S-1} a_k(x) (\partial_x^k V)(q^{m_{0,k}} \xi, x/q^{m_{1,k}}),$$

for all $V \in \mathbb{C}[[\xi, \xi^{-1}, x]]$, where $a_k(x) = \sum_{s \in I_k} a_{ks} x^s \in \mathbb{C}[x]$, with I_k be a finite subset of $\mathbb{N}$, for $0 \leq k \leq S-1$.

We make the following hypothesis.

Assumption (A) For all $0 \leq k \leq S-1$, for all $s \in I_k$, we have

$$s - k \geq 2m_{0,k} \quad , \quad m_{1,k} \geq s + S - k.$$

Remark: This assumption can be weakened to be

$$s + S - k \geq 2m_{0,k} \quad , \quad m_{1,k} \geq s + S - k,$$

for all $0 \leq k \leq S-1$, for all $s \in I_k$ if one asks additional conditions on the growth properties of the initial contidions in (4). This constraints would be related to the constant T in $\mathbb{E}_{q,(T,X)}$.

We consider the operator $\mathcal{A}$ from $\mathbb{C}[[\xi, \xi^{-1}, x]]$ into $\mathbb{C}[[\xi, \xi^{-1}, x]]$ defined by

$$\mathcal{A}(V(\xi, x)) = V(\xi, x) - \mathcal{D}(\partial_x^{-S} V(\xi, x)) = \sum_{k=0}^{S-1} a_k(x) (\partial_x^{k-S} V)(q^{m_{0,k}} \xi, x/q^{m_{1,k}})$$

for all $V \in \mathbb{C}[[\xi, \xi^{-1}, x]]$.

From Lemma 1, we deduce the following

Lemma 4 *Let $T > 0$. Then, there exists $X > 0$ such that $\mathcal{A}$ is a linear bounded operator from $(\mathbb{E}_{q,(T,X)}, || \cdot ||_{(T,X)})$ into itself. Moreover, we have that*

$$||\mathcal{A}(V(\xi, x))||_{(T,X)} \leq \frac{1}{2} ||V(\xi, x)||_{(T,X)},$$

for all $V \in \mathbb{E}_{q,(T,X)}$.

From Lemma 4, we deduce the next

Corollary 1 *Let $T > 0$. Then, there exists $X > 0$ such that $\mathcal{D} \circ \partial_x^{-S}$ is an invertible linear operator from $(\mathbb{E}_{q,(T,X)}, ||.||_{(T,X)})$ into itself. In particular, there exists $C > 0$ such that*

$$||\mathcal{D}(\partial_x^{-S} b(\xi, x))||_{(T,X)} \leq C ||b(\xi, x)||_{(T,X)}$$

for all $b(\xi, x) \in \mathbb{E}_{q,(T,X)}$.

Theorem 1 *Let $S \geq 1$ be an integer. For all $0 \leq k \leq S-1$, let $m_{0,k}, m_{1,k}$ be positive integers and $a_k(x) = \sum_{s \in I_k} a_{ks} x^s \in \mathbb{C}[x]$. We make the hypothesis that the assumption* **(A)** *holds.*
We consider the following functional equation

$$\partial_x^S V(\xi, x) = \sum_{k=0}^{S-1} a_k(x)(\partial_x^k V)(q^{m_{0,k}}\xi, x/q^{m_{1,k}}) \tag{19}$$

with initial conditions

$$(\partial_x^j V)(\xi, 0) = \phi_j(\xi) \quad , \quad 0 \leq j \leq S-1. \tag{20}$$

We assume that $\phi_j(\xi) \in \mathbb{E}_{q,(T,X_0)}$, for $0 \leq j \leq S-1$, where $X_0 > 0$ and $T > 0$. Then, there exists $X > 0$ such that the problem (19), (20) has a unique solution $V(\xi, x) \in \mathbb{E}_{q,(T,X)}$. Moreover, there exists $C > 0$ (depending on S,q,$a_k(x)$,$m_{0,k}$,$m_{1,k}$, for $0 \leq k \leq S-1$ and X_0,T) such that

$$||V(\xi, x)||_{(T,X)} \leq C \sum_{j=0}^{S-1} ||\phi_j(\xi)||_{(T,X_0)}.$$

Proof A formal series $V(\xi, x) \in \mathbb{C}[[\xi, \xi^{-1}, x]]$ which satisfies (20) can be written in the form $V(\xi, x) = \partial_x^{-S} U(\xi, x) + I(\xi, x)$ where

$$I(\xi, x) = \sum_{j=0}^{S-1} \phi_j(\xi) \frac{x^j}{j!}$$

and $U(\xi, x) \in \mathbb{C}[[\xi, \xi^{-1}, x]]$. A formal series $V(\xi, x) \in \mathbb{C}[[\xi, \xi^{-1}, x]]$ is a solution of the problem (19), (20) if and only if $U(\xi, x)$ satisfies the equation

$$\mathcal{D}(\partial_x^{-S} U(\xi, x)) = -\mathcal{D}(I(\xi, x)). \tag{21}$$

By construction, we have that

$$-\mathcal{D}(I(\xi, x)) = \sum_{k=0}^{S-1} \sum_{j=k}^{S-1} \sum_{s \in I_k} \frac{a_{ks}}{q^{m_{1,k}(j-k)}(j-k)!} x^{s+j-k} \phi_j(q^{m_{0,k}}\xi)$$

From Lemma 2 (taking $T_0 = T_1 := T$) and the assumption **(A)**, there exists $X_1 > 0$ such that

$$x^{s+j-k} \phi_j(q^{m_{0,k}}\xi) \in \mathbb{E}_{q,(T,X_1)}$$

for all $0 \leq k \leq S-1$, all $k \leq j \leq S-1$, all $s \in I_k$. Moreover, there exists $C_1 > 0$ (depending on I_k,j,$m_{0,k}$,X_0,T) such that

$$||x^{s+j-k} \phi_j(q^{m_{0,k}}\xi)||_{(T,X_1)} \leq C_1 ||\phi_j(\xi)||_{(T,X_0)}. \tag{22}$$

We deduce that $\mathcal{D}(I(\xi, x)) \in \mathbb{E}_{q,(T,X_1)}$ and from (22) there exists a constant $C_1' > 0$ (depending on q,$a_k(x)$,$m_{0,k}$,$m_{1,k}$, for $0 \leq k \leq S-1$ and X_0,T) such that

$$||\mathcal{D}(I(\xi, x))||_{(T,X_1)} \leq C_1' \sum_{j=0}^{S-1} ||\phi_j(\xi)||_{(T,X_0)}. \tag{23}$$

From Corollary 1, we deduce that the equation (21) has a unique solution $U(\xi,x) \in \mathbb{E}_{q,(T,X_1)}$. Moreover, there exists a constant $C_2 > 0$ (depending on q,$a_k(x)$,$m_{0,k}$,$m_{1,k}$, for $0 \leq k \leq S-1$) such that

$$||U(\xi,x)||_{(T,X_1)} \leq C_2 ||\mathcal{D}(I(\xi,x))||_{(T,X_1)}. \tag{24}$$

Take $T_1 = T_0 := T$ in Lemma 3. We derive there exists $X_2 < X_1$ such that $\partial_x^{-S} U(\xi,x) \in \mathbb{E}_{q,(T,X_2)}$. Moreover, there exists a constant $C_3 > 0$ (depending on q,S,T,X_1) such that

$$||\partial_x^{-S} U(\xi,x)||_{(T,X_2)} \leq C_3 ||U(\xi,x)||_{(T,X_1)} \tag{25}$$

From Lemma 2 (with $T_0 = T_1 := T$), there exists $X_3 < X_2$ such that $I(\xi,x) \in \mathbb{E}_{q,(T,X_3)}$. Moreover, there exists a constant $C_4 > 0$ (depending on S,q,T,X_0) such that

$$||I(\xi,x)||_{(T,X_3)} \leq C_4 \sum_{j=0}^{S-1} ||\phi_j(\xi)||_{(T,X_0)}. \tag{26}$$

Finally, the formal series $V(\xi,x) = \partial_x^{-S} U(\xi,x) + I(\xi,x)$, solution of the problem (19), (20), belongs to $\mathbb{E}_{q,(T,X_3)}$. Moreover, from the inequalities (23), (24), (25) and (26), we get a constant C_5 (depending on S,q,$a_k(x)$,$m_{0,k}$,$m_{1,k}$, for $0 \leq k \leq S-1$ and X_0,T) such that

$$||V(\xi,x)||_{(T,X_3)} \leq C_5 \sum_{j=0}^{S-1} ||\phi_j(\xi)||_{(T,X_0)}.$$

□

4 A Cauchy problem in analytic spaces of $q-$exponential growth

Let $S \geq 1$, $r_1, r_2 \geq 0$ be integers. For all $0 \leq k \leq S-1$, let $m_{0,k}$,$m_{1,k}$ be positive integers and $b_k(z) = \sum_{s \in I_k} b_{ks} z^s$ be a polynomial in z, where I_k is a subset of $\mathbb{N}$.

Lemma 5 *For every choice of formal series $\hat{X}_j \in \mathbb{C}[[t]]$, $0 \leq j \leq S-1$, the Cauchy problem (3), (4) has a unique solution in the form of a formal power series $\hat{X}(t,z) = \sum_{h \geq 0} \hat{X}_h(t) \frac{z^h}{h!}$, where $\hat{X}_h \in \mathbb{C}[[t]]$ for every $h \geq 0$.*

Proof Let us put $\hat{X}_h(t) = \sum_{m \geq 0} f_{m,h} t^m$, $h \geq 0$, $f_{m,h} \in \mathbb{C}$. The values $f_{m,h}$, $m \geq 0$, $0 \leq h \leq S-1$, are given by the initial conditions. The coefficients $f_{m,S}$, $m \geq 0$ can be found by substituting $\hat{X}$ in (3) and looking at the coefficients of z^0. We can repeat this argument as the second index in $f_{m,h}$ increases. □

With the help of the $q-$Laplace transform, we reformulate our problem. Consider the Cauchy problem

$$((z\partial_z + 1)^{r_1} \tau^{r_2} + 1) \partial_z^S \hat{W}(\tau,z) = \sum_{k=0}^{S-1} b_k(z) \tau^{m_{0,k}} (\partial_z^k \hat{W})(\tau, zq^{-m_{1,k}}) \tag{27}$$

with initial conditions

$$(\partial_z^j \hat{W})(\tau,0) = \hat{W}_j(\tau) \in \mathbb{C}[[\tau]] \ , \quad 0 \leq j \leq S-1. \tag{28}$$

Lemma 6 *The formal series* $\hat{X}(t,z)=\sum_{h\geq 0}\hat{X}_h(t)\frac{z^h}{h!}$, *where* $\hat{X}_h\in\mathbb{C}[[t]]$ *for every* $h\geq 0$, *satisfies the Cauchy problem (3), (4) if, and only if, the formal series* $\hat{W}(\tau,z)=\sum_{h\geq 0}\hat{\mathcal{B}}_q\hat{X}_h(\tau)\frac{z^h}{h!}$ *satisfies the Cauchy problem (27), (28) with* $W_j(\tau)=\hat{\mathcal{B}}_q\hat{X}_j$, $0\leq j\leq S-1$.
Conversely, $\hat{W}(\tau,z)=\sum_{h\geq 0}\hat{W}_h(\tau)\frac{z^h}{h!}$, *with* $\hat{W}_h\in\mathbb{C}[[\tau]]$ *for every* $h\geq 0$, *satisfies the Cauchy problem (27), (28) if, and only if, the formal series* $\hat{X}(t,z)=\sum_{h\geq 0}\hat{\mathcal{L}}_q\hat{W}_h(t)\frac{z^h}{h!}$ *satisfies the Cauchy problem (3), (4) with* $\hat{X}_j(t)=\hat{\mathcal{L}}_q\hat{W}_j(t)$ *for* $0\leq j\leq S-1$.

Proof It suffices to insert each series in the corresponding Cauchy problem and apply (17). □

Let V be an open and bounded set in $\mathbb{C}^*$, and $q\in\mathbb{C}$ with $|q|>1$. In the following result we study the $q-$exponential growth of the coefficients of a solution to the Cauchy problem (27), (28). We will depart from initial conditions W_j, $0\leq j\leq S-1$, holomorphic in $Vq^{\mathbb{Z}}$. We make the assumption **(A)** in the previous Section, so that we may apply Theorem 1, and we also suitably choose q and V in order to deal with a small divisors problem.

Theorem 2 *Let the assumption* **(A)** *(of Section 3) be fulfilled by the sets* I_k *and the integers* $m_{0,k}$,$m_{1,k}$, *for* $0\leq k\leq S-1$.
1) We make the following assumptions on q *and on the open set* V: q *is of the form* $q=|q|e^{i\theta}$, *with* $\theta=\frac{2\pi}{br_2}$ *for some* $b\in\mathbb{N}$, $b\geq 1$. *If we denote* $V^{r_2}:=\{x^{r_2}/x\in V\}$, *we assume that there exists* $\varepsilon\in(0,\min\{\pi/b,\pi/2\})$ *such that*

$$V^{r_2}\bigcap\Big(\bigcup_{l=0}^{b-1}S(-\pi+\frac{2\pi l}{b},2\varepsilon)\Big)=\emptyset,$$

where $S(d,\varphi)$ *stands for the unbounded sector in* $\mathbb{C}$ *with vertex at 0, bisected by direction* d *and with opening* φ.
2) The following assumptions on the initial conditions hold: Let $T>0$. *There exists a constant* $K_0>0$ *such that*

$$\sup_{x\in V}|W_j(xq^l)|\leq K_0|q|^{\frac{1}{4}l^2}(\frac{1}{T})^l\frac{1}{1+l^2}\quad,\quad\sup_{x\in V}|W_j(xq^{-l})|\leq K_0T^l\frac{1}{1+l^2}\tag{29}$$

for all $0\leq j\leq S-1$, *all* $l\geq 0$.
Then, there exists a unique solution of (27), (28)

$$(\tau,z)\mapsto W(\tau,z)=\sum_{h\geq 0}W_h(\tau)\frac{z^h}{h!}$$

which is holomorphic on $Vq^{\mathbb{Z}}\times\mathbb{C}$. *Moreover, for all* $\rho>0$, *there exist two constants* $C,T>0$ *(depending on* ρ,S,$|q|$,$b_k(z)$,$m_{0,k}$,$m_{1,k}$, *for* $0\leq k\leq S-1$ *and* T*) such that*

$$\sup_{x\in V,z\in D(0,\rho)}|W(xq^l,z)|\leq CK_0|q|^{\frac{1}{2}l^2}(\frac{1}{T})^l\quad,\quad\sup_{x\in V,z\in D(0,\rho)}|W(xq^{-l},z)|\leq CK_0T^l\tag{30}$$

for all $l\geq 0$ *(where* $K_0>0$ *is defined in (29)).*

Proof From the hypothesis 1) in the statement, there exists $\delta > 0$ such that

$$|(h+1)^{r_1} x^{r_2} q^{r_2 l} + 1| > \delta \tag{31}$$

for all $l \in \mathbb{Z}$, all $h \geq 0$, all $x \in V$.
Remark: Condition 1) in the previous statement could be replaced by a more general condition, namely: Let q and V be such that (31) is verified for some $\delta > 0$ and for all $l \in \mathbb{Z}$, all $h \geq 0$, all $x \in V$. However, we preferred to use 1) because of its easy geometrical interpretation.

We consider the sequence of functions $W_h(\tau)$, $h \geq S$, defined as follows

$$\frac{W_{h+S}(xq^l)}{h!} = \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1 \in I_k} \frac{b_{kh_1} x^{m_{0,k}} q^{m_{0,k} l}}{((h+1)^{r_1} x^{r_2} q^{r_2 l} + 1)} \frac{W_{h_2+k}(xq^l)}{h_2! q^{m_{1,k} h_2}} \tag{32}$$

for all $h \geq 0$, all $l \in \mathbb{Z}$, all $x \in V$. One checks that the sequence $W_h(\tau)$, $h \geq 0$, of holomorphic functions on $Vq^{\mathbb{Z}}$, satisfies the recursion (32) if and only if the formal series $W(\tau, z) = \sum_{h \geq 0} W_h(\tau) \frac{z^h}{h!}$ in the z variable, satisfies the problem (27), (28). From this we deduce that the solution W, if it exists, is unique.
According to (29) and (32), we can recursively prove that the sequence $(w_{l,h})_{l \in \mathbb{Z}, h \geq 0}$ defined by

$$w_{l,h} = \sup_{x \in V} |W_h(xq^l)|, \tag{33}$$

for all $l \in \mathbb{Z}$, all $h \geq 0$, consists of positive real numbers. Due to (31), the sequence $(w_{l,h})_{l \in \mathbb{Z}, h \geq 0}$ satisfies the following inequalities: There exists $r > 0$ (depending on $m_{0,k}$, V) such that

$$\frac{w_{l,h+S}}{h!} \leq \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1 \in I_k} \frac{|b_{kh_1}| r |q|^{m_{0,k} l}}{\delta} \frac{w_{l,h_2+k}}{h_2! |q|^{m_{1,k} h_2}}$$

for all $l \in \mathbb{Z}$, all $h \geq 0$.
We consider the sequence of real numbers $(v_{l,h})_{l \in \mathbb{Z}, h \geq 0}$ defined by the following recursion

$$\frac{v_{l,h+S}}{h!} = \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1 \in I_k} \frac{|b_{kh_1}| r |q|^{m_{0,k} l}}{\delta} \frac{v_{l,h_2+k}}{h_2! |q|^{m_{1,k} h_2}} \tag{34}$$

with initial conditions $v_{l,j} = w_{l,j}$, for $0 \leq j \leq S-1$, all $l \in \mathbb{Z}$. By construction, we have that

$$w_{l,h} \leq v_{l,h} \tag{35}$$

for all $l \in \mathbb{Z}$, all $h \geq 0$.

In the following, we put $a_k(x) = \sum_{s \in I_k} (|b_{ks}| r/\delta) x^s$ for $0 \leq k \leq S-1$ and we consider the formal Laurent series $V(\xi, x) = \sum_{l \in \mathbb{Z}, h \geq 0} v_{l,h} \xi^l \frac{x^h}{h!}$. From the recursion (34), we get that $V(\xi, x)$ satisfies the following Cauchy problem

$$\partial_x^S V(\xi, x) = \sum_{k=0}^{S-1} a_k(x) (\partial_x^k V)(\xi |q|^{m_{0,k}}, x/|q|^{m_{1,k}}) \tag{36}$$

with initial conditions

$$(\partial_x^j V)(\xi, 0) = \phi_j(\xi) := \sum_{l \in \mathbb{Z}} w_{l,j} \xi^l \tag{37}$$

From the hypothesis (29), we get that $\phi_j(\xi)$ belongs to $\mathbb{E}_{|q|,(T,X_0)}$, for all $X_0 > 0$. By hypothesis, the assumption **(A)** holds for the sets I_k and the numbers $m_{0,k}$,$m_{1,k}$. From Theorem 1, we deduce that the unique solution $V(\xi,x)$ of the problem (36), (37) satisfies $V(\xi,x) \in \mathbb{E}_{|q|,(T,X)}$ for a real number $X > 0$. Moreover, there exists a constant $C > 0$ (depending on S,$|q|$,$a_k(x)$,$m_{0,k}$,$m_{1,k}$, for $0 \leq k \leq S-1$ and X_0,T) such that

$$||V(\xi,x)||_{(T,X)} \leq C \sum_{j=0}^{S-1} ||\phi_j(\xi)||_{(T,X_0)}. \tag{38}$$

From the inequality $P(l,h) \leq \frac{l^2}{2} - \frac{h^2}{4}$, for all $l \in \mathbb{Z}$, $h \geq 0$, and (38) we get that there exists a constant $C' > 0$ (depending on S,$|q|$,$a_k(x)$,$m_{0,k}$,$m_{1,k}$, for $0 \leq k \leq S-1$ and X_0,T) such that

$$|v_{l,h}| \leq K_0 C' |q|^{\frac{l^2}{2}} |q|^{\frac{-h^2}{4}} h! (\frac{1}{T})^l (\frac{1}{X})^h \quad , \quad |v_{-l,h}| \leq K_0 C' |q|^{\frac{-h^2}{2}} T^l h! (\frac{1}{X})^h \tag{39}$$

for all $l \geq 0$, all $h \geq 0$, where K_0 is the constant introduced in (29). From the inequalities (35) and (39), we get that

$$\sup_{x \in V, z \in D(0,\rho)} |W(xq^l,z)| \leq K_0 C' |q|^{\frac{l^2}{2}} (\frac{1}{T})^l (\sum_{h \geq 0} |q|^{-h^2/4} (\frac{\rho}{X})^h),$$
$$\sup_{x \in V, z \in D(0,\rho)} |W(xq^{-l},z)| \leq K_0 C' T^l (\sum_{h \geq 0} |q|^{-h^2/2} (\frac{\rho}{X})^h)$$

for all $l \geq 0$, all $\rho > 0$. So that the estimates (30) hold. □

5 Second auxiliary Cauchy problem

Our second approach to the auxiliary problem is to assume the initial conditions W_h, $0 \leq h \leq S-1$, of (27), (28) are holomorphic in suitably small neighbourhoods of 0. Our purpose is to obtain information on the rate of decreasing of the derivatives of the functions W_h, $h \geq 0$, coefficients of the solution constructed in Theorem 2, near the origin. This will be done in the next Section, where we will need the second auxiliary Cauchy problem we deal with in this Section.

Definition 3 *Let $q > 1$ be given. Let us consider the space $\mathbb{H}_{(T,X)}$ of formal power series*

$$V(\xi,x) = \sum_{l \geq 0, h \geq 0} v_{l,h} \xi^l \frac{x^h}{h!} \in \mathbb{C}[[\xi,x]]$$

such that

$$|V(\xi,x)|'_{(T,X)} := \sum_{l \geq 0, h \geq 0} |v_{l,h}| T^l q^{h^2/2} \frac{X^h}{h!} < \infty,$$

where T,X are positive real numbers.

The space $(\mathbb{H}_{(T,X)}, |\cdot|'_{(T,X)})$ is a Banach algebra.
Remark: We have a continuous inclusion $(\mathbb{H}_{(T,X')}, |\cdot|'_{(T,X')}) \hookrightarrow (\mathbb{H}_{(T,X)}, |\cdot|'_{(T,X)})$ whenever $0 < X \leq X'$.

The procedure followed in this Section matches point by point with the one used in Section 3 so details are omitted. In this Section, only the second inequality of assumption **(A)** must hold. It deserves pointing out that the series $R(\xi) := \sum_{\ell \geq 0} 2^{\ell+\xi}\xi^{\ell}$, involved in the following result, belongs to $\mathbb{H}_{(T,X)}$ if and only if $T < 1/2$.

Theorem 3 *Let us consider the Cauchy problem*

$$\partial_x^S V(\xi, x) = \sum_{k=0}^{S-1} c_k(x) R(\xi)(\partial_x^k V)(\xi, x/|q|^{m_{1,k}}) \tag{40}$$

with initial conditions

$$(\partial_x^j V)(\xi, 0) = \phi_j(\xi), \qquad 0 \leq j \leq S-1, \tag{41}$$

and assume that $\phi_j(\xi) \in \mathbb{H}_{(T,X_0)}$, $0 \leq j \leq S-1$, where $X_0 > 0$ and $0 < T < \frac{1}{2}$. Then, there exists $X_1 > 0$ such that the problem (40), (41) has a unique solution $V(\xi, x) \in \mathbb{H}_{(T,X_1)}$. Moreover, there exists $C > 0$ (depending on S, q, X_0, T, and $c_k(x), m_{1,k}$ for $0 \leq k \leq S-1$) such that

$$|V(\xi, x)|'_{(T,X_1)} \leq C \sum_{j=0}^{S-1} |\phi_j(\xi)|'_{(T,X_0)}.$$

6 Estimates for the derivatives of W_j near the origin

In the Cauchy problem (27), (28) we consider initial conditions W_j which are holomorphic functions respectively defined in open sets containing the closed disc

$$\overline{D}_j = \{\tau : |\tau| \leq 1/(2(j+1)^{r_1/r_2})\}, \quad 0 \leq j \leq S-1$$

(for the sake of brevity, we say that W_j is holomorphic in $\overline{D}_j$). Then, Cauchy's integral formula for the derivatives allows us to obtain constants $A_j > 0$ such that for every natural number $n \geq 0$ we have $\max_{\tau \in \overline{D}_j} |\partial^n W_j(\tau)| \leq A_j^n n!$. So, the assumptions in the following result are not restrictive.

Theorem 4 *Consider the Cauchy problem (27), (28). Suppose $W_j(\tau)$, $0 \leq j \leq S-1$, are holomorphic functions in $\overline{D}_j$ such that there exist constants $T, K > 0$ such that*

$$\max_{\tau \in \overline{D}_j} |\partial^n W_j(\tau)| \leq K \left(\frac{1}{T}\right)^n \frac{n!}{1+n^2}, \quad n \geq 0, \quad j = 0, 1, ..., S-1.$$

Then there exists a formal solution of (27), (28), $W(\tau, z) = \sum_{h \geq 0} W_h(\tau)\frac{z^h}{h!}$, where W_h is a holomorphic function in $\overline{D}_h = \{\tau : |\tau| \leq 1/(2(h+1)^{r_1/r_2})\}$, $h \geq S$. Moreover, there exists a constant $X_1 > 0$ such that

$$\sup_{\tau \in \overline{D}_j} |\partial^n W_j(\tau)| \leq C_1 \left(\frac{1}{T}\right)^n \left(\frac{1}{X_1}\right)^j n! j! (j+1)^{r_1 n/r_2} |q|^{-j^2/2}, \tag{42}$$

for every $n, j \geq 0$, where C_1 is a positive constant (depending on S,q,T,$b_k(z)$ and $m_{1,k}$, for $0 \leq k \leq S-1$).

Proof We look for formal series solutions of (27), (28) of the form $W(\tau,z) = \sum_{h\geq 0} W_h(\tau)\frac{z^h}{h!}$, what leads to the equalities

$$\frac{W_{h+S}(\tau)}{h!} = \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1\in I_k} \frac{b_{kh_1}\tau^{m_{0,k}}}{((h+1)^{r_1}\tau^{r_2}+1)} \frac{W_{h_2+k}(\tau)}{h_2! q^{m_{1,k}h_2}}, \tag{43}$$

for all $h \geq 0$. These equations recursively define in a unique way the sequence $\{W_h\}_{h\geq 0}$, and we easily see that W_h is holomorphic in $\overline{D}_h = \{\tau : |\tau| \leq 1/(2(h+1)^{r_1/r_2})\}$, $h \geq 0$. We aim at estimating the rate of growth of the derivatives of W_h in $\overline{D}_h$.

Let n_0 be a natural number. Differentiating n_0 times in (43) we get

$$\frac{\partial^{n_0} W_{h+S}(\tau)}{h!} = \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1\in I_k} b_{kh_1} \sum_{l_1+l_2=n_0} \frac{n_0!}{l_1! l_2!} \partial^{l_1}\Big(\frac{\tau^{m_{0,k}}}{((h+1)^{r_1}\tau^{r_2}+1)}\Big) \frac{\partial^{l_2} W_{h_2+k}(\tau)}{h_2! q^{m_{1,k}h_2}}. \tag{44}$$

It is clear that

$$\partial^{l_1}\Big(\frac{\tau^{m_{0,k}}}{((h+1)^{r_1}\tau^{r_2}+1)}\Big) = \sum_{\lambda_1+\lambda_2=l_1, \lambda_1\leq m_{0,k}} \frac{l_1!}{\lambda_1!\lambda_2!} \frac{m_{0,k}!}{(m_{0,k}-\lambda_1)!} \tau^{m_{0,k}-\lambda_1} \partial^{\lambda_2}\Big(\frac{1}{((h+1)^{r_1}\tau^{r_2}+1)}\Big). \tag{45}$$

Following the proof of Lemma 7 in [15], we get that for all $\lambda_2 \geq 0$, all $\tau \in \overline{D}_h$,

$$\Big|\partial^{\lambda_2}\Big(\frac{1}{((h+1)^{r_1}\tau^{r_2}+1)}\Big)\Big| \leq \lambda_2! 2^{\lambda_2+1}(h+1)^{\frac{r_1}{r_2}\lambda_2}. \tag{46}$$

We take (46) into (45) and then into (44) so one has

$$\begin{aligned} \frac{|\partial^{n_0} W_{h+S}(\tau)|}{n_0! h! (h+S+1)^{\frac{r_1}{r_2}n_0}} &\leq \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1\in I_k} |b_{kh_1}| \sum_{l_1+l_2=n_0} 2^{l_1+1} \frac{(h+1)^{\frac{r_1}{r_2}n_0}}{(h+S+1)^{\frac{r_1}{r_2}n_0}} \\ &\quad \times \frac{(h_2+k+1)^{\frac{r_1}{r_2}l_2}}{(h+1)^{\frac{r_1}{r_2}l_2}} \frac{|\partial^{l_2} W_{h_2+k}(\tau)|}{l_2! h_2! (h_2+k+1)^{\frac{r_1}{r_2}l_2} |q|^{m_{1,k}h_2}}. \end{aligned} \tag{47}$$

Let us put

$$w_{n_0,h} := \sup_{\tau\in\overline{D}_h} \frac{|\partial^{n_0} W_h(\tau)|}{(h+1)^{r_1 n_0/r_2}}, \qquad n_0 \geq 0,\ h \geq 0.$$

From (47) we deduce that

$$\frac{w_{n_0,h+S}}{n_0! h!} \leq \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1\in I_k} |b_{kh_1}| \sum_{l_1+l_2=n_0} 2^{l_1+1} \frac{w_{l_2,h_2+k}}{l_2! h_2! |q|^{m_{1,k}h_2}}.$$

Now we define a multi-sequence $\{v_{l,h}\}_{l,h}$ by

$$v_{l,h} = w_{l,h}, \qquad l \geq 0,\ 0 \leq h \leq S-1,$$

and by the following recurrence relations for $n_0 \geq 0$, $h \geq 0$:

$$\frac{v_{n_0,h+S}}{n_0! h!} = \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1\in I_k} |b_{kh_1}| \sum_{l_1+l_2=n_0} 2^{l_1+1} \frac{v_{l_2,h_2+k}}{l_2! h_2! |q|^{m_{1,k}h_2}}. \tag{48}$$

It is clear that $w_{l,h} \leq v_{l,h}$ for all $l \geq 0$, all $h \geq 0$. Let us consider the functions

$$R(\xi) = \sum_{l=0}^{\infty} 2^{l+1}\xi^l = \frac{2}{1-2\xi}, \qquad \text{and} \qquad c_k(x) = \sum_{s \in I_k} |b_{ks}| x^s, \qquad 0 \leq k \leq S-1.$$

Due to the recursions (48), one can check that the formal power series $\hat{V}(\xi,x) = \sum_{l,h\geq 0} \frac{v_{l,h}}{l!}\xi^l \frac{x^h}{h!}$ is a formal solution of the Cauchy problem (40), (41) with initial conditions

$$(\partial_x^j V)(\xi,0) = \phi_j(\xi) := \sum_{l\geq 0} \frac{w_{l,j}}{l!}\xi^l, \qquad 0 \leq j \leq S-1.$$

It is immediate to check that, for any $X_0 > 0$, we have $\phi_j(\xi) \in \mathbb{H}_{(T,X_0)}$ for $0 \leq j \leq S-1$. From Theorem 3 and the fact that $\hat{V}(\xi,x) = \sum_{n,j\geq 0} \frac{v_{n,j}}{n!}\xi^n \frac{x^j}{j!}$ is the unique formal solution of (40), (41), we can find $X_1 > 0$ such that

$$w_{n,j} \leq v_{n,j} \leq C\Big(\frac{1}{T}\Big)^n \Big(\frac{1}{X_1}\Big)^j n!j!q^{-j^2/2}\Big[\sum_{l=0}^{S-1} |\phi_l(\xi)|'_{(T,X_0)}\Big] \leq C_1\Big(\frac{1}{T}\Big)^n\Big(\frac{1}{X_1}\Big)^j n!j!|q|^{-j^2/2}.$$

for certain $C, C_1 > 0$ (depending on S,q,$c_k(x)$,$m_{1,k}$, for $0 \leq k \leq S-1$ and X_0,T). We conclude by the very definition of the multi-sequence $\{w_{n,j}\}_{n,j\geq 0}$. $\square$

7 Analytic solutions of the Cauchy problem with Fuchsian and irregular singularities

Let W_h be the initial data in the Cauchy problem (27), (28), and suppose they are subject to the hypotheses of Theorem 2 and to the hypotheses in Theorem 4. Those results give us a sequence of functions $\{W_h\}_{h\geq 0}$, holomorphic in $Vq^{\mathbb{Z}} \cup D_h$ for each $h \geq 0$, and such that the series

$$W(\tau,z) = \sum_{h\geq 0} W_h(\tau)\frac{z^h}{h!}$$

defines a holomorphic function on $Vq^{\mathbb{Z}} \times \mathbb{C}$ which solves the Cauchy problem.

Moreover, from (33), (35) and (39) in the proof of Theorem 2 we know that

$$\sup_{x\in V} |W_h(xq^l)| \leq K_0 C' |q|^{\frac{l^2}{2}} |q|^{\frac{-h^2}{4}} h! (\frac{1}{T})^l (\frac{1}{X})^h \tag{49}$$

for all $l, h \geq 0$.

Let us choose $\lambda \in V$ and $\delta > 0$. By (49) we see that every W_h verifies estimates as those in (8). If we choose an integer $n(h)$ in such a way that $\lambda q^{n(h)} \in D_h$, then, according to Proposition 1, the $q-$Laplace transform of W_h in the direction $\lambda q^{n(h)} q^{\mathbb{Z}}$, which clearly equals $\lambda q^{\mathbb{Z}}$, is given by

$$\mathcal{L}_q^{\lambda q^{n(h)}}(W_h)(t) = \sum_{m\in\mathbb{Z}} \frac{W_h(q^m \lambda q^{n(h)})}{\Theta(\frac{q^m\lambda q^{n(h)}}{t})} = \sum_{m\in\mathbb{Z}} \frac{W_h(q^m\lambda)}{\Theta(\frac{q^m\lambda}{t})},$$

so that it deserves to be denoted by $\mathcal{L}_q^{\lambda}(W_h)(t)$. This function is well defined and holomorphic in the set $\mathcal{T}_{\lambda q^{n(h)},q,\delta,r(h)}$, which is equal to $\mathcal{T}_{\lambda,q,\delta,r(h)}$, whenever $r(h) < |\lambda q^{n(h)} q^{1/2}|T$. We will show that these radii $r(h)$ can be taken independent of h, equal to $r_0 = |\lambda q^{1/2}|T/|q| = |\lambda q^{-1/2}|T$ for every $h \geq 0$, and we will obtain precise estimates for the corresponding $q-$asymptotic expansions.

Let us assume that the function W_h has the following Taylor expansion at 0,

$$W_h(\tau) = \sum_{n \geq 0} \frac{f_{n,h}}{q^{n(n-1)/2}} \tau^n, \tag{50}$$

where $f_{n,h} \in \mathbb{C}$, $n, h \geq 0$, and $\tau \in \overline{D}_h$.

Proposition 3 *In the situation assumed in this Section, there exist constants $B(h), D(h) > 0$ (to be specified) such that*

$$|\mathcal{L}_q^\lambda(W_h)(t) - \sum_{m=0}^{n-1} f_{m,h} t^m| \leq D(h) B(h)^n |q|^{n(n-1)/2} |t|^n \tag{51}$$

for all $n \geq 1$, for all $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$.

Proof
According to the estimates (42) in Theorem 4, we can write

$$\left|\frac{f_{n,h}}{q^{n(n-1)/2}}\right| = \left|\frac{1}{n!}\partial^n W_h(0)\right| \leq C_1 \Big(\frac{1}{T}\Big)^n \Big(\frac{1}{X_1}\Big)^h h!(h+1)^{r_1 n/r_2} |q|^{-h^2/2} = C(h) A(h)^n \tag{52}$$

for every $n, h \geq 0$, where we have put, for short,

$$C(h) = C_1 \Big(\frac{1}{X_1}\Big)^h h! |q|^{-h^2/2}, \qquad A(h) = \frac{1}{T}(h+1)^{r_1/r_2}, \qquad h \geq 0. \tag{53}$$

For each $h \geq 0$ we define $m_h := \max\{m \in \mathbb{Z} : |q^m \lambda| < \frac{1}{2A(h)}\}$, so that

$$|q^m \lambda| A(h) < \frac{1}{2}, \qquad m \leq m_h. \tag{54}$$

Also, we recall from Theorem 3 that $T < 1/2$, so $\frac{1}{A(h)} < \frac{1}{2(h+1)^{\frac{r_1}{r_2}}}$, and we deduce that

$$q^m \lambda \in D_h, \qquad m \leq m_h. \tag{55}$$

Moreover, by the very definition of m_h, we have that

$$m_h + 1 \geq \frac{-\log(2|\lambda| A(h))}{\log(|q|)}, \qquad |q|^{m_h+1} \geq \frac{1}{2|\lambda| A(h)}. \tag{56}$$

Let $K \geq 0$ be a fixed integer. Firstly, we give estimates for $\sum_{m > m_h} W_h(q^m\lambda)/\Theta(q^m\lambda/t)$. Using (49), (12) and (14), we get

$$\left|\frac{W_h(q^m\lambda)}{\Theta(q^m\lambda/t)}\right| \leq \frac{C' K_0 h!}{K_1 \delta} \Big(\frac{1}{|\lambda|}\Big)^K |q|^{K(K-1)/2} |t|^K \Big(\frac{|t|}{T|\lambda||q|^{1/2}}\Big)^m \Big(\frac{1}{X}\Big)^h |q|^{-h^2/4}$$

for all $m > m_h$, all $t \in \mathcal{R}_{\lambda,q,\delta}$. For every $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$ we have $|t| < r_0 < T|\lambda||q|^{-1/2}$. Using (56), we obtain that

$$\sum_{m > m_h} \Big(\frac{|t|}{T|\lambda||q|^{1/2}}\Big)^m \leq \sum_{m > m_h} \Big(\frac{1}{|q|}\Big)^m = \frac{1}{1 - |q|^{-1}} \frac{1}{|q|^{m_h+1}} \leq \frac{2|\lambda| A(h)}{1 - |q|^{-1}},$$

hence

$$\sum_{m>m_h}|\frac{W_h(q^m\lambda)}{\Theta(q^m\lambda/t)}| \leq \frac{2|\lambda|C'K_0}{K_1\delta(1-|q|^{-1})}A(h)h!\big(\frac{1}{X}\big)^h|q|^{-h^2/4}(\frac{1}{|\lambda|})^K|q|^{K(K-1)/2}|t|^K \tag{57}$$

for all $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$.

In a second step, we give estimates for the sum $\sum_{m\leq m_h} W_h(q^m\lambda)/\Theta(q^m\lambda/t) - \sum_{n=0}^K f_{n,h}t^n$, where the $f_{n,h}$ are defined in the Taylor expansion (50). Taking into account (50) and (55), we can formally write

$$\begin{aligned}\sum_{m\leq m_h} W_h(q^m\lambda)/\Theta(q^m\lambda/t) - \sum_{n=0}^K f_n t^n = \sum_{m\leq m_h}\frac{1}{\Theta(q^m\lambda/t)}\left(\sum_{n\geq K+1}\frac{f_{n,h}}{q^{n(n-1)/2}}(q^m\lambda)^n\right)\\ -\sum_{n=0}^K\frac{f_{n,h}}{q^{n(n-1)/2}}\left(\sum_{m>m_h}\frac{(q^m\lambda)^n}{\Theta(q^m\lambda/t)}\right)\end{aligned} \tag{58}$$

for all $t \in \mathbb{C}^*$.

From (58) and (52), we deduce that

$$|\sum_{m\leq m_h} W_h(q^m\lambda)/\Theta(q^m\lambda/t) - \sum_{n=0}^K f_{n,h}t^n| \leq \mathcal{A}(t) + \mathcal{B}(t) \tag{59}$$

where

$$\mathcal{A}(t) = \sum_{m\leq m_h}\frac{1}{|\Theta(q^m\lambda/t)|}\left(\sum_{n\geq K+1} C(h)A(h)^n(|q|^m|\lambda|)^n\right)$$

and

$$\mathcal{B}(t) = \sum_{n=0}^K C(h)A(h)^n\left(\sum_{m>m_h}\frac{|(q^m\lambda)^n|}{|\Theta(q^m\lambda/t)|}\right),$$

for all $t \in \mathbb{C}^*$.

We give estimates for $\mathcal{A}(t)$. Taking into account (54) and (13), we deduce

$$\begin{aligned}\mathcal{A}(t) &\leq \frac{C(h)A(h)}{1-A(h)|q|^{m_h}|\lambda|}\frac{|\lambda|}{K_1\delta}\frac{|q|^{m_h}}{1-|q|^{-1}}A(h)^K|q|^{K(K-1)/2}|t|^K\\ &\leq \frac{C(h)}{K_1\delta(1-|q|^{-1})}A(h)^K|q|^{K(K-1)/2}|t|^K\end{aligned} \tag{60}$$

for all $t \in \mathcal{R}_{\lambda,q,\delta}$.

In the next step, we get estimates for $\mathcal{B}(t)$. From (13), we have that

$$|\Theta(q^m\lambda/t)| \geq K_1\delta|q|^{-(K+1)K/2}|\frac{q^m\lambda}{t}|^{K+1}$$

for all $m > m_h$, all $t \in \mathcal{R}_{\lambda,q,\delta}$. This yields

$$\sum_{m>m_h}\frac{|(q^m\lambda)^n|}{|\Theta(q^m\lambda/t)|} \leq \frac{|\lambda|^n}{K_1\delta}(\frac{1}{|\lambda|})^{K+1}|q|^{(K+1)K/2}|t|^{K+1}\frac{(\frac{1}{|q|^{K+1-n}})^{m_h+1}}{1-\frac{1}{|q|^{K+1-n}}}. \tag{61}$$

From $|q|^{K+1-n} \geq |q|$ for $0 \leq n \leq K$, if we write $(\frac{1}{|q|^{K+1-n}})^{m_h+1} = \left(\frac{1}{|q|^{m_h+1}}\right)^{K+1}(|q|^{m_h+1})^n$ in (61) we get

$$\mathcal{B}(t) \leq \frac{C(h)}{K_1\delta(1-|q|^{-1})}(\frac{1}{|\lambda||q|^{m_h+1}})^{K+1}|q|^{(K+1)K/2}|t|^{K+1}\sum_{n=0}^{K}(2A(h)|\lambda||q|^{m_h+1})^n.$$

By (56) we know that $2A(h)|\lambda||q|^{m_h+1} \geq 1$. For every real number $x \geq 1$ we have

$$\sum_{n=0}^{K} x^n \leq \sum_{n=0}^{K}\binom{K}{n}x^n x^{K-n} = (2x)^K,$$

and we deduce that $\sum_{n=0}^{K}(2A(h)|\lambda||q|^{m_h+1})^n \leq (4A(h)|\lambda||q|^{m_h+1})^K$. Also, we have $|t| < |\lambda q^{-1/2}|T$ whenever $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$, and $(K+1)K/2 = K + K(K-1)/2$. Gathering all these facts and using (56), we deduce that

$$\begin{aligned} \mathcal{B}(t) &\leq \frac{|t|C(h)}{K_1\delta(1-|q|^{-1})|\lambda||q|^{m_h+1}}(4A(h)|q|)^K|q|^{K(K-1)/2}|t|^K \\ &\leq \frac{2|\lambda|TA(h)C(h)}{K_1\delta|q|^{1/2}(1-|q|^{-1})}(4A(h)|q|)^K|q|^{K(K-1)/2}|t|^K \end{aligned} \tag{62}$$

for all $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$.

Finally, using the estimates

$$\begin{aligned} |\sum_{m\in\mathbb{Z}} W_h(q^m\lambda)/\Theta(q^m\lambda/t) - \sum_{n=0}^{K} f_{n,h}t^n| &\leq |\sum_{m\leq m_h} W_h(q^m\lambda)/\Theta(q^m\lambda/t) - \sum_{n=0}^{K} f_{n,h}t^n| \\ &\quad + |\sum_{m>m_h} W_h(q^m\lambda)/\Theta(q^m\lambda/t)| \end{aligned}$$

we deduce from (57), (59), (60), (62) that

$$|\mathcal{L}_q^{\lambda}(W_h)(t) - \sum_{n=0}^{K} f_{n,h}t^n| \leq D_1(h)B_1(h)^K|q|^{K(K-1)/2}|t|^K \tag{63}$$

for all $K \geq 0$, for all $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$, with

$$B_1(h) = B_1(h+1)^{r_1/r_2}, \qquad D_1(h) = B_2(h+1)^{r_1/r_2}h!B_3^h|q|^{-h^2/4}, \tag{64}$$

where B_1, B_2 and B_3 are positive constants that do not depend on h. In order to conclude, it suffices to write, for $K \geq 1$,

$$|\mathcal{L}_q^{\lambda}(W_h)(t) - \sum_{n=0}^{K-1} f_{n,h}t^n| \leq |\mathcal{L}_q^{\lambda}(W_h)(t) - \sum_{n=0}^{K} f_{n,h}t^n| + |f_{K,h}t^K|,$$

and take into account (63) and (52). According to the expressions (53) and (64), one obtains the estimates (51) with

$$B(h) = A_1(h+1)^{r_1/r_2}, \qquad D(h) = A_2(h+1)^{r_1/r_2}h!A_3^h|q|^{-h^2/4}, \tag{65}$$

where A_1, A_2 and A_3 are again positive constants that do not depend on h.

□

We are ready to obtain our main result.

Theorem 5 *Suppose $\hat{X}_j(t)=\sum_{m\geq 0} f_{m,j}t^m \in \mathbb{C}[[t]]$, $0\leq j\leq S-1$, are given initial conditions for the Cauchy problem (3), (4), and let*

$$\hat{X}(t,z)=\sum_{h\geq 0}\hat{X}_h(t)\frac{z^h}{h!}=\sum_{h\geq 0}\sum_{m\geq 0} f_{m,h}t^m\frac{z^h}{h!}$$

be the only formal series solution of the problem (see Lemma 5). We suppose that the series $\hat{X}_j(t)$, $0\leq j\leq S-1$, are $q-$Gevrey of order 1, and that their formal $q-$Borel transforms of order 1, $W_j(\tau)=\hat{\mathcal{B}}_q\hat{X}_j(\tau)$, which are holomorphic functions around 0, indeed satisfy the assumptions of Theorems 2 and 4. We also assume that the rest of hypotheses of Theorem 2 are satisfied. Let

$$W(\tau,z)=\sum_{h\geq 0}W_h(\tau)\frac{z^h}{h!}$$

be the solution of the Cauchy problem (27), (28), corresponding to the initial conditions W_j, $0\leq j\leq S-1$. Then, we have that:

1) The function $X(t,z)=\sum_{h\geq 0}\mathcal{L}_q^{\lambda}(W_h)(t)\frac{z^h}{h!}$ is holomorphic in $\mathcal{T}_{\lambda,q,\delta,r_0}\times\mathbb{C}$.

2) The function $X(t,z)$ solves the Cauchy problem (3), (4).

3) If $r_1\geq 1$, given $R>0$ there exist constants $\tilde{C}>0$, $\tilde{D}>0$ such that for every $n\in\mathbb{N}$, $n\geq 1$, one has

$$\Big|X(t,z)-\sum_{h\geq 0}\sum_{m=0}^{n-1} f_{m,h}t^m\frac{z^h}{h!}\Big|\leq \tilde{C}\tilde{D}^n\Gamma(\frac{r_1}{r_2}(n+1))|q|^{n(n-1)/2}|t|^n \tag{66}$$

for every $t\in\mathcal{T}_{\lambda,q,\delta,r_0}$, $z\in D(0,R)$.

If $r_1=0$, given $R>0$ there exist constants $\tilde{C}>0$, $\tilde{D}>0$ such that for every $n\in\mathbb{N}$, $n\geq 1$, one has

$$\Big|X(t,z)-\sum_{h\geq 0}\sum_{m=0}^{n-1} f_{m,h}t^m\frac{z^h}{h!}\Big|\leq \tilde{C}\tilde{D}^n|q|^{n(n-1)/2}|t|^n \tag{67}$$

for every $t\in\mathcal{T}_{\lambda,q,\delta,r_0}$, $z\in D(0,R)$.

Remark: Due to the estimates (66) and (67), we may say that the function $X(t,z)$ admits the series $\sum_{h\geq 0}\sum_{m\geq 0} f_{m,h}t^m\frac{z^h}{h!}$ as $q-$asymptotic expansion of order 1 in t, uniformly for z in the compact subsets of $\mathbb{C}$. It may be noted that, because of the small divisors problem we have dealt with, a new factor appears in the estimates, in terms of the Eulerian Gamma function. The value r_1/r_2 may be thought of as a sub-order, or a second-level order, in the asymptotic expansion.

Proof 1) In view of (51), for $n=1$, and (65) we have that

$$|\mathcal{L}_q^{\lambda}(W_h)(t)-f_{0,h}|\leq D(h)B(h)|t|\leq A_1(h+1)^{2r_1/r_2}A_2h!A_3^h|q|^{-h^2/4}r_0$$

for every $h\geq 0$, every $t\in\mathcal{T}_{\lambda,q,\delta,r_0}$. On the other hand, by (52) we have that

$$|f_{0,h}|\leq C_1\Big(\frac{1}{X_1}\Big)^h h!|q|^{-h^2/2}$$

for every $h \geq 0$. So, we conclude that there exist $A_4, A_5 > 0$ such that

$$|\mathcal{L}_q^\lambda(W_h)(t)| \leq A_4 A_5^h h! |q|^{-h^2/4}$$

for every $h \geq 0$, every $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$. Then, for $z \in D(0,R)$ we have

$$|\sum_{h\geq 0} \mathcal{L}_q^\lambda(W_h)(t)\frac{z^h}{h!}| \leq \sum_{h\geq 0} A_4(A_5R)^h|q|^{-h^2/4} < \infty,$$

so that the series converges and the function it defines is holomorphic in $\mathcal{T}_{\lambda,q,\delta,r_0} \times \mathbb{C}$.
2) Since the series $\sum_{h\geq 0} W_h(\tau)\frac{z^h}{h!}$ is a solution of (27), (28), one can guarantee that $X(t,z)$ is a solution of the Cauchy problem (3), (4) by Proposition 2.
3) For every $n \geq 1$, every $(t,z) \in \mathcal{T}_{\lambda,q,\delta,r_0} \times D(0,R)$, the sum

$$\sum_{h\geq 0}\sum_{m=0}^{n-1} f_{m,h}t^m\frac{z^h}{h!}$$

is convergent, as we see from (52). One may take into account (51) and (65) and write

$$\begin{aligned}
\Big|X(t,z) - \sum_{h\geq 0}\sum_{m=0}^{n-1} f_{m,h}t^m\frac{z^h}{h!}\Big| &\leq \sum_{h\geq 0}|\mathcal{L}_q^\lambda(W_h)(t) - \sum_{m=0}^{n-1} f_{m,h}t^m|\frac{R^h}{h!} \\
&\leq A_2A_1^n|q|^{n(n-1)/2}|t|^n\sum_{h\geq 0}(h+1)^{r_1(n+1)/r_2}(A_3R)^h|q|^{-h^2/4} \\
&= \frac{A_2}{A_3R}A_1^n|q|^{n(n-1)/2}|t|^n\sum_{h\geq 1}h^{r_1(n+1)/r_2}(A_3R)^h|q|^{-(h-1)^2/4}. \qquad (68)
\end{aligned}$$

In case $r_1 = 0$, the conclusion easily follows, since the last sum is convergent and independent of n. In case $r_1 \geq 1$, we follow an idea of B. Braaksma and L. Stolovitch [5]. Let $\varepsilon > 0$, and let γ be a contour that goes from $\infty e^{-i\pi}$ to $-\varepsilon$ along the negative real axis, then it turns once around 0 in the positive sense, and it goes from $-\varepsilon$ to $\infty e^{i\pi}$ again along the negative real axis. For $\mu := \frac{r_1(n+1)}{r_2} > 0$, Hankel's formula allows us to write $\frac{h^\mu}{\Gamma(\mu+1)} = \frac{1}{2\pi i}\int_\gamma e^{hs}s^{-\mu-1}\,ds$, so that the sum in (68) may be written as

$$\begin{aligned}
(69)\quad \frac{\Gamma(\mu+1)}{2\pi i}&\sum_{h\geq 1}(A_3R)^h|q|^{-(h-1)^2/4}\int_\gamma e^{hs}s^{-\mu-1}\,ds \\
&= \frac{\Gamma(\mu+1)}{2\pi i}\sum_{h\geq 1}\int_\gamma s^{-\mu-1}|q|^{-(h-1)^2/4}(A_3Re^s)^h\,ds.
\end{aligned}$$

We consider now the entire function $F(z) = \sum_{h\geq 1}|q|^{-(h-1)^2/4}z^h$, $z \in \mathbb{C}$. The series converges uniformly in every closed disc. Observe that as s runs over γ, its real part remains bounded above, and the same is valid for the modulus of A_3Re^s. So, we may write

$$F(A_3Re^s) = \sum_{h\geq 1}|q|^{-(h-1)^2/4}(A_3Re^s)^h$$

uniformly in γ, and the dominated convergence theorem ensures that

$$(70)\quad \sum_{h\geq 1}\int_{\gamma} s^{-\mu-1}|q|^{-(h-1)^2/4}(A_3Re^s)^h\,ds = \int_{\gamma} s^{-\mu-1}\sum_{h\geq 1}|q|^{-(h-1)^2/4}(A_3Re^s)^h\,ds$$

$$= \int_{\gamma} s^{-\mu-1}F(A_3Re^s)\,ds.$$

Moreover, $F(A_3Re^s)$ remains bounded as s runs over γ, say by $M > 0$, and it is easy to obtain, estimating on each of the three parts of γ, that

$$(71)\qquad |\int_{\gamma} s^{-\mu-1}F(A_3Re^s)\,ds| \leq 2\frac{M}{\mu\varepsilon^{\mu}} + \frac{2\pi M}{\varepsilon^{\mu}} \leq \frac{\tilde{M}^{\mu}}{\mu\varepsilon^{\mu}},$$

where $\tilde{M} > 0$ is some suitable constant independent of h. Gathering (68), (69), (70), (71) and from the definition of μ one can conclude. □

Remark: The case $r_1 = 0$, as it may be seen in the last Theorem, deserves some attention, since the Fuchsian singularity at $z = 0$ does not appear any more. The most important consequence of this fact is the disappearance of the small divisors phenomenon we had in general.
Moreover, the condition 1) in Theorem 2, concerning the argument of q and the set V, can be relaxed. Indeed, the estimates (31) hold if one assumes that there exists $\delta > 0$ such that

$$\operatorname{dist}(V^{r_2}q^{r_2\mathbb{Z}}, \{-1\}) > \delta,$$

where dist is the Euclidean distance between two sets in $\mathbb{C}$. For example, this is valid when V is such that there exist R_1, R_2 with $0 < R_1 \leq |x^{r_2}| \leq R_2$ for all $x \in V$, and suppose that $R_2 < |q|R_1$ and $|q|^{r_2 j}R_2 < 1 < |q|^{r_2(j+1)}R_1$ for some $j \in \mathbb{Z}$.
In Theorem 4 all the functions W_h are holomorphic in a common disc, say $\overline{D}$, and there exists a constant $X_1 > 0$ such that

$$\sup_{\tau\in\overline{D}}|\partial^n W_j(\tau)| \leq C_1\Big(\frac{1}{T}\Big)^n\Big(\frac{1}{X_1}\Big)^j n!j!|q|^{-j^2/2}$$

for every $n, j \geq 0$. The proof of Proposition 3 admits some simplification, and one obtains that

$$|\mathcal{L}_q^{\lambda}(W_h)(t) - \sum_{m=0}^{n-1} f_{m,h}t^m| \leq A_2 h! A_3^h |q|^{-h^2/4} A_1^n |q|^{n(n-1)/2}|t|^n,$$

for every $h \geq 0$, $n \geq 1$. Finally, no sub-order appears in the $q-$asymptotic expansion of the solution $X(t,z)$.

Some examples of equations solved in the present work:
Let $S = 2$, $m_{00} = 1$, $m_{10} = 5$, $m_{01} = 0$, $m_{11} = 2$ with $I_0 = \{2,3\}$ and $I_1 = \{1\}$. We fix $b_{02}, b_{03}, b_{11} \in \mathbb{C}$.

If we take $r_1 = 0$ and $r_2 = 1$, the equation (3) turns

$$t\partial_z^2 X(qt,z) + \partial_z^2 X(t,z) = (b_{02} + b_{03}z)z^2 tX(qt, zq^{-5}) + b_{11}z\partial_z X(t, zq^{-2}),$$

whilst for $r_1 = 1$ and $r_2 = 1$, the problem considered is

$$tz\partial_z^3 X(qt,z) + t\partial_z^2 X(qt,z) + \partial_z^2 X(t,z) = (b_{02} + b_{03}z)z^2 tX(qt, zq^{-5}) + b_{11}z\partial_z X(t, zq^{-2}).$$

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