EIGENVALUE BOUNDS FOR STARK OPERATORS WITH COMPLEX POTENTIALS

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Abstract. We consider the 3-dimensional Stark operator perturbed by a complex-valued potential. We obtain an estimate for the number of eigenvalues of this operator as well as for the sum of imaginary parts of eigenvalues situated in the upper half-plane.

1. Introduction and main results

Let H_0 be the free Stark operator

$$H_0 = -\Delta + x_1, \tag{1.1}$$

acting in the space $L^2(\mathbb{R}^3)$. In the formula above, x_1 denotes the function whose value at a point $x \in \mathbb{R}^3$ coincides with the first coordinate of x. Since H_0 is an unbounded operator, one has to specify its domain of definition. For this purpose we simply mention that H_0 is essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)$. We study the spectral properties of the operator

$$H = H_0 + V,$$

where the potential V is a bounded complex-valued function, satisfying

$$\int_{\mathbb{R}^3} |V(x)|^r dx < \infty, \qquad \text{for some} \quad r > 0. \tag{1.2}$$

While the interest of mathematicians in the theory of non-selfadjoint operators of this type is quite new, Stark operators with real potentials have been studied thoroughly in mathematical physics for a long time. Among the classical results applicable to the self-adjoint case are the theorems of Avron and Herbst [2] who considered scattering for the pair of operators H and H_0 in the case where V is a short-range potential. In particular, it was established that the spectrum of H is purely absolutely continuous and covers the real line $\mathbb{R}$ (besides [2], see Herbst [10]). It was proved in [2], [10] and [37] that for a short-range potential V, the wave operators

$$\Omega_\pm = s - \lim e^{itH} e^{-itH_0}, \qquad \text{as} \qquad t \to \pm\infty,$$

exist and are unitary. Further development of the methods used to study Stark operators led to the theory of scattering of several particles in an external constant electric field (see the papers [13] and [21]). Besides the results related to the scattering theory, the mathematical literature on Stark operators contains numerous statements about the distribution of resonances in the models involving a constant electric field. Here, we only mention the article [11] and the recent paper [22], which can be also used for finding other relevant references.

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Let us now describe the main results of the present paper devoted to the non-selfadjoint case. Under the condition (1.2), the spectrum $\sigma(H)$ of H coincides with the union of the line $\mathbb{R}$ and the discrete set of complex eigenvalues that might accumulate only to real points. We denote by $\{\lambda_j\}_{j=1}^\infty$ the sequence of eigenvalues of H in $\mathbb{C}\setminus\mathbb{R}$ enumerated in an arbitrary order. The number of times an eigenvalue appears in the sequence $\{\lambda_j\}_{j=1}^\infty$ coincides with the algebraic multiplicity of the eigenvalue. We will show that the condition (1.2) with $r<2$ guarantees that

$$\sum_j |\operatorname{Im}\lambda_j| < \infty.$$

Theorem 1.1. *Let $4<p\leqslant 5$. Let V be a bounded complex-valued function satisfying the condition* (1.2) *with $r=p/(p-2)$. Then the eigenvalues λ_j of the operator H obey the estimate*

$$\sum_j |\operatorname{Im}\lambda_j| \leqslant C_p\Big[\Big(\int_{\mathbb{R}^3}|V|^{p/2}dx\Big)^2+\Big(\int_{\mathbb{R}^3}|V|^{p/(p-2)}dx\Big)^{p-2}\Big]. \tag{1.3}$$

The constant $C_p>0$ in this inequality is independent of V.

As a consequence of the method used in the proof of Theorem 1.1 we will obtain the following statement, where V might decay slower than a potential satisfying (1.2) with $r<2$.

Theorem 1.2. *Let $q>1$ and $4<p<q+3$. Let V be a bounded complex-valued potential such that*

$$\int_{\mathbb{R}^3}|V(x)|^{p/2}dx<\infty.$$

Then the spectrum of H is discrete in $\mathbb{C}\setminus\mathbb{R}$ and the eigenvalues λ_j of the operator H satisfy the estimate

$$\sum_j |\operatorname{Im}\lambda_j|^q \leqslant C_{p,q}\Big(\int_{\mathbb{R}^3}|V(x)|^{p/2}dx\Big)^{2q/(p-3)} \tag{1.4}$$

with a positive constant $C_{p,q}>0$ depending only on p and q.

Remark. Estimates (1.3) and (1.4) also hold for eigenvalues of the operator $-\Delta+V$ in $d=3$.

The next theorem gives a very interesting bound on the number of eigenvalues of H in the half-plane $\{\lambda\in\mathbb{C}:\ \operatorname{Re}\lambda<\alpha\}$ under the condition

$$\int_{\mathbb{R}^3}|V(x)|^{p/2}\rho(x)dx<\infty \tag{1.5}$$

where the weight ρ is the exponentially growing as $x_1\to-\infty$ function, given by

$$\rho(x)=(1+e^{-px_1/2})(1+|x_1|)^2 \quad\text{and}\quad p>5. \tag{1.6}$$

Functions V satisfying the condition (1.5) decay exponentially fast in some integral sense in the direction of the negative x_1-axis. However, such potentials might decay slowly in other directions. For instance, any function obeying

$$|V(x)|\leqslant\frac{C}{(1+e^{-x_1})(1+|x|)^s},\qquad s>0,\ C>0,$$

satisfies (1.5) with $p > 10/s$. While the usual Schrödinger operator $-\Delta + V$ perturbed by such a potential might have infinitely many non-real eigenvalues in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \alpha\}$, our theorem says that the number of non-real eigenvalues of the Stark operator H in this half-plane is still finite.

Theorem 1.3. *Let $\delta > 0$ and $p > 5$. Let V be a bounded complex-valued function satisfying (1.5) with $\rho(x)$ defined by (1.6) and let $\alpha > 0$. Then the number $N(\alpha)$ of non-real eigenvalues of the operator H situated in the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \alpha\}$ obeys the estimate*

$$N(\alpha) \leqslant C_{\alpha,p,\delta}\Big(\int_{\mathbb{R}^3} |V(x)|^{p/2}\rho(x)dx\Big)^{2(1+\delta)}, \tag{1.7}$$

where

$$C_{\alpha,p,\delta} = C_{p,\delta}\min_{\varepsilon>0}\Big[\varepsilon^{-2}e^{(1+\delta)p(\alpha+\varepsilon)}\Big(\frac{\alpha+\varepsilon}{\varepsilon^{1+2\delta}} + (1+\varepsilon^2)e^{2(1+\delta)p\varepsilon^2}\Big)\Big],$$

and $C_{p,\delta} > 0$ is a constant that depends only on p and δ.

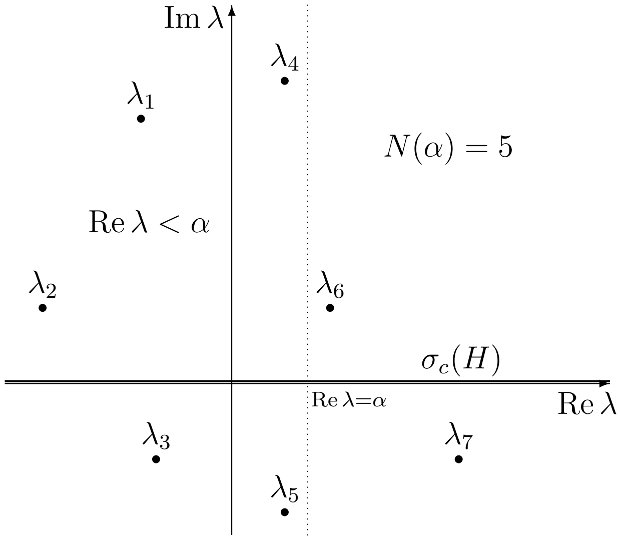

FIGURE 1. Eigenvalues of H

Remarks. 1) A similar estimate holds for the number of resonances of H contained in a region $\{\lambda \in \mathbb{C} : -\infty < \operatorname{Re}\lambda < \alpha, \ -\beta < \operatorname{Im}\lambda \leqslant 0\}$ (the constant in such an estimate depends on the region).

2) The statement of Theorem 1.3 holds with ρ replaced by

$$\tilde\rho(x) = (1 + e^{-\epsilon x_1})(1 + |x_1|)^2,$$

where $\epsilon > 0$ is an arbitrarily small number. Obviously, the constant in the corresponding estimate for the number of eigenvalues will be different. In particular, it will depend on ϵ.

One can combine this theorem with the fact that all eigenvalues are situated in a disk of a finite radius, to obtain an estimate for the total number of non-real eigenvalues.

Theorem 1.4. *1) Let $V \in L^\infty(\mathbb{R}^3)$. There exists a universal constant $C > 0$, such that all eigenvalues $\lambda \in \mathbb{C} \setminus \mathbb{R}$ of the operator H are situated in the disk*

$$|\lambda| \leqslant C\Big(\int_{\mathbb{R}^3} (1+|x|)^4 |V(x)|\,dx + \Big(\int_{\mathbb{R}^3} |V|^2 dx\Big)^{1/2}\Big)^4. \tag{1.8}$$

In particular, the conditions (1.5) *and* (1.8) *imply that the total number N of non-real eigenvalues of the operator H is finite*

$$(1.5) \text{ and } (1.8) \implies N < \infty$$

and coincides with $N(\alpha)$, where α equals the right hand side of (1.8)*.*

2) If H is the Stark operator perturbed by a potential

$$V \in L^{q/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), \text{where} \quad q < 3, \tag{1.9}$$

then non-real eigenvalues λ_j of H are contained in a disk of a finite radius. In particular, if V satisfies both hypothesis (1.5) *and* (1.9)*, then H has finitely many eigenvalues in $\mathbb{C} \setminus \mathbb{R}$*

$$(1.5) \text{ and } (1.9) \implies N < \infty.$$

Remark. In the same way, one can show that the condition $V \in L^{d/2}(\mathbb{R}^d)$ with $d \geqslant 3$ implies that all non-real eigenvalues z_j of $-\Delta + V$ are contained in a disk of a finite radius and $\sum_j |\mathrm{Im}\sqrt{z_j}| < \infty$ (see [8]).

Let us say a couple of words about our approach to the problem. It is well known that eigenvalues of most important differential operators can be described as zeros of the corresponding perturbation determinants, which depend analytically on the spectral parameter. The latter observation allows one to turn the analysis of eigenvalues into the study of zeros of analytic functions. Similar ideas were successfully used in the paper [8] by Frank and Sabin for the study of the eigenvalues of the Schrödinger operator perturbed by a decaying potential. Among other related papers are the articles [5], [7]. The problem pertaining to the Stark operator is however more complicated compared to the one involving the usual Schrödinger equation, simply because the free Stark operator is not diagonalized by the Fourier transformation.

We would also like to point out that the approach based on the study of the perturbation determinant is not the only method of obtaining eigenvalue estimates in the non-selfadjoint case. For instance, the authors of [6] and [27] use a completely different technique to estimate eigenvalues of a Schrödinger operator.

Our present paper has a complicated structure. The proof of Theorem 1.1 is given in Sections 3 and 4. Theorem 1.2 will be proved in Sections 5, 6 and 7. The two following Sections 8 and 9 contain a proof of Theorem 1.3 establishing a bound for the number of non-real eigenvalues contained in the half-plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < \alpha\}$. The estimate (1.8) of the radius of the disk containing all eigenvalues will be justified in the last Section 10.

Notations. We denote by C various possibly different constants whose values are irrelevant. The upper half-plane $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda > 0\}$ will be denoted by the symbol $\mathbb{C}_+$. By $\mathcal{B}$ and $\mathfrak{S}_\infty$ we denote the classes of bounded and compact operators, respectively. The symbols $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are used to denote the trace class and the Hilbert-Schmidt class equipped with the norms

$\|\cdot\|_{\mathfrak{S}_1}$ and $\|\cdot\|_{\mathfrak{S}_2}$, respectively. More generally, $\mathfrak{S}_p$ denotes the class of compact operators K obeying

$$||K||_{\mathfrak{S}_p}^p = \mathrm{tr}\Big(K^*K\Big)^{p/2} < \infty, \qquad p \geqslant 1.$$

Note that if $K \in \mathfrak{S}_p$ for some $p \geqslant 1$, then $K \in \mathfrak{S}_q$ for $q > p$ and

$$||K||_{\mathfrak{S}_q} \leqslant ||K||_{\mathfrak{S}_p}.$$

For a self-adjoint operator $T = T^*$ the symbol $E_T(\cdot)$ denotes its (operator-valued) spectral measure.

2. Preliminaries

Very often, eigenvalues of closed operators can be described as zeros of analytic functions. The latter circumstance allows one to use known results on the distribution of zeros of holomorphic functions to obtain bounds on the eigenvalues of a given operator. In particular, the eigenvalues of H coincide with zeros of the so called perturbation determinant $D_n(\lambda)$, which depends analytically on λ.

The definition of $D_n(\lambda)$ requires that we find two functions W_1 and W_2 having the properties

$$V = W_2 W_1, \qquad |W_1| = |W_2|,$$

and set

$$Y_0(\lambda) = W_1\, R_0(\lambda)\, W_2, \qquad R_0(\lambda) = (H_0 - \lambda)^{-1}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.10}$$

The condition (1.2) implies that $Y_0(\lambda)$ is an $\mathfrak{S}_{2r}$- operator whenever $r > 3/2$ and $\mathrm{Im}\lambda \neq 0$. Therefore, we can define the determinants

$$D_n(\lambda) = \det{}_n(I + Y_0(\lambda)), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{2.11}$$

for integer $n \geqslant 2r$. The standard way to describe $\det_n(I + K)$ in terms of eigenvalues z_j of a compact operator $K \in \mathfrak{S}_n$ is to define it as

$$\det{}_n(I + K) = \prod_j (1 + z_j) \exp\Big(\sum_{m=1}^{n-1} \frac{(-1)^m z_j^m}{m}\Big), \qquad n \geqslant 2;$$

$$\det(I + K) = \prod_j (1 + z_j), \qquad n = 1.$$

The following relations can be found in Section 3 of the book [35]. If $X, Y \in \mathcal{B}$ and both products XY, YX belong to $\mathfrak{S}_n$, then

$$\det{}_n(I + XY) = \det{}_n(I + YX). \tag{2.12}$$

The mapping $X \to \det(I + X)$ is continuous on $\mathfrak{S}_1$, which is guaranteed by the inequality

$$|\det(I + X) - \det(I + Y)| \leqslant \|X - Y\|_{\mathfrak{S}_1} e^{1 + \|X\|_{\mathfrak{S}_1} + \|Y\|_{\mathfrak{S}_1}}. \tag{2.13}$$

Moreover, there exists a constant $C_n > 0$ (see [9]) depending only on n such that

$$|\det{}_n(I + X)| \leqslant e^{C_n \|X\|_{\mathfrak{S}_n}^n}, \qquad \forall X \in \mathfrak{S}_n. \tag{2.14}$$

While the inequality in the proposition below is less known compared to (2.14), it is still a very useful estimate of the n-th determinant.

Proposition 2.1. *Let $n \geqslant 2$. Then for any $n-1 \leqslant p \leqslant n$, there exists a constant $C_{p,n} > 0$ depending only on p and n such that*

$$|\det_n(I+X)| \leqslant e^{C_{p,n}\|X\|^p_{\mathfrak{S}_p}}, \qquad \forall X \in \mathfrak{S}_p, \qquad n \geqslant 2. \tag{2.15}$$

Proof. We need to show that

$$\ln(|\det_n(I+X)|) \leqslant C_{p,n}\|X\|^p_{\mathfrak{S}_p}$$

with some constant $C_{p,n} > 0$. For that purpose, it is sufficient to prove that

$$\mathrm{Re}\,\ln(1+z) + \mathrm{Re}\sum_{m=1}^{n-1}\frac{(-1)^m z^m}{m} \leqslant C_{p,n}|z|^p \tag{2.16}$$

for all $z \in \mathbb{C}$. The inequality (2.16) is obvious for very large and very small $|z|$. Therefore, it holds for all z lying outside of a small neighborhood of the point -1. On the other hand, the left hand side of (2.16) is negative if z is sufficiently close to -1. Consequently, it holds everywhere. ■

If an operator-valued function $X : \Omega \to \mathfrak{S}_1$ is analytic on a domain $\Omega \subset \mathbb{C}$ and $(I+X(z))^{-1} \in \mathcal{B}$ for all $z \in \Omega$, then the function $F(z) = \det(I + X(z))$ is also analytic and its derivative satisfies the relation

$$F'(z) = F(z)\,\mathrm{Tr}\Big((I + X(z))^{-1}X'(z)\Big), \qquad z \in \Omega. \tag{2.17}$$

Similarly, if an operator-valued function $X : \Omega \to \mathfrak{S}_n$, (here, $n \geqslant 2$) is analytic on a domain $\Omega \subset \mathbb{C}$ and $(I + X(z))^{-1} \in \mathcal{B}$ for all $z \in \Omega$, then the function $F(z) = \det_n(I + X(z))$ is analytic and its derivative equals

$$F'(z) = F(z)\,\mathrm{Tr}\Big(\big((I + X(z))^{-1} - \sum_{j=0}^{n-2}(-1)^j X^j\big)X'(z)\Big), \qquad z \in \Omega. \tag{2.18}$$

Let $n \geqslant 2r$ be integer. We will show that if V is a bounded function satisfying (1.2) with $r > 3/2$, then $Y_0(\lambda)$ is an $\mathfrak{S}_{2r}$-operator for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The latter condition implies that the function $D_n(\lambda)$ is analytic on the open domain $\mathbb{C} \setminus \mathbb{R}$. The following statement is known as the Birman-Schwinger principle (for more detailed description, see [7]).

Lemma 2.2. *Let $V \in L^\infty(\mathbb{R}^3)$ satisfy (1.2) with $r > 3/2$. Let $n \geqslant 2r$ be integer. The point $\lambda \in \mathbb{C}\setminus\mathbb{R}$ is an eigenvalue of the operator H if and only if λ is a zero of $D_n(\lambda)$. The algebraic multiplicity of each eigenvalue $\lambda \in \sigma(H)\setminus\mathbb{R}$ coincides with the multiplicity of the corresponding zero of the function $D_n(\cdot)$.*

3. Estimates of the norms of the Birman-Schwinger operator

Let $H_0 = -\Delta + x_1$ be the Stark operator. We are going to use the representation of $\exp(-itH_0)$ as a product of different factors, one of which is $\exp(it\Delta)$. One of such formulas was discovered in [2] and is given by

$$e^{-itH_0} = e^{-itx_1}e^{it\Delta}e^{t^2\frac{\partial}{\partial x_1}}e^{-i\frac{t^3}{3}}, \qquad \forall\, t \in \mathbb{R}. \tag{3.1}$$

Another representation of $\exp(-itH_0)$ is

$$e^{-itH_0} = e^{-i\frac{t^3}{12}}\big(e^{-itx_1/2}e^{it\Delta}e^{-itx_1/2}\big), \qquad \forall\, t \in \mathbb{R}.$$

What makes this formula useful is that the integral kernel of the operator $e^{it\Delta}$ on $L^2(\mathbb{R}^3)$ is given explicitly

$$(e^{it\Delta})(x,y) = \frac{e^{-i3\pi/4}}{(4\pi t)^{3/2}} e^{i|x-y|^2/4t}, \qquad t>0, \tag{3.2}$$

$x,y\in\mathbb{R}^3$. Now, we can work with $R_0(\lambda)=(H_0-\lambda)^{-1}$, once we recall how to express it in terms of e^{-itH_0}. Indeed, let

$$\mathfrak{R}(\lambda,\zeta) = \int_0^\infty e^{-it(H_0-\lambda)} t^{\zeta-1} dt, \tag{3.3}$$

for all $\operatorname{Re}\zeta>0$. If $\operatorname{Im}\lambda>0$, then the integral in (3.3) converges (absolutely) in the operator-norm topology. Moreover,

$$R_0(\lambda) = i\,\mathfrak{R}(\lambda,1).$$

It turns out that if $\operatorname{Re}\zeta>3/2$, then $\mathfrak{R}(\lambda,\zeta)$ is an integral operator that can be applied at least to functions from $C_0^\infty(\mathbb{R}^3)$. Before proving this fact, we introduce the following convenient notation

$$\Lambda = \lambda - 2^{-1}(x_1+y_1),$$

which will be used throughout the paper.

Proposition 3.1. *The operator e^{-itH_0} is representable in the form*

$$e^{-itH_0} = e^{-i\frac{t^3}{12}}\big(e^{-itx_1/2}e^{it\Delta}e^{-itx_1/2}\big), \qquad \forall\, t\in\mathbb{R}. \tag{3.4}$$

The integral kernel $r_\zeta(x,y,\lambda)$ of the operator $\mathfrak{R}(\lambda,\zeta)$ equals

$$r_\zeta(x,y,\lambda) = \frac{e^{-i\frac{3\pi}{4}}}{\sqrt{(4\pi)^3}} \int_0^\infty e^{\frac{i}{4t}|x-y|^2} e^{-i\frac{t^3}{12}} e^{it\Lambda} t^{\zeta-1}\frac{dt}{t^{3/2}}, \qquad \operatorname{Re}\zeta>3/2, \tag{3.5}$$

for $x,y\in\mathbb{R}^3$, $\operatorname{Re}\zeta>3/2$ and $\lambda\in\mathbb{C}_+$.

Proof. The formula (3.4) which implies (3.5) can be proved by direct differentiation. Indeed, for any $f\in C_0^\infty(\mathbb{R}^3)$,

$$\frac{d}{dt}\Big(e^{-itx_1/2}e^{it\Delta}e^{-itx_1/2}\Big)f =$$

$$-\frac{ix_1}{2}\Big(e^{-itx_1/2}e^{it\Delta}e^{-itx_1/2}\Big)f + i\Big(e^{-itx_1/2}\Delta e^{it\Delta}e^{-itx_1/2}\Big)f - \frac{i}{2}\Big(e^{-itx_1/2}e^{it\Delta}x_1e^{-itx_1/2}\Big)f =$$

$$-iH_0\Big(e^{-itx_1/2}e^{it\Delta}e^{-itx_1/2}\Big)f + i\Big(\big[e^{-itx_1/2},\Delta\big]e^{it\Delta}e^{-itx_1/2}\Big)f - \frac{i}{2}\Big(e^{-itx_1/2}\big[e^{it\Delta},x_1\big]e^{-itx_1/2}\Big)f.$$

It remains to note that

$$\big[e^{-itx_1/2},\Delta\big] = e^{-itx_1/2}\big(it\frac{\partial}{\partial x_1}+t^2/4\big), \qquad \text{and} \qquad \big[e^{it\Delta},x_1\big] = 2it\frac{\partial}{\partial x_1}e^{it\Delta}.$$

■

Let us now define the operators

$$\mathfrak{R}_1(\lambda,\zeta) = \int_0^1 e^{-it(H_0-\lambda)}t^{\zeta-1}dt \quad \text{and} \quad \mathfrak{R}_2(\lambda,\zeta) = \int_1^\infty e^{-it(H_0-\lambda)}t^{\zeta-1}dt.$$

Proposition 3.2. *The operators* $\mathfrak{R}_1(\lambda,\zeta)$ *and* $\mathfrak{R}_2(\lambda,\zeta)$ *are bounded if* $\operatorname{Re}\zeta>0$ *and* $\operatorname{Re}\zeta<0$ *correspondingly. Moreover,*

$$||\mathfrak{R}_1(\lambda,\zeta)||\leqslant\frac{1}{\operatorname{Re}\zeta},\qquad \operatorname{Re}\zeta>0 \tag{3.6}$$

$$||\mathfrak{R}_2(\lambda,\zeta)||\leqslant\frac{1}{|\operatorname{Re}\zeta|},\qquad \operatorname{Re}\zeta<0 \tag{3.7}$$

The integral kernels of the operators $\mathfrak{R}_1(\lambda,\zeta)$ *and* $\mathfrak{R}_2(\lambda,\zeta)$ *equal*

$$\begin{aligned}\rho_1(x,y;\lambda,\zeta)&=\frac{e^{-i\frac{3\pi}{4}}}{\sqrt{(4\pi)^3}}\int_0^1 e^{\frac{i}{4t}|x-y|^2}e^{-i\frac{t^3}{12}}e^{it\Lambda}t^{\zeta-1}\frac{dt}{t^{3/2}},\quad \textit{and}\\ \rho_2(x,y;\lambda,\zeta)&=\frac{e^{-i\frac{3\pi}{4}}}{\sqrt{(4\pi)^3}}\int_1^\infty e^{\frac{i}{4t}|x-y|^2}e^{-i\frac{t^3}{12}}e^{it\Lambda}t^{\zeta-1}\frac{dt}{t^{3/2}}\end{aligned} \tag{3.8}$$

for $x,y\in\mathbb{R}^3$ *and* $\lambda\in\mathbb{C}_+$. *There exists a finite* $C_\zeta(p)>0$ *depending only on* $\operatorname{Re}\zeta$ *and* $p\geqslant 1$ *such that*

$$\begin{aligned}\int|\rho_1(x,y;\lambda,\zeta)|^2d\lambda<C_\zeta(2),\quad \forall\operatorname{Re}\zeta>2\quad \textit{and}\\ \int|\rho_2(x,y;\lambda,\zeta)|^pd\lambda<C_\zeta(p),\quad \forall p<\frac{2}{2\operatorname{Re}\zeta-3},\quad 3/2\leqslant\operatorname{Re}\zeta<2.\end{aligned} \tag{3.9}$$

There exists another $\tilde C_\zeta>0$ *depending only on* $\operatorname{Re}\zeta$ *such that*

$$\begin{aligned}\sup_{x,y,\lambda}|\rho_1(x,y;\lambda,\zeta)|<\tilde C_\zeta,\quad \forall\operatorname{Re}\zeta>3/2\quad \textit{and}\\ \sup_{x,y,\lambda}|\rho_2(x,y;\lambda,\zeta)|<\tilde C_\zeta,\quad \forall\operatorname{Re}\zeta<3/2.\end{aligned} \tag{3.10}$$

Proof. All statements of this proposition are trivial. One only needs to explain relations (3.9), which follow from the fact that, as functions of λ, the kernels ρ_1 and ρ_2 are Fourier transforms of functions that could be estimated by $t^{\zeta-5/2}\chi(t)$, where χ is the characteristic function of either $[0,1]$ or $[1,\infty)$. In this sense, proving the estimate involving ρ_2 is more difficult, because, additionally, one needs to observe that $t^{\zeta-5/2}\chi(t)$ belongs to $L^q[1,\infty)$ with $q=p/(p-1)$. ■

The following result helps us to turn the information provided by (3.9) into the information about the integral of the norms of the Birman-Schwinger operators.

Proposition 3.3. *Let* $\eta(x,y,\lambda)$ *be a measurable function on* $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}$ *such that*

$$||\eta||^p_{\infty,p}=\sup_{x,y}\int_{\mathbb{R}}|\eta(x,y,\lambda)|^pd\lambda<\infty,\qquad p\geqslant 2. \tag{3.11}$$

Let T_λ *be the integral operator, whose kernel is* $\eta(\cdot,\cdot,\lambda)$. *Finally, let* W_1 *and* W_2 *be two functions from the space* $L^2(\mathbb{R}^3)$. *Then* $W_1T_\lambda W_2$ *is a Hilbert-Schmidt operator for almost every* $\lambda\in\mathbb{R}$. *Moreover,*

$$\int_{\mathbb{R}}||W_1T_\lambda W_2||^p_{\mathfrak{S}_2}\,d\lambda\leqslant||\eta||^p_{\infty,p}||W_1||^p_{L^2}\cdot||W_2||^p_{L^2}. \tag{3.12}$$

Proof. The case $p = 2$ is obvious. Let us assume now that $p > 2$. Then

$$\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|W_1(x)|^2|\eta(x,y,\lambda)|^2|W_2(y)|^2dxdy\Big)^{p/2}d\lambda \leqslant$$
$$\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|W_1(x)|^2|W_2(y)|^2dxdy\Big)^{p/2-1}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|W_1(x)|^2|\eta(x,y,\lambda)|^p|W_2(y)|^2dxdy\Big)d\lambda.$$

The statement of the proposition follows. ■

Corollary 3.4. *Let $\varepsilon \in (0, 1/2)$. Then*

$$||W_1\mathfrak{R}_2(\lambda,\zeta)W_2|| \leqslant \frac{1}{\varepsilon}\,||W_1||_{L^\infty}\cdot||W_2||_{L^\infty}, \qquad \operatorname{Re}\zeta = -\varepsilon, \tag{3.13}$$

and

$$\int_{\mathbb{R}}||W_1\mathfrak{R}_2(\lambda,\zeta)W_2||^p_{\mathfrak{S}_2}\,d\lambda \leqslant C_\zeta(p)||W_1||^p_{L^2}\cdot||W_2||^p_{L^2}, \quad \forall p < \frac{2}{1-2\varepsilon}, \quad \operatorname{Re}\zeta = 2-\varepsilon. \tag{3.14}$$

Interpolating between the two cases considered in this corollary, we obtain the following very important statement.

Lemma 3.5. *Let $\varepsilon \in (0\,1/2)$ and let $\theta = (1+\varepsilon)/2$.*

$$\int_{\mathbb{R}}||W_1\mathfrak{R}_2(\lambda,1)W_2||^{p/\theta}_{\mathfrak{S}_{2/\theta}}\,d\lambda \leqslant \varepsilon^{p-p/\theta}C_{2-\varepsilon}(p)||W_1||^{p/\theta}_{L^{2/\theta}}\cdot||W_2||^{p/\theta}_{L^{2/\theta}}, \quad \forall p < \frac{2}{1-2\varepsilon}. \tag{3.15}$$

Proof. Let us take an arbitrary measurable operator-valued function $G(\cdot)$ such that

$$||G||_{dual} := \Big(\int_{\mathbb{R}}||G(\lambda)||^{\frac{p}{p-\theta}}_{\mathfrak{S}_{\frac{2}{2-\theta}}}\,d\lambda\Big)^{\frac{p-\theta}{p}} < \infty.$$

For each $\lambda \in \mathbb{R}$, the value $G(\lambda)$ is an operator in $L^2(\mathbb{R}^3)$. Set now $g(\lambda) = ||G(\lambda)||_{\mathfrak{S}_{\frac{2}{2-\theta}}}$ and $Q(\lambda) = g(\lambda)^{-1}G(\lambda)$. Then the $\mathfrak{S}_{\frac{2}{2-\theta}}$-norm of $Q(\lambda) \in \mathfrak{S}_{\frac{2}{2-\theta}}$ equals 1 and $g \in L^{\frac{p}{p-\theta}}(\mathbb{R})$. Now we define

$$f(\zeta) = \int_{\mathbb{R}}\operatorname{Tr}\big[|W_1|^{\frac{\zeta+\varepsilon}{2\theta}}\mathfrak{R}_2(\lambda,\zeta)|W_2|^{\frac{\zeta+\varepsilon}{2\theta}}Q(\lambda)|Q(\lambda)|^{-1+\frac{4-\zeta-\varepsilon}{4-2\theta}}\big]|g(\lambda)|^{\frac{2p-(\zeta+\varepsilon)}{2(p-\theta)}}\,d\lambda.$$

Observe that, for any $t \in \mathbb{R}$,

$$f(-\varepsilon+it) = \int_{\mathbb{R}}\operatorname{Tr}\big[|W_1|^{\frac{it}{2\theta}}\mathfrak{R}_2(\lambda,-\varepsilon+it)|W_2|^{\frac{it}{2\theta}}Q|Q|^{-1+\frac{4-it}{4-2\theta}}\big]|g(\lambda)|^{\frac{2p-it}{2(p-\theta)}}d\lambda.$$

Hence, due to(3.13),

$$|f(-\varepsilon+it)| \leqslant \varepsilon^{-1}||g||^{\frac{p}{p-\theta}}_{L_{\frac{p}{p-\theta}}} \tag{3.16}$$

Similarly, since the relation

$$f(2-\varepsilon+it) = \int_{\mathbb{R}}\operatorname{Tr}\big[|W_1|^{\frac{2+it}{2\theta}}\mathfrak{R}_2(\lambda,2-\varepsilon+it)|W_2|^{\frac{2+it}{2\theta}}Q|Q|^{-1+\frac{2-it}{4-2\theta}}\big]|g(\lambda)|^{\frac{2p-2-it}{2(p-\theta)}}d\lambda.$$

implies the inequality

$$|f(2-\varepsilon+it)| \leqslant ||g||^{\frac{p-1}{p-\theta}}_{L_{\frac{p}{p-\theta}}}\Big(\int_{\mathbb{R}}||\,|W_1|^{\frac{2+it}{2\theta}}\mathfrak{R}_2(\lambda,2-\varepsilon+it)|W_2|^{\frac{2+it}{2\theta}}\,||^p_{\mathfrak{S}_2}d\lambda\Big)^{1/p},$$

we obtain from (3.14) that

$$|f(2-\varepsilon+it)| \leqslant C_{2-\varepsilon}^{1/p}(p)||g||_{L_{\frac{p}{p-\theta}}}^{\frac{p-1}{p-\theta}} ||W_1||_{L^{\frac{2}{\theta}}}^{1/\theta} ||W_2||_{L^{\frac{2}{\theta}}}^{1/\theta}. \tag{3.17}$$

It follows now from (3.16) and (3.17) by the three lines theorem, that

$$|f(1)| \leqslant \varepsilon^{\theta-1} C_{2-\varepsilon}^{\theta/p}(p)||g||_{L_{\frac{p}{p-\theta}}} ||W_1||_{L^{\frac{2}{\theta}}} ||W_2||_{L^{\frac{2}{\theta}}}. \tag{3.18}$$

On the other hand,

$$f(1) = \int_{\mathbb{R}} \mathrm{Tr}\big[|W_1|\mathfrak{R}_2(\lambda,1)|W_2|G(\lambda)\big]\,d\lambda.$$

Therefore, (3.18) will turn into (3.15), once we take $G(\lambda) = w(\lambda)\Big||W_1|\mathfrak{R}_2(\lambda,1)|W_2|\Big|^{2/\theta-1}\Omega^*(\lambda)$ with

$$\Omega(\lambda) = |W_1|\mathfrak{R}_2(\lambda,1)|W_2| \cdot \Big||W_1|\mathfrak{R}_2(\lambda,1)|W_2|\Big|^{-1} \quad \text{and}$$
$$\omega(\lambda) = |||W_1|\mathfrak{R}_2(\lambda,1)|W_2|||_{\mathfrak{S}_{2/\theta}}^{(p-2)/\theta}.$$

■

Observe now, that p/θ in (3.15) is any number satisfying

$$\frac{2}{1+\varepsilon} \leqslant \frac{p}{\theta} < \frac{4}{(1+\varepsilon)(1-2\varepsilon)} = \frac{4}{1-\varepsilon-2\varepsilon^2}.$$

In particular, we can choose $p/\theta = 4/(1-\varepsilon)$. Thus, we obtain the following

Theorem 3.6. *Let $\varepsilon \in (0,1/2)$, let $p = \frac{4}{1-\varepsilon}$ and let $q = \frac{4}{1+\varepsilon}$. Then there exists a constant $C(\varepsilon) > 0$ such that*

$$\int_{\mathbb{R}} ||W_1\mathfrak{R}_2(\lambda,1)W_2||_{\mathfrak{S}_q}^p\,d\lambda \leqslant C(\varepsilon)||W_1||_{L^q}^p \cdot ||W_2||_{L^q}^p. \tag{3.19}$$

We work with the operator $W_1\mathfrak{R}_1(\lambda,\zeta)W_2$ in the same way. First, we formulate the following consequence of Proposition 3.2.

Corollary 3.7. *Let $\varepsilon \in (0,1/2)$. Then*

$$||W_1\mathfrak{R}_1(\lambda,\zeta)W_2|| \leqslant \frac{1}{\varepsilon}\,||W_1||_{L^\infty} \cdot ||W_2||_{L^\infty}, \qquad \mathrm{Re}\,\zeta = \varepsilon, \tag{3.20}$$

and

$$\int_{\mathbb{R}} ||W_1\mathfrak{R}_1(\lambda,\zeta)W_2||_{\mathfrak{S}_2}^2\,d\lambda \leqslant C_{2+\varepsilon}(2)||W_1||_{L^2}^2 \cdot ||W_2||_{L^2}^2, \qquad \mathrm{Re}\,\zeta = 2+\varepsilon. \tag{3.21}$$

Interpolating between these two cases, we derive the estimate

$$\int_{\mathbb{R}} ||W_1\mathfrak{R}_1(\lambda,1)W_2||_{\mathfrak{S}_{2/\theta}}^{2/\theta}\,d\lambda \leqslant C_\varepsilon||W_1||_{L^{2/\theta}}^{2/\theta} \cdot ||W_2||_{L^{2/\theta}}^{2/\theta}, \tag{3.22}$$

where $\theta \in (0,1)$ is the number satisfying the relation $\theta(2+\varepsilon) + (1-\theta)\varepsilon = 1$. Put differently $\theta = (1-\varepsilon)/2$. Observe now, that

$$\frac{2}{\theta} = \frac{4}{1-\varepsilon}.$$

Thus, we obtain the following theorem.

Theorem 3.8. *Let $\varepsilon \in (0, 1/2)$ and let $p = \frac{4}{1-\varepsilon}$. Then there exists a constant $C_\varepsilon > 0$ such that*

$$\int_{\mathbb{R}} ||W_1 \mathfrak{R}_1(\lambda, 1) W_2||^p_{\mathfrak{S}_p} \, d\lambda \leqslant C_\varepsilon ||W_1||^p_{L^p} \cdot ||W_2||^p_{L^p}. \tag{3.23}$$

Finally, since

$$R_0(\lambda) = i\big[\mathfrak{R}_1(\lambda, 1) + \mathfrak{R}_2(\lambda, 1)\big],$$

we conclude that the following assertion can be made about the Birman-Schwinger operators.

Theorem 3.9. *Let $\tilde{\varepsilon} \in (0, 1/2)$, let $p = \frac{4}{1-\tilde{\varepsilon}}$ and let $q = \frac{4}{1+\tilde{\varepsilon}}$. Then there exists a constant $C(\tilde{\varepsilon}) > 0$ such that*

$$\int_{\mathbb{R}} ||W_1 R_0(\lambda + i\tau) W_2||^p_{\mathfrak{S}_p} \, d\lambda \leqslant C(\tilde{\varepsilon}) \Big(||W_1||^p_{L^p} \cdot ||W_2||^p_{L^p} + ||W_1||^p_{L^q} \cdot ||W_2||^p_{L^q} \Big), \tag{3.24}$$

for any $\tau \geqslant 0$.

Theorem 3.9 would not be so useful in applications without the following result.

Proposition 3.10. *Let $a(\cdot)$ be an analytic function on $\mathbb{C}_+ = \{\operatorname{Im} \lambda > 0\}$ satisfying*

$$a(\lambda) = 1 + o(|\lambda|^{-1}) \qquad \text{as } |\lambda| \to \infty \text{ in } \mathbb{C}_+. \tag{3.25}$$

Assume that there is a family of positive functions $f_\varepsilon \in L^1(\mathbb{R})$, $0 < \varepsilon < \varepsilon_0$, such that

$$\ln |a(\lambda + i\varepsilon)| \leqslant f_\varepsilon(\lambda), \qquad \forall \lambda \in \mathbb{R}, \qquad \forall \varepsilon \in (0, \varepsilon_0). \tag{3.26}$$

Then the zeros λ_j of $a(\cdot)$ in $\mathbb{C}_+$, repeated according to multiplicities, satisfy

$$\sum_j \operatorname{Im} \lambda_j \leqslant \frac{1}{2\pi} \sup_{0<\varepsilon<\varepsilon_0} \int_{\mathbb{R}} f_\varepsilon(\lambda) \, d\lambda. \tag{3.27}$$

Proof. Consider first the function $a_\varepsilon(\lambda) = a(\lambda + i\varepsilon)$ for $0 < \varepsilon < \varepsilon_0$. Note that zeros of a_ε are the points $\lambda_j - i\varepsilon$. Consequently, the Blaschke product for a_ε in $\mathbb{C}_+$ is

$$B_\varepsilon(\lambda) = \prod_{\operatorname{Im} \lambda_j > \varepsilon} \frac{\lambda + i\varepsilon - \lambda_j}{\lambda - i\varepsilon - \overline{\lambda}_j}.$$

Since $a_\varepsilon(\lambda)/B_\varepsilon(\lambda)$ is analytic and non-zero in $\mathbb{C}_+ = \{\operatorname{Im} \lambda > 0\}$, the function $\log(a_\varepsilon(\lambda)/B_\varepsilon(\lambda))$ exists and is analytic there. For $R > 0$ we denote by C_R the contour which consists of the interval $[-R, R]$, traversed from left to right, and the circular part $\Gamma_R := \{\lambda \in \mathbb{C} : |\lambda| = R, \operatorname{Im} \lambda > 0\}$, traversed counterclockwise. Then

$$\int_{C_R} \log \frac{a_\varepsilon(\lambda)}{B_\varepsilon(\lambda)} \, d\lambda = 0,$$

and, therefore,

$$\operatorname{Re} \int_{-R}^{R} \log \frac{a_\varepsilon(x)}{B_\varepsilon(x)} \, dx + \operatorname{Re} \int_{\Gamma_R} \log \frac{a_\varepsilon(\lambda)}{B_\varepsilon(\lambda)} \, d\lambda = 0. \tag{3.28}$$

We note that $|B_\varepsilon(x)| = 1$ if $x \in \mathbb{R}$ and, therefore,

$$\begin{aligned} \operatorname{Re} \int_{-R}^{R} \log \frac{a_\varepsilon(x)}{B_\varepsilon(x)}\, dx &= \int_{-R}^{R} \ln \left| \frac{a_\varepsilon(x)}{B_\varepsilon(x)} \right| dx \\ &= \int_{-R}^{R} \ln |a_\varepsilon(x)|\, dx\,. \end{aligned} \tag{3.29}$$

(We denote by ln the natural logarithm to distinguish it from the particular branch of the complex logarithm log chosen before.) On the other hand, by (3.25) and $B_\varepsilon(\lambda) = 1 + O(|\lambda|^{-1})$ (note that the zeros λ_j are contained in a bounded set as a consequence of (3.25)), both $\log a_\varepsilon(\lambda)$ and $\log B_\varepsilon(\lambda)$ are well-defined for all sufficiently large $|\lambda|$ and we have, for all sufficiently large R,

$$\operatorname{Re} \int_{\Gamma_R} \log \frac{a_\varepsilon(\lambda)}{B_\varepsilon(\lambda)}\, d\lambda = \operatorname{Re} \int_{\Gamma_R} \log a_\varepsilon(\lambda)\, d\lambda - \operatorname{Re} \int_{\Gamma_R} \log B_\varepsilon(\lambda)\, d\lambda\,. \tag{3.30}$$

We conclude from (3.28), (3.29) and (3.30) that

$$\operatorname{Re} \int_{\Gamma_R} \log B_\varepsilon(\lambda)\, d\lambda = \int_{-R}^{R} \ln |a_\varepsilon(x)|\, dx + \operatorname{Re} \int_{\Gamma_R} \log a_\varepsilon(\lambda)\, d\lambda \tag{3.31}$$

for all sufficiently large R. We assume that $|\lambda_j - i\varepsilon| < R$ for all j. Since

$$\log B_\varepsilon(\lambda) = 2i \sum_{\operatorname{Im} \lambda_j > \varepsilon} \frac{\varepsilon - \operatorname{Im} \lambda_j}{\lambda} + O((\lambda)^{-2})\,,$$

we get

$$\begin{aligned} &\int_{\Gamma_R} \log B_\varepsilon(\lambda)\, d\lambda = \\ &\qquad = -2\pi \sum_{\operatorname{Im} \lambda_j > \varepsilon} (\varepsilon - \operatorname{Im} \lambda_j) + O(R^{-1}) \qquad \text{as } R \to \infty. \end{aligned} \tag{3.32}$$

On the other hand, by (3.25),

$$\operatorname{Re} \int_{\Gamma_R} \log a_\varepsilon(\lambda)\, d\lambda = o(1) \qquad \text{as } R \to \infty\,. \tag{3.33}$$

Moreover, by (3.26),

$$\int_{-R}^{R} \ln |a_\varepsilon(x)|\, dx \leqslant \int_{-R}^{R} f_\varepsilon(\lambda) d\lambda \leqslant \int_{-\infty}^{\infty} f_\varepsilon(\lambda) d\lambda\,. \tag{3.34}$$

Relations (3.31), (3.32), (3.33) and (3.34) imply

$$\sum_j (\operatorname{Im} \lambda_j - \varepsilon)_+ \leqslant \frac{1}{2\pi} \int_{\mathbb{R}} f_\varepsilon(\lambda)\, d\lambda \leqslant \frac{1}{2\pi} \sup_{0<\varepsilon<\varepsilon_0} \int_{\mathbb{R}} f_\varepsilon(\lambda)\, d\lambda. \tag{3.35}$$

Inequality (3.27) now follows from (3.35) by the monotone convergence theorem. ∎

4. Proof of Theorem 1.1

It is sufficient to consider the case $V \in C_0^\infty(\mathbb{R}^3)$. We apply Proposition 3.10 with

$$a(\lambda) = \det_5(I + W_1 R_0(\lambda) W_2)$$

and $f_\varepsilon(\lambda) = C||W_1 R_0(\lambda + i\varepsilon)W_2||^p_{\mathfrak{S}_p}$ where C is the constant from (2.15) with $n = 5$ and $4 < p \leqslant 5$. Note that the zeros of $a(\lambda)$ are eigenvalues of H.

We use Theorem 3.9 to conclude that

$$\int_{\mathbb{R}} f_\varepsilon(\lambda) d\lambda \leqslant C\Big[\Big(\int_{\mathbb{R}^3} |V|^{p/2} dx\Big)^2 + \Big(\int_{\mathbb{R}^3} |V|^{q/2} dx\Big)^{2p/q}\Big].$$

where $q = 4/(1+\tilde\varepsilon)$ and $\tilde\varepsilon \in (0, 1/5]$ is such that $p = 4/(1-\tilde\varepsilon)$. Also, it is established in the last section that $Y_0(\lambda)$ satisfies the following estimate.

Theorem 4.1. *Let* $\mathrm{Im}\lambda \geqslant 0$. *Then for any* $p > 11$, *there exists a positive constant* $C_p > 0$ *depending only on p such that*

$$||Y_0(\lambda)||_{\mathfrak{S}_4} \leqslant \frac{C_p}{1+|\lambda|^{1/4}} \Big(\int_{\mathbb{R}^3} (1+|x|)^p |V|^2 dx\Big)^{1/2}. \tag{4.36}$$

It is clear from this theorem that $a(\lambda) = 1 + O(|\lambda|^{-5/4})$, as $|\lambda| \to \infty$. So, all conditions of Proposition 3.10 are fulfilled, and therefore, Theorem 1.1 follows. ■

5. Non-integral bounds for the Birman-Schwinger operator

Another consequence of Proposition 3.2 is the following statement

Proposition 5.1. *Let* $\varepsilon > 0$. *Let* W_1 *and* W_2 *be two functions on* $\mathbb{R}^3$. *Then*

$$\begin{aligned} ||W_1 \mathfrak{R}_1(\lambda,\zeta) W_2|| &\leqslant \frac{1}{\varepsilon}||W_1||_{L^\infty}||W_2||_{L^\infty}, \qquad \mathrm{Re}\,\zeta = \varepsilon \\ ||W_1 \mathfrak{R}_2(\lambda,\zeta) W_2|| &\leqslant \frac{1}{\varepsilon}||W_1||_{L^\infty}||W_2||_{L^\infty}, \qquad \mathrm{Re}\,\zeta = -\varepsilon. \end{aligned} \tag{5.37}$$

Moreover, there is a constant $C_\varepsilon > 0$ *such that*

$$\begin{aligned} ||W_1 \mathfrak{R}_1(\lambda,\zeta) W_2||_{\mathfrak{S}_2} &\leqslant C_\varepsilon ||W_1||_{L^2}||W_2||_{L^2}, \qquad \mathrm{Re}\,\zeta = 3/2+\varepsilon, \\ ||W_1 \mathfrak{R}_2(\lambda,\zeta) W_2||_{\mathfrak{S}_2} &\leqslant C_\varepsilon ||W_1||_{L^2}||W_2||_{L^2}, \qquad \mathrm{Re}\,\zeta = 3/2-\varepsilon. \end{aligned} \tag{5.38}$$

The standard interpolation (that has been used already in this paper) leads to

Theorem 5.2. *Let* $\varepsilon \in (0, 1/2)$. *Let* W_1 *and* W_2 *be two functions on* $\mathbb{R}^3$. *Then*

$$\begin{aligned} ||W_1 \mathfrak{R}_1(\lambda,1) W_2||_{\mathfrak{S}_{2/\theta}} &\leqslant \tilde C_\varepsilon ||W_1||_{L^{2/\theta}}||W_2||_{L^{2/\theta}}, \qquad \theta = 2(1-\varepsilon)/3, \\ ||W_1 \mathfrak{R}_2(\lambda,1) W_2||_{\mathfrak{S}_{2/\tilde\theta}} &\leqslant \tilde C_\varepsilon ||W_1||_{L^{2/\tilde\theta}}||W_2||_{L^{2/\tilde\theta}}, \qquad \tilde\theta = 2(1+\varepsilon)/3. \end{aligned} \tag{5.39}$$

Finally, applying the triangle inequality, we obtain

Corollary 5.3. *Let* $\varepsilon \in (0, 1/2)$. *Let* W_1 *and* W_2 *be two functions on* $\mathbb{R}^3$. *Then there exists a positive constant* $C(\varepsilon) > 0$, *such that*

$$\begin{aligned} ||W_1 R_0(\lambda) W_2||_{\mathfrak{S}_p} &\leqslant C(\varepsilon)\big(||W_1||_{L^p}||W_2||_{L^p} + ||W_1||_{L^q}||W_2||_{L^q}\big), \\ &\textit{where} \qquad p = 3/(1-\varepsilon), \quad q = 3/(1+\varepsilon). \end{aligned} \tag{5.40}$$

6. Interpolation for $\operatorname{Im}\lambda > 0$.

Our starting point is the formula

$$\mathfrak{R}(\lambda,\zeta) = \int_0^\infty e^{-it(H_0-\lambda)} t^{\zeta-1} dt, \tag{6.41}$$

where $\operatorname{Re}\zeta > 0$. If $\operatorname{Im}\lambda > 0$, then the integrals in (6.41) converge (absolutely) in the operator-norm topology. Moreover,

$$R_0(\lambda) = i\,\mathfrak{R}(\lambda,1).$$

We remind the reader that the integral kernel $r_\zeta(x,y,\lambda)$ of the operator $\mathfrak{R}(\lambda,\zeta)$ equals

$$r_\zeta(x,y,\lambda) = \frac{e^{-i\frac{3\pi}{4}}}{\sqrt{(4\pi)^3}} \int_0^\infty e^{\frac{i}{4t}|x-y|^2} e^{-i\frac{t^3}{12}} e^{it\Lambda} t^{\zeta-1} \frac{dt}{t^{3/2}}, \tag{6.42}$$

where

$$\Lambda = \lambda - 2^{-1}(x_1+y_1), \qquad x,y\in\mathbb{R}^3, \quad \text{and} \quad \lambda\in\mathbb{C}_+.$$

Proposition 6.1. *The operator $\mathfrak{R}(\lambda,\zeta)$ is bounded if $\operatorname{Re}\zeta > 0$ and $\operatorname{Im}\lambda > 0$. Moreover, there exists a positive constant $C_\zeta > 0$ depending only on $\operatorname{Re}\zeta$, such that*

$$||\mathfrak{R}(\lambda,\zeta)|| \leqslant \frac{C_\zeta}{|\operatorname{Im}\lambda|^{\operatorname{Re}\zeta}}, \qquad \operatorname{Re}\zeta > 0. \tag{6.43}$$

There exists a finite $C_\zeta > 0$ depending only on $\operatorname{Re}\zeta$ such that

$$\Big(\int_{\mathbb{R}} |r_\zeta(x,y,\lambda+i\gamma)|^2 d\lambda\Big)^{1/2} < \frac{C_\zeta}{\gamma^{\operatorname{Re}\zeta-2}}, \qquad \forall \operatorname{Re}\zeta > 2. \tag{6.44}$$

There exists another $\tilde C_\zeta > 0$ depending only on $\operatorname{Re}\zeta$ such that

$$\sup_{x,y,\lambda} |r_\zeta(x,y,\lambda)| < \frac{\tilde C_\zeta}{|\operatorname{Im}\lambda|^{\operatorname{Re}\zeta-3/2}}, \qquad \forall \operatorname{Re}\zeta > 3/2. \tag{6.45}$$

Let us now turn this information into the information about the Birman-Schwinger operators.

Corollary 6.2. *Let $\varepsilon > 0$ and $\tau \geqslant 2+\varepsilon$. Then*

$$||W_1\mathfrak{R}(\lambda,\zeta)W_2|| \leqslant \frac{C_\varepsilon}{|\operatorname{Im}\lambda|^\varepsilon}\, ||W_1||_{L^\infty}\cdot||W_2||_{L^\infty}, \qquad \operatorname{Re}\zeta = \varepsilon, \tag{6.46}$$

and

$$\Big(\int_{\mathbb{R}} ||W_1\mathfrak{R}(\lambda+i\gamma,\zeta)W_2||^2_{\mathfrak{S}_2}\, d\lambda\Big)^{1/2} \leqslant \frac{C_\tau}{\gamma^{\tau-2}}||W_1||_{L^2}\cdot||W_2||_{L^2}, \quad \forall \operatorname{Re}\zeta = \tau. \tag{6.47}$$

Interpolating between the two cases considered in this corollary, we obtain the following very important statement.

Lemma 6.3. *Let $0<\varepsilon<1$, $\tau \geqslant 2+\varepsilon$ and let $\theta = (1-\varepsilon)/(\tau-\varepsilon)$. Then*

$$\int_{\mathbb{R}} ||W_1\mathfrak{R}(\lambda+i\gamma,1)W_2||^{2/\theta}_{\mathfrak{S}_{2/\theta}}\, d\lambda \leqslant \frac{C_\varepsilon^{2/\theta-2}C_\tau^2}{\gamma^{2/\theta-4}}||W_1||^{2/\theta}_{L^{2/\theta}}\cdot||W_2||^{2/\theta}_{L^{2/\theta}}. \tag{6.48}$$

Proof. Let us take an arbitrary measurable operator-valued function $Q(\cdot)$ such that

$$||Q||_{dual} := \Big(\int_{\mathbb{R}} ||Q(\lambda)||^{\frac{2}{2-\theta}}_{\mathfrak{S}_{\frac{2}{2-\theta}}} d\lambda\Big)^{\frac{2-\theta}{2}} < \infty.$$

For each $\lambda \in \mathbb{R}$, the value $Q(\lambda)$ is an operator in $L^2(\mathbb{R}^3)$. Now we define

$$f(\zeta) = \int_{\mathbb{R}} \mathrm{Tr}\big[|W_1|^{\frac{\zeta-\varepsilon}{1-\varepsilon}} \mathfrak{R}(\lambda + i\gamma, \zeta)|W_2|^{\frac{\zeta-\varepsilon}{1-\varepsilon}} Q(\lambda)|Q(\lambda)|^{-1+\frac{2\tau-\zeta-\varepsilon}{(\tau-\varepsilon)(2-\theta)}}\big] d\lambda.$$

Observe that, for any $t \in \mathbb{R}$,

$$f(\varepsilon + it) = \int_{\mathbb{R}} \mathrm{Tr}\big[|W_1|^{\frac{it}{1-\varepsilon}} \mathfrak{R}(\lambda + i\gamma, \varepsilon + it)|W_2|^{\frac{it}{1-\varepsilon}} Q(\lambda)|Q(\lambda)|^{-1+\frac{2(\tau-\varepsilon)-it}{(\tau-\varepsilon)(2-\theta)}}\big] d\lambda.$$

Hence, due to (6.46),

$$|f(\varepsilon + it)| \leqslant \frac{C_\varepsilon}{\gamma^\varepsilon} ||Q||^{\frac{2}{2-\theta}}_{dual} \tag{6.49}$$

Similarly, since the relation

$$f(\tau + it) = \int_{\mathbb{R}} \mathrm{Tr}\big[|W_1|^{\frac{\tau-\varepsilon+it}{1-\varepsilon}} \mathfrak{R}(\lambda + i\gamma, \tau + it)|W_2|^{\frac{\tau-\varepsilon+it}{1-\varepsilon}} Q(\lambda)|Q(\lambda)|^{-1+\frac{\tau-\varepsilon-it}{(\tau-\varepsilon)(2-\theta)}}\big] d\lambda.$$

implies the inequality

$$|f(\tau + it)| \leqslant ||Q||^{\frac{1}{2-\theta}}_{dual}\Big(\int_{\mathbb{R}} ||\,|W_1|^{\frac{\tau-\varepsilon+it}{1-\varepsilon}} \mathfrak{R}(\lambda + i\gamma, \tau + it)|W_2|^{\frac{\tau-\varepsilon+it}{1-\varepsilon}}||^2_{\mathfrak{S}_2} d\lambda\Big)^{1/2},$$

we obtain from (6.47) that

$$|f(\tau + it)| \leqslant \frac{C_\tau}{\gamma^{\tau-2}} ||Q||^{\frac{1}{2-\theta}}_{dual} ||W_1||^{1/\theta}_{L^{\frac{2}{\theta}}} ||W_2||^{1/\theta}_{L^{\frac{2}{\theta}}}. \tag{6.50}$$

It follows now from (3.16) and (3.17) by the three lines theorem, that

$$|f(1)| \leqslant \frac{C_\varepsilon^{1-\theta} C_\tau^\theta}{\gamma^{1-2\theta}} ||Q||_{dual} ||W_1||_{L^{\frac{2}{\theta}}} ||W_2||_{L^{\frac{2}{\theta}}}. \tag{6.51}$$

On the other hand,

$$f(1) = \int_{\mathbb{R}} \mathrm{Tr}\big[|W_1|\mathfrak{R}(\lambda + i\gamma, 1)|W_2|Q(\lambda)\big] d\lambda.$$

Therefore, (6.51) will turn into (6.52), once we take $Q(\lambda) = \Big||W_1|\mathfrak{R}(\lambda + i\gamma, 1)|W_2|\Big|^{2/\theta-1} \Omega^*(\lambda)$ with

$$\Omega(\lambda) = |W_1|\mathfrak{R}(\lambda + i\gamma, 1)|W_2| \cdot \Big||W_1|\mathfrak{R}(\lambda + i\gamma, 1)|W_2|\Big|^{-1}.$$

■

It is more convenient to formulate Lemma 6.3 in the following way.

Theorem 6.4. *Let $p > 4$ and $\gamma > 0$. Then there exists a constant $C_p > 0$ depending only on p such that*

$$\int_{\mathbb{R}} ||W_1 R_0(\lambda + i\gamma)W_2||^p_{\mathfrak{S}_p} d\lambda \leqslant \frac{C_p}{\gamma^{p-4}} ||W_1||^p_{L^p} \cdot ||W_2||^p_{L^p}. \tag{6.52}$$

Another consequence of Proposition 6.1 is the following statement

Proposition 6.5. *Let $\varepsilon > 0$ and let $\tau \geqslant 3/2 + \varepsilon$. Let W_1 and W_2 be two functions on $\mathbb{R}^3$. Then there exists a constant $C_\varepsilon > 0$ such that*

$$||W_1 \mathfrak{R}(\lambda, \zeta) W_2|| \leqslant \frac{C_\varepsilon}{|\mathrm{Im}\lambda|^\varepsilon} ||W_1||_{L^\infty} ||W_2||_{L^\infty}, \qquad \mathrm{Re}\,\zeta = \varepsilon \tag{6.53}$$

Moreover, there is a constant $C_\tau > 0$ such that

$$||W_1 \mathfrak{R}_1(\lambda, \zeta) W_2||_{\mathfrak{S}_2} \leqslant \frac{C_\tau}{|\mathrm{Im}\lambda|^{\tau - 3/2}} ||W_1||_{L^2} ||W_2||_{L^2}, \qquad \mathrm{Re}\,\zeta = \tau. \tag{6.54}$$

The standard interpolation (that has been used already in this paper) leads to

Lemma 6.6. *Let $\varepsilon \in (0,1)$, $\tau \geqslant 3/2 + \varepsilon$ and $\theta = (1-\varepsilon)/(\tau - \varepsilon)$. Let W_1 and W_2 be two functions on $\mathbb{R}^3$. Then*

$$||W_1 \mathfrak{R}(\lambda, 1) W_2||_{\mathfrak{S}_{2/\theta}} \leqslant \frac{C_\varepsilon^{1-\theta} C_\tau^\theta}{|\mathrm{Im}\lambda|^{1 - 3\theta/2}} ||W_1||_{L^{2/\theta}} ||W_2||_{L^{2/\theta}}. \tag{6.55}$$

Put differently, we obtain

Theorem 6.7. *Let $p > 3$. Let W_1 and W_2 be two functions on $\mathbb{R}^3$. Then there exists a positive constant $C_p > 0$, such that*

$$||W_1 R_0(\lambda) W_2||_{\mathfrak{S}_p}^p \leqslant \frac{C_p}{|\mathrm{Im}\lambda|^{p-3}} ||W_1||_{L^p}^p ||W_2||_{L^p}^p \tag{6.56}$$

7. Proof of Theorem 1.2

In this section, we establish some bounds on the sums of the powers of imaginary parts of the eigenvalues of the operator H. First, we prove the following statement which could be viewed as a generalization of Theorem 1.1.

Theorem 7.1. *Let $p > 4$. Let $V \in L^{p/2}(\mathbb{R}^3)$ be a bounded complex-valued potential. Then there exists a positive constant $C_p > 0$ depending only on p, such that for any $\gamma > 0$, the eigenvalues λ_j of the operator H satisfy the estimate*

$$\sum_j (\mathrm{Im}\,\lambda_j - \gamma)_+ \leqslant \frac{C_p}{\gamma^{p-4}} \Big[\int_{\mathbb{R}^3} |V(x)|^{p/2}\, dx \Big]^2, \tag{7.57}$$

Proof. It is sufficient to consider the case $V \in C_0^\infty(\mathbb{R}^3)$. We apply Proposition 3.10 with

$$a(\lambda) = \mathrm{det}_n \Big(I + Y_0(\lambda + i\gamma) \Big), \qquad n - 1 \leqslant p \leqslant n,$$

and $f_\varepsilon(\lambda) = C||Y_0(\lambda + i(\gamma + \varepsilon))||_{\mathfrak{S}_p}^p$ where C is the constant from (2.15). Note that a point $\lambda \in \mathbb{C}_+$ is a zero of $a(\lambda)$ if and only if $\lambda + i\gamma$ is an eigenvalue of H. Note also that Theorem 6.4 implies the inequality

$$\int_{\mathbb{R}} f_\varepsilon(\lambda) d\lambda \leqslant \frac{C_p}{\gamma^{p-4}} \Big[\int_{\mathbb{R}^3} |V(x)|^{p/2}\, dx \Big]^2.$$

It is also clear that $a(\lambda) = 1 + O(|\lambda|^{-5/4})$, as $|\lambda| \to \infty$. So, all conditions of Proposition 3.10 are fulfilled. Therefore, Theorem 7.1 follows. ■

Proof of Theorem 1.2. It is enough to consider the case $V \in C_0^\infty(\mathbb{R}^3)$. First, we observe that according to Theorem 6.7, there exists a positive constant $c_p > 0$ depending only on p

such that $|\mathrm{Im}\lambda_j| < c_p\Big[\int_{\mathbb{R}^3}|V(x)|^{p/2}\,dx\Big]^{2/(p-3)}$ for all j. Now we multiply (7.57) by γ^{q-2} and integrate the resulting inequality with respect to γ from 0 to $\gamma_0 = c_p\Big[\int_{\mathbb{R}^3}|V(x)|^{p/2}\,dx\Big]^{2/(p-3)}$

$$\sum_j \int_0^\infty (\mathrm{Im}\,\lambda_j-\gamma)_+\gamma^{q-2}d\gamma \leqslant \Big[\int_{\mathbb{R}^3}|V(x)|^{p/2}\,dx\Big]^2\int_0^{\gamma_0}\frac{C_p}{\gamma^{p-4}}\gamma^{q-2}d\gamma, \tag{7.58}$$

A simple change of the variable in the corresponding integral leads to the equality

$$\int_0^\infty (\mathrm{Im}\,\lambda_j-\gamma)_+\gamma^{q-2}d\gamma = |\mathrm{Im}\,\lambda_j|^q\int_0^\infty(1-\gamma)_+\gamma^{q-2}d\gamma. \tag{7.59}$$

■

8. Resolvent operator. Revised

Let $H_0 = -\Delta + x_1$ be the free Stark operator. The representation

$$e^{-itH_0} = e^{-i\frac{t^3}{12}}\Big(e^{-itx_1/2}e^{it\Delta}e^{-itx_1/2}\Big), \qquad \forall\, t\in\mathbb{R},$$

also implies that the closure of $e^{zx_1}e^{-zH_0}$ can be written in the form described below.

Proposition 8.1. *Let* $\mathrm{Re}\ z \geqslant 0$. *Then the operator* e^{-zH_0} *maps* $D(e^{-zH_0})$ *onto a subset of* $D(e^{zx_1})$. *Moreover, if* $K(z)$ *is the closure of* $e^{zx_1}e^{-zH_0}$, *then* $K(z)$ *is bounded and*

$$\big(K(z)f,g\big) = e^{\frac{z^3}{12}}\Big(\big(e^{z\Delta}e^{-zx_1/2}\big)f, e^{\bar{z}x_1/2}g\Big), \qquad \forall f,g\in C_0^\infty(\mathbb{R}^3),\quad \forall\ \mathrm{Re}\ z\geqslant 0. \tag{8.60}$$

Note that (8.60) could be formally written in the following (dubious) form

$$e^{-zH_0} = e^{\frac{z^3}{12}}\Big(e^{-zx_1/2}e^{z\Delta}e^{-zx_1/2}\Big), \qquad \forall\ \mathrm{Re}\ z\geqslant 0. \tag{8.61}$$

However, the factors in this product are unbounded operators and, therefore, this equality needs a justification.

Proof. The formula (8.60) follows from the observation that the quantity $\big(e^{-zH_0}E_{H_0}(a,b)f,g\big)$ depends analytically on z for any $-\infty<a<b<\infty$, $f\in L^2(\mathbb{R}^3)$ and $g\in C_0^\infty(\mathbb{R}^3)$. On the other hand, for the same a, b, f and g, the quantity

$$\big(e^{z\Delta - z^2\frac{\partial}{\partial x_1}}e^{\frac{z^3}{3}}E_{H_0}(a,b)f, e^{-\bar{z}x_1}g\big)$$

depends analytically on z in the right half-plane $\{z:\ \mathrm{Re}\,z>0\}$. Due to the fact that it is also continuous up to the boundary of the half-plane, we obtain from (3.1) that

$$\big(e^{-zH_0}E_{H_0}(a,b)f,g\big) = \big(e^{z\Delta - z^2\frac{\partial}{\partial x_1}}e^{\frac{z^3}{3}}E_{H_0}(a,b)f, e^{-\bar{z}x_1}g\big), \qquad \mathrm{Re}\,z\geqslant 0,$$

simply because this relation holds for $z=it$ where t is real. Setting $u=e^{-\bar{z}x_1}g$, we obtain

$$\big(e^{-zH_0}E_{H_0}(a,b)f, e^{\bar{z}x_1}u\big) = \big(e^{z\Delta - z^2\frac{\partial}{\partial x_1}}e^{\frac{z^3}{3}}E_{H_0}(a,b)f, u\big), \qquad \mathrm{Re}\,z\geqslant 0,\ \forall u\in C_0^\infty(\mathbb{R}^3).$$

One can drop the spectral projection $E_{H_0}(a,b)$, under the condition that $f\in D(e^{-zH_0})$. The resulting relation will immediately imply that $e^{-zH_0}f\in D(e^{zx_1})$ and

$$e^{zx_1}e^{-zH_0}f = e^{z\Delta - z^2\frac{\partial}{\partial x_1}}e^{\frac{z^3}{3}}f, \qquad \mathrm{Re}\,z\geqslant 0.$$

The latter relation leads to (8.60), because

$$e^{\frac{z^3}{3}}\big(e^{z\Delta-z^2\frac{\partial}{\partial x_1}}f,g\big)=e^{\frac{z^3}{12}}\big(e^{z\Delta}e^{-zx_1/2}f,e^{\bar z x_1/2}g\big)\qquad \forall\, f,g\in C_0^\infty(\mathbb{R}^3). \tag{8.62}$$

■

The next statement follows from (8.62).

Corollary 8.2. *Let* Re $z\geqslant 0$. *Then the operator* $e^{z\Delta}e^{-zx_1/2}$ *maps* $C_0^\infty(\mathbb{R}^3)$ *onto a subset of* $D(e^{zx_1/2})$. *Moreover,*

$$e^{\frac{z^3}{12}}\big(e^{zx_1/2}e^{z\Delta}e^{-zx_1/2}\big)f=K(z)f,\qquad \forall f\in C_0^\infty(\mathbb{R}^3),\quad \forall\ \mathrm{Re}\ z\geqslant 0. \tag{8.63}$$

As we see, e^{-tH_0} is not a continuous operator. However, (the closure of) the product $e^{zx_1}e^{-zH_0}$ could be viewed as a bounded operator for all $\mathrm{Re}\, z\geqslant 0$, due to the fact that Δ is negative.

Observe now that for any $-\infty<a<b<\infty$, the product of the resolvent operator $R_0(\lambda)=(H_0-\lambda)^{-1}$ and the spectral projection $E_{H_0}(a,b)$ can be written as the sum of two integrals

$$R_0(\lambda)E_{H_0}(a,b)=\int_0^1 e^{-t(H_0-\lambda)}E_{H_0}(a,b)dt+i\int_0^\infty e^{-i(t-i)(H_0-\lambda)}E_{H_0}(a,b)dt. \tag{8.64}$$

While the first integral converges for all λ, the second integral in the right hand side of (8.64) converges (absolutely) in the operator-norm topology only for $\mathrm{Im}\lambda>0$. We will often drop the projection $E_{H_0}(a,b)$ and write formally that

$$R_0(\lambda)=\int_0^1 e^{-t(H_0-\lambda)}dt+i\int_0^\infty e^{-i(t-i)(H_0-\lambda)}dt. \tag{8.65}$$

Remark. One can also represent the product of the resolvent operator and the spectral projection in the following form

$$R_0(\lambda)E_{H_0}(a,b)=\int_0^\epsilon e^{-t(H_0-\lambda)}E_{H_0}(a,b)dt+i\int_0^\infty e^{-(\epsilon+it)(H_0-\lambda)}E_{H_0}(a,b)dt$$

for any $\epsilon>0$. The resulting formula leads to the statement mentioned in one of the remarks following Theorem 1.3. Its proof is a counterpart of the argument of this section.

Proposition 8.3. *Let* W_1 *and* W_2 *be two* $C_0^\infty(\mathbb{R}^3)$*-functions. Let* $\mathrm{Im}\,\lambda>0$. *Then*

$$W_1R_0(\lambda)W_2=\int_0^1 e^{t^3/12}\Big((W_1e^{-tx_1/2})e^{t\Delta}(e^{-tx_1/2}W_2)\Big)e^{t\lambda}dt+ \\ i\int_0^\infty e^{(1+it)^3/12}\Big((W_1e^{-(1+it)x_1/2})e^{(1+it)\Delta}(e^{-(1+it)x_1/2}W_2)\Big)e^{(1+it)\lambda}dt. \tag{8.66}$$

Proof. Let $K(z)$ be the closure of $e^{zx_1}e^{-zH_0}$ (initially defined on $D(e^{-zH_0})$). One can easily prove that $||K(z)|| \leqslant e^{(\mathrm{Re} z)^3/3}$ for all $\mathrm{Re} z > 0$. Also (8.63) leads to

$$e^{\frac{z^3}{12}}\big((e^{-zx_1/2}W_1)e^{z\Delta}(e^{-zx_1/2}W_2)\big) = (W_1e^{-zx_1})K(z)W_2, \qquad \forall \text{ Re } z \geqslant 0.$$

Therefore, it is sufficient to show that

$$W_1R_0(\lambda)W_2 = \int_0^1 \Big((W_1e^{-tx_1})K(t)W_2\Big)e^{t\lambda}dt + i\int_0^\infty \Big((W_1e^{-(1+it)x_1})K(1+it)W_2\Big)e^{(1+it)\lambda}dt.$$

The latter follows from the definition of $K(z)$. Indeed, for any $-\infty < a < b < \infty$,

$$W_1R_0(\lambda)E_{H_0}(a,b)W_2 = \int_0^1 \Big((W_1e^{-tx_1})K(t)E_{H_0}(a,b)W_2\Big)e^{t\lambda}dt + i\int_0^\infty \Big((W_1e^{-(1+it)x_1})K(1+it)E_{H_0}(a,b)W_2\Big)e^{(1+it)\lambda}dt.$$

It remains to pass to the limit as $a \to -\infty$ and $b \to \infty$. The limit exists in both sides in the strong operator topology sense, i.e. one needs to apply operators to an arbitrary vector and then pass to the limit. ∎

We will show now that the two terms in the right hand side of (8.66) could be viewed as the values of two families of operators $\mathfrak{T}_0(\lambda,\zeta)$ and $\mathfrak{T}(\lambda,\zeta)$ at $\zeta = 1$. In particular, we will have

$$Y_0(\lambda) = \mathfrak{T}_0(\lambda,1) + \mathfrak{T}(\lambda,1).$$

These families $\mathfrak{T}_0(\lambda,\zeta)$ and $\mathfrak{T}(\lambda,\zeta)$ will depend on ζ analytically, which will allow us to interpolate.

Let us deal with the first term in the right hand side of (8.66). Assume that W_1 and W_2 are C_0^∞-functions on $\mathbb{R}^3$. In order to use an interpolation, we introduce the family of bounded operators

$$\mathfrak{T}_0(\lambda,\zeta) = e^{\zeta^2-1}\int_0^1 e^{t^3/12}\Big(W_1e^{-tx_1/2}e^{t\Delta}e^{-tx_1/2}W_2\Big)e^{t\lambda}t^{\zeta-1}dt.$$

depending analytically on the parameter ζ. Set also

$$\mathfrak{T}_*(\lambda,\zeta) = (1+|x_1|)^{-(\zeta-\varepsilon)/(\tau-\varepsilon)}\mathfrak{T}_0(\lambda,\zeta)(1+|x_1|)^{-(\zeta-\varepsilon)/(\tau-\varepsilon)}, \qquad 0<\varepsilon<1,\ \tau>5/2,$$

and define $\mathcal{P}(x) = 1 + e^{-x_1/2}$. It is easy to see that if $\mathrm{Re}\,\zeta = \varepsilon \in (0,1)$, then

$$||\mathfrak{T}_*(\lambda,\zeta)|| \leqslant \frac{(1+e^{\mathrm{Re}\,\lambda})}{\varepsilon}||\mathcal{P}W_1||_{L^\infty}\cdot||\mathcal{P}W_2||_{L^\infty}. \tag{8.67}$$

Let us denote the kernel of the operator $\mathfrak{T}_0(\lambda,\zeta)$ by $\nu(x,y;\lambda,\zeta)$. Recall again that the integral kernel of the operator $e^{t\Delta}$ on $L^2(\mathbb{R}^3)$ is the function

$$(e^{t\Delta})(x,y) = \frac{1}{(4\pi t)^{3/2}}e^{-|x-y|^2/4t}, \qquad t>0, \tag{8.68}$$

where $x,y\in\mathbb{R}^3$. Obviously, the function

$$\nu(x,y,\lambda,z) = \frac{e^{\zeta^2-1}W_1(x)W_2(y)}{\sqrt{(4\pi)^3}}\int_0^1 e^{\frac{-1}{4t}|x-y|^2}e^{\frac{t^3}{12}-2^{-1}t(x_1+y_1))}e^{\lambda t}\,\frac{t^{\zeta-1}dt}{t^{3/2}}, \tag{8.69}$$

satisfies the inequality

$$|\nu(x,y;\lambda,\zeta)| \leqslant C_{\mathrm{Re}\,\zeta}(1+e^{\mathrm{Re}\lambda})\mathcal{P}(x)\mathcal{P}(y)|W_1(x)|\cdot|W_2(y)| \quad \text{for all} \quad \mathrm{Re}\,\zeta>3/2. \tag{8.70}$$

It turns out that ν decays as $|\lambda|\to\infty$ in the half-plane $\{\lambda:\ \mathrm{Re}\,\lambda<\alpha\}$ for any $\alpha\in\mathbb{R}$. In order to obtain an estimate that shows such a behavior of ν, we write ν in the form

$$\nu(x,y,\lambda,z)=\frac{e^{\zeta^2-1}W_1(x)W_2(y)}{\lambda\sqrt{(4\pi)^3}}\int_0^1 e^{\frac{-1}{4t}|x-y|^2}e^{\frac{t^3}{12}-2^{-1}t(x_1+y_1))}\Big(\frac{de^{\lambda t}}{dt}\Big)\frac{t^{\zeta-1}dt}{t^{3/2}}, \tag{8.71}$$

for $x,y\in\mathbb{R}^3$ and $\lambda\in\mathbb{C}_+\setminus\{0\}$. Integrating by parts in (8.71), we obtain

$$\nu(x,y;\lambda,\zeta)=$$

$$\frac{e^{\zeta^2-1}W_1(x)W_2(y)}{\lambda\sqrt{(4\pi)^3}}\int_0^1 e^{\frac{-1}{4t}|x-y|^2+\frac{t^3}{12}+t\,\Lambda}\Big(2^{-1}(x_1+y_1)+\Big(\frac{5}{2}-\zeta\Big)t^{-1}-\frac{|x-y|^2+t^4}{4t^2}\Big)\frac{t^{\zeta-1}dt}{t^{3/2}}+$$

$$+\frac{e^{\zeta^2-1}W_1(x)W_2(y)}{\lambda\sqrt{(4\pi)^3}}e^{\frac{-1}{4}|x-y|^2+\frac{1}{12}+\Lambda} \tag{8.72}$$

for $x,y\in\mathbb{R}^3$ and $\lambda\in\mathbb{C}_+$. The formula (8.72) (combined with the inequality (8.70)) leads to the estimate

Lemma 8.4. *Let $\mathcal{P}_1(x)=(1+|x_1|)(1+e^{-x_1/2})$. There exists a positive constant $C_\tau>0$ such that*

$$|\nu(x,y;\lambda,\zeta)| \leqslant C_\tau\frac{(1+e^{\mathrm{Re}\lambda})}{1+|\lambda|}\mathcal{P}_1(x)\mathcal{P}_1(y)|W_1(x)|\cdot|W_2(y)|, \qquad \mathrm{Re}\,\zeta=\tau>5/2 \tag{8.73}$$

Corollary 8.5. *The Hilbert-Schmidt norm of the operator $\mathfrak{T}_*(\lambda,\zeta)$ satisfies the estimate*

$$||\mathfrak{T}_*(\lambda,\zeta)||_{\mathfrak{S}_2} \leqslant C_\tau\frac{(1+e^{\mathrm{Re}\,\lambda})}{1+|\lambda|}||\mathcal{P}W_1||_{L^2}\cdot||\mathcal{P}W_2||_{L^2}, \qquad \mathrm{Re}\,\zeta=\tau>5/2. \tag{8.74}$$

Interpolating between (8.67) and (8.74) we obtain

Proposition 8.6. *Let $p>5$ and let $\mathcal{P}(x)=1+e^{-x_1/2}$. Then the $\mathfrak{S}_p-$ norm of the operator $\mathfrak{T}_*(\lambda,1)$ satisfies the estimate*

$$||\mathfrak{T}_*(\lambda,1)||^p_{\mathfrak{S}_p} \leqslant C_p\frac{(1+e^{\mathrm{Re}\,\lambda})^p}{(1+|\lambda|)^2}||\mathcal{P}W_1||^p_{L^p}\cdot||\mathcal{P}W_2||^p_{L^p}. \tag{8.75}$$

Proof. We use (8.67) with $0<\varepsilon<1$ and (8.74) with $\tau>5/2$. The previously used interpolation technique leads to (8.75) with $p=2/\theta$ where $\theta\in(0,1)$ satisfies the relation $\varepsilon(1-\theta)+\tau\theta=1$. Put differently, $\theta=(1-\varepsilon)/(\tau-\varepsilon)$, which implies that p can be any number greater than 5. ∎

This proposition immediately implies

Proposition 8.7. *Let V be a complex-valued function on $\mathbb{R}^3$ and let $p>5$. Assume that W_1 and W_2 satisfy the relations $W_1=|V|^{1/2}$ and $V=W_1W_2$. Then the $\mathfrak{S}_p-$ norm of the operator $\mathfrak{T}_0(\lambda,1)$ satisfies the estimate*

$$||\mathfrak{T}_0(\lambda,1)||^p_{\mathfrak{S}_p} \leqslant C_p\frac{(1+e^{\mathrm{Re}\,\lambda})^p}{(1+|\lambda|)^2}\Big(\int_{\mathbb{R}^3}(1+e^{-x_1/2})^p(1+|x_1|)^2|V|^{p/2}dx\Big)^2. \tag{8.76}$$

Similarly, one can deal with the second term in the right hand side of (8.66). Following the steps of our work with the first term in (8.66), we introduce the operators

$$\mathfrak{T}(\lambda,\zeta) = i\int_0^\infty e^{-i\frac{(t-i)^3}{12}}\Big(W_1 e^{-(1+it)x_1/2}e^{(1+it)\Delta}e^{-(1+it)x_1/2}W_2\Big)e^{(1+it)\lambda}t^{\zeta-1}dt$$

for $\operatorname{Re}\zeta > 0$. First, we observe that the following statement holds true.

Proposition 8.8. *Let $\varepsilon \in (0,1)$. Let W_1 and W_2 be two functions on $\mathbb{R}^3$. Let $\psi(x) = e^{-x_1/2}$. Then*

$$||\mathfrak{T}(\lambda,\zeta)|| \leqslant C_\varepsilon \frac{e^{\operatorname{Re}\lambda + ((\operatorname{Im}\lambda)_- + 1)^2}}{\big(1+(\operatorname{Im}\lambda)_+\big)^\varepsilon}||\psi W_1||_{L^\infty}||\psi W_2||_{L^\infty}, \qquad \text{for all} \quad \operatorname{Re}\zeta = \varepsilon. \tag{8.77}$$

Proof. One only needs to estimate the integral

$$\int_0^\infty \big|e^{-i\frac{(t-i)^3}{12}}e^{(1+it)\lambda}t^{\zeta-1}\big|dt = e^{\operatorname{Re}\lambda}\int_0^\infty e^{\frac{-3t^2+1}{12}}e^{-t\operatorname{Im}\lambda}t^{\varepsilon-1}dt.$$

Assume that $\operatorname{Im}\lambda > 0$. Then

$$e^{\operatorname{Re}\lambda}\int_0^\infty e^{\frac{-3t^2+1}{12}}e^{-t\operatorname{Im}\lambda}t^{\varepsilon-1}dt \leqslant Ce^{\operatorname{Re}\lambda}\int_0^\infty e^{-t\operatorname{Im}\lambda}t^{\varepsilon-1}dt \leqslant \frac{C_\varepsilon e^{\operatorname{Re}\lambda}}{(\operatorname{Im}\lambda)_+^\varepsilon}.$$

If $\operatorname{Im}\lambda < 0$, then

$$e^{\operatorname{Re}\lambda}\int_0^\infty e^{\frac{-3t^2+1}{12}}e^{-t\operatorname{Im}\lambda}t^{\varepsilon-1}dt \leqslant e^{\operatorname{Re}\lambda}\int_0^1 e^{\frac{-3t^2+1}{12}}e^{-t\operatorname{Im}\lambda}t^{\varepsilon-1}dt + e^{\operatorname{Re}\lambda}\int_1^\infty e^{\frac{-3t^2+1}{12}}e^{-t\operatorname{Im}\lambda}dt.$$

Both integrals in the right hand side can be estimated in a very simple way:

$$\int_0^1 e^{\frac{-3t^2+1}{12}}e^{-t\operatorname{Im}\lambda}t^{\varepsilon-1}dt \leqslant C_\varepsilon e^{(\operatorname{Im}\lambda)_-},$$

and

$$\int_1^\infty e^{\frac{-3t^2+1}{12}}e^{-t\operatorname{Im}\lambda}dt \leqslant \int_{-\infty}^\infty e^{\frac{-3t^2+1}{12}}e^{-t\operatorname{Im}\lambda}dt \leqslant Ce^{(\operatorname{Im}\lambda)^2}.$$

■

One can also provide a proof of the following statement.

Proposition 8.9. *The integral kernel of the operator $\mathfrak{T}(\lambda,\zeta)$ is the function*

$$\eta(x,y;\lambda,\zeta) = \frac{e^{-i\frac{\pi}{4}}W_1(x)W_2(y)}{\sqrt{(4\pi)^3}}\int_0^\infty e^{\frac{it-1}{4(t^2+1)}|x-y|^2}e^{-i\frac{(t-i)^3}{12}}e^{i(t-i)\Lambda}\frac{t^{\zeta-1}dt}{(t-i)^{3/2}},$$

where the agreement about the choice of the branch of $(t-i)^{3/2}$ is that $(t-i)^{3/2}\big|_{t=0} = e^{-i3\pi/4}$.

Proof. We use (8.61) to obtain

$$\frac{\eta(x,y;\lambda,\zeta)}{W_1(x)W_2(y)} = (2\pi)^{-3}i\int_0^\infty e^{i(t-i)\Lambda - \frac{i(t-i)^3}{12}}\Big(\int_{\mathbb{R}^3}e^{i\tilde{p}(x-y)}e^{-i(t-i)|\tilde{p}|^2}d\tilde{p}\Big)t^{\zeta-1}dt.$$

Now, the statement of the proposition follows from the fact that

$$(2\pi)^{-3}\int_{\mathbb{R}^3}e^{i\tilde{p}(x-y)}e^{-i(t-i)|\tilde{p}|^2}d\tilde{p} = \frac{e^{-i3\pi/4}}{\sqrt{(4\pi)^3}}\frac{e^{\frac{i}{4(t-i)}|x-y|^2}}{(t-i)^{3/2}}.$$

■

Note that

$$\Big|\frac{\eta(x,y;\lambda,\zeta)}{W_1(x)W_2(y)}\Big| \leqslant \frac{1}{\sqrt{(4\pi)^3}}\int_0^\infty e^{\frac{(1-3t^2)}{12}}\big|e^{(1+it)\,\Lambda}\big|\, t^{\mathrm{Re}\,\zeta-1}dt = \frac{e^{\mathrm{Re}\,\Lambda}}{\sqrt{(4\pi)^3}}\int_0^\infty e^{\frac{(1-3t^2)}{12}}e^{-t\mathrm{Im}\,\Lambda}\,t^{\mathrm{Re}\,\zeta-1}dt.$$

Consequently, we can state the following result.

Proposition 8.10. *Let $\tau = \mathrm{Re}\,\zeta > 3/2$ and let $\psi(x) = e^{-x_1/2}$. Then*

$$\Big|\frac{\eta(x,y;\lambda,\zeta)}{W_1(x)W_2(y)}\Big| \leqslant C_\tau \frac{e^{\mathrm{Re}\,\lambda+2(\mathrm{Im}\lambda)_-^2}}{(1+(\mathrm{Im}\lambda)_+)^\tau}\psi(x)\psi(y). \tag{8.78}$$

Proof. Let $\mathrm{Im}\,\lambda < 0$. Then by the Schwarz inequality,

$$\Big|\frac{\eta(x,y;\lambda,\zeta)}{W_1(x)W_2(y)}\Big| \leqslant C_\tau e^{\mathrm{Re}\,\Lambda}\Big(\int_0^\infty e^{\frac{(1-3t^2)}{12}}e^{-2t\mathrm{Im}\,\lambda}\,dt\Big)^{1/2} \leqslant C_\tau e^{\mathrm{Re}\,\Lambda+2(\mathrm{Im}\lambda)_-^2}.$$

If $\mathrm{Im}\,\lambda > 0$, then

$$\Big|\frac{\eta(x,y;\lambda,\zeta)}{W_1(x)W_2(y)}\Big| \leqslant \frac{e^{\mathrm{Re}\,\Lambda}e^{\frac{1}{12}}}{\sqrt{(4\pi)^3}}\int_0^\infty e^{-t\mathrm{Im}\,\lambda}\,t^{\tau-1}dt \leqslant C_\tau\frac{e^{\mathrm{Re}\,\Lambda}}{(\mathrm{Im}\,\lambda)^\tau}.$$

■

Corollary 8.11. *Let $\tau = \mathrm{Re}\,\zeta > 3/2$ and let $\psi(x) = e^{-x_1/2}$. Then*

$$||\mathfrak{T}(\lambda,\zeta)||_{\mathfrak{S}_2} \leqslant C_\tau\frac{e^{\mathrm{Re}\,\lambda+2(\mathrm{Im}\lambda)_-^2}}{(1+(\mathrm{Im}\lambda)_+)^\tau}||\psi W_1||_{L^2}||\psi W_2||_{L^2}. \tag{8.79}$$

Interpolating between (8.77) and (8.79), we derive

Proposition 8.12. *Let $p > 3$ and let $\psi(x) = e^{-x_1/2}$. Then there exists a constant depending only on p such that*

$$||\mathfrak{T}(\lambda,1)||_{\mathfrak{S}_p} \leqslant C_p\frac{e^{\mathrm{Re}\,\lambda+2((\mathrm{Im}\lambda)_-+1)^2}}{(1+(\mathrm{Im}\lambda)_+)}||\psi W_1||_{L^p}||\psi W_2||_{L^p}. \tag{8.80}$$

Finally, combining (8.76) and (8.80), we obtain by the triangle inequality that the $\mathfrak{S}_p$-norm of $Y_0(\lambda)$ could be estimated as follows:

Theorem 8.13. *Let V be a complex-valued function on $\mathbb{R}^3$ and let $p > 5$. Assume that W_1 and W_2 satisfy the relations $W_1 = |V|^{1/2}$ and $V = W_1W_2$. Then the $\mathfrak{S}_p-$ norm of the operator $Y_0(\lambda)$ satisfies the estimate*

$$||Y_0(\lambda)||_{\mathfrak{S}_p}^p \leqslant C_p\Big[\frac{1+e^{p\mathrm{Re}\,\lambda}}{(1+|\lambda|)^2} + \frac{e^{p\mathrm{Re}\,\lambda+2p(\mathrm{Im}\lambda)_-^2}}{(1+(\mathrm{Im}\lambda)_+)^p}\Big]\Big(\int_{\mathbb{R}^3}(1+e^{-x_1/2})^p(1+|x_1|)^2|V|^{p/2}dx\Big)^2. \tag{8.81}$$

9. Jensen's inequality for a function analytic in a corner and its applications

Here we prove the following result about zeros of an analytic function.

Proposition 9.1. *Let $\varepsilon > 0$ and $\alpha > 0$ be two positive numbers. Let $a(z)$ be an analytic function on the domain $\Omega = \{z \in \mathbb{C} : \ \operatorname{Re} z \leqslant \alpha + \varepsilon, \ \operatorname{Im} z \geqslant -\varepsilon\}$, having the asymptotics $a(z) = 1 + o(|z|^{-2})$ as $|z| \to \infty$ in Ω. Assume also that*

$$\ln |a(z)| \leqslant \Big(\frac{(1 + e^{p\operatorname{Re} z})}{(1 + |z|)^2} + \frac{e^{2p(\operatorname{Im} z)_-^2 + p\operatorname{Re} z}}{(1 + (\operatorname{Im} z)_+)^p} \Big)^{1+\delta} \cdot M, \qquad \text{if} \quad z \in \Omega, \tag{9.82}$$

for some $M > 0$, , $p > 5$ and $\delta > 0$, which are independent of z. Then the number N of zeros of $a(z)$ in the domain $\{z \in \mathbb{C} : \ \operatorname{Re} z \leqslant \alpha, \ \operatorname{Im} z \geqslant 0\}$ satisfies

$$N \leqslant \varepsilon^{-2} C_{p,\delta} \cdot M \Big[e^{(1+\delta)p(\alpha+\varepsilon)} \Big(\frac{\alpha + \varepsilon}{\varepsilon^{1+2\delta}} + (1 + \varepsilon^2) e^{2(1+\delta)p\varepsilon^2} \Big) \Big], \tag{9.83}$$

where $C_{p,\delta} > 0$ is independent of α, ε and M.

Proof. The function $\log(a(z))$ it is not analytic in Ω, due to the possibility of having zeros of $a(z)$ in Ω. To get rid of the zeros, we introduce the following Blaschke product:

$$B(z) = \prod_j \frac{(z - \alpha + (i-1)\varepsilon)^2 - (z_j - \alpha + (i-1)\varepsilon)^2}{(z - \alpha + (i-1)\varepsilon)^2 - (\bar{z}_j - \alpha - (1+i)\varepsilon)^2},$$

where z_j are zeros of $a(z)$. It is easy to see that the function $\log[a(z)/B(z)]$ is analytic on Ω, because $B(z)$ vanishes exactly at the points $z = z_j$. On the other hand, $|B(z)| = 1$ for all z that belong to the boundary of Ω.

Let $C_R = \{z \in \Omega : \ |z - \alpha| = R\}$, let I_R be the interval $\{z \in \Omega : -\sqrt{R^2 - \varepsilon^2} \leqslant \operatorname{Re} z - \alpha \leqslant \varepsilon, \text{ and } \operatorname{Im} z = -\varepsilon\}$, and let J_R be the interval $\{z \in \Omega : \operatorname{Re} z = \alpha + \varepsilon, \text{ and } -\varepsilon \leqslant \operatorname{Im} z \leqslant \sqrt{R^2 - \varepsilon^2}\}$. Define $\Gamma_R = C_R \cup I_R \cup J_R$ as a traversed counterclockwise contour. Then

$$\int_{\Gamma_R} \log[a(z)/B(z)](z - \alpha + (i-1)\varepsilon) dz = 0.$$

Consequently,

$$\lim_{R\to\infty} \operatorname{Re} \int_{C_R} \log[B(z)/a(z)](z - \alpha + (i-1)\varepsilon) dz = \lim_{R\to\infty} \int_{I_R} \log |a(z)|(z - \alpha + (i-1)\varepsilon) dz + \lim_{R\to\infty} \int_{J_R} \log |a(z)|(z - \alpha + (i-1)\varepsilon) dz,$$

which implies that

$$\lim_{R\to\infty} \operatorname{Re} \int_{C_R} \log[B(z)](z - \alpha + (i-1)\varepsilon) dz = \lim_{R\to\infty} \int_{I_R} \log |a(z)|(z - \alpha + (i-1)\varepsilon) dz + \lim_{R\to\infty} \int_{J_R} \log |a(z)|(z - \alpha + (i-1)\varepsilon) dz. \tag{9.84}$$

On the other hand, due to the expansion

$$\log[B(z)](z - \alpha + (i-1)\varepsilon) = -\frac{2i}{z} \sum_j \operatorname{Im} (z_j - \alpha + (i-1)\varepsilon)^2 + O(1/|z|^2), \qquad \text{as} \quad |z| \to \infty,$$

the limit of the integral in the left hand side can be easily computed. Namely,

$$\lim_{R\to\infty} \operatorname{Re} \int_{C_R} \log[B(z)](z - \alpha + (i-1)\varepsilon) dz = \pi \sum_j \operatorname{Im} (z_j - \alpha + (i-1)\varepsilon)^2.$$

Therefore,

$$\lim_{R\to\infty} \mathrm{Re}\int_{C_R} \log[B(z)](z-\alpha+(i-1)\varepsilon)dz = 2\pi\sum_j (\mathrm{Im}\, z_j+\varepsilon)(\mathrm{Re} z_j-(\alpha+\varepsilon)) \leqslant -2\pi\varepsilon^2 N.$$

Taking into account the condition (9.82), we obtain from (9.84) that

$$2\pi\varepsilon^2 N \leqslant M\int_{-\infty}^{\alpha+\varepsilon}\Big(\frac{1+e^{pt}}{(1+|t-i\varepsilon|)^2}+e^{2p\varepsilon^2}e^{pt}\Big)^{1+\delta}|t-\alpha-\varepsilon|dt+ \\ M\int_{-\varepsilon}^{\infty}\Big(\frac{1+e^{p(\alpha+\varepsilon)}}{(1+|\alpha+\varepsilon+it|)^2}+\frac{e^{p(\alpha+\varepsilon)}e^{2pt_-^2}}{(1+t_+)^p}\Big)^{1+\delta}(t+\varepsilon)dt. \tag{9.85}$$

Note that

$$\int_{-\varepsilon}^{\infty}\Big(\frac{1+e^{p(\alpha+\varepsilon)}}{(1+|\alpha+\varepsilon+it|)^2}+\frac{e^{p(\alpha+\varepsilon)}e^{2pt_-^2}}{(1+t_+)^p}\Big)^{1+\delta}(t+\varepsilon)dt \leqslant \\ C_{\delta,p}\Big[\frac{1+e^{(1+\delta)p(\alpha+\varepsilon)}}{(\alpha+\varepsilon)^{2\delta}}+e^{(1+\delta)p(\alpha+\varepsilon)}(1+\varepsilon+e^{2p(1+\delta)\varepsilon^2}\varepsilon^2)\Big]$$

and

$$\int_{-\infty}^{\alpha+\varepsilon}\Big(\frac{1+e^{pt}}{(1+|t-i\varepsilon|)^2}+e^{2p\varepsilon^2}e^{pt}\Big)^{1+\delta}|t-\alpha-\varepsilon|dt \leqslant \\ C_{p,\delta}\Big[\frac{(1+e^{(1+\delta)p(\alpha+\varepsilon)})(\alpha+\varepsilon)}{\varepsilon^{1+2\delta}}+e^{2(1+\delta)p\varepsilon^2}e^{(1+\delta)p(\alpha+\varepsilon)}\Big].$$

Consequently, (9.85) can be written in the form

$$2\pi\varepsilon^2 N \leqslant C_{\delta,p}\cdot M\Big[e^{(1+\delta)p(\alpha+\varepsilon)}\Big(\frac{\alpha+\varepsilon}{\varepsilon^{1+2\delta}}+(1+\varepsilon^2)e^{2(1+\delta)p\varepsilon^2}\Big)\Big].$$

The proof is completed. ■

We now can apply this proposition to the function

$$a(z)=\det{}_n(I+Y_0(z)).$$

where n is such that $n-1\leqslant p(1+\delta)\leqslant n$ Let's remind the reader that according to Theorem 8.13 combined with the inequality (2.15), there exists a positive constant $C_{p,\delta}>0$ depending on p and δ such that (9.82) holds with

$$M = C_{p,\delta}\Big(\int_{\mathbb{R}^3}(1+e^{-x_1/2})^p(1+|x_1|)^2|V|^{p/2}dx\Big)^{2(1+\delta)}.$$

Thus, Theorem 1.3 follows from Proposition 9.1.

10. Individual eigenvalue bounds. Proof of Theorem 1.4

Here we obtain an estimate of the Hilbert-Schmidt norm of the Birman-Schwinger operator that allows us to say something about the location of eigenvalues of H in the complex plane.

Let $H_0=-\Delta+x_1$ be the free Stark operator. We are going to use the representation of $\exp(-itH_0)$ as a product of different factors, one of which is $\exp(it\Delta)$. Namely,

$$e^{-itH_0}=e^{-i\frac{t^3}{12}}\Big(e^{-itx_1/2}e^{it\Delta}e^{-itx_1/2}\Big), \qquad \forall\, t\in\mathbb{R}.$$

On the other hand, the resolvent operator $R_0(\lambda) = (H_0 - \lambda)^{-1}$ can be written as the integral

$$R_0(\lambda) = i \int_0^\infty e^{-it(H_0-\lambda)} dt.$$

If $\operatorname{Im}\lambda > 0$, then this integral converges (absolutely) in the operator-norm topology. We remind the reader that

$$\Lambda = \lambda - 2^{-1}(x_1 + y_1).$$

Proposition 10.1. *The integral kernel $r_0(x, y, \lambda)$ of the operator $R_0(\lambda)$ equals*

$$r_0(x, y, \lambda) = \frac{e^{i\sqrt{\Lambda}|x-y|}}{4\pi|x-y|} + \frac{e^{-i\frac{\pi}{4}}}{\sqrt{(4\pi)^3}} \int_0^\infty e^{\frac{i}{4t}|x-y|^2} \Big(e^{-i\frac{t^3}{12}} - 1\Big) e^{it\Lambda} \frac{dt}{t^{3/2}}, \tag{10.86}$$

for $x, y \in \mathbb{R}^3$ and $\lambda \in \mathbb{C}_+$.

Proof. Indeed, since

$$R_0(\lambda) = i \int_0^\infty e^{-i\frac{t^3}{12}} \Big(e^{-itx_1/2} e^{it\Delta} e^{-itx_1/2}\Big) e^{it\lambda} dt,$$

we come to the conclusion that

$$R_0(\lambda) = i \int_0^\infty e^{-itx_1/2} e^{it\Delta} e^{-itx_1/2} e^{it\lambda} dt + i \int_0^\infty (e^{-i\frac{t^3}{12}} - 1)\Big(e^{-itx_1/2} e^{it\Delta} e^{-itx_1/2}\Big) e^{it\lambda} dt. \tag{10.87}$$

It remains to observe that the two terms in (10.86) are the integral kernels of the operators in the right hand side of (10.87). ■

The following estimate plays a key role in the arguments of this section.

Lemma 10.2. *Let W_1 and W_2 be two functions from the space $L^3(\mathbb{R}^3)$. Then*

$$\frac{1}{(4\pi)^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|W_1(x)|^2 |W_2(y)|^2}{|x-y|^2} dx dy \leqslant C^2 ||W_1||_{L^3}^2 ||W_2||_{L^3}^2. \tag{10.88}$$

where $C > 0$ is independent of W_1 and W_2.

Proof. Note that the function $W_1(x)W_2(y)/(4\pi|x-y|)$ is the integral kernel of the operator $T = W_1(-\Delta)^{-1}W_2$. According to the Cwikel-Lieb-Rozenblum inequality (see [4],[28] and [34]), the number $n(s, T)$ of singular values of T lying to the right of $s > 0$ satisfies the relation

$$n(s, T) \leqslant C s^{-3/2} ||W_1||_{L^3}^{3/2} ||W_2||_{L^3}^{3/2}$$

with a constant $C > 0$ independent of W_1 and W_2. In particular, it implies the bound $||T|| \leqslant C^{2/3} ||W_1||_{L^3} ||W_2||_{L^3}$. It remains to note that

$$\frac{1}{(4\pi)^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|W_1(x)|^2 |W_2(y)|^2}{|x-y|^2} dx dy = ||T||_{\mathfrak{S}_2}^2 = 2 \int_0^{||T||} n(s, T) s ds.$$

■

Corollary 10.3. *Let W_1 and W_2 be two functions from the space $L^3(\mathbb{R}^3)$. Then*

$$||W_1(-\Delta - \lambda)^{-1} W_2||_{\mathfrak{S}_2}^2 \leqslant C^2 ||W_1||_{L^3}^2 ||W_2||_{L^3}^2, \qquad \forall \lambda \in \overline{\mathbb{C}}_+,$$

where C is the same as in (10.88).

Proof. The function $e^{i\sqrt{\lambda}|x-y|}/(4\pi|x-y|)$ is the kernel of the operator $(-\Delta-\lambda)^{-1}$. Consequently,

$$||W_1(-\Delta-\lambda)^{-1}W_2||^2_{\mathfrak{S}_2} \leqslant \frac{1}{(4\pi)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|W_1(x)|^2|W_2(y)|^2}{|x-y|^2}dxdy.$$

■

Let us now introduce the following convenient notations

$$\mu(\lambda,x,y)=\frac{e^{-i\frac{\pi}{4}}}{\sqrt{(4\pi)^3}}\int_0^\infty e^{\frac{i}{4t}|x-y|^2}\Big(e^{-i\frac{t^3}{12}}-1\Big)e^{it\Lambda}\frac{dt}{t^{3/2}},$$

and

$$\mu_1(\lambda,x,y)=\mu_0(\Lambda,x,y)-\mu_0(\lambda,x,y),$$

where

$$\mu_0(\lambda,x,y):=\frac{1}{4\pi|x-y|}e^{i\sqrt{\lambda}|x-y|}.$$

In these notations,

$$r_0(x,y,\lambda)=\mu_0(\lambda,x,y)+\mu_1(\lambda,x,y)+\mu(\lambda,x,y).$$

This representation of the integral kernel leads to the corresponding decomposition of the resolvent operator

$$R_0(\lambda)=\mathfrak{F}_0(\lambda)+\mathfrak{F}_1(\lambda)+\mathfrak{F}(\lambda).$$

We also need to introduce the characteristic function $\chi_\lambda(x)$ of the set $\{x\in\mathbb{R}^3:\ \ |x_1|<|\lambda|/2\}$.

Lemma 10.4. *Let V, W_1 and W_3 be three functions on $\mathbb{R}^3$ such that $V=W_2W_1$, and $|W_1|=|W_2|$. Let $\mathfrak{F}_1(\lambda)$ be the integral operator on $L^2(\mathbb{R}^3)$ with the kernel $\mu_1(\lambda,x,y)$. Let C_0 be the best constant in* (10.88). *Then*

$$||W_1\mathfrak{F}_1(\lambda)W_2-\chi_\lambda W_1\mathfrak{F}_1(\lambda)W_2\chi_\lambda||_{\mathfrak{S}_2}\leqslant\frac{2^{9/4}C_0}{(2+|\lambda|)^{1/4}}\Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/4}|V|^{3/2}dx\Big)^{2/3}.$$

Proof. Note that due to the fact that square of the Hilbert-Schmidt norm of an operator is the integral of the square of its kernel, we have

$$||W_1\mathfrak{F}_1(\lambda)W_2||_{\mathfrak{S}_2}\leqslant 2||W_1\mathfrak{F}_0(0)W_2||_{\mathfrak{S}_2}\leqslant 2C_0||W_1||_{L^3}||W_2||_{L^3}.$$

The statement of the lemma follows from the simple fact that

$$||W_1(1-\chi_\lambda)||_{L^3}+||W_2(1-\chi_\lambda)||_{L^3}\leqslant\frac{2^{5/4}}{(2+|\lambda|)^{1/4}}\Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/4}|V|^{3/2}dx\Big)^{1/3}.$$

The proof is completed. ■

Lemma 10.5. *Let $\mathfrak{F}_1(\lambda)$ be the operator with the kernel $\mu_1(\lambda,x,y)$. Let $\lambda\in\overline{\mathbb{C}}_+\setminus\{0\}$. Then*

$$||\chi_\lambda W_1\mathfrak{F}_1(\lambda)W_2\chi_\lambda||_{\mathfrak{S}_2}\leqslant\frac{1}{4\pi(1+|\lambda|)^{1/4}}\Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/2}|V|dx\Big).$$

Proof. It is easy to see that the kernel μ_1 satisfies the estimate

$$|\chi_\lambda(x)\mu_1(\lambda,x,y)\chi_\lambda(y)| \leqslant \frac{1}{4\pi(1+|\lambda|)^{1/4}}\big((1+|x_1|)(1+|y_1|)\big)^{3/4}. \tag{10.89}$$

Indeed, since

$$\mu_1(\lambda,x,y) = \frac{e^{i\sqrt{\Lambda}|x-y|}}{4\pi|x-y|} - \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}, \qquad \Lambda = \lambda - (x_1+y_1)/2,$$

we obtain that

$$|\chi_\lambda(x)\mu_1(\lambda,x,y))\chi_\lambda(y)| \leqslant \frac{\left|\sqrt{\Lambda}-\sqrt{\lambda}\right|\cdot|x-y|}{4\pi|x-y|} \leqslant \frac{\sqrt{2}}{16\pi|\lambda|^{1/2}}|x_1+y_1|,$$

for $(|x_1|+|y_1|) < |\lambda|$. The latter implies that

$$|\chi_\lambda(x)\mu_1(\lambda,x,y))\chi_\lambda(y)| \leqslant \frac{1}{8\pi}, \qquad \text{for} \quad |\lambda| \leqslant 1.$$

That proves (10.89) for $|\lambda| \leqslant 1$. The estimate (10.89) in the case $|\lambda| > 1$ follows from the inequality

$$\frac{\sqrt{2}}{16\pi|\lambda|^{1/2}}|x_1+y_1| \leqslant \frac{1}{4\pi(1+|\lambda|)^{1/4}}|x_1+y_1|^{3/4}, \qquad \text{for } |\lambda| > 1.$$

The statement of the lemma immediately follows from (10.89) and the definition of the Hilbert-Schmidt norm. ■

Corollary 10.6. *Let $\mathfrak{F}_1(\lambda)$ be the operator with the kernel $\mu_1(\lambda,x,y)$. Let C_0 be the best constant in* (10.88). *Let $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$. Then*

$$||W_1\mathfrak{F}_1(\lambda)W_2||_{\mathfrak{S}_2} \leqslant \frac{2^{9/4}C_0}{(2+|\lambda|)^{1/4}}\Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/4}|V|^{3/2}dx\Big)^{2/3} + \frac{1}{4\pi(1+|\lambda|)^{1/4}}\Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/2}|V|dx\Big).$$

Let us now consider the operator $W_1\mathfrak{F}_0(\lambda)W_2$, where $\mathfrak{F}_0(\lambda)$ is the operator with the integral kernel $\mu_0(\lambda,x,y)$. According to Theorem 12 of the paper [8], we can state the following:

Theorem 10.7. *Let $\lambda \in \mathbb{C}\setminus[0,\infty)$. The $\mathfrak{S}_p$-norms of the operator $W_1\mathfrak{F}_0(\lambda)W_2$ satisfy the estimates*

$$||W_1\mathfrak{F}_0(\lambda)W_2||_{\mathfrak{S}_p} \leqslant C_q|\lambda|^{-1+3/(2q)}||W_1||_{L^{2q}}||W_1||_{L^{2q}},$$

with $3/2 \leqslant q \leqslant 2$ and $p = 2q/(3-q)$. In particular,

$$||W_1\mathfrak{F}_0(\lambda)W_2||_{\mathfrak{S}_4} \leqslant C|\lambda|^{-1/4}\Big(\int_{\mathbb{R}^3}|V|^2dx\Big)^{1/2}, \tag{10.90}$$

where the positive constant C is independent of V and λ.

Finally, we are going to obtain an estimate for the Hilbert-Schmidt norm of the operator $W_1\mathfrak{F}(\lambda)W_2$. Let μ be the function

$$\mu(\lambda,x,y) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{(4\pi)^3}}\int_0^\infty e^{\frac{i}{4t}|x-y|^2}\Big(e^{-i\frac{t^3}{12}}-1\Big)e^{it\Lambda}\frac{dt}{t^{3/2}},$$

where $x,y\in\mathbb{R}^3$ and $\lambda \in \overline{\mathbb{C}}_+$.

Proposition 10.8. *There exists a universal constant $C > 0$ such that*

$$|\mu(\lambda, x, y)| \leqslant C \frac{(1+|x|)^2(1+|y|)^2}{(1+|\lambda|)^{1/4}}, \qquad \text{for all } x, y \in \mathbb{R}^3 \text{ and } \lambda \in \overline{\mathbb{C}}_+. \tag{10.91}$$

Proof. First, note that the function $\mu(\lambda, x, y)$ is bounded by a constant independent of the variables λ, x and y. The latter implies that one needs to prove (10.91) only for $|\lambda| > 1$. For this purpose we set $\beta = |\lambda|^{1/2}$ and write $\mu(\lambda, x, y)$ as the sum of two integrals

$$\begin{aligned} \mu(\lambda, x, y) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{(4\pi)^3}} \int_0^\beta e^{\frac{i}{4t}|x-y|^2} \Big(e^{-i\frac{t^3}{12}} - 1\Big) e^{it\Lambda} \frac{dt}{t^{3/2}} + \\ \frac{e^{-i\frac{\pi}{4}}}{\sqrt{(4\pi)^3}} \int_\beta^\infty e^{\frac{i}{4t}|x-y|^2} \Big(e^{-i\frac{t^3}{12}} - 1\Big) e^{it\Lambda} \frac{dt}{t^{3/2}}. \end{aligned} \tag{10.92}$$

The second integral can be easily estimated by $C/|\lambda|^{1/4}$, while the first integral equals

$$\frac{ie^{-i\frac{\pi}{4}}}{\lambda\sqrt{(4\pi)^3}} \int_0^\beta e^{it\lambda} \frac{d}{dt} \Big[\frac{e^{\frac{i}{4t}|x-y|^2} \Big(e^{-i\frac{t^3}{12}} - 1\Big) e^{-it(x_1+y_1)/2}}{t^{3/2}} \Big] dt + O(1/|\lambda|^{7/4}),$$

as $|\lambda| \to \infty$. Now we observe that

$$\Big| \frac{d}{dt} \Big[\frac{e^{\frac{i}{4t}|x-y|^2} \Big(e^{-i\frac{t^3}{12}} - 1\Big) e^{-it(x_1+y_1)/2}}{t^{3/2}} \Big] \Big| \leqslant C(t^{-1/2}|x-y|^2 + |x_1| + |y_1| + 1 + t^{1/2})$$

and integrate the expression in the right hand side with respect to t from 0 to β. As a result, we will obtain that

$$|\mu(\lambda, x, y)| \leqslant C(|\lambda|^{-3/4}|x-y|^2 + (|x_1| + |y_1| + 1)|\lambda|^{-1/2} + |\lambda|^{-1/4}) \qquad \text{for } |\lambda| > 1,$$

which definitely implies that

$$|\mu(\lambda, x, y)| \leqslant C|\lambda|^{-1/4}\Big(|x-y|^2 + |x_1| + |y_1| + 1\Big), \qquad \text{for } |\lambda| > 1.$$

It remains to note that $|x-y|^2 + |x_1| + |y_1| + 1 \leqslant (1+|x|)^2(1+|y|)^2$. The proof is completed. ■

Corollary 10.9. *Let $\mathfrak{F}(\lambda)$ be the operator on $L^2(\mathbb{R}^3)$ with the integral kernel $\mu(\lambda, x, y)$. Let W_1, $W_2 \in C_0^\infty(\mathbb{R}^3)$ be two functions such that $|W_1| = |W_2|$ and let $V = W_1 W_2$. Then*

$$||W_1 \mathfrak{F}(\lambda) W_2||_{\mathfrak{S}_2} \leqslant \frac{C}{(1+|\lambda|)^{1/4}} \int_{\mathbb{R}^3} (1+|x|)^4 |V(x)|\, dx, \tag{10.93}$$

where C is the same as in (10.91)

Corollary 10.10. *There exists a universal constant $C > 0$, such that all eigenvalues $\lambda \in \mathbb{C}\setminus\mathbb{R}$ of the operator H are situated in the disk*

$$|\lambda|^{1/4} \leqslant C\Big(\int_{\mathbb{R}^3} (1+|x|)^4 |V(x)|\, dx + \Big(\int_{\mathbb{R}^3} |V|^2 dx\Big)^{1/2}\Big).$$

Moreover, there is a universal constant $C_1 > 0$ such that the condition

$$\int_{\mathbb{R}^3} (1+|x|)^4 |V(x)|\, dx + \Big(\int_{\mathbb{R}^3} |V|^2 dx\Big)^{1/2} < C_1$$

implies that the spectrum of H coincides with the real line $\mathbb{R}$.

Proof. It is very well known that all non-real eigenvalues of H are situated in the set

$$\{\lambda \in \mathbb{C} \setminus \mathbb{R} : \; ||Y_0(\lambda)|| \geqslant 1)\}.$$

Obviously,

$$||Y_0(\lambda)|| = ||W_1 R_0(\lambda) W_2|| \leqslant ||W_1 \mathfrak{F}_0(\lambda) W_2||_{\mathfrak{S}_4} + ||W_1 \mathfrak{F}_1(\lambda) W_2||_{\mathfrak{S}_2} + ||W_1 \mathfrak{F}(\lambda) W_2||_{\mathfrak{S}_2}.$$

On the other hand, according to Theorem 10.7 combined with Corollary 10.3,

$$||W_1 \mathfrak{F}_0(\lambda) W_2||_{\mathfrak{S}_4} \leqslant C \frac{1}{1+|\lambda|^{1/4}} \Big[\Big(\int_{\mathbb{R}^3} |V|^2 dx \Big)^{1/2} + \Big(\int_{\mathbb{R}^3} |V|^{3/2} dx \Big)^{2/3} \Big].$$

Due to Corollary 10.6, we also have

$$||W_1 \mathfrak{F}_1(\lambda) W_2||_{\mathfrak{S}_2} \leqslant C \frac{1}{1+|\lambda|^{1/4}} \Big[\Big(\int_{\mathbb{R}^3} (1+|x_1|)^{3/2} |V| dx \Big) + \Big(\int_{\mathbb{R}^3} (1+|x_1|)^{3/4} |V|^{3/2} dx \Big)^{2/3} \Big].$$

Finally, Corollary 10.9 gives us the estimate

$$||W_1 \mathfrak{F}(\lambda) W_2||_{\mathfrak{S}_2} \leqslant \frac{C}{(1+|\lambda|)^{1/4}} \int_{\mathbb{R}^3} (1+|x|)^4 |V(x)|\, dx.$$

Consequently,

$$||Y_0(\lambda)|| \leqslant \frac{C}{(1+|\lambda|)^{1/4}} \Big[\int_{\mathbb{R}^3} (1+|x|)^4 |V(x)|\, dx + \Big(\int_{\mathbb{R}^3} |V|^2 dx \Big)^{1/2} \Big].$$

The latter implies both statements of Corollary 10.10. ■

As a consequence of the method, we obtain the following estimate with a very short expression in the right hand side:

Theorem 10.11. *Let* $\mathrm{Im}\lambda \geqslant 0$. *Then for any* $p > 11$, *there exists a positive constant* $C_p > 0$ *depending only on* p *such that*

$$||Y_0(\lambda)||_{\mathfrak{S}_4} \leqslant \frac{C_p}{1+|\lambda|^{1/4}} \Big(\int_{\mathbb{R}^3} (1+|x|)^p |V|^2 dx \Big)^{1/2}. \tag{10.94}$$

Proof. It is enough to note that

$$\int_{\mathbb{R}^3} (1+|x|)^4 |V(x)|\, dx + \Big(\int_{\mathbb{R}^3} |V|^2 dx \Big)^{1/2} \leqslant C_p \Big(\int_{\mathbb{R}^3} (1+|x|)^p |V|^2 dx \Big)^{1/2}.$$

■

Let us prove that that all eigenvalues of H are contained in a disk of a finite radius under the condition $V \in L^{q/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ where $q < 3$. First, we find ε so that $q = 3/(1+\varepsilon)$ and set $p = 3/(1-\varepsilon)$. Without loss of generality, we can assume that ε is very small.

According to (6.56),

$$||W_1 R_0(\lambda) W_2||_{\mathfrak{S}_p} \leqslant C \Big(||W_1||_{L^p} ||W_2||_{L^p} + ||W_1||_{L^q} ||W_2||_{L^q} \Big), \qquad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Choose $\delta > 0$ so small that $C\delta(||W_1||_{L^p} + ||W_1||_{L^q} + ||W_2||_{L^p} + ||W_2||_{L^q} + 2\delta) < 1/2$. Let now $\tilde{W}_j$ be bounded compactly supported approximations of W_j having the property that

$$||\tilde{W}_j - W_j||_{L^p} + ||\tilde{W}_j - W_j||_{L^q} < \delta.$$

Then

$$||W_1R_0(\lambda)W_2-\tilde W_1R_0(\lambda)\tilde W_2||_{\mathfrak{S}_p}\leqslant C\delta\Big(||W_1||_{L^2}+||W_1||_{L^3}+||W_2||_{L^2}+||W_2||_{L^3}+2\delta\Big).\quad (10.95)$$

The right hand side in (10.95) is smaller than 1/2. Set now $\tilde Y_0(\lambda)=\tilde W_1R_0(\lambda)\tilde W_2$. Then according to the methods that led us to (10.94),

$$||\tilde Y_0(\lambda)||_{\mathfrak{S}_4}=O(|\lambda|^{-1/4}),\qquad \text{as}\quad |\lambda|\to\infty.\quad (10.96)$$

It remains to note that $||Y_0(\lambda)||\leqslant\frac12+||\tilde Y_0(\lambda)||$. ■

We provide an extensive list of mathematical articles [1]-[3], [10]-[24], [26], [29]- [31], [33], [36], [37] containing the important work on Stark operators, which are operators with the potential corresponding to a constant electric field, and the work related to the study of the Stark effect. Our list includes the titles of the books [32] and [35] containing the relevant theory of Schrödinger operators and perturbation determinants. Finally, the paper [25] is mentioned because it indicates the possible direction of the related follow up research.

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