Gap opening in two-dimensional periodic systems

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Abstract

We present a new method of gap control in two-dimensional periodic systems with the perturbation consisting of a second-order differential operator and a family of narrow potential 'walls' separating the period cells in on direction. We show that under appropriate assumptions one can open gaps around points determined by dispersion curves of the associated 'waveguide' system, in general any finite number of them, and to control their widths in terms of the perturbation parameter. Moreover, the distinctive feature of those gaps is that their edge values are attained by the corresponding band functions at internal points of the Brillouin zone.

1 Introduction

Spectral properties of second-order differential operators with periodic coefficients are of considerable interest from more than one reason. On the one hand, it is an interesting mathematical problem with a rich structure. At the same time such operators are important in description of physical systems, in the first place crystals of various types. To illustrate how involved these problem can be mathematically, it is enough to recall the famous Bethe-Sommerfeld conjecture claiming that in system periodic in more than one direction the number of open spectral gaps is finite [22]. The reasoning that led to it was so natural that the physics community adopted it immediately, however, it took decades to establish its validity in a rigorous mathematical way, cf. [7, 13, 17, 18, 19, 20, 21, 23] and references therein. Nowadays it is done for a large number of systems including a more complicated behavior in some borderline situations [3].

On the other hand, there are situation where intuition may mislead you. It was widely believed, for instance, that band edges correspond to quasimomenta laying at the boundary of the corresponding Brillouin zone or at the center of this zone. Was this the case, the task of finding the spectral bands would be easier as

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the manifold to explore would have one dimension less. It was shown, however, that such a claim does not hold generally in system periodic in more than one direction [11] and this result also extends under additional assumptions to systems periodic in one dimension [10]. The corresponding counterexamples used (discrete or metric) quantum graphs rather than Schrödinger operators. For those an example of dispersion curves with extrema in the interior of the Brillouin zone was constructed in [6], while in higher dimensions the question remained up to now open.

This is not the only issue we are going to discuss here. Our goal in the present paper is also to address another important problem, the gap control. Recall that this question acquired importance recently in connection with the progress in the physics of metamaterials, in other words, artificially prepared periodically structured substances. As the band structure plays decisive role in their conductivity properties of the materials, one wants to know whether it is possible to open a gap at a prescribed energy value and to control its width. Various models in which this goal can be achieved have been constructed. In one dimension one can use, for instance, an array of 'cells' connected by narrow 'windows' [1, 2], a double waveguide with a periodically perturbed 'membrane' [5] or a waveguide with a periodic perturbation [4, 15]. An alternative way proposed is to place into a waveguide a periodic array of small δ' traps [8].

In higher dimensions there are fewer results. Khrabustovskyi constructed a model in which gaps can be opened with the help of a lattice of small 'pierced resonators' [14] and in [9] a similar result was obtained by means of a lattice of δ' traps. The goal of this paper is to present a new method of gap control in a two-dimensional system with a periodic perturbation described by a 'small' second order differential operator consisting of raising high and narrow 'potential walls' in one direction. We compare this system to the family of parallel waveguides, with the wall replaced by the Dirichlet condition, and show that under appropriate assumption gaps may open around the points where the dispersion curved of the waveguide cross, in general any finite number of them. Moreover, we are able to control the gap width in terms of the perturbation parameter. Equally important, the construction answers at the same time the question mentioned above: we show that the edge values of the opened gaps are attained by the band functions at internal points of the Brillouin zone.

Let us describe briefly the contents of the paper. In the next section we formulate the problem properly and state our main results as Theorem 2.1. The rest, Sections 3 and 4 is devoted to the proof.

2 Formulation of the problem and main result

Let $x \in \mathbb{R}^2$ be a point expressed through its Cartesian coordinate, $x = (x_1, x_2)$, and let $\mathcal{H}_0$ be the negative Laplacian in $\mathbb{R}^2$. The operator $\mathcal{H}_0$ is self-adjoint in $L_2(\mathbb{R}^2)$ with the domain $W_2^2(\mathbb{R}^2)$. By Γ we denote the rectangular lattice $a_1\mathbb{Z} \times a_2\mathbb{Z}$, where a_1, a_2 are positive real constants, and $\square := \{x : \; 0 < x_1 < a_1, \, 0 < x_2 < a_2\}$ stands for the corresponding periodicity cell.

Next we introduce the following operator in $L_2(\mathbb{R}^2)$,

$$\mathcal{L} := \frac{\partial}{\partial x_1} A_{11}(x) \frac{\partial}{\partial x_1} + \mathrm{i} \sum_{j=1}^{2} \left(A_j(x) \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} A_j(x) \right) + A_0(x) \tag{2.1}$$

with the domain $W_2^2(\mathbb{R}^2)$, where $A_{11}, A_j \in C^1(\mathbb{R}^2)$, $A_0 \in C(\mathbb{R}^2)$ are real functions periodic with respect to the lattice Γ. The functions A_{11}, A_j and A_0 are assumed to vanish in the vicinity of the boundary $\partial\square$. The operator $\mathcal{L}$ is obviously symmetric.

By V we denote a real function of the variable x_2 satisfying the conditions

$$\operatorname{supp} V \subseteq [-a_3, a_3], \quad V \in C[-a_3, a_3], \quad V(x_2) \geqslant c_0 > 0 \ \text{ in } [-a_3, a_3], \tag{2.2}$$

where a_3 and c_0 are fixed constants. Let us note that the compact support assumption is made here for convenience only, the argument would go through with slight modifications without it as well.

The main object of our investigation is the operator in $L_2(\mathbb{R}^2)$ defined as

$$\mathcal{H}_\varepsilon := \mathcal{H}_0 + \varepsilon^\alpha \mathcal{L} + \varepsilon^{-\frac{3}{2}} V_\varepsilon, \tag{2.3}$$

$$V_\varepsilon = V_\varepsilon(x_2) := \sum_{p \in \mathbb{Z}} V\left(\frac{x_2 - p a_2}{\varepsilon} \right), \tag{2.4}$$

where ε is a small real parameter and $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ is a fixed constant. It is easy to see that the operator $\mathcal{H}_\varepsilon$ is self-adjoint on the domain $W_2^2(\mathbb{R}^2)$. Moreover, the operator $\mathcal{H}_\varepsilon$ is periodic with respect to the lattice Γ, and consequently, its spectrum has a band-gap structure. The main aim of the present work is to study the existence of the spectral gaps in the spectrum of $\mathcal{H}_\varepsilon$ for the parameter small ε enough.

In order to state our main result, we need to introduce additional notations. We begin with the standard formula for the spectrum of the operator $\mathcal{H}_\varepsilon$,

$$\sigma(\mathcal{H}_\varepsilon) = \bigcup_{k \in \mathbb{N}} \{ E_\varepsilon^{(k)}(\tau) : \tau \in \square^* \}, \tag{2.5}$$

where $\tau = (\tau_1, \tau_2)$ is the quasi-momentum and

$$\square^* := \left\{ \tau : \ |\tau_1| \leqslant \frac{\pi}{a_1}, \ |\tau_2| \leqslant \frac{\pi}{a_2} \right\}$$

is the basic cell of the lattice dual to Γ, or in the physical terminology the Brillouin zone. The symbols $E_\varepsilon^{(k)}(\tau)$ stand for the band functions, that is, these are the eigenvalues of the operator $\mathcal{H}_\varepsilon(\tau)$ in $L_2(\square)$ with the differential expression

$$\mathcal{H}_\varepsilon(\tau) = -\Delta + \varepsilon^\alpha \mathcal{L} + \varepsilon^{-\frac{3}{2}} V_\varepsilon(x) \tag{2.6}$$

subject to the standard quasiperiodic boundary conditions, in other words, the domain of the operator $\mathcal{H}_\varepsilon(\tau)$ consists of the functions $u \in W_2^2(\square)$ which satisfy the boundary conditions

$$\mathrm{e}^{\mathrm{i}\tau_j a_j} u\big|_{x_j=0} = u\big|_{x_j=a_j}, \quad \mathrm{e}^{\mathrm{i}\tau_j a_j} \frac{\partial u}{\partial x_j}\bigg|_{x_j=0} = \frac{\partial u}{\partial x_j}\bigg|_{x_j=a_j}, \quad j = 1, 2. \tag{2.7}$$

The eigenvalues $E_\varepsilon^{(k)}(\tau)$ are supposed to be arranged in the ascending order with the multiplicity taken into account. Let $E_0 \in \left(\frac{\pi^2}{a_2^2}, \frac{9\pi^2}{a_2^2}\right)$ be a point such that

$$E_0 = E_0^{(n,1)}(\tau_0) = \left(\tau_0 + \frac{2\pi n}{a_1}\right)^2 + \frac{\pi^2}{a_2^2} = E_0^{(m,2)}(\tau_0) = \left(\tau_0 + \frac{2\pi m}{a_1}\right)^2 + \frac{4\pi^2}{a_2^2} \quad (2.8)$$

for some $\tau_0 \in \left[-\frac{\pi}{a_1}, \frac{\pi}{a_1}\right]$ and some integer n, m. It is easy to see that triples for which the condition (2.8) is satisfied always exist, and moreover, we may suppose without loss of generality that at least one of n and m is nonzero. It is also clear that condition (2.8) holds also with τ_0, n, m replaced by $-\tau_0$, $-n$, $-m$. We denote

$$\psi_0^{(n,p)}(x,\tau_2) := \frac{\sqrt{2}}{\sqrt{a_1 a_2}}\, \mathrm{e}^{\mathrm{i}\frac{2\pi n}{a_1}x_1}\, \mathrm{e}^{-\mathrm{i}\tau_2 x_2} \sin\frac{\pi p}{a_2}x_2$$

and

$$\mathcal{L}(\tau_1) := -\left(\mathrm{i}\frac{\partial}{\partial x_1} - \tau_1\right) A_{11}(x) \left(\mathrm{i}\frac{\partial}{\partial x_1} - \tau_1\right) + \mathrm{i}\sum_{j=1}^{2}\left(A_j(x)\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}A_j(x)\right) + A_0(x).$$

Next we define the functions

$$\beta_\pm(\tau_1) := \pm\frac{|M_{12}^0(\tau_1)|}{|k_3(\tau_1)|}\sqrt{k_3^2(\tau_1) - k_1^2(\tau_1)} - \frac{k_1(\tau_1)k_4(\tau_1)}{k_3(\tau_1)} + k_2(\tau_1), \quad (2.9)$$

where

$$\begin{aligned}
&k_1(\tau_1) := -\frac{2\pi(n+m)}{a_1} - \tau_1, \quad k_2(\tau_1) := -\frac{M_{11}^{(0)}(\tau_1) + M_{22}^{(0)}(\tau_1)}{2}, \\
&k_3(\tau_1) := \frac{2\pi}{a_1}(n-m), \qquad\quad k_4(\tau_1) := \frac{M_{22}^{(0)}(\tau_1) - M_{11}^{(0)}(\tau_1)}{2}.
\end{aligned} \quad (2.10)$$

$$\begin{aligned}
M^{(0)}(\tau_1) &:= \begin{pmatrix} M_{11}^{(0)}(\tau_1) & M_{12}^{(0)}(\tau_1) \\ M_{21}^{(0)}(\tau_1) & M_{22}^{(0)}(\tau_1) \end{pmatrix} \\
&= \begin{pmatrix} \left(\psi_0^{(n,1)}, \mathcal{L}(\tau_1)\psi_0^{(n,1)}\right)_{L_2(\square)} & \left(\psi_0^{(n,1)}, \mathcal{L}(\tau_1)\psi_0^{(m,2)}\right)_{L_2(\square)} \\ \left(\psi_0^{(m,2)}, \mathcal{L}(\tau_1)\psi_0^{(n,1)}\right)_{L_2(\square)} & \left(\psi_0^{(m,2)}, \mathcal{L}(\tau_1)\psi_0^{(m,2)}\right)_{L_2(\square)} \end{pmatrix}, \\
M^{(1)}(\tau_1) &:= \begin{pmatrix} \frac{2\pi n}{a_1} + \tau_1 & 0 \\ 0 & \frac{2\pi m}{a_1} + \tau_1 \end{pmatrix}
\end{aligned} \quad (2.11)$$

if $\tau_1 \geqslant 0$, and

$$\begin{aligned}
&k_1(\tau_1) := \frac{2\pi(n+m)}{a_1} - \tau_1, \quad k_2(\tau_1) := -\frac{M_{11}^{(0)}(\tau_1) + M_{22}^{(0)}(\tau_1)}{2}, \\
&k_3(\tau_1) := \frac{2\pi}{a_1}(m-n), \qquad\quad k_4(\tau_1) := \frac{M_{22}^{(0)}(\tau_1) - M_{11}^{(0)}(\tau_1)}{2},
\end{aligned} \quad (2.12)$$

$$\begin{aligned}
M^{(0)}(\tau_1) &:= \begin{pmatrix} M_{11}^{(0)}(\tau_1) & M_{12}^{(0)}(\tau_1) \\ M_{21}^{(0)}(\tau_1) & M_{22}^{(0)}(\tau_1) \end{pmatrix} \\
&= \begin{pmatrix} \left(\psi_0^{(-n,1)}, \mathcal{L}(\tau_1)\psi_0^{(-n,1)}\right)_{L_2(\square)} & \left(\psi_0^{(-n,1)}, \mathcal{L}(\tau_1)\psi_0^{(-m,2)}\right)_{L_2(\square)} \\ \left(\psi_0^{(-m,2)}, \mathcal{L}(\tau_1)\psi_0^{(-n,1)}\right)_{L_2(\square)} & \left(\psi_0^{(-m,2)}, \mathcal{L}(\tau_1)\psi_0^{(-m,2)}\right)_{L_2(\square)} \end{pmatrix},
\end{aligned}$$

$$M^{(1)}(\tau_1) := \begin{pmatrix} -\frac{2\pi n}{a_1} + \tau_1 & 0 \\ 0 & -\frac{2\pi m}{a_1} + \tau_1 \end{pmatrix} \tag{2.13}$$

if $\tau_1 < 0$. We observe that although the functions $\psi_0^{(n,1)}$ and $\psi_0^{(m,2)}$ depend on τ_2, this is *not* the case for the above introduced functions and matrices. We put

$$\beta_l := \max\{\beta_-(\tau_0), \beta_-(-\tau_0)\}, \qquad \beta_r := \min\{\beta_+(\tau_0), \beta_+(-\tau_0)\}. \tag{2.14}$$

By $\tau_{1,l}^*$ we denote one of the values $\pm\tau_0$, at which the maximum is attained in the formula for β_l. In the same way, by $\tau_{1,r}^*$ we denote one of the values $\pm\tau_0$, at which the minimum is attained in the formula for β_r.
We define

$$\begin{aligned} t_r &= -\frac{k_1(\tau_{1,r}^*)\big|M_{12}^{(0)}(\tau_{1,r}^*)\big|}{\big|k_3(\tau_{1,r}^*)\big|\sqrt{k_3^2(\tau_{1,r}^*) - k_1^2(\tau_{1,r}^*)}} - \frac{k_4(\tau_{1,r}^*)}{k_3(\tau_{1,r}^*)}, \\ t_l &= \frac{k_1(\tau_{1,l}^*)\big|M_{12}^{(0)}(\tau_{1,l}^*)\big|}{\big|k_3(\tau_0)\big|\sqrt{k_3^2(\tau_{1,l}^*) - k_1^2(\tau_{1,l}^*)}} - \frac{k_4(\tau_{1,l}^*0)}{k_3(\tau_{1,l}^*)} \end{aligned} \tag{2.15}$$

The numbers

$$\begin{aligned} &\frac{M_{11}^{(0)}(\tau_{1,l/r}^*) + M_{22}^{(0)}(\tau_{1,l/r}^*)}{2} - 2t_{l/r}\left(\frac{\pi(n+m)}{a_1} + \tau_{1,l/r}^*\right) \\ &\pm\left(\left(\frac{M_{11}^{(0)}(\tau_{1,l/r}^*) - M_{22}^{(0)}(\tau_{1,l/r}^*)}{2} - t_{l/r}\frac{\pi(n-m)}{a_1}\right)^2 + \big|M_{12}^{(0)}(\tau_{1,l/r}^*)\big|^2\right)^{\frac{1}{2}} \end{aligned}$$

are the eigenvalues of the matrix $M^{(0)}(\tau_{1,l/r}^*) - 2t_{l/r}M^{(1)}(\tau_{1,l/r}^*)$ and we denote by $e_{l/r} = \begin{pmatrix} e_{l/r}^{(1)} \\ e_{l/r}^{(2)} \end{pmatrix}$ the associated eigenvectors orthonormalized in $\mathbb{C}^2$. Furthermore, we put

$$\lambda_r := -\frac{8\pi^2}{a_2^3\langle V\rangle}\max\left\{|e_r^{(1)}|^2, |e_r^{(2)}|^2\right\}, \quad \lambda_l := -\frac{8\pi^2}{a_2^3\langle V\rangle}\min\left\{|e_l^{(1)}|^2, |e_l^{(2)}|^2\right\},$$

which is well defined because $\langle V\rangle := \int_{-a_3}^{a_3} v(x)\,\mathrm{d}x$ is positive by assumption.

Now we are in position to formulate our main result.

Theorem 2.1. *Assume that the following inequalities hold,*

$$M_{12}^{(0)}(\pm\tau_0) \neq 0, \quad \left(\tau_0 + \frac{2\pi n}{a_1}\right)\left(\tau_0 + \frac{2\pi m}{a_1}\right) < 0 \quad \text{and} \quad \beta_l < \beta_r. \tag{2.16}$$

Then there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the spectrum of the operator $\mathcal{H}_\varepsilon$ has a gap $\big(\eta_l(\varepsilon), \eta_r(\varepsilon)\big)$ with the edges that behave asymptotically as

$$\eta_l(\varepsilon) = E_0 + \varepsilon^\alpha\beta_l + \varepsilon^{\frac{1}{2}}\lambda_{\frac{1}{2}}^l + O(\varepsilon^{2\alpha}), \quad \eta_r(\varepsilon) = E_0 + \varepsilon^\alpha\beta_r + \varepsilon^{\frac{1}{2}}\lambda_{\frac{1}{2}}^r + O\big(\varepsilon^{2\alpha}\big), \tag{2.17}$$

and the corresponding band functions $E_{l/r}(\varepsilon,\tau)$ of the operator $\mathcal{H}_\varepsilon$ attain their extrema, i.e. the gap edges $\big(\eta_l(\varepsilon),\eta_r(\varepsilon)\big)$ at the points $\tau_{l/r}(\varepsilon)$,

$$\min_\tau E_r(\varepsilon,\tau)=E_r\big(\varepsilon,\tau_r(\varepsilon)\big),\quad \min_\tau E_l(\varepsilon,\tau)=E_l\big(\varepsilon,\tau_l(\varepsilon)\big), \tag{2.18}$$

for which the identities

$$\begin{aligned}
&\tau_{l/r}(\varepsilon)=(\tau_{1,l/r},\tau_{2,l/r})+\varepsilon^\alpha(\gamma_{l/r},0)+O(\varepsilon^{1/2})\\
&\gamma_l:=\frac{k_1(\tau_{1,l})|M_{12}^0(\tau_{1,l})|}{|k_3(\tau_{1,l})|\sqrt{k_3(\tau_{1,l})^2-k_1^2(\tau_{1,l})}}-\frac{k_4(\tau_{1,l})}{k_3(\tau_{1,l})},\\
&\gamma_r:=-\frac{k_1(\tau_{1,r})|M_{12}^0(\tau_{1,r})|}{|k_3(\tau_{1,r})|\sqrt{k_3(\tau_{1,r})^2-k_1^2(\tau_{1,r})}}-\frac{k_4(\tau_{1,r})}{k_3(\tau_{1,r})},
\end{aligned} \tag{2.19}$$

are valid. Here $\tau_{2,r}\in\{-\frac{\pi}{a_2},0,\frac{\pi}{a_2}\}$ if $|e_r^{(1)}|>|e_r^{(2)}|$ and $\tau_{2,r}\in\{-\frac{\pi}{2a_2},\frac{\pi}{2a_2}\}$ if $|e_r^{(1)}|<|e_r^{(2)}|$. If $|e_r^{(1)}|=|e_r^{(2)}|$, then $\tau_{2,r}$ is undetermined. For $\tau_{2,l}$ we have similar relations, namely $\tau_{2,l}\in\{-\frac{\pi}{a_2},0,\frac{\pi}{a_2}\}$ if $|e_l^{(1)}|<|e_l^{(2)}|$ and $\tau_{2,l}\in\{-\frac{\pi}{2a_2},\frac{\pi}{2a_2}\}$ if $|e_l^{(1)}|>|e_l^{(2)}|$. If $|e_l^{(1)}|=|e_l^{(2)}|$, then $\tau_{2,l}$ is undetermined.

Let us discuss briefly the meaning of this theorem. It states that once the inequalities (2.16) are satisfied, the band spectrum of the operator $\mathcal{H}_\varepsilon$ has a small gap around the point E_0 for all sufficiently small ε. The edges of this gap are attained by the values of the corresponding band functions at points τ_l and τ_r; let us denote the coordinates of these points by $\tau_l/r^{(j)}$, $j=1,2$. The first coordinates $\tau_{l/r}^{(j)}$ are close to $+\tau_0$ or $-\tau_0$ depending on which of these points the minimum and the maximum in (2.14). The number τ_0 is defined by condition (2.8) and as we see easily, varying numbers a_1 and a_2, we can satisfy identity (2.8) for arbitrary prescribed τ_0 and E_0. This means that we can open a gap around a prescribed value E_0 so that its edges are attained by the values of the corresponding band functions at the points $\tau_{l/r}$ with the first coordinate being close to the prescribed τ_0. On the second coordinate $\tau_{l/r}^{(2)}$ we can not say much. The only information we can provide is that the second coordinate is close to one of the values $\{0,\pm\frac{\pi}{a_2}\}$ or $\{\pm\frac{\pi}{2a_2}\}$ if $|e_{l/r}^{(1)}|\neq|e_{l/r}^{(2)}|$. Thus the points $\tau_{l/r}$ are close to one of the following points: $(\pm\tau_0,\frac{\pi}{a_2})$, $(\pm\tau_0,-\frac{\pi}{a_2})$, $(\pm\tau_0,0)$, $(\pm\tau_0,\frac{\pi}{2a_2})$, $(\pm\tau_0,-\frac{\pi}{2a_2})$. In this list, the first two points are located at the boundary of the Brillouin zone, the third one is located at the middle line $(\tau_1,0)$, while the two last points are located at the intermediate lines $(\tau_1,\pm\frac{\pi}{2a_2})$. If $|e_{l/r}^{(1)}|=|e_{l/r}^{(2)}|$, our analysis provides no information about localization of $\tau_{l/r}^{(2)}$. In our opinion, this degenerate case could hide a situation, when the second coordinate is close to some value different from those listed above.

We should also stress that if there are several triples (m,n,τ_0) obeying (2.8) with different values E_0, and for each of them there exists a corresponding gap in the spectrum of $\mathcal{H}_\varepsilon$ with the above described properties. In particular, this implies that choosing a_1 large enough, we can open *arbitrarily many gaps* in the spectrum of the operator $\mathcal{H}_\varepsilon$.

3 Approximation of the band functions

This section is devoted to the preliminary part of the proof of Theorem 2.1. The main idea here is to approximate the band functions $E_\varepsilon^{(k)}(\tau)$ by similar band functions corresponding to a simpler operator.

By $\mathcal{H}_0(\tau_1)$ we denote the Laplacian in $\square$ subject to the quasiperiodic boundary conditions of the type (2.7) on the 'longitudinal' boundaries of $\square$, $j=1$, while on

$$\gamma := \{x:\ 0<x_1<a_1,\, x_2=0\} \cup \{x:\ 0<x_1<a_1,\, x_2=a_2\}$$

we impose the Dirichlet condition. Its eigenvalues and the associated eigenfunctions are easily seen to be

$$\begin{aligned} E_0^{(n,p)}(\tau_1) &= \left(\tau_1+\frac{2\pi n}{a_1}\right)^2+\frac{\pi^2p^2}{a_2^2}, \\ \Psi_0^{(n,p)}(x,\tau_1) &= \frac{\sqrt{2}}{\sqrt{a_1a_2}}\,\mathrm{e}^{\mathrm{i}\left(\tau_1+\frac{2\pi n}{a_1}\right)x_1}\sin\frac{\pi p}{a_2}x_2. \end{aligned} \tag{3.1}$$

The main statement we are going to prove in this section is as follows.

Lemma 3.1. *Given any fixed E, there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0]$ and all (n,p) obeying*

$$E_0^{(n,p)}(\tau_1)\leqslant E \tag{3.2}$$

the estimates

$$\left|E_\varepsilon^{(k)}(\tau)-E_0^{(n,p)}(\tau_1)\right|\leqslant C\varepsilon^\alpha, \tag{3.3}$$

hold, where $E_\varepsilon^{(k)}(\tau)$ are the band functions appearing in (2.5) and C is a constant independent of ε, μ, n, p and τ but dependent on E. The eigenvalues $E_0^{(n,p)}$ in (3.3) are assumed to be arranged in the ascending order counting the multiplicities and this makes the correspondence between the superscripts k and (n,p) in (3.3).

The proof of this lemma consists of several steps; let us briefly describe them.

We begin with introducing one more operator $L_2(\square)$ which we denote as $\mathcal{H}_V(\tau)$. The differential expression for $\mathcal{H}_V(\tau)$ is given by formula (2.6) with $\mathcal{L}=0$ and the boundary conditions are quasiperiodic ones. Since the potential V depends on x_2 only, we can find the eigenvalues and the associated eigenfunctions of the operator $\mathcal{H}_V(\tau)$ by separation of variables,

$$\begin{aligned} E_V^{(n,p)}(\tau) &= \left(\tau_1+\frac{2\pi n}{a_1}\right)^2+\lambda_\varepsilon^{(p)}(\tau_2), \\ \Psi_V^{(n,p)}(x,\tau) &= \frac{1}{\sqrt{a_1}}\,\mathrm{e}^{\mathrm{i}\left(\tau_1+\frac{2\pi n}{a_1}\right)x_1}\,\Psi_\varepsilon^{(p)}(x_2,\tau_2), \end{aligned} \tag{3.4}$$

where $\lambda_\varepsilon^{(p)}$ and $\Psi_\varepsilon^{(p)}$ are the eigenvalues and the associated eigenfunctions of the operator

$$\mathcal{A}_\varepsilon(\tau_2) = -\frac{d^2}{dx_2^2}+\varepsilon^{-\frac{3}{2}}V_\varepsilon \quad \text{in}\ \ L_2(0,a_2) \tag{3.5}$$

subject to the quasiperiodic boundary conditions. The eigenvalues $\lambda_\varepsilon^{(p)}$ are taken in the ascending order counting the multiplicities and the associated eigenfunctions are chosen being normalized in $L_2(0,a_2)$.

The first step in the proof of Lemma 3.1 is to estimate the differences between the eigenvalues $E_V^{(n,p)}(\tau)$ and $E_0^{(n,p)}$. A precise formulation of the sought result is formulated in the following lemma.

Lemma 3.2. *Given any fixed E, there is an $\varepsilon_0>0$ such that for all $\varepsilon\leqslant\varepsilon_0$ and all all (n,p) obeying (3.2) the estimates*

$$\left|E_V^{(n,p)}(\tau)-E_0^{(n,p)}(\tau)\right|\leqslant C\varepsilon^{\frac{1}{2}}, \tag{3.6}$$

hold, where C is a constant independent of ε, μ, n, p and τ but dependent on E.

At the second step, we estimate the differences between the band functions $E_\varepsilon^{(k)}(\tau)$ and $E_V^{(n,p)}$ and this will lead us to the statement of Lemma 3.1.

The proof of Lemma 3.2 consists again of two parts, which we present below as separate Subsections 3.1, 3.2. The second part of the proof of Lemma 3.1 will then follow in Subsection 3.3.

3.1 Approximation of the resolvent of $\mathcal{A}_\varepsilon$

To begin with we consider the operator

$$\mathcal{A}_0:=-\frac{d^2}{dx_2^2}\quad\text{in}\quad L_2(0,a_2) \tag{3.7}$$

subject to Dirichlet condition at the interval endpoints. Our aim is to show that $\mathcal{A}_\varepsilon$ converges to $\mathcal{A}_0$ as $\varepsilon\to 0$ in the norm resolvent sense and to estimate the corresponding convergence rate.

Given an $f\in L_2(0,a_2)$, we denote $u_\varepsilon:=\big(\mathcal{A}_\varepsilon(\tau_2)-\mathrm{i}\big)^{-1}f$, $u_0:=(\mathcal{A}_0-\mathrm{i})^{-1}f$, and $v_\varepsilon:=u_\varepsilon-u_0$. We write the boundary value problems for u_ε and u_0, multiply the equations by v_ε and integrate the obtained expressions by parts over $(0,a_2)$. This standard procedure yields the following integral identities,

$$\begin{aligned}&(\nabla u_\varepsilon,\nabla v_\varepsilon)_{L_2(0,a_2)}+\varepsilon^{-\frac{3}{2}}(V_\varepsilon u_\varepsilon,v_\varepsilon)_{L_2(0,a_2)}-\mathrm{i}(u_\varepsilon,v_\varepsilon)_{L_2(0,a_2)}=(f,v_\varepsilon)_{L_2(0,a_2)},\\ &(\nabla u_0,\nabla v_\varepsilon)_{L_2(0,a_2)}-\overline{u_\varepsilon(0)}\left(\mathrm{e}^{\mathrm{i}\tau_2 a_2}u_0'(a_2)-u_0'(0)\right)\\ &\qquad\qquad\qquad-\mathrm{i}(u_\varepsilon,v_\varepsilon)_{L_2(0,a_2)}=(f,v_\varepsilon)_{L_2(0,a_2)}.\end{aligned} \tag{3.8}$$

Subtracting these relations from the each other, we get

$$\begin{aligned}\|\nabla v_\varepsilon\|_{L_2(0,a_2)}^2-&\mathrm{i}\|v_\varepsilon\|_{L_2(0,a_2)}^2+\varepsilon^{-\frac{3}{2}}(V_\varepsilon v_\varepsilon,v_\varepsilon)_{L_2(0,a_2)}\\ &=-\varepsilon^{-\frac{3}{2}}(V_\varepsilon u_0,v_\varepsilon)_{L_2(0,a_2)}-\overline{u_\varepsilon(0)}\left(\mathrm{e}^{\mathrm{i}\tau_2 a_2}u_0'(a_2)-u_0'(0)\right),\end{aligned}$$

an consequently, by virtue of conditions (2.2) we obtain

$$\begin{aligned}&\|v_\varepsilon\|_{W_2^1(0,a_2)}^2\leqslant 2C\varepsilon^{-\frac{3}{2}}\|u_0\|_{L_2(S_\varepsilon)}\|v_\varepsilon\|_{L_2(S_\varepsilon)}+2|u_\varepsilon(0)|\left(|u_0'(0)|+|u_0'(a_2)|\right),\\ &S_\varepsilon:=\{x:\ 0<x_2<a_3\varepsilon\}\cup\{x:\ a_2-a_3\varepsilon<x_2<a_2\},\quad C:=\max_{x_2}V(x_2).\end{aligned} \tag{3.9}$$

To proceed with estimating in (3.9), we shall need the following auxiliary result.

Lemma 3.3. *The functions u_0, u_ε, and v_ε satisfy the inequalities*

$$|u_0'(0)| + |u_0'(a_2)|| \leqslant C\|f\|_{L_2(0,a_2)}, \tag{3.10}$$

$$\|u_0\|_{L_2(S_\varepsilon)} \leqslant C\varepsilon^{\frac{3}{2}}\|f\|_{L_2(0,a_2)}, \tag{3.11}$$

$$\|v_\varepsilon\|_{L_2(S_\varepsilon)} \leqslant C\varepsilon^{\frac{3}{4}}\|f\|_{L_2(0,a_2)}, \tag{3.12}$$

$$|u_\varepsilon(0)| \leqslant C\varepsilon^{\frac{1}{4}}\|f\|_{L_2(0,a_2)} \tag{3.13}$$

for all ε small enough, where C is a constant independent of ε and f.

Proof. Throughout the proof the symbol C stands for various positive constants independent of x, ε, f, v_ε, u_0, and u_ε the values of which are not essential. Using standard smoothness improving theorems, we infer that

$$\|u_0\|_{W_2^2(0,a_2)} \leqslant C\|f\|_{L_2(0,a_2)}. \tag{3.14}$$

Hence, by the embedding of $W_2^1(0,a_2)$ into $C[0,a_2]$, we get

$$|u_0'(0)| + |u_0'(a_2)| \leqslant C\|u_0\|_{W_2^2(0,a_2)} \leqslant C\|f\|_{L_2(0,a_2)},$$

which proves (3.10). Since the function u_0 vanishes as $x_2 = 0$ by assumption, for almost all $x_2 \in [0, a_3\varepsilon]$ we have

$$u_0(x_2) = \int\limits_0^{x_2} u_0'(t)\,dt, \qquad u_0'(x_2) = -\int\limits_{a_2}^{x_2} \big(\chi(t)u_0'(t)\big)'\,dt, \tag{3.15}$$

where $\chi = \chi(t)$ is a fixed infinitely differentiable cut-off function equal to one if $t < \frac{a_2}{3}$ and vanishing for $t > \frac{2a_2}{3}$. The first of these relation implies by virtue of Cauchy-Schwarz inequality that

$$|u_0(x_2)|^2 \leqslant x_2 \int\limits_0^{x_2} |u_0'(t)|^2\,dt,$$

and since $x_2 \leqslant a_3\varepsilon$ holds by assumption, we arrive at the estimate

$$\begin{aligned}\int\limits_0^{a_3\varepsilon} |u_0(x_2)|^2\,dx_2 \leqslant \int\limits_0^{a_3\varepsilon} x_2\,dx_2 \int\limits_0^{a_3\varepsilon} |u_0'|^2\,dx_2 \leqslant C\varepsilon^2 \int\limits_0^{a_3\varepsilon} \|u_0'(x_2)\|^2_{W_2^1(0,a_2)}\,dx_2\\ \leqslant C\varepsilon^3\|u_0'\|^2_{W_2^1(0,a_2)}\end{aligned}$$

which in combination with (3.14) yields (3.11).

Next we write the integral identity for u_ε analogous to (3.8) employing now this function as the test one,

$$\|\nabla u_\varepsilon\|^2_{L_2(0,a_2)} + \varepsilon^{-\frac{3}{2}}(V_\varepsilon u_\varepsilon, u_\varepsilon)_{L_2(0,a_2)} - \mathrm{i}\|u_\varepsilon\|^2_{L_2(0,a_2)} = (f, u_\varepsilon)_{L_2(0,a_2)}.$$

In view of condition (2.2) this implies

$$\|u_\varepsilon\|^2_{W_2^1(0,a_2)} + c_0\varepsilon^{-\frac{3}{2}}\|u_\varepsilon\|^2_{L_2(S_\varepsilon)} \leqslant 2\|f\|_{L_2(0,a_2)}\|u_\varepsilon\|_{L_2(0,a_2)}. \tag{3.16}$$

Thus we find

$$\|u_\varepsilon\|_{W_2^1(0,a_2)} \leqslant 2\|f\|_{L_2(0,a_2)}, \tag{3.17}$$

and

$$\|u_\varepsilon\|_{L_2(S_\varepsilon)} \leqslant 2c_0^{-\frac{1}{2}}\varepsilon^{\frac{3}{4}}\|f\|_{L_2(0,a_2)}. \tag{3.18}$$

This estimate in combination with (3.11) and the definition of v_ε yields (3.12).

It remains to prove (3.13). To this aim we denote

$$S_\varepsilon^+ := \{x_2 : \ 0 < x_2 < a_3\varepsilon\}$$

and use integration by parts to rewrite the norm in question as follows,

$$\begin{aligned}
\|u_\varepsilon\|^2_{L_2(S_\varepsilon^+)} &= \int\limits_0^{a_3\varepsilon} |u_\varepsilon|^2\,dx_2 = a_3\varepsilon|u_\varepsilon(0)|^2 + 2\int\limits_0^{a_3\varepsilon} (a_3\varepsilon - x_2)\operatorname{Re}\overline{u_\varepsilon}u_\varepsilon'\,dx_2 \\
&= a_3\varepsilon|u_\varepsilon(0)|^2 + 2\int\limits_{S_\varepsilon^+} (a_3\varepsilon - x_2)\operatorname{Re}\overline{u_\varepsilon}u_\varepsilon'\,dx_2.
\end{aligned}$$

Using next (3.18) and Cauchy-Schwarz inequality together with the fact that the norm $\|u_\varepsilon'\|_{L_2(S_\varepsilon^+)}$ is bounded by (3.17) we arrive at the bound

$$a_3\varepsilon|u_\varepsilon(0)|^2 \leqslant \|u_\varepsilon\|^2_{L_2(S_\varepsilon^+)} + 2a_3\varepsilon\|u_\varepsilon\|_{L_2(S_\varepsilon^+)}\|u_\varepsilon'\|_{L_2(S_\varepsilon^+)} \leqslant C\varepsilon^{\frac{3}{2}}\|f\|^2_{L_2(0,a_2)}$$

which implies (3.13). Note that in view of the quasiperiodic boundary conditions $|u_\varepsilon(a_2)|$ satisfies the same inequality. $\square$

The proven lemma allows us to proceed with the estimate in (3.9) arriving thus at the inequality

$$\|v_\varepsilon\|^2_{W_2^1(0,a_2)} \leqslant C\varepsilon^{\frac{1}{4}}\|f\|^2_{L_2(0,a_2)},$$

where C is here and in the following is a constant independent of ε and f. This yields

$$\|v_\varepsilon\|_{W_2^1(0,a_2)} \leqslant C\varepsilon^{\frac{1}{8}}\|f\|_{L_2(0,a_2)},$$

which is in view of the definition of v_ε equivalent to the estimate

$$\left\|\big(\mathcal{A}_\varepsilon(\tau_2) - \mathrm{i}\big)^{-1} - (\mathcal{A}_0 - \mathrm{i})^{-1}\right\|_{L_2(0,a_2)\to W_2^1(0,a_2)} \leqslant C\varepsilon^{\frac{1}{8}}, \tag{3.19}$$

in which $\|\cdot\|_{L_2(0,a_2)\to W_2^1(0,a_2)}$ denotes the norm of an operator acting from $L_2(0,a_2)$ to $W_2^1(0,a_2)$.

3.2 Approximation of the eigenvalues of $\mathcal{A}_\varepsilon$

Let us now complete the proof of Lemma 3.2. The estimate (3.19) and formula (3.4) for $E_V^{(n,p)}$ imply immediately that the eigenvalues $E_V^{(n,p)}$ converge to $E_0^{(n,p)}$. In particular, given $E > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ all the eigenvalues $E_V^{(n,p)}(\tau)$ with (n,p) such that $E_0^{(n,p)}(\tau)$ does not obey (3.2) satisfy the lower bound

$$E_V^{(n,p)}(\tau) > \frac{E}{2}.$$

Consider next the eigenvalues $E_0^{(n,p)}(\tau)$ that do obey (3.2). There are finitely many of them and our aim is to prove estimate (3.6) for each pair (n,p) for which the inequality (3.2) is satisfied.

The main idea to achieve this goal is to construct the asymptotic expansions of $\lambda_\varepsilon^{(p)}$ as $\varepsilon \to +0$. Here we employ a standard approach consisting of two steps. At the first step, we construct the asymptotic expansions formally using the method of matching asymptotic expansions [12]. The next step consists of justifying the formal asymptotics by estimating the error term. We fix m and adopt the following Ansatz for $\lambda_\varepsilon^{(p)}$,

$$\lambda_\varepsilon^{(p)}(\tau_2) = \lambda_0^{(p)} + \varepsilon^{\frac{1}{2}}\lambda_{\frac{1}{2}}^{(p)}(\tau_2) + \varepsilon\lambda_1^{(p)}(\tau_2) + \ldots, \qquad \lambda_0^{(p)} := \frac{\pi^2 p^2}{a_2^2}. \tag{3.20}$$

The asymptotics for the associated eigenfunction $\Psi_\varepsilon^{(p)}(x_2,\tau_2)$ is constructed as a combination of 'external' and 'internal' expansions. The former reads as

$$\begin{aligned}
&\Psi_{\varepsilon,\mathrm{ex}}^{(p)}(x_2,\tau_2) = \Psi_0^{(p)}(x_2) + \varepsilon^{\frac{1}{2}}\Psi_{\frac{1}{2}}^{(p)}(x_2,\tau_2) + \varepsilon\Psi_1^{(p)}(x_2,\tau_2) + \ldots, \qquad (3.21)\\
&\Psi_0^{(p)}(x_2) := \frac{\sqrt{2}}{\sqrt{a_2}}\sin\frac{\pi p x_2}{a_2},
\end{aligned}$$

while the internal expansion is constructed as follows,

$$\begin{aligned}
&\Psi_{\varepsilon,\mathrm{in}}^{(p)}(\xi,x_2,\tau_2) = e^{\mathrm{i}\tau_2 x_2}\Big(\varepsilon^{\frac{1}{2}}\Psi_{\frac{1}{2},\mathrm{in}}^{(p)}(\xi,\tau_2) + \varepsilon\Psi_{1,\mathrm{in}}^{(p)}(\xi,\tau_2) + \ldots\Big), \qquad (3.22)\\
&\xi = \frac{x_2}{\varepsilon} \ \text{ for } \ 0 < x_2 < 2\varepsilon^{\frac{1}{2}} \quad \text{and} \quad \xi = \frac{x_2 - a_2}{\varepsilon} \ \text{ for } \ a_2 - 2\varepsilon^{\frac{1}{2}} < x_2 < a_2.
\end{aligned}$$

To be explicit, the external expansion is employed to approximate the eigenfunction outside small neighborhoods of the points $x_2 = 0$ and $x_2 = a_2$, while the internal refers to the behavior in the vicinity of the mentioned points. The form of the Ansatz (3.22) ensures that the quasiperiodic conditions are satisfied.

The final approximation for the eigenfunction $\Psi_\varepsilon^{(p)}$ is obtained by matching the two expansions as follows,

$$\begin{aligned}
\Psi_\varepsilon^{(p)}(x_2,\tau_2) =& \Psi_{\varepsilon,\mathrm{ex}}^{(p)}(x_2,\tau_2)\chi\big(x_2\varepsilon^{-\frac{1}{2}}\big)\chi\big((a_2-x_2)\varepsilon^{-\frac{1}{2}}\big)\\
&+ \Psi_{\varepsilon,\mathrm{in}}^{(p)}\big(x_2\varepsilon^{-1},\tau_2\big)\Big(1-\chi\big(x_2\varepsilon^{-\frac{1}{2}}\big)\Big)\\
&+ \Psi_{\varepsilon,\mathrm{in}}^{(p)}\big((x_2-a_2)\varepsilon^{-1},\tau_2\big)\Big(1-\chi\big((a_2-x_2)\varepsilon^{-\frac{1}{2}}\big)\Big),
\end{aligned} \tag{3.23}$$

where $\chi = \chi(t)$ is an infinitely differentiable function that is equal to one for $t > 2$ and vanishes for $t < 1$. We substitute expansions (3.20), (3.27), (3.22) into the eigenvalue equation

$$\left(-\frac{d^2}{dx_2^2} + \varepsilon^{-\frac{3}{2}}V_\varepsilon\right)\Psi_\varepsilon^{(p)} = \lambda_\varepsilon^{(p)}\Psi_\varepsilon^{(p)}$$

and identify the coefficients at the same powers of ε. This yields the equations

$$-\frac{d^2\Psi_\beta^{(p)}}{dx_2^2} = \lambda_0^{(p)}\Psi_\beta^{(p)} + \lambda_\beta^{(p)}\Psi_0^{(p)} \quad \text{in} \quad (0,a_2), \qquad \beta \in \left\{\tfrac{1}{2}, 1\right\}, \tag{3.24}$$

and

$$-\frac{d^2\Psi_{\frac{1}{2},\mathrm{in}}^{(p)}}{d\xi^2} = 0 \qquad\qquad \text{in } \mathbb{R}, \tag{3.25}$$

$$-\frac{d^2\Psi_{1,\mathrm{in}}^{(p)}}{d\xi^2} + V(\xi)\Psi_{\frac{1}{2},\mathrm{in}}^{(p)} = 0 \qquad \text{in } \mathbb{R}. \tag{3.26}$$

The expansions (3.21), (3.22) are to be matched in the intermediate zones, namely for $\varepsilon^{\frac{1}{2}} < x_2 < 2\varepsilon^{\frac{1}{2}}$ and $a_2 - 2\varepsilon^{\frac{1}{2}} < x_2 < a_2 - \varepsilon^{\frac{1}{2}}$. The asymptotic behavior of the external expansion as $x_2 \to 0+$ and $x_2 \to a_2-$ should coincide there with the asymptotic behavior of the internal expansion as $\xi \to \pm\infty$. To be more precise, we match in this way the expansion $e^{-\mathrm{i}\tau_2 x_2}\Psi_{\varepsilon,\mathrm{ex}}^{(p)}$ and $\Psi_{\varepsilon,\mathrm{in}}^{(p)}$. In view of the definition of $\Psi_0^{(p)}$ this yields

$$\begin{aligned}
&\Psi_0^{(p)}(x_2) = \frac{\sqrt{2}\pi p}{a_2^{\frac{3}{2}}}x_2 + \mathcal{O}(x_2^3), && x_2 \to 0+,\\
&\Psi_0^{(p)}(x_2) = \frac{(-1)^m\sqrt{2}\pi p}{a_2^{\frac{3}{2}}}(x_2 - a_2) + \mathcal{O}\big((a_2 - x_2)^3\big), && x_2 \to a_2-,\\
&\mathrm{e}^{-\mathrm{i}\tau_2 x_2} = 1 - \mathrm{i}\tau_2 x_2 + \mathcal{O}(x_2^2), && x_2 \to 0+,\\
&\mathrm{e}^{-\mathrm{i}\tau_2 x_2} = e^{-\mathrm{i}\tau_2 a_2}\Big(1 - \mathrm{i}\tau_2(x_2 - a_2) + \mathcal{O}\big((x_2 - a_2)^2\big)\Big), && x_2 \to 0+,\\
&\Psi_\beta^{(p)}(x_2) = \Psi_\beta^{(p)}(0) + \mathcal{O}(x_2), && x_2 \to 0+,\\
&\Psi_\beta^{(p)}(x_2) = \Psi_\beta^{(p)}(a_2) + \mathcal{O}(a_2 - x_2), && x_2 \to a_2-,
\end{aligned} \tag{3.27}$$

where $\beta \in \left\{\frac{1}{2}, 1\right\}$. The matching conditions are

$$\begin{aligned}
&\Psi_{\frac{1}{2},\mathrm{in}}^{(p)}(\xi) = \Psi_{\frac{1}{2}}^{(p)}(0) + o(1), && \xi \to +\infty,\\
&\Psi_{\frac{1}{2},\mathrm{in}}^{(p)}(\xi) = \Psi_{\frac{1}{2}}^{(p)}(a_2)e^{-\mathrm{i}\tau_2 a_2} + o(1), && \xi \to -\infty,
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
&\Psi_{1,\mathrm{in}}^{(p)}(\xi) = \frac{\sqrt{2}\pi p}{a_2^{\frac{3}{2}}}\xi + \Psi_1^{(p)}(0) + o(1), && \xi \to +\infty,\\
&\Psi_{1,\mathrm{in}}^{(p)}(\xi) = \left(\frac{(-1)^m\sqrt{2}\pi p}{a_2^{\frac{3}{2}}}\xi + \Psi_1^{(p)}(a_2)\right)e^{-\mathrm{i}\tau_2 a_2} + o(1), && \xi \to -\infty.
\end{aligned} \tag{3.29}$$

The only solution to equation (3.25) satisfying (3.28) is a constant, that is,

$$\Psi_{\frac{1}{2},\mathrm{in}}^{(p)}(\xi) \equiv K_{\frac{1}{2}}, \qquad K_{\frac{1}{2}} := \Psi_{\frac{1}{2}}^{(p)}(0) = \mathrm{e}^{-\mathrm{i}\tau_2 a_2}\Psi_{\frac{1}{2}}^{(p)}(a_2), \tag{3.30}$$

and the second identity is to be regarded as the solvability condition. The general solution to equation (3.26) is given by the formula

$$\Psi^{(p)}_{1,\mathrm{in}}(\xi)=\tilde\Psi^{(p)}_{1,\mathrm{in}}(\xi)+K_1,\quad \tilde\Psi^{(p)}_{1,\mathrm{in}}(\xi):=\frac{K_{\frac12}}{2}\int\limits_{\mathbb{R}}|\xi-z|V(z)\,dz+C_1\xi. \tag{3.31}$$

In view of the formulæ

$$\begin{aligned}&\int\limits_{\mathbb{R}}|\xi-z|V(z)\,dz=\pm\langle V\rangle\xi\pm\langle zV\rangle,\quad \xi\to\pm\infty,\\ &\langle V\rangle:=\int\limits_{\mathbb{R}}V(z)\,dz,\qquad \langle zV\rangle:=\int\limits_{\mathbb{R}}zV(z)\,dz,\end{aligned} \tag{3.32}$$

and conditions (3.29) we obtain

$$\frac{K_{\frac12}}{2}\langle V\rangle+C_1=\frac{\sqrt{2}\pi p}{a_2^{\frac32}},\qquad -\frac{K_{\frac12}}{2}\langle V\rangle+C_1=\frac{(-1)^m\sqrt{2}\pi p}{a_2^{\frac32}}e^{-\mathrm{i}\tau_2 a_2},$$

arriving thus finally at

$$K_{\frac12}=\frac{1}{\langle V\rangle}\frac{\sqrt{2}\pi p\big(1-(-1)^m e^{-\mathrm{i}\tau_2a_2}\big)}{a_2^{\frac32}},\qquad C_1=\frac{\pi p\big(1+(-1)^m e^{-\mathrm{i}\tau_2a_2}\big)}{\sqrt{2}a_2^{\frac32}}. \tag{3.33}$$

The solvability condition of problem (3.24), (3.30), (3.33) for $\Psi^{(p)}_{\frac12}$ is obtained in the standard way: equation (3.24) should be multiplied by $\Psi^{(p)}_0$ and integrated by parts twice over $(0,a_2)$. This gives an expression for $\lambda^{(p)}_{\frac12}$,

$$\lambda^{(p)}_{\frac12}(\tau_2)=-\frac{2\pi^2p^2\big|1-(-1)^m e^{-\mathrm{i}\tau_2a_2}\big|^2}{a_2^3\langle V\rangle}. \tag{3.34}$$

The corresponding solution to problem (3.24), (3.30), (3.33) for $\Psi^{(p)}_{\frac12}$ reads as

$$\Psi^{(p)}_{\frac12}(x_2,\tau_2)=\frac{\lambda^{(p)}_{\frac12}(\tau_2)}{\sqrt{2a_2}\lambda^{(p)}_0}\left(\frac{\pi p}{a_2}x_2\cos\frac{\pi p}{a_2}x_2+\frac12\sin\frac{\pi p}{a_2}x_2\right)+K_{\frac12}\cos\frac{\pi p}{a_2}x_2. \tag{3.35}$$

This solution is orthogonal to $\Psi^{(p)}_0$ in $L_2(0,a_2)$ as a consequence of the assumed normalization of the perturbed eigenfunction.

To justify the obtained asymptotics (3.20), (3.21), (3.22), (3.23), the standard argument can be used. Namely, in the same way as above we construct sufficiently many terms in the expansions so that the truncated series of (3.20), (3.23) solve the eigenvalue equation up to an error of order $O(\varepsilon^{\frac12})$. Then we apply Vishik-Lyusternik's lemma, see, for instance, [16, Sect. III.1.1, Lemma 1.1] or [24, Sect. 9, Lemma 13] which this gives the sought asymptotics,

$$\lambda^{(p)}_\varepsilon(\tau_2)=\lambda^{(p)}_0+\mathcal{O}(\varepsilon^{\frac12}),$$

where the error term is uniform in τ_2. The obtained expansions together with formulæ (3.4), (3.1) complete the proof of Lemma 3.2.

3.3 Approximation of band functions $E_\varepsilon^{(k)}$

The goal of this subsection we complete the proof of Lemma 3.1. The key point will be the following estimate,

$$\|(\mathcal{H}_\varepsilon(\tau)-\mathrm{i})^{-1}-(\mathcal{H}_V(\tau)-\mathrm{i})^{-1}\|_{L_2(\square)\to W_2^1(\square)}\leqslant C\varepsilon^\alpha, \tag{3.36}$$

where C is a constant independent of ε and τ. Once this inequality is established, it infers a statement similar to Lemma 3.1. Specifically, estimate (3.36) yields that to any fixed E there exists an $\varepsilon_0>0$ such that for all $\varepsilon\leqslant\varepsilon_0$ and all (n,p) obeying (3.2) the estimates

$$\left|E_\varepsilon^{(k)}(\tau)-E_V^{(n,p)}(\tau)\right|\leqslant C\varepsilon^\alpha \tag{3.37}$$

hold, where C is a constant independent of ε, μ, n, p, and τ but dependent on E, where the eigenvalues $E_0^{(n,p)}$ are assumed to be arranged in the ascending order counting the multiplicities. Estimate (3.37) and Lemma 3.2 prove Lemma 3.1. The rest of this subsection is thus devoted to proving inequality (3.36).

Given an $f\in L_2(\square)$, we denote $w_\varepsilon:=(\mathcal{H}_\varepsilon-\mathrm{i})^{-1}f$, $w_V:=(\mathcal{H}_V-\mathrm{i})^{-1}f$, and $w:=w_\varepsilon-w_V$. We write the integral identities for w_ε and w_V employing w as the test function,

$$\begin{aligned}
&(\nabla w_V,\nabla w)_{L_2(\square)}+\varepsilon^{-\frac{3}{2}}(V_\varepsilon w_V,w)_{L_2(\square)}-\mathrm{i}(w_V,w)_{L_2(\square)}=(f,w)_{L_2(\square)},\\
&(\nabla w_\varepsilon,\nabla w)_{L_2(\square)}+\varepsilon^{-\frac{3}{2}}(V_\varepsilon w_\varepsilon,w)_{L_2(\square)}+\varepsilon^\alpha\mathfrak{h}(w_\varepsilon,w)-\mathrm{i}(w_\varepsilon,w)_{L_2(\square)}=(f,w)_{L_2(\square)},\\
&\mathfrak{h}(w_\varepsilon,w):=\sum_{i,j=1}^{2}\left(A_{ij}\frac{\partial w_\varepsilon}{\partial x_j},\frac{\partial w}{\partial x_i}\right)_{L_2(\square)}+\mathrm{i}\sum_{j=1}^{2}\left(A_j\frac{\partial w_\varepsilon}{\partial x_j},w\right)_{L_2(\square)}\\
&\qquad\qquad-\mathrm{i}\sum_{j=1}^{2}\left(w_\varepsilon,A_j\frac{\partial w}{\partial x_j}\right)_{L_2(\square)}+(A_0w_\varepsilon,w)_{L_2(\square)}.
\end{aligned}$$

Subtracting the first two identities from each other and employing the representation $\mathfrak{h}(w_\varepsilon,w)=\mathfrak{h}(w,w)+$, we obtain

$$\|\nabla w\|_{L_2(\square)}^2+\varepsilon^{-\frac{3}{2}}(V_\varepsilon w,w)_{L_2(\square)}+\varepsilon^\alpha\mathfrak{h}(w,w)-\mathrm{i}\|w\|_{L_2(\square)}^2=-\varepsilon^\alpha\mathfrak{h}(w_V,w). \tag{3.38}$$

In view of the assumed positivity of the function V_ε, a similar identity for w_V with the test function w_V,

$$\|\nabla w_V\|_{L_2(\square)}^2+\varepsilon^{-\frac{3}{2}}(V_\varepsilon w_V,w_V)_{L_2(\square)}-\mathrm{i}\|w\|_{L_2(\square)}^2=(f,w_V)_{L_2(\square)},$$

implies the *apriori* estimate

$$\|w_V\|_{W_2^1(\square)}\leqslant C\|f\|_{L_2(\square)}\,; \tag{3.39}$$

the symbol C stands again for various inessential constants independent of ε, τ, and f. This allows us to estimate the right hand in (3.38) as

$$\varepsilon^\alpha|\mathfrak{h}(w_V,w)|\leqslant C\varepsilon^\alpha\|w_V\|_{W_2^1(\square)}\|w\|_{W_2^1(\square)}. \tag{3.40}$$

The real part of left-hand side in (3.38) can be estimated from below as

$$\begin{aligned}\|\nabla w\|^2_{L_2(\square)}+\varepsilon^{-\frac{3}{2}}(V_\varepsilon w,w)_{L_2(\square)}+\varepsilon^\alpha\mathfrak{h}(w,w)\geqslant&\|\nabla w\|^2_{L_2(\square)}+\varepsilon^\alpha\mathfrak{h}(w,w)\\ \geqslant&\frac{1}{2}\|\nabla w\|^2_{L_2(\square)}-C\varepsilon^\alpha\|w\|^2_{L_2(\square)}.\end{aligned}$$

Using this result, (3.40), and taking the real and the imaginary part of (3.38), we get

$$\begin{aligned}&\|w\|^2_{L_2(\square)}\leqslant C\varepsilon^\alpha\|f\|_{L_2(\square)}\|w\|_{W_2^1(\square)},\\ &\frac{1}{2}\|\nabla w\|^2_{L_2(\square)}-C\varepsilon^\alpha\|w\|^2_{L_2(\square)}\leqslant C\varepsilon^\alpha\|f\|_{L_2(\square)}\|w\|_{W_2^1(\square)},\end{aligned}$$

and consequently,

$$\|w\|_{W_2^1(\square)}\leqslant C\varepsilon^\alpha\|f\|_{L_2(\square)}$$

which finally proves (3.36).

4 Proof of Theorem 2.1

Now we are in position to prove our main result, Theorem 2.1. In view of Lemma 3.1, the band functions $E_\varepsilon^{(k)}(\tau)$ converge to the eigenvalues $E_0^{(n,p)}$ as $\varepsilon\to 0+$. Considering a small fixed neighbourhood of the point E_0, we immediately conclude from Lemma 3.1 that there are constants $C_1>0$, $C_2>0$ such that as $|\tau_1-\tau_0|<C_1\varepsilon^\alpha$, the segment $[E_0-C_2\varepsilon^\alpha,E_0+C_2\varepsilon^\alpha]$ contains no eigenvalues of the operator $\mathcal{H}_\varepsilon(\tau)$ except exactly one pair of them converging to $E_0^{(n,1)}(\tau)$ and $E_0^{(m,2)}(\tau)$, respectively, as $\varepsilon\to 0+$. Let us analyze the behavior of these two eigenvalues.

We first rewrite the eigenvalue equation by changing the eigenfunction $\psi\mapsto \mathrm{e}^{\mathrm{i}(\tau_1x_1+\tau_2x_2)}\psi$. This leads to a new equation

$$\tilde{\mathcal{H}}_\varepsilon(\tau)\psi=E\psi, \tag{4.1}$$

where $\tilde{\mathcal{H}}_\varepsilon$ is a self-adjoint operator in $L_2(\square)$ with the differential expression

$$\begin{aligned}\tilde{\mathcal{H}}_\varepsilon(\tau)=&\sum_{j=1}^{2}\left(\mathrm{i}\frac{\partial}{\partial x_j}-\tau_j\right)^2-\varepsilon^\alpha\left(\mathrm{i}\frac{\partial}{\partial x_1}-\tau_1\right)A_{11}(x)\left(\mathrm{i}\frac{\partial}{\partial x_1}-\tau_1\right)\\ &+\mathrm{i}\varepsilon^\alpha\sum_{j=1}^{2}\left(A_j(x)\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_j}A_j(x)\right)+\varepsilon^\alpha A_0(x)+\varepsilon^{-\frac{3}{2}}V_\varepsilon(x)\end{aligned}$$

subject to periodic boundary conditions on the boundary $\partial\square$.

Next we introduce an auxiliary parameter t setting $\tau_1=t_0+\varepsilon^\alpha t$ and we denote the perturbed eigenvalues in question by $E_\varepsilon^\pm(t)$. The parameter t ranges over some segment $[-C,C]$ for a sufficiently large C. The associated eigenfunctions of the operator $\tilde{\mathcal{H}}_\varepsilon$ are denoted by $\Psi_\varepsilon^\pm(x,t)$. We are going to construct the first terms in the asymptotic expansions of $E_\varepsilon^\pm(t)$. This will be done using the same scheme as in Subsection 3.2 taking into consideration the presence of the perturbation $\varepsilon^\alpha\mathcal{L}$ in the operator.

We adopt the following Ansätze for the eigenvalues $E_\varepsilon^\pm(t,\tau_2)$,

$$E_\varepsilon^\pm(t,\tau_2) = E_0 + \varepsilon^\alpha \lambda_\alpha^\pm(t) + \varepsilon^{\frac{1}{2}}\lambda_1^\pm(t,\tau_2) + \dots \tag{4.2}$$

The asymptotics for the associated eigenfunctions are again constructed as a combination of the external and inner expansions. The former is introduced as

$$\Psi_{\varepsilon,\mathrm{ex}}^\pm(x,t,\tau_2) = \Psi_0^\pm(x,t,\tau_2) + \varepsilon^\alpha \Psi_\alpha^\pm(x,t,\tau_2) + \varepsilon^{\frac{1}{2}}\Psi_{\frac{1}{2}}^\pm(x,t,\tau_2) + \dots, \tag{4.3}$$

where

$$\Psi_0^\pm(x,t,\tau_2) := \frac{\sqrt{2}}{\sqrt{a_1 a_2}}\left(c_n^\pm(t,\tau_2)\psi_0^{(n,1)}(x,\tau_2) + c_m^\pm(t,\tau_2)\psi_0^{(m,2)}(x,\tau_2)\right), \tag{4.4}$$

and $c_j^\pm(t,\tau_2)$ are constants to be determined. The inner expansion is of the form

$$\Psi_{\varepsilon,\mathrm{in}}^\pm(x,t,\tau_2) = \varepsilon^{\frac{1}{2}}\Psi_{\frac{1}{2},\mathrm{in}}^\pm(\xi,x_1,t,\tau_2) + \varepsilon\Psi_{1,\mathrm{in}}^\pm(\xi,x_1,t,\tau_2) + \dots, \tag{4.5}$$

where the variable ξ is the same as in (3.22). The approximation for the eigenfunctions is defined via the external and inner expansion matching as in (3.23):

$$\begin{aligned}\Psi_\varepsilon^\pm(x,t,\tau_2) =& \Psi_{\varepsilon,\mathrm{ex}}^\pm(x,t,\tau_2)\chi\big(x_2\varepsilon^{-\frac{1}{2}}\big)\chi\big((a_2-x_2)\varepsilon^{-\frac{1}{2}}\big)\\ &+ \Psi_{\varepsilon,\mathrm{in}}^\pm\big(x_2\varepsilon^{-1},x_1,t,\tau_2\big)\Big(1-\chi\big(x_2\varepsilon^{-\frac{1}{2}}\big)\Big)\\ &+ \Psi_{\varepsilon,\mathrm{in}}^\pm\big((x_2-a_2)\varepsilon^{-1},x_1,t,\tau_2\big)\Big(1-\chi\big((a_2-x_2)\varepsilon^{-\frac{1}{2}}\big)\Big).\end{aligned} \tag{4.6}$$

We substitute Ansätze (4.2), (4.3), (4.5) into equation (4.1) and collect the coefficients at the same powers of ε. This gives the equations

$$\tilde{\mathcal{H}}_0\Psi_\alpha^\pm - E_0\Psi_\alpha^\pm = 2t\left(\mathrm{i}\frac{\partial}{\partial x_1} - \tau_0\right)\Psi_0^\pm - \mathcal{L}(\tau_0)\Psi_0^\pm + \lambda_\alpha^\pm\Psi_0^\pm, \tag{4.7}$$

$$\tilde{\mathcal{H}}_0\Psi_{\frac{1}{2}}^\pm - E_0\Psi_{\frac{1}{2}}^\pm = \lambda_{\frac{1}{2}}^\pm\Psi_0^\pm, \tag{4.8}$$

both in $\square$, where the differential expression

$$\tilde{\mathcal{H}}_0 := \left(\mathrm{i}\frac{\partial}{\partial x_1} - \tau_0\right)^2 + \left(\mathrm{i}\frac{\partial}{\partial x_2} - \tau_2\right)^2$$

has to be amended with boundary conditions. On the lateral boundaries we postulate the periodic ones,

$$\Psi_\beta^\pm\big|_{x_1=0} = \Psi_\beta^\pm\big|_{x_1=a_1}, \qquad \left.\frac{\partial\Psi_\beta^\pm}{\partial x_1}\right|_{x_1=0} = \left.\frac{\partial\Psi_\beta^\pm}{\partial x_1}\right|_{x_1=a_1}, \qquad \beta\in\{\alpha,\tfrac{1}{2}\}, \tag{4.9}$$

recall that $\beta\in\left\{\frac{1}{2},1\right\}$. In contrast, Dirichlet boundary condition are assumed for $\Psi_\alpha^\pm$,

$$\Psi_\alpha^\pm = 0 \quad \text{on} \ \ \gamma. \tag{4.10}$$

The boundary condition for $\Psi_{\frac{1}{2}}^\pm$ on γ will be determined later, by matching with the inner expansion.

Remark 1. The homogeneous Dirichlet condition for $\Psi_\alpha^\pm$ have been postulated from the following reason. We could have assumed that this function has some unknown values on γ to be determined by the matching with the inner expansion as it was done in Subsection 3.2. Then one would have to introduce an additional term $\varepsilon^{\frac{1}{2}+\alpha}\Psi^{(m)}_{\frac{1}{2}+\alpha}$ in the external expansion and terms $\varepsilon^\alpha\Psi^\pm_{\alpha,\mathrm{in}}+\varepsilon^{\frac{1}{2}+\alpha}\Psi^\pm_{\frac{1}{2}+\alpha,\mathrm{in}}$ in the inner expansion. Matching of $\Psi_\alpha^\pm$ with these extra terms then implies that this function has to vanish on γ and all these extra terms are zero. With this fact in mind we adopt from the beginning the homogeneous Dirichlet condition on γ for $\Psi_\alpha^\pm$ obtaining in this way a simpler Ansatz for the perturbed eigenfunctions.

In view of the condition (4.10), the solvability criterion for problem (4.7), (4.9), (4.10) reduces to the orthogonality of the right-hand in the equation to the functions $\psi_0^{(n,1)}$ and $\psi_0^{(m,2)}$ in $L_2(\square)$. These requirements can be written as the system of linear equations,

$$\begin{aligned}&M(t)C^\pm(t)=\lambda_\alpha^\pm(t)C^\pm(t), \qquad M(t):=M^{(0)}(\tau_0)-2tM^{(1)}(\tau_0),\\ &C^\pm(t):=\begin{pmatrix}c_n(t)\\ c_m(t)\end{pmatrix}.\end{aligned}\tag{4.11}$$

Hence $\lambda_\alpha^\pm$ are the eigenvalues of the matrix M and $C^\pm$ are the associated eigenfunctions. Since the matrix M is Hermitian, we can choose the vectors $C^\pm$ to be orthonormal in $\mathbb{C}^2$. It is straightforward to find the eigenvalues $\lambda_\alpha^\pm$ explicitly,

$$\begin{aligned}\lambda_\alpha^\pm(t):=&\frac{M_{11}^{(0)}(\tau_0)+M_{22}^{(0)}(\tau_0)}{2}-2t\left(\frac{\pi(n+m)}{a_1}+\tau_0\right)\\ &\pm\left(\left(\frac{M_{11}^{(0)}(\tau_0)-M_{22}^{(0)}(\tau_0)}{2}-t\frac{\pi(n-m)}{a_1}\right)^2+\left|M_{12}^{(0)}(\tau_0)\right|^2\right)^{\frac{1}{2}}.\end{aligned}\tag{4.12}$$

Let us next determine the boundary conditions to be imposed on $\Psi^\pm_{\frac{1}{2}}$ on the boundary γ. We substitute expansions (4.5) and (4.2) into eigenvalue equation (4.2), pass to the variable ξ, and collect the coefficients at the same powers of ε. This leads us to the equations for the coefficients of the inner expansion,

$$-\frac{d^2\Psi^\pm_{\frac{1}{2},\mathrm{in}}}{d\xi}=0 \qquad \text{in } \mathbb{R},\tag{4.13}$$

$$-\frac{d^2\Psi^\pm_{1,\mathrm{in}}}{d\xi}+V\Psi^\pm_{\frac{1}{2},\mathrm{in}}=0 \quad \text{in } \mathbb{R}.\tag{4.14}$$

We expand the functions $\Psi_0^\pm$, $\Psi_\alpha^\pm$ and $\Psi^\pm_{\frac{1}{2}}$ as $x_2\to0+$ and $x_2\to a_2-$,

$$\begin{aligned}&\Psi_0^\pm(x,t)=\phi_{0,+}^\pm(x_1,t)x_2+\mathcal{O}(x_2^3), \qquad x_2\to0+,\\ &\Psi_0^\pm(x,t)=\phi_{0,-}^\pm(x_1,t)(x_2-a_2)+\mathcal{O}\big((x_2-a_2)^3\big), \qquad x_2\to a_2-,\\ &\phi_{0,+}^\pm(x_1,t):=\frac{\sqrt{2}\pi}{a_1^{\frac{1}{2}}a_2^{\frac{3}{2}}}\left(c_n^\pm(t)e^{\mathrm{i}\frac{2\pi n}{a_1}x_1}+2c_p^\pm(t)\,\mathrm{e}^{\mathrm{i}\frac{2\pi m}{a_1}x_1}\right),\end{aligned}$$

$$
\begin{aligned}
&\phi_{0,-}^{\pm}(x_1,t) := \frac{\sqrt{2}\pi}{a_1^{\frac{1}{2}} a_2^{\frac{3}{2}}}\Big(- c_n^{\pm}(t) e^{\mathrm{i}\frac{2\pi n}{a_1}x_1} + 2c_p^{\pm}(t)\, \mathrm{e}^{\mathrm{i}\frac{2\pi m}{a_1}x_1}\Big)\, \mathrm{e}^{-\mathrm{i}\tau_2 a_2},\\
&\Psi_\alpha^{\pm}(x,t,\tau_2) = \mathcal{O}(x_2), \quad x_2\to 0+,\\
&\Psi_\alpha^{\pm}(x,t,\tau_2) = \mathcal{O}(x_2-a_2), \quad x_2\to a_2-,\\
&\Psi_\beta^{\pm}(x,t,\tau_2) = \Psi_\beta^{\pm}(x_1,0,t,\tau_2)+\mathcal{O}(x_2), \quad x_2\to 0+,\\
&\Psi_\beta^{\pm}(x,t,\tau_2) = \Psi_\beta^{\pm}(x_1,a_2,t,\tau_2)+\mathcal{O}(x_2-a_2), \quad x_2\to a_2-, \quad \beta\in\{\tfrac{1}{2},1\}.
\end{aligned}
$$

Rewriting these formulae in the variable ξ, by matching condition we conclude that the coefficients of the inner expansion should behave at infinity as follows,

$$
\begin{aligned}
&\Psi_{\frac{1}{2},\mathrm{in}}^{\pm}(\xi,x_1,t,\tau_2) = \Psi_{\frac{1}{2}}^{\pm}(x_1,0,t,\tau_2)+o(1), \quad \xi\to+\infty,\\
&\Psi_{\frac{1}{2},\mathrm{in}}^{\pm}(\xi,x_1,t,\tau_2) = \Psi_{\frac{1}{2}}^{\pm}(x_1,a_2,t,\tau_2)+o(1), \quad \xi\to-\infty,
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
&\Psi_{1,\mathrm{in}}^{\pm}(\xi,x_1,t,\tau_2) = \phi_{0,+}^{\pm}(x_1,t)\xi+\mathcal{O}(1), \quad \xi\to+\infty,\\
&\Psi_{1,\mathrm{in}}^{\pm}(\xi,x_1,t,\tau_2) = \phi_{0,-}^{\pm}(x_1,t)\xi+o(1), \quad \xi\to-\infty.
\end{aligned}
\tag{4.16}
$$

Problem (4.13), (4.15) is solvable if and only if

$$
\Psi_{\frac{1}{2}}^{\pm}(x_1,0,t,\tau_2) = \Psi_{\frac{1}{2}}^{\pm}(x_1,a_2,t,\tau_2) =: \phi_{\frac{1}{2}}^{\pm}(x_1,t,\tau_2) \tag{4.17}
$$

and its solution reads as

$$
\Psi_{\frac{1}{2}}(\xi,x_1,t,\tau_2) = \phi_{\frac{1}{2}}^{\pm}(x_1,t,\tau_2). \tag{4.18}
$$

Next we proceed to equation (4.14) writing its solution as

$$
\Psi_{1,\mathrm{in}}^{\pm}(\xi,x_1,t,\tau_2) = \frac{\phi_{\frac{1}{2}}(x_1,t,\tau_2)}{2}\int\limits_{\mathbb{R}} |\xi-z|V(z)\,dz + T_1(x_1,t,\tau)\xi + T_0(x_1,t,\tau_2),
$$

where T_0, T_1 are functions independent of ξ. The behavior of the function $\Psi_{1,\mathrm{in}}^{\pm}$ at infinity can be expressed using formulæ (3.32); comparing the result with (4.16), we get

$$
\frac{\phi_{\frac{1}{2}}^{\pm}}{2}\langle V\rangle + T_1 = \phi_{0,+}^{\pm}, \qquad -\frac{\phi_{\frac{1}{2}}^{\pm}}{2}\langle V\rangle + T_1 = \phi_{0,-}^{\pm},
$$

which determines $\phi_{\frac{1}{2}}^{\pm}$,

$$
\phi_{\frac{1}{2}}^{\pm}(x_1,t,\tau_2) = \frac{\phi_{0,+}^{\pm}-\phi_{0,-}^{\pm}}{\langle V\rangle}. \tag{4.19}
$$

The solvability of the problem determined by boundary conditions (4.9), (4.10), (4.17), (4.19) is obtained in the standard way, that is, equation (4.9) is multiplied by $\psi_0^{(n,1)}$ and $\psi_0^{(m,2)}$ and integrated twice by parts over $\square$ taking the indicated conditions into account. This yields $\lambda_{\frac{1}{2}}^{\pm}$ in the form of the following expression

$$
\begin{aligned}
\lambda_{\frac{1}{2}}^{\pm}(t,\tau_2) &= -\frac{2\pi^2}{a_1 a_2^3\langle V\rangle}\int\limits_0^{a_1}\Big| c_n^{\pm}(t) e^{\mathrm{i}\frac{2\pi n}{a_1}x_1}(1+\mathrm{e}^{-\mathrm{i}\tau_2 a_2}) + c_p^{\pm}(t)\,\mathrm{e}^{\mathrm{i}\frac{2\pi m}{a_1}x_1}(1-\mathrm{e}^{-\mathrm{i}\tau_2 a_2})\Big|^2 dx_1\\
&= -\frac{8\pi^2}{a_2^3\langle V\rangle}\Big(\big(c_n^{\pm}(t)\big)^2\cos^2\tau_2 a_2 + \big(c_p^{\pm}(t)\big)^2\sin^2\tau_2 a_2\Big).
\end{aligned}
\tag{4.20}
$$

The justification of asymptotics (4.2) can be done in the same way is in Subsection 3.2: we need to construct sufficiently many terms in the expansion to get an error of order $\mathcal{O}(\varepsilon^{2\alpha})$ and then we can apply the Vishik-Lyusternik's lemma. Finally, this allows us to conclude that the asymptotic expansions of the eigenvalues $E_\varepsilon^\pm(t,\tau_2)$ are

$$E_\varepsilon^\pm(t,\tau_2)=E_0+\varepsilon^\alpha\lambda_\alpha^\pm(t)+\varepsilon^{\frac{1}{2}}\lambda_{\frac{1}{2}}^\pm(t,\tau_2)+O(\varepsilon^{2\alpha}) \tag{4.21}$$

uniformly in t and τ_2. In order to complete the proof of Theorem 2.1, it is sufficient to calculate the extrema of the leading terms in the above asymptotics and compare them mutually.

Let us inspect the behavior of the said eigenvalues with respect to t and τ_2. By straightforward calculations one can check that the extrema of the functions $\lambda_\alpha^\pm$ are given by the formulæ

$$\begin{gathered}\min_{\mathbb{R}}\lambda_\alpha^+(t)=\lambda_\alpha^+(t_+),\qquad \max_{\mathbb{R}}\lambda_\alpha^-(t)=\lambda_\alpha^-(t_-),\qquad \lambda_\alpha^\pm(t_\pm)=\beta_\pm(\tau_0),\\ t_\pm=\mp\frac{k_1(\tau_0)\big|M_{12}^{(0)}(\tau_0)\big|}{\big|k_3(\tau_0)\big|\sqrt{k_3^2(\tau_0)-k_1^2(\tau_0)}}-\frac{k_4(\tau_0)}{k_3(\tau_0)},\end{gathered}$$

where $\beta_\pm$ are the functions from (2.9). It follows immediately from formula (4.20) that for each t, the extrema of $\lambda_{\frac{1}{2}}^\pm(t,\tau_2)$ are attained at the points $\tau_2=-\frac{\pi}{a_2}$, $\tau_2=0$, $\tau_2=\frac{\pi}{a_2}$ if $|c_n^\pm(t)|>|c_p^\pm(t)|$ and they are attained at the points $\tau_2=\pm\frac{\pi}{2a_2}$ if $|c_n^\pm(t)|<|c_p^\pm(t)|$. If $|c_n^\pm(t)|=|c_p^\pm(t)|$, the function $\lambda_{\frac{1}{2}}^\pm(t,\tau_2)$ is independent of τ_2. Finally, we have

$$\begin{aligned}\min_{\tau_2}\lambda_{\frac{1}{2}}^+(t,\tau_2)&=-\frac{8\pi^2}{a_2^3\langle V\rangle}\max\left\{\big(c_n^+(t)\big)^2,\big(c_p^+(t)\big)^2\right\},\\ \max_{\tau_2}\lambda_{\frac{1}{2}}^-(t,\tau_2)&=-\frac{8\pi^2}{a_2^3\langle V\rangle}\min\left\{\big(c_n^-(t)\big)^2,\big(c_p^-(t)\big)^2\right\}.\end{aligned}$$

Comparing now the minimum of E_ε^+ and the maximum of E_ε^-, we see that under the assumptions of Theorem 2.1 there is a gap $\big(\eta_l(\varepsilon),\eta_r(\varepsilon)\big)$ in the spectrum of the operator $\mathcal{H}_\varepsilon$ with the properties described in the statement of this theorem.

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