Some notes on the parametric Borel summability for linear singularly perturbed Cauchy problems with linear fractional transforms

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Abstract

This paper is a slightly modified, abridged version of the work [8]. It deals with some questions made to the authors during the school and conference Complex Differential and Difference Equations, held in Bedlewo (Poland) during the first two weeks of september, 2018.

We study linear singularly perturbed Cauchy problems under the action of partial differential operators and linear fractional transforms. The asymptotic behavior of the holomorphic solutions is determined with respect to the perturbation parameter near the origin. Moreover, two asymptotic levels are distinguished: Gevrey and 1^+.

Key words: asymptotic expansion, Borel-Laplace transform, Cauchy problem, difference equation, integro-differential equation, linear partial differential equation, singular perturbation.
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1 Introduction

The main purpose of this revision is to give answer to certain questions and fruitful mathematical discussions with participants of the school and conference Complex Differential and Difference Equations, held in Bedlewo (Poland) during the first two weeks of september, 2018, where the work [8] was presented. Moreover, we provide some additional motivation and future problems in this line of research. We have decided to simplify technical difficulties for the sake of clarity, which can be found in detail in [8].

As a first motivation of our results, we first indicate an example appearing in [2]. The authors consider the next difference equation

$$h(s+1) - \frac{a}{s}h(s) = \frac{1}{s}, \tag{1}$$

for given $a > 0$. Then, they construct a formal power series solution $\hat{h}(s) = \sum_{n\geq 1} h_n s^{-n}$ and exhibit sharp estimates on their coefficients:

$$|h_n| \leq K \left(\frac{n}{\log(n)}\right)^n A^n, \quad n \geq 2,$$

for some constants $A, K > 0$. Such estimates on the formal solutions are natural when dealing with difference equations under the so-called 1^+ phenomena, in the sense of G. Immink (see [4]). We observe that these are finer bounds than classical Gevrey estimates of order 1 (of the kind

$K\left(\frac{n}{e}\right)^n A^n$, $n \geq 1$). In a second step, the authors provide actual solutions of (1), expressed as Laplace transforms

$$h_{\mathcal{P}}(s) = \int_{\mathcal{P}} \left(\sum_{n\geq 1} \frac{h_n}{(n-1)!} t^{n-1} \right) e^{-st} dt,$$

where $u(t) := \sum_{n\geq 1} \frac{h_n}{(n-1)!} t^{n-1}$ is the Borel transform of the formal solution. The convolution equation

$$e^{-t}u(t) - a\int_0^t u(x)dx = 1,$$

where the path $\mathcal{P}$ connects the origin and infinity in $\mathbb{C}$, has a unique analytic solution $u(t) = e^{t-a}e^{ae^t}$. Such solution turns out to be analytic in $\{t \in \mathbb{C} : \mathrm{Re}(t) \leq 0\}$. Therefore, for all directions $d \in (\pi/2, 3\pi/2)$, the usual Laplace transform

$$h_{L_d}(s) = \int_{L_d} u(t)e^{-st}dt, \tag{2}$$

for $L_d = \mathbb{R}_+ e^{\sqrt{-1}d}$ represent solutions of (1), holomorphic on unbounded sectors $S_{d,R} = \{s \in \mathbb{C}^\star : -\pi/2 < \arg(s) + d < \pi/2, |s| > R\}$, for all $R > 0$.

On the other hand, $u(t)$ suffers super-exponential growth on every halfline $L_d \subseteq \{\mathrm{Re}(t) > 0\}$, and the series is not convergent. However, the authors noticed a finer structure in $u(t)$ of super exponential growth and decay on certain horizontal strips: $u(t)$ has super exponential decay on the disjoint strips $H_k = \{t \in \mathbb{C} : \mathrm{Re}(t) \geq 0, \mathrm{Im}(t) \in (\pi/2, 3\pi/2) + 2k\pi\}$ and super exponential growth on $J_k = \{t \in \mathbb{C} : \mathrm{Re}(t) \geq 0, \mathrm{Im}(t) \in (-\pi/2, \pi/2) + 2k\pi\}$.

As a result, for every $k \in \mathbb{Z}$, a piecewise linear path $\mathcal{P}_k$ can be chosen in such a way that the Laplace-like transform

$$h_k(s) = \int_{\mathcal{P}_k} u(t)e^{-st}dt$$

represents a solution of (1), holomorphic on $S_{0,R} = \{s \in \mathbb{C}^\star : \arg(s) \in (-\pi/2, \pi/2), |s| > R\}$, for all $R > 0$. The function h_k is an entire function. The functions h_k and h_{L_d} both admit $\hat{h}$ as Gevrey asymptotic expansion of order 1 on their domains of holomorphy, say $\mathcal{D}$, meaning there exist $K, A > 0$ with

$$\left| h_\star(s) - \sum_{n=1}^{N-1} h_n s^{-n} \right| \leq KA^N \left(\frac{N}{e}\right)^N |s|^{-N},$$

for $\star \in \{k, L_d\}$ and all $N \geq 1$, $s \in \mathcal{D}$.

This approach is now adapted to study the parametric Borel summability for some families of singularly perturbed PDEs involving Moebius transforms.

2 Main Cauchy problem

We consider the Cauchy problem

$$P(\epsilon t^2\partial_t)\partial_z^S u(t,z,\epsilon) = \sum_{\underline{k}=(k_0,k_1,k_2)\in\mathcal{A}} c_{\underline{k}}(z,\epsilon) \left((t^2\partial_t)^{k_0}\partial_z^{k_1} u \right) \left(\frac{t}{1+k_2\epsilon t}, z, \epsilon\right), \tag{3}$$

and initial data

$$(\partial_z^j u)(t,0,\epsilon) = \varphi_j(t,\epsilon), \quad 0 \leq j \leq S-1. \tag{4}$$

Here, $S \geq 1$ is an integer, $P(X) \in \mathbb{C}[X]$ with all its roots belonging to $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, and $\mathcal{A}$ is a finite subset of $\mathbb{N}^3$. For all $\underline{k} \in \mathcal{A}$, $c_{\underline{k}}(z,\epsilon)$ is a holomorphic function in a neighborhood of $(0,0)$. Moreover, we assume the existence of $\xi > 0$ and $b > 1$ such that for all $\underline{k} = (k_0, k_1, k_2) \in \mathcal{A}$, one has

$$S \geq k_1 + bk_0 + \frac{bk_2}{\xi} \quad , \quad S > k_1. \tag{5}$$

Remark: The change of variable $t = 1/s$ makes equation (3) turn into a singularly perturbed linear PDE combined with small shifts of the form $T_{k_2,\epsilon}X(s,z,\epsilon) = X(s + k_2\epsilon, z, \epsilon)$, with $u(t,z,\epsilon) = X(1/t, z, \epsilon)$.

In the context of differential equations, most results are devoted to problems of the form

$$\epsilon \partial_t x(t,\epsilon) = f(t, \epsilon, x(t,\epsilon), x(t \pm \delta, \epsilon)),$$

for some vector valued function f, a small real parameter $\epsilon > 0$, and $\delta > 0$ which may depend on ϵ. Solutions of such problems are related with asymptotic expansions of the form

$$x(t,\epsilon) = \sum_{\ell=0}^{n-1} x_\ell(t)\epsilon^\ell + R_n(t,\epsilon), \quad n \geq 2,$$

as $\epsilon \to 0$, see [3, 11]. In the framework of PDEs, some asymptotic analysis has been performed for reaction-diffusion equations with small delay

$$\partial_t u - \epsilon^2 \partial_x^2 u = f(u(t,z,\epsilon), u(t-s,x,\epsilon), \epsilon),$$

under certain initial and boundary conditions to be determined for small enough $\epsilon, s > 0$. Actual solutions are constructed. Moreover, asymptotic relations are stated with respect to the perturbation parameter, see [12, 9].

The construction of the initial data in (4) is as follows. We consider two sets of bounded setors $\{\mathcal{E}_k^-\}_{k \in \{-n,\dots,n\}}$ for some $n \geq 1$, and $\{\mathcal{E}_p\}_{0 \leq p \leq \iota-1}$ and an open and bounded sector $\mathcal{T}$ with bisecting direction $d = 0$, built in such a way that the next statements hold:

- $\pi - \arg(t) - \arg(\epsilon) \in (-\pi/2 + \delta, \pi/2 - \delta)$ for some $\delta > 0$ and all $\epsilon \in \mathcal{E}_k^-$, for $k \in \{-n, \dots, n\}$, $t \in \mathcal{T}$.
- Let S_p for $0 \leq p \leq \iota - 1$ be a family of unbounded sectors centered at 0 with bisecting direction $d_p \in (-\pi/2, \pi/2)$ such that the roots of $P(x)$ fall outside S_p, and $P(0) \neq 0$. Then, for every $t \in \mathcal{T}$ and $\epsilon \in \mathcal{E}_p$ there exists $\gamma_p \in \mathbb{R}$ with $e^{\sqrt{-1}\gamma_p} \in S_p$, and
$$\gamma_p - \arg(t) - \arg(\epsilon) \in \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right),$$
for some $\delta > 0$.
- The family $\underline{\mathcal{E}} = \{\mathcal{E}_k^-\}_{k \in \{-n,\dots,n\}} \cup \{\mathcal{E}_p\}_{0 \leq p \leq \iota-1}$ forms a good covering in $\mathbb{C}^\star$, meaning that their elements cover a punctured disc centered at the origin, two neighboring sectors intersect, and every three of them have empty intersection.

The initial data $\varphi_j(t,\epsilon)$, displayed in (4), consists on a family of holomorphic functions $\varphi_{j,k}(t,\epsilon)$ on $\mathcal{T} \times \mathcal{E}_k^-$, $k \in \{-n, \dots, n\}$ and $\varphi_{j,p}(t,\epsilon)$ on $\mathcal{T} \times \mathcal{E}_p$, $0 \leq p \leq \iota - 1$. For the sake of clarity, we have omitted the precise shape of these functions, which can be found in detail in [8].

Under assumption (5), and suitable control on the growth of the initial data, a first collection of actual solutions of (3) and (4) is built as a classical Laplace transform of a function, $w_p(\tau, z, \epsilon)$, holomorphic on $(S_p \cup D(0,r)) \times D(0,\rho) \times (D(0,\epsilon_0) \setminus \{0\})$, under exponential bounds

$$|w_p(\tau, z, \epsilon)| \leq C_p |\tau| e^{\frac{\sigma_1}{|\epsilon|}|\tau|},$$

for some $C_p, \sigma_1 > 0$. Indeed, we have

$$u_p(t, z, \epsilon) = \int_{L_{\gamma_p}} w_p(u, z, \epsilon) e^{-\frac{u}{\epsilon t}} \frac{du}{u},$$

along $L_p = \mathbb{R}_+ e^{\sqrt{-1}\gamma_p}$, defining a holomorphic function on $\mathcal{T} \times D(0,\rho) \times \mathcal{E}_p$.

On the other hand, one can proceed in the way of example (1), and construct a second set of actual solutions of (3) and (4) obtained as special Laplace transforms

$$u_k(t, z, \epsilon) = \int_{\mathcal{P}_k} w_{HJ_n}(u, z, \epsilon) e^{-\frac{u}{\epsilon t}} \frac{du}{u}, \tag{6}$$

defining a bounded holomorphic function on $\mathcal{T} \times D(0,\rho) \times \mathcal{E}_k^-$, for each $k \in \{-n, \ldots, n\}$. Here, $HJ_n = \cup(H_k \cup J_k)$, where H_k and J_k are horizontal infinite strips that are consecutively overlapping. The function $w_{HJ_n}(\tau, z, \epsilon)$ defines a holomorphic map on $HJ_n \times D(0,\rho) \times (D(0,\epsilon_0) \setminus \{0\})$ under super-exponential decay estimates:

$$|w_{HJ_n}(\tau, z, \epsilon)| \leq C_{H_k} |\tau| \exp\left(\frac{\sigma_1}{|\epsilon|}|\tau| - \sigma_2 e^{\sigma_3|\tau|}\right),$$

for some $C_{H_k}, \sigma_1, \sigma_2, \sigma_3 > 0$, all $\tau \in H_k$, $z \in D(0,\rho)$ and $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$, and

$$|w_{HJ_n}(\tau, z, \epsilon)| \leq C_{J_k} |\tau| \exp\left(\frac{\sigma_1}{|\epsilon|}|\tau| - s_2 e^{s_3|\tau|}\right),$$

for some $C_{J_k}, s_2, s_3 > 0$, all $\tau \in J_k$, $z \in D(0,\rho)$ and $z \in D(0,\epsilon_0) \setminus \{0\}$. As above, $\mathcal{P}_k$ stands for a piecewise linear path (see Figure 1).

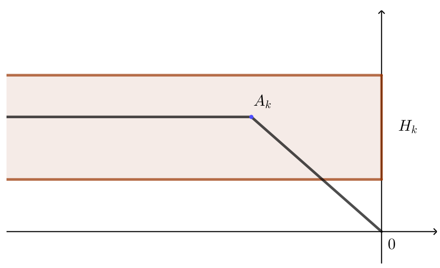

Figure 1: Path $\mathcal{P}_k$, for $A_k \in H_k$

3 Asymptotic behavior of the solution

In this section, we give some information on the asymptotic expansion of the solutions u_p and u_k with respect to ϵ near the origin.

We observe that a fine structure involving a double layer of Gevrey asymptotics arise. Namely, the functions u_k for $-n \leq k \leq n$, and u_p for $0 \leq p \leq \iota - 1$, can be decomposed as a sum of two holomorphic functions

$$u_\star(t,z,\epsilon) = u_\star^1(t,z,\epsilon) + u_\star^2(t,z,\epsilon),$$

for $\star \in \{k,p\}$, for which two formal power series

$$\hat{u}^j(t,z,\epsilon) = \sum_{\ell \geq 0} u_\ell^j(t,z)\epsilon^\ell \in \mathcal{O}_b(\mathcal{T} \times D(0,\rho))[[\epsilon]], \tag{7}$$

$j = 1,2$ exist, with the property that

$$\left| u_\star^1 - \sum_{\ell=1}^{N-1} u_\ell^1(t,z)\epsilon^\ell \right| \leq KA^N \left(\frac{N}{e}\right)^N |\epsilon|^N, \tag{8}$$

for every $N \geq 1$, all $\epsilon \in \mathcal{E}_k^-$ in case $\star = k$ and all $\epsilon \in \mathcal{E}_p$ if $\star = p$, $z \in D(0,\rho)$, $t \in \mathcal{T}$, for some $K, A > 0$. Then, it holds that Gevrey asymptotics of order 1 are obtained.

Also, one has

$$\left| u_\star^2 - \sum_{\ell=1}^{N-1} u_\ell^2(t,z)\epsilon^\ell \right| \leq KA^N \left(\frac{N}{\log(N)}\right)^N |\epsilon|^N, \tag{9}$$

for all $N \geq 2$, $t \in \mathcal{T}$, $z \in D(0,\rho)$, and all $\epsilon \in \mathcal{E}_k^-$ in case $\star = k$ and all $\epsilon \in \mathcal{E}_p$ if $\star = p$. In this case, Gevrey bounds are attained within the so-called level 1^+.

Furthermore, we can provide information about the unicity of the expansions in (7) and (8). More precisely, provided that the sector S_{p_0} has opening larger than π, the function $u_{p_0}^1(t,z,\epsilon)$ is the unique holomorphic function satisfying (8) on $\mathcal{E}_{p_0}$ and is reconstructed by means of the classical procedure of Borel-Laplace summation. This function turns out to be the 1-sum of $\hat{u}^1(t,z,\epsilon)$ on $\mathcal{E}_{p_0}$ in the classical sense (see [1] for a detailed definition). In addition to that, the functions $u_{\pm n}^2$ and u_p^2 are the restriction of a common holomorphic function $u^2(t,z,\epsilon)$ on $\mathcal{T} \times D(0,\rho) \times (\mathcal{E}_{-n}^- \cup \mathcal{E}_n^- \bigcup_{\rho=0}^{\iota-1} \mathcal{E}_p)$, which is the unique holomorphic function that admits $\hat{u}^2(t,z,\epsilon)$ as Gevrey asymptotic expansion of level 1^+. Moreover, $u^2(t,z,\epsilon)$ can be constructed via an analog of Borel-Laplace procedure in the framework of $\mathbb{M}-$summability for the strong regular sequence $\mathbb{M} = (M_n)_{n \geq 0}$ where $M_n = \left(\frac{n}{\log(n+2)}\right)^n$, see Section 5 or [7] for more details.

The proof of the main asymptotic result leans on a multilevel version of Ramis-Sibuya Theorem. This is a two leveled version of the classical cohomological result, which can be found in [10]. In this respect, we provide upper estimates for the following consecutive differences via adequate path deformation:

(α) $|u_{p+1}(t,z,\epsilon) - u_p(t,z,\epsilon)|$, $0 \leq p \leq \iota - 2$,

(β) $|u_{k+1}(t,z,\epsilon) - u_k(t,z,\epsilon)|$, $-n < k < n$,

(γ) $|u_{-n}(t,z,\epsilon) - u_0(t,z,\epsilon)|$,

(δ) $|u_n(t,z,\epsilon) - u_{\iota-1}(t,z,\epsilon)|$.

In case (α), the path $L_{\gamma_{p+1}} - L_{\gamma_p}$ is deformed into three pieces: two halflines and an arc of circle with radius $r/2$. (α) is upper bounded by $K_p \exp(-M_p/|\epsilon|)$ for some $K_p, M_p > 0$ and $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$. The path $\mathcal{P}_{k+1} - \mathcal{P}_k$ is deformed as shown in Figure 2.

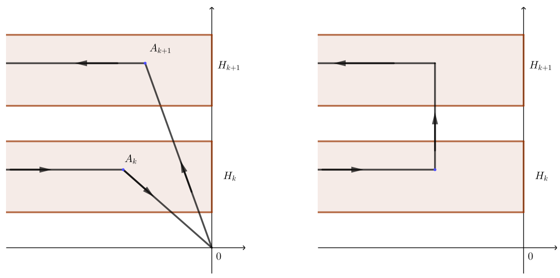

Figure 2: $\mathcal{P}_{k+1} - \mathcal{P}_k$ (left) and its deformation (right)

This configuration yields upper bounds for (β) of the form $K_k \exp\left(-\frac{M_k}{|\epsilon|}\log\left(\frac{L_k}{|\epsilon|}\right)\right)$, for some positive constants K_k, M_k, and $L_k > 1$, for all $\epsilon \in \mathcal{E}_{k+1}^{-} \cap \mathcal{E}_k^{-}$.

Finally, the case (γ) and (δ) can be considered in an analogous way. The path $\mathcal{P}_{-n} - L_{d_0}$ can be deformed as displayed in Figure 3, giving rise to an exponential decay, i.e. (γ) and (δ) are upper bounded by $K_{-n,0}\exp(-M_{n,0}/|\epsilon|)$, for $\epsilon \in \mathcal{E}_{d_0} \cap \mathcal{E}_{-n}^{-}$.

Figure 3: $\mathcal{P}_{-n} - L_{\gamma_0}$ (left) and its deformation (right)

4 Further results and future work

The set of Cauchy problems studied with this technique can be enlarged, allowing a more general form of the coefficients $c_{\underline{k}}(z,\epsilon)$, and differential operators involved in (3). For this purpose, we consider the Cauchy problem

$$P_2(t,z,\epsilon,\partial_t,\partial_z)y(t,z,\epsilon) = u(t,z,\epsilon), \tag{10}$$

under the action of the initial data

$$(\partial_z^j y)(t,0,\epsilon) = \Psi_j(t,\epsilon), \qquad 0 \le j \le S_B - 1, \tag{11}$$

where P_2 is the operator given by

$$P_2 y(t,z,\epsilon) := P_B(\epsilon t^2\partial_t)\partial_z^{S_B} y(t,z,\epsilon) - \sum_{\ell=(\ell_0,\ell_1,\ell_2)\in\mathcal{B}} d_\ell(z,\epsilon) t^{\ell_0}\partial_t^{\ell_1}\partial_z^{\ell_2} y(t,z,\epsilon),$$

u belongs to the set of solutions $u_k(t,z,\epsilon)$, $-n \leq k \leq n$, $u_p(t,z,\epsilon)$, $0 \leq p \leq \iota - 1$ of (3), (4). In this problem, $P_B(X) \in \mathbb{C}[X]$ shares properties with P in the previous section. $\mathcal{B}$ stands for a finite subset of $\mathbb{N}^3$, and the coefficients d_ℓ turn out to be holomorphic and bounded functions on $D(0,\rho) \times D(0,\epsilon_0)$; the initial data Ψ_j consist on holomorphic functions $\Psi_{j,k}$ defined on $\mathcal{T} \times \mathcal{E}_k^-$, $-n \leq k \leq n$ in the case that $u = u_k$, and $\Psi_{j,p}$ holomorphic on $\mathcal{T} \times \mathcal{E}_p$, for $0 \leq p \leq \iota - 1$ for $u = u_p$. The precise expression of the initial data is omitted for the sake of clarity.

The importance of this novel problem is that one is able to construct solutions and describe their asymptotic behavior of the coupled problem joining (10), (11) together with (3), (4). This new situation is concerned with a singularly perturbed Cauchy problem combining Moebius transforms and preserving the same structure as the problem considered in Section 2:

$$P(\epsilon t^2 \partial_t) P_B(\epsilon t^2 \partial_t) \partial_z^{S+S_B} y(t,z,\epsilon) = \sum_{q=(\ell_1,\ell_2,k_2) \in \mathcal{C} \subseteq \mathbb{N}^3} e_q(t,z,\epsilon) \partial_t^{\ell_1} \partial_z^{\ell_2} y\left(\frac{t}{1+k_2\epsilon t}, z, \epsilon\right), \tag{12}$$

under initial data

$$(\partial_z^j y)(t,0,\epsilon) = \psi_j(t,\epsilon), \qquad 0 \leq j \leq S_B - 1, \tag{13}$$

together with

$$(\partial_z^j P_2(t,z,\epsilon,\partial_t,\partial_z) y)(t,0,\epsilon) = \varphi_j(t,\epsilon), \qquad 0 \leq j \leq S - 1, \tag{14}$$

where the coefficients $e_q(t,z,\epsilon)$ are holomorphic near the origin, and polynomial with respect to t. Two families of actual solutions to (10), (11), can be built. One of them, $y_p(t,z,\epsilon)$ for $0 \leq p \leq \iota - 1$, is constructed by means of the usual Laplace transform, and is a holomorphic and bounded function on $\mathcal{T} \times D(0,\rho) \times \mathcal{E}_p$. The second family, $y_k(t,z,\epsilon)$ for $-n \leq k \leq n$, is described by means of a special Laplace transform, as shown in (6). Their elements are holomorphic functions on $\mathcal{T} \times D(0,\rho) \times \mathcal{E}_k^-$.

The asymptotic description of the analytic solutions of this problem leads us to analogous results as those enumerated in Section 2 in the case of the Cauchy problem (3), (4).

Several problems arise and can be considered for a future research. In this work, we have dealt with Cauchy problems in which the equations involved are linear with nonconstant coefficients. The nonlinear framework seems to be interesting. However, technical issues should be considered. One of them is the definition of convolution operators acting on functions on horizontal bands in the complex plane. Recent results on the resurgence of solutions of differential equations (see [5], and the extended version available in [6]) may provide the right direction in order to make progresses. A brilliant reference on this topic is also the chapter contained in this book of proceedings by Prof. Kamimoto, where the author gives details on his course during the school Complex Differential and Difference Equations, held in Bedlewo (Poland), 2018.

5 On the $\mathbb{M}$-summability

A sequence $\mathbb{M} = (M_p)_{p\geq 0}$ is said to be a strongly regular sequence if $M_p^2 \leq M_{p-1}M_{p+1}$ holds for all $p \geq 1$; there exists $A > 0$ such that $M_{p+\ell} \leq A^{p+\ell} M_p M_\ell$ for all $p, \ell \geq 0$; there exists $B > 0$ such that $\sum_{\ell \geq p} M_\ell/((\ell+1)M_{\ell+1}) \leq BM_p/M_{p+1}$, for all $p \geq 0$.

Under the assumption that the sequence $m = (m_p)_{p\geq 0}$ given by $m_p = M_{p+1}/M_p$ satisfies that

$$\lim_{p\to\infty} p \log(m_{p+1}/m_p)$$

exists (this is the case of the sequence assotiated to 1^+ level), then a function $M(r)$ defined by $\sup_{p\geq 0}\log(r^p/M_p)$, for $r>0$ and $M(0)=0$ and a positive number $\omega(\mathbb{M})$ are defined. More precisely, $\omega(\mathbb{M}) = (\limsup_{n\to\infty}\frac{\log(n)}{\log(M_{n+1}/M_n)})^{-1}\in(0,\infty)$. It is worth mentioning that in the framework of the strongly regular sequence associated to the so called 1^+ level, we have $\omega(\mathbb{M})=1$.

Let $d\in\mathbb{R}$. A formal power series $\hat{f}(z)=\sum_{n\geq 0}f_nz^n$ is $\mathbb{M}-$summable in direction d if there exists a bounded sector S of bisecting direction d and opening $\gamma>\pi\omega(\mathbb{M})$ and a function $f\in\mathcal{O}_b(S)$ with

$$\left|f(z)-\sum_{p=0}^{n-1}f_pz^p\right|\leq CA^nM_n|z|^n,$$

for all $z\in S$, $n\geq 1$, for some constant $A>0$.

In case such f exists, then it is unique under the previous properties. Moreover, it can be built from $\hat{f}(z)$. More precisely, the kernel function $e(z)$ of a Laplace-like operator is constructed under the hypotheses of being a holomorphic function on an unbounded sector with bisecting direction $d=0$, and such that for all proper subsector there exist $C,K>0$ with $|e(z)|\leq Ce^{-M(|z|/K)}$ for all z in such subsector.

The moment sequence $m_e=(m_e(p))_{p\geq 0}$, defined by $m_e(p)=\int_0^{+\infty}t^{p-1}e(t)dt$, for all $p\geq 0$, is equivalent to the initial sequence $\mathbb{M}=(M_p)_{p\geq 0}$ in the sense there exist $L,H>0$ with $L^pM_p\leq m_e(p)\leq H^pM_p$, for all $p\geq 0$. Then, the formal $e-$Borel transform $\sum_{p\geq 0}f_p/m_e(p)\tau^p$ converges in a neighborhood of the origin, and admits analytic continuation on an unbounded sector S, with $|g(\tau)|\leq Ce^{M(|z|/K)}$, for $z\in S$, and some $C,K>0$.

The function f can be written in the form of an $e-$Laplace transform on a finite sector of bisecting direction d and opening larger than $\pi\omega(\mathbb{M})$,

$$f(z)=\int_{L_\gamma}g(u)e(\frac{u}{z})\frac{du}{u},$$

along any halfline $L_\gamma=\mathbb{R}_+e^{\sqrt{-1}\gamma}\subseteq S\cup\{0\}$.

A very interesting course held during the conference Complex Differential and Difference Equations was given by S. Kamimoto. This course dealt with the question of $\mathbb{M}$-summability from a cohomological point of view, in one and several levels. This and other questions can be found in a chapter in this same volume of proceedings.

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