ON PERIODIC SOLUTIONS TO SOME HAMILTONIAN SYSTEM WITH NON-COMPACT CONFIGURATION SPACE

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Abstract. A classical mechanics Lagrangian system with discrete symmetry is considered. The configuration space is a cylinder $\mathbb{R}^m\times\mathbb{T}^n$. A large class of nonhomotopic periodic solutions has been found.

1. Statement of the Problem and the Main Result

Introduce some notations. Let $x=(x^1,\ldots,x^m)$ and $\varphi=(\varphi^1,\ldots,\varphi^n)$ be points of the standard $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Then let z stand for the point $(x,\varphi)\in\mathbb{R}^{m+n}$. By $|\cdot|$ denote the standard Euclidean norm of $\mathbb{R}^k$, $\quad k=m,m+n$ that is $|x|^2=\sum_{i=1}^k(x^i)^2$.

The main object of our study is the following Lagrangian system

$$L(t,z,\dot z)=\frac{1}{2}g_{ij}\dot z^i\dot z^j+a_i\dot z^i-V,\quad z=(z^1,\ldots,z^{m+n}),\tag{1.1}$$

here we use the Einstein summation convention. The functions g_{ij},a_i,V depend on (t,z) and belong to $C^2(\mathbb{R}^{m+n+1})$; moreover all these functions are $2\pi-$periodic in each variable φ^j and $\omega-$periodic in the variable t, $\quad\omega>0$. For all $(t,z)\in\mathbb{R}^{m+n+1}$ it follows that $g_{ij}=g_{ji}$.

We also assume that there are positive constants C,M,A,K such that for all (t,z) and $\xi\in\mathbb{R}^m$ we have

$$|a_i(t,z)\xi^i|\le|\xi|(C+M|x|),\quad V(t,z)\le A|x|^2,\quad \frac{1}{2}g_{ij}(t,z)\xi^i\xi^j\ge K|\xi|^2.\tag{1.2}$$

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Theorem 1.1. *Assume that*
1) all the functions are even:

$$g_{ij}(-t,-z)=g_{ij}(t,z),\quad a_i(-t,-z)=a_i(t,z),\quad V(-t,-z)=V(t,z);$$

2) the following inequality holds

$$K-\frac{M\omega}{\sqrt{2}}-\frac{A\omega^2}{2}>0. \tag{1.3}$$

Then for each $\nu=(\nu_1,\dots,\nu_n)\in\mathbb{Z}^n$ system (1.1) has a solution $z(t)=(x(t),\varphi(t))\in C^3(\mathbb{R},\mathbb{R}^{m+n})$ such that
1) the function z is odd: $z(-t)=-z(t)$;
2) $x(t+\omega)=x(t),\quad \varphi(t+\omega)=\varphi(t)+2\pi\nu$.

The second condition of the theorem implies two remarks. First, if all the functions do not depend on t then we can choose ω to be arbitrary small and inequality (1.3) is satisfied. Taking a vanishing sequence of ω, we obtain infinitely many periodic solutions.

Second, assume that $M=0$ and $V(t,z)\le A_1|x|^\alpha,\quad \alpha<2$. Choose a constant $A>0$ small such that inequality (1.3) is satisfied then choose a constant $c_1>0$ such that for all $|x|$ one has $A_1|x|^\alpha\le A|x|^2+c_1$. Now the second condition of the theorem is satisfied for the new potential

$$V_1=V-c_1\le A|x|^2.$$

1.1. **Examples.** Our first example is as follows.

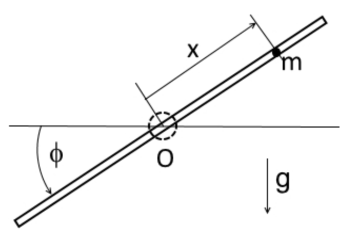

FIGURE 1. the tube and the ball

A thin tube can rotate freely in the vertical plane about a fixed horizontal axis passing through its centre O. A moment of inertia of the tube about this axis is equal to J. The mass of the tube is distributed symmetrically such that tube's centre of mass is placed at the point O.

Inside the tube there is a small ball which can slide without friction. The mass of the ball is m. The ball can pass by the point O and fall out from the ends of the tube.

The system undergoes the standard gravity field $\boldsymbol{g}$.

It seems to be evident that for typical motion the ball reaches an end of the tube and falls down out the tube. It is surprisingly, at least for the first glance, that this system has very many periodic solutions such that the tube turns around several times during the period.

The sense of generalized coordinates ϕ, x is clear from Figure 1.

The Lagrangian of this system is as follows

$$L(x,\phi,\dot x,\dot\phi)=\frac{1}{2}\Big(mx^2+J\Big)\dot\phi^2+\frac{1}{2}m\dot x^2-mgx\sin\phi. \tag{1.4}$$

Form theorem 1.1 it follows that for any constant $\omega>0$ system (1.4) has a solution $\phi(t),x(t),\quad t\in\mathbb{R}$ such that

1) $x(t)=-x(-t),\quad \phi(t)=-\phi(-t)$;

2) $x(t+\omega)=x(t),\quad \phi(t+\omega)=\phi(t)+2\pi$.

This result shows that for any $\omega>0$ the system has an $\omega-$periodic motion such that the tube turns around once during the period. The length of the tube should be chosen properly.

Our second example is a counterexample. Let us show that the first condition of the theorem 1.1 can not be omitted.

Consider a mass point m that slides on a right circular cylinder of radius r. The surface of the cylinder is perfectly smooth. The axis x of the cylinder is parallel to the gravity $\boldsymbol{g}$ and directed upwards.

The Lagrangian of this system is

$$L(x,\varphi,\dot x,\dot\varphi)=\frac{m}{2}\Big(r^2\dot\varphi^2+\dot x^2\Big)-mgx. \tag{1.5}$$

All the conditions except the evenness are satisfied but it is clear this system does not have periodic solutions.

2. Proof of Theorem 1.1

In this section we use several standard facts from functional analysis and the Sobolev spaces theory [2], [1].

1. Recall that the Sobolev space $H^1_{\mathrm{loc}}(\mathbb{R})$ consists of functions $u(t),t\in\mathbb{R}$ such that $u,\dot u\in L^2_{\mathrm{loc}}(\mathbb{R})$. The following embedding holds $H^1_{\mathrm{loc}}(\mathbb{R})\subset C(\mathbb{R})$.

Lemma 2.1. *Let $u\in H^1_{\mathrm{loc}}(\mathbb{R})$ and $u(0)=0$. Then for any $a>0$ we have*

$$\|u\|^2_{L^2(0,a)}\le\frac{a^2}{2}\|\dot u\|^2_{L^2(0,a)},\quad \|u\|^2_{C[0,a]}\le a\|\dot u\|^2_{L^2(0,a)}.$$

Here and below the notation $\|\dot u\|_{L^2(0,a)}$ implies that

$$\Big\|\dot u\Big|_{(0,a)}\Big\|_{L^2(0,a)}$$

the same is concerned to $\|u\|_{C[0,a]}$ etc.

This Lemma is absolutely standard, nevertheless just for completeness of exposition sake we bring a sketch of its proof.

Proof of Lemma 2.1. From formula

$$u(t)=\int_0^t \dot u(s)ds \tag{2.1}$$

it follows that

$$\int_0^a u^2(s)ds=\int_0^a\Big(\int_0^t \dot u(s)ds\Big)^2dt.$$

It remains to observe that by the Cauchy inequality

$$\Big|\int_0^t \dot u(s)ds\Big|\le\int_0^t|\dot u(s)|ds\le\|\dot u\|_{L^2(0,a)}\Big(\int_0^t ds\Big)^{1/2},\quad t\in[0,a].$$

This implies the first inequality of the lemma. The second inequality also follows from formula (2.1) and the Cauchy inequality.

The Lemma is proved.

2. Here we collect several spaces those are needed in the sequel.

Definition 1. *By X denote a space of functions $u\in H^1_{\mathrm{loc}}(\mathbb{R})$ such that for all $t\in\mathbb{R}$ the following conditions hold*

$$u(-t)=-u(t),\quad u(t+\omega)=u(t).$$

By virtue of lemma 2.1, the mapping $u\mapsto\|\dot u\|_{L^2(0,\omega)}$ determines a norm in X. This norm is denoted by $\|u\|$. The norm $\|\cdot\|$ is equivalent to the standard norm of $H^1[0,\omega]$. The space $(X,\|\cdot\|)$ is a Banach space. Since the norm $\|\cdot\|$ is generated by an inner product

$$(u,v)_X=\int_0^\omega \dot u(t)\dot v(t)dt$$

the space X is also a real Hilbert space, particularly this implies that X is a reflexive Banach space.

Definition 2. *Let Φ stand for the space $\{ct+u(t)\mid c\in\mathbb{R},\quad u\in X\}$.*

By the same argument, $(\Phi,\|\cdot\|)$ is a reflexive Banach space. Observe also that $\Phi=\mathbb{R}\oplus X$ and by direct calculation we get

$$\|\psi\|^2=\omega c^2+\|u\|^2,\quad \psi(t)=ct+u(t)\in\Phi.$$

Observe that $X\subset\Phi$.

Definition 3. *Let E stand for the space*

$$X^m\times\Phi^n=\{z(t)=(x^1,\dots,x^m,\varphi^1,\dots,\varphi^n)(t)\mid x^i\in X,\quad \varphi^j\in\Phi\}.$$

The space E is also a real Hilbert space with an inner product defined as follows
$$(z, y)_E = \int_0^\omega \sum_{i=1}^{m+n} \dot{x}^i(t)\dot{y}^i(t)dt,$$
where $z = (z^k), y = (y^k) \in E, \quad k = 1, \dots, m+n$.

We denote the corresponding norm in E by the same symbol and write
$$\|z\|^2 = \big\||z|\big\|^2 = \sum_{k=1}^{m+n} \|z^k\|^2.$$
The space E is also a reflexive Banach space.

Introduce the following set
$$E_0 = \Big\{(x, \varphi) \in E \Big| \varphi^j = \frac{2\pi\nu_j}{\omega}t + u_j, \quad u_j \in X, \quad j = 1, \dots, n\Big\}.$$
This set is a closed plane of codimension n in E.

If $\varphi \in E_0$ then $\varphi(t+\omega) = \varphi(t) + 2\pi\nu$.

Definition 4. *Let Y stand for the space*

$\big\{u \in L^2_{\mathrm{loc}}(\mathbb{R}) \mid u(t) = u(-t), \quad u(t+\omega) = u(t) \quad$ *almost everywhere in* $\mathbb{R}\big\}$.

This is a real Hilbert space with respect to the inner product
$$(u, v)_Y = \int_0^\omega u(t)v(t)dt.$$
Let us endow the space Y^{m+n} with the real Hilbert space structure by means of the inner product
$$(u, v)_{Y^{m+n}} = \int_0^\omega \sum_{i=1}^{m+n} u^i(t)u^i(t)dt,$$
where
$$u = (u^i), \quad v = (v^i), \quad u^i, v^i \in Y, \quad i = 1, \dots, m+n.$$

3. Introduce the Action Functional $S : E_0 \to \mathbb{R}$,
$$S(z) = \int_0^\omega L(t, z, \dot{z})dt.$$
Our next goal is to prove that this functional attains its minimum.

Observe that $|x| \le |z|$ then by using estimates (1.2) we get
$$S(z) \ge \int_0^\omega \big(K|\dot{z}|^2 - |\dot{z}|(C + M|z|) - A|z|^2\big)dt.$$

From the Cauchy inequality and Lemma 2.1 it follows that

$$\int_0^\omega |\dot z||z|dt \le \frac{\omega}{\sqrt{2}}|z\|^2,$$
$$\int_0^\omega |z|^2 dt \le \frac{\omega^2}{2}\|z\|^2,$$
$$\int_0^\omega |\dot z|dt \le \sqrt{\omega}\|z\|.$$

We finally yield

$$S(z) \ge \Big(K - \frac{M\omega}{\sqrt{2}} - \frac{A\omega^2}{2}\Big)\|z\|^2 - C\sqrt{\omega}\|z\|.$$

By formula (1.3) the functional S is coercive:

$$S(z) \to \infty \tag{2.2}$$

as $\|z\| \to \infty$.

Note that the Action Functional which corresponds to system (1.5) is also coercive but, as we see above, property (2.2) by itself does not imply existence results.

4. Let $\{z_k\} \subset E_0$ be a minimizing sequence:

$$S(z_k) \to \inf_{z\in E_0} S(z)$$

as $k \to \infty$.

By formula (2.2) the sequence $\{z_k\}$ is bounded: $\sup_k \|z_k\| < \infty$. Since the space E is reflexive, this sequence contains a weakly convergent subsequence. Denote this subsequence in the same way: $z_k \to z_*$ weakly in E.

Moreover, the space $H^1[0,\omega]$ is compactly embedded in $C[0,\omega]$. Thus extracting a subsequence from the subsequence and keeping the same notation we also have

$$\max_{t\in[0,\omega]} |z_k(t) - z_*(t)| \to 0, \tag{2.3}$$

as $k \to \infty$.

The set E_0 is convex and strongly closed therefore it is weakly closed: $z_* \in E_0$.

5. Let us show that $\inf_{z\in E_0} S(z) = S(z_*)$.

Lemma 2.2. *Let a sequence $\{u_k\} \subset \Phi$ weakly converges to $u \in \Phi$ (or $u_k, u \subset X$ and $u_k \to u$ weakly in X); and also* $\max_{t\in[0,\omega]} |u_k(t) - u(t)| \to 0$.

Then for any $f \in C(\mathbb{R})$ and for any $v \in L^2(0,\omega)$ it follows that

$$\int_0^\omega f(u_k)\dot u_k v dt \to \int_0^\omega f(u)\dot u v dt.$$

Indeed,

$$\int_0^\omega f(u_k)\dot u_k v dt$$
$$= \int_0^\omega \big(f(u_k) - f(u)\big)\dot u_k v dt + \int_0^\omega f(u)\dot u_k v dt.$$

The function f is uniformly continuous in a compact set

$$\Big[\min_{t\in[0,\omega]}\{u(t)\} - c, \max_{t\in[0,\omega]}\{u(t)\} + c\Big]$$

with some constant $c > 0$. Consequently we obtain

$$\max_{t\in[0,\omega]} |f(u_k(t)) - f(u(t))| \to 0.$$

Since the sequence $\{u_k\}$ is weakly convergent it is bounded:

$$\sup_k \|u_k\| < \infty$$

particularly, we get

$$\|\dot u_k\|_{L^2(0,\omega)} < \infty.$$

So that

$$\Big|\int_0^\omega \big(f(u_k) - f(u)\big)\dot u_k v dt\Big|$$
$$\le \big\|v\big(f(u_k) - f(u)\big)\big\|_{L^2(0,\omega)} \cdot \|\dot u_k\|_{L^2(0,\omega)} \to 0.$$

To finish the proof it remains to observe that a function

$$u \mapsto \int_0^\omega f(u)\dot u v dt$$

belongs to Φ' (or to X'). Indeed,

$$\Big|\int_0^\omega f(u)\dot u v dt\Big| \le \max_{t\in[0,\omega]} |f(u(t))| \cdot \|v\|_{L^2(0,\omega)}\|u\|.$$

6. The following lemma is proved similarly.

Lemma 2.3. *Let a sequence $\{u_k\} \subset \Phi$ (or $\{u_k\} \subset X$) be such that*

$$\max_{t\in[0,\omega]} |u_k(t) - u(t)| \to 0.$$

Then for any $f \in C(\mathbb{R})$ and for any $v \in L^1(0,\omega)$ it follows that

$$\int_0^\omega f(u_k) v dt \to \int_0^\omega f(u) v dt.$$

7. Introduce a function $p_k(\xi) = L(t, z_k, \dot z_* + \xi)$. Here the variable t is supposed to be fixed for a while. The function p_k is a quadratic polynomial of $\xi \in \mathbb{R}^{m+n}$, so that

$$p_k(\xi) = L(t, z_k, \dot z_*) + \frac{\partial L}{\partial \dot z^i}(t, z_k, \dot z_*)\xi^i + \frac{1}{2}\frac{\partial^2 L}{\partial \dot z^j \partial \dot z^i}(t, z_k, \dot z_*)\xi^i\xi^j.$$

The last term in this formula is non-negative:

$$\frac{\partial^2 L}{\partial \dot z^j \partial \dot z^i}(t, z_k, \dot z_*)\xi^i\xi^j = g_{ij}(t, z_k, \dot z_*)\xi^i\xi^j \geq 0.$$

We consequently obtain

$$p_k(\xi) \geq L(t, z_k, \dot z_*) + \frac{\partial L}{\partial \dot z^i}(t, z_k, \dot z_*)\xi^i.$$

It follows that

$$\begin{aligned} S(z_k) = \int_0^\omega p_k(\dot z_k - \dot z_*)dt \geq \int_0^\omega L(t, z_k, \dot z_*)dt \\ + \int_0^\omega \frac{\partial L}{\partial \dot z^i}(t, z_k, \dot z_*)(\dot z_k^i - \dot z_*^i)dt. \end{aligned} \tag{2.4}$$

From lemma 2.2 and lemma 2.3 it follows that

$$\int_0^\omega L(t, z_k, \dot z_*)dt \to \int_0^\omega L(t, z_*, \dot z_*)dt,$$

and

$$\int_0^\omega \frac{\partial L}{\partial \dot z^i}(t, z_k, \dot z_*)(\dot z_k^i - \dot z_*^i)dt \to 0.$$

Passing to the limit as $k \to \infty$ in (2.4) we finally yield

$$\inf_{z \in E_0} S(z) \geq S(z_*).$$

8. Thus for any $v \in X^{m+n}$ it follows that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(z_* + \varepsilon v) = 0.$$

Every element $v \in X^{m+n}$ is presented as follows

$$v(t) = \int_0^t y(s)ds,$$

where $y \in Y^{m+n}$ is such that

$$\int_0^\omega y(s)ds = 0.$$

Introduce a linear operator $h : Y^{m+n} \to \mathbb{R}^{m+n}$ by the formula

$$h(y) = \int_0^\omega y(s)ds.$$

Define a linear functional $q : Y^{m+n} \to \mathbb{R}$ by the formula

$$q(y) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S\Big(z_* + \varepsilon \int_0^t y(s)ds\Big).$$

Now all our observations are resumed as follows

$$\ker h \subseteq \ker q.$$

Therefore, there exists a linear functional $\lambda : \mathbb{R}^{m+n} \to \mathbb{R}$ such that

$$q = \lambda h.$$

9. Let us rewrite the last formula explicitly. There are real constants λ_k such that for any $y^k \in Y$ one has

$$\int_0^\omega \Big(\frac{\partial L}{\partial \dot z^k}(t, z_*, \dot z_*) y^k(t) + \frac{\partial L}{\partial z^k}(t, z_*, \dot z_*) \int_0^t y^k(s)ds\Big)dt$$
$$= \lambda_k \int_0^\omega y^k(s)ds.$$

By the Fubini theorem we obtain

$$\int_0^\omega \frac{\partial L}{\partial \dot z^k}(t, z_*, \dot z_*) y^k(t)dt + \int_0^\omega y^k(s) \int_s^\omega \frac{\partial L}{\partial z^k}(t, z_*, \dot z_*)dtds$$
$$= \lambda_k \int_0^\omega y^k(s)ds. \tag{2.5}$$

In this formula it does not matter whether the function y^k is periodic and even, we use this function only in the interval $(0, \omega)$. Any function from $L^2(0, \omega)$ can be extended up to a function of Y.

Therefore, equation (2.5) is rewritten as the following system

$$\frac{\partial L}{\partial \dot z^k}(t, z_*(t), \dot z_*(t)) + \int_t^\omega \frac{\partial L}{\partial z^k}(s, z_*(s), \dot z_*(s))ds = \lambda_k, \tag{2.6}$$

here $k = 1, \dots, m+n$. Equalities (2.6) hold for almost all $t \in (0, \omega)$.

10. Let us show that if $w \in L^2_{\mathrm{loc}}(\mathbb{R})$ is an $\omega-$periodic and odd function:

$$w(-t) = -w(t), \quad w(t+\omega) = w(t) \quad \text{almost everywhere},$$

then a function $t \mapsto \int_t^\omega w(\tau)d\tau$ belongs to Y.

Indeed, $\int_t^\omega w(\tau)d\tau = \int_t^0 w(\tau)d\tau + \int_0^\omega w(t)dt$. Due to oddity and periodicity

$$\int_0^\omega w(t)dt = \int_{-\omega/2}^{\omega/2} w(\tau)d\tau = 0.$$

Thus $\int_t^0 w(\tau)d\tau \in Y$.

11. The function
$$\frac{\partial L}{\partial z^k}(t, z_*, \dot z_*)dt$$
is $\omega-$periodic and odd in t. From the argument above we conclude that
$$\int_t^\omega \frac{\partial L}{\partial z^k}(\tau, z_*, \dot z_*)d\tau \in Y.$$

Since the left side of (2.6) is $\omega-$periodic, this equation holds for almost all $t \in \mathbb{R}$.

If we formally differentiate in t both sides of equations (2.6) we obtain the Lagrange equations with Lagrangian L.

12. Present equation (2.6) in the form
$$\dot z_*^j(t) = g^{kj}(t, z_*(t))\Big(\lambda_k - \int_t^\omega \frac{\partial L}{\partial z^k}(s, z_*(s), \dot z_*(s))ds - a_k(t, z_*(t))\Big). \tag{2.7}$$
Recall that by the Sobolev embedding theorem, $z_* \in X \subset C(\mathbb{R})$. Thus the right side of equality (2.7) belongs to $C(\mathbb{R})$. Therefore, $z_* \in C^1(\mathbb{R})$. Applying the same argument two times again, we get $z_* \in C^3(\mathbb{R})$.

The theorem is proved.

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