On the well-posedness of the magnetic, semi-relativistic Schrödinger-Poisson system

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Abstract. We prove global existence and uniqueness of strong solutions for the Schrödinger-Poisson system in the repulsive Coulomb case with relativistic, magnetic kinetic energy.

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1 Introduction

In the present work, we show the global well-posedness of the semi-relativistic, magnetic Schrödinger-Poisson system on a finite domain. Such system is relevant to the description of many-body semi-relativistic quantum particles in the mean-field limit (for example, in heated plasma), when the particles move with extremely high velocities and interact with an external magnetic field. Consider such semi-relativistic quantum particles localized in domain $\Omega \subset \mathbb{R}^3$ which is an open, bounded set with the Lebesgue measure $|\Omega| < \infty$ and a C^2 boundary. The particles interact by the electrostatic field they collectively generate. In the mean-field limit, the density matrix $\rho(t)$ that describes the *mixed* state of the system satisfies the Hartree-von Neumann equation

$$\begin{cases} i\partial_t \rho(t) = [H_{A,V}, \rho(t)], \quad x \in \Omega, \quad t \geq 0 \\ -\Delta V = n(t,x), \quad n(t,x) = \rho(t,x,x), \ \rho(0) = \rho_0 \end{cases}, \tag{1.1}$$

satisfying Dirichlet boundary conditions, $\rho(t,x,y) = 0$ if x or $y \in \partial\Omega$, for $t \geq 0$. The Hamiltonian is given by

$$H_{A,V} := T_{A,m} + V(t,x) \tag{1.2}$$

with the magnetic, relativistic kinetic energy operator

$$T_{A,m} := \sqrt{(-i\nabla + A)^2 + m^2} - m \tag{1.3}$$

defined by means of the spectral calculus. Analogously to [6], we assume that $A(x) \in \mathbb{C}^1(\bar{\Omega}, \mathbb{R}^3)$ and $divA = 0$. In the present work $(-i\nabla + A)^2$ stands for the magnetic Dirichlet Laplacian on $L^2(\Omega)$, and $m > 0$ is the particle mass; see [4, 5] for a derivation of such system of equations in the *non-relativistic, non-magnetic* case. Due to the fact that $\rho(t)$ is a nonnegative, self-adjoint trace-class operator acting on $L^2(\Omega)$, its kernel can, for every $t \in \mathbb{R}_+$, be decomposed with respect to an orthonormal basis of $L^2(\Omega)$. The kernel of the initial data ρ_0 can be written in the form

$$\rho_0(x,y) = \sum_{k\in\mathbb{N}} \lambda_k \psi_{0,k}(x)\overline{\psi_{0,k}(y)}. \tag{1.4}$$

Here $\{\psi_{0,k}\}_{k\in\mathbb{N}}$ stands for an orthonormal basis of $L^2(\Omega)$, with $\psi_k|_{\partial\Omega} = 0$ for all $k \in \mathbb{N}$, and coefficients

$$\underline{\lambda} := \{\lambda_k\}_{k\in\mathbb{N}} \in \ell^1\ ,\ \ \lambda_k \geq 0\ ,\ \sum_{k\in\mathbb{N}} \lambda_k = 1. \tag{1.5}$$

As proven below, there exists a one-parameter family of complete orthonormal bases of $L^2(\Omega)$, $\{\psi_k(t)\}_{k\in\mathbb{N}}$, with $\psi_k(t)|_{\partial\Omega} = 0$ for all $k \in \mathbb{N}$, and for $t \in \mathbb{R}_+$, such that the kernel of the solution $\rho(t)$ to (1.1) can be written as

$$\rho(t,x,y) = \sum_{k\in\mathbb{N}} \lambda_k \psi_k(t,x)\overline{\psi_k(t,y)}. \tag{1.6}$$

Notably, the coefficients $\underline{\lambda}$ are *independent* of t, and thus the same as those in ρ_0, which is because the operators $-iH_{A,V}$ and $\rho(t)$ form a Lax pair in problem (1.1). When substituting (1.6) in (1.1), the one- parameter family of orthonormal vectors $\{\psi_k(t)\}_{k\in\mathbb{N}}$ is seen to satisfy the semi-relativistic, magnetic Schrödinger-Poisson system

$$i\frac{\partial \psi_k}{\partial t} = T_{A,m}\psi_k + V[\Psi]\psi_k, \quad k \in \mathbb{N}, \tag{1.7}$$

$$-\Delta V[\Psi] = n[\Psi], \quad \Psi := \{\psi_k\}_{k=1}^{\infty}, \tag{1.8}$$

$$n[\Psi](t,x) = \sum_{k=1}^{\infty} \lambda_k |\psi_k(t,x)|^2, \tag{1.9}$$

with initial data

$$\psi_k(t=0,x)=\psi_{0,k}(x),\ \ k\in\mathbb{N}. \tag{1.10}$$

Our potential function $V[\Psi]$ is a solution of the Poisson equation (1.8). On both $V[\Psi]$ and $\psi_k(t)$, for all $k\in\mathbb{N}$, we impose Dirichlet boundary conditions

$$\psi_k(t,x)\ ,\ \ V(x,t)=0,\ \ t\geq 0,\ \ \forall x\in\partial\Omega. \tag{1.11}$$

As we prove in Lemma 6 further down, solutions of (1.7)-(1.9) preserve the orthonormality of $\{\psi_k(t)\}_{k\in\mathbb{N}}$. Let us introduce the magnetic Sobolev norms for functions:

$$\|f\|^2_{H_A^{\frac{1}{2}}(\Omega)}:=\|f\|^2_{L^2(\Omega)}+\||-i\nabla+A|^{\frac{1}{2}}f\|^2_{L^2(\Omega)}. \tag{1.12}$$

$$\|f\|^2_{H_A^1(\Omega)}:=\|f\|^2_{L^2(\Omega)}+\|(-i\nabla+A)f\|^2_{L^2(\Omega)}. \tag{1.13}$$

Here, $|-i\nabla+A|$ denotes the operator $\sqrt{(-i\nabla+A)^2}$, and has the meaning of the relativistic kinetic energy of a particle with zero mass in the presence of a magnetic field. The standard Sobolev norms $\|f\|^2_{H^{\frac{1}{2}}(\Omega)}$ and $\|f\|^2_{H^1(\Omega)}$ will be used when the magnetic vector potential $A(x)$ vanishes. The state space for our magnetic, semi-relativistic Schrödinger-Poisson system is defined as

$$\mathcal{L}:=\{(\Psi,\underline{\lambda})\mid\Psi=\{\psi_k\}_{k=1}^{\infty}\subset H^{\frac{1}{2}}_{0,A}(\Omega)\cap H^1_A(\Omega)\ \ \textit{is a complete orthonormal system in } L^2(\Omega),$$

$$\underline{\lambda}=\{\lambda_k\}_{k=1}^{\infty}\in\ell^1,\quad \lambda_k\geq 0,\ \ k\in\mathbb{N},\quad \sum_{k=1}^{\infty}\lambda_k\int_{\Omega}|(-i\nabla+A)\psi_k|^2dx<\infty\}.$$

For fixed $\underline{\lambda}\in\ell^1$, $\lambda_k\geq 0$, and for sequences of square integrable functions $\Phi:=\{\phi_k\}_{k=1}^{\infty}$ and $\Psi:=\{\psi_k\}_{k=1}^{\infty}$, we define the inner product

$$(\Phi,\Psi)_{X_\Omega}:=\sum_{k=1}^{\infty}\lambda_k(\phi_k,\psi_k)_{L^2(\Omega)},$$

which induces the norm

$$\|\Phi\|_{X_\Omega}:=(\sum_{k=1}^{\infty}\lambda_k\|\phi_k\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Let us introduce the corresponding Hilbert space

$$X_\Omega:=\{\Phi=\{\phi_k\}_{k=1}^{\infty}\mid\phi_k\in L^2(\Omega),\ \forall\ k\in\mathbb{N},\ \|\Phi\|_{X_\Omega}<\infty\}.$$

Our main result is as follows.

Theorem 1. *Let* $A(x)\in\mathbb{C}^1(\bar{\Omega},\mathbb{R}^3)$ *and* $divA=0$. *For every initial state* $(\Psi(x,0),\underline{\lambda})\in\mathcal{L}$, *there is a unique mild solution* $\Psi(x,t)$, $t\in[0,\infty)$, *of (1.7)-(1.10) with* $(\Psi(x,t),\underline{\lambda})\in\mathcal{L}$. *This is also a unique strong global solution in* X_Ω, *i.e.* $\Psi\in\mathbb{C}([0,\infty);Z_{\Omega,A})\cap\mathbb{C}^1([0,\infty);X_\Omega)$.

Proving the global well-posedness of the Schrödinger-Poisson system plays a crucial role in establishing the existence and nonlinear stability of stationary states, i.e. the nonlinear bound states of the Schrödinger-Poisson system, which was done in the non-relativistic, non-magnetic case in [8, 12], in the magnetic, non-relativistic case in [6], in the semi-relativistic case without a magnetic field in [1] and [2]. The global well posedness of the non-relativistic, magnetic Schrödinger-Poisson system in the whole $\mathbb{R}^3$ was established in [7] on the assumption that the vector potential $A(x)$ is smooth and bounded. The problem in one dimension was studied in [15]. The semiclassical limit of such system with the relativistic kinetic energy was treated in [3]. The global well-posedness for a single semi-relativistic Hartree problem in $\mathbb{R}^3$ was proved in [9]. In the present article, we study the infinite system of equations in a bounded set with Dirichlet boundary conditions, and, as distinct from [9], we do not use the regularization of the Poisson equation. Furthermore, both the results of [9] and Theorem 1 above do not rely on Strichartz type estimates. Note that operator (1.3) is crucial for the studies of the relativistic stability of matter in the presence of a magnetic field (see e.g. [11]).

2 Proof of global well-posedness

We make a fixed choice of $\underline{\lambda} = \{\lambda_k\}_{k=1}^{\infty} \in \ell^1$, with $\lambda_k \geq 0$ and $\sum_{k\in\mathbb{N}} \lambda_k = 1$, standing for the sequence of coefficients determined by the initial data ρ_0 of the Hartree-von Neumann equation (1.1) via (1.6), for $t = 0$.

Let us introduce the inner products $(\cdot,\cdot)_{Y_{\Omega,A}}$ and $(\cdot,\cdot)_{Z_{\Omega,A}}$ which induce the generalized inhomogenous magnetic Sobolev norms

$$\|\Phi\|_{Y_{\Omega,A}} := \Big(\sum_{k=1}^{\infty} \lambda_k \|\phi_k\|^2_{H_A^{\frac{1}{2}}(\Omega)}\Big)^{\frac{1}{2}} \ \text{ and } \ \|\Phi\|_{Z_{\Omega,A}} := \Big(\sum_{k=1}^{\infty} \lambda_k \|\phi_k\|^2_{H_A^1(\Omega)}\Big)^{\frac{1}{2}},$$

and define the corresponding Hilbert spaces

$$Y_{\Omega,A} := \{\Phi = \{\phi_k\}_{k=1}^{\infty} \mid \phi_k \in H^{\frac{1}{2}}_{0,A}(\Omega), \ \forall\, k \in \mathbb{N}, \ \|\Phi\|_{Y_{\Omega,A}} < \infty\}$$

and

$$Z_{\Omega,A} := \{\Phi = \{\phi_k\}_{k=1}^{\infty} \mid \phi_k \in H^{\frac{1}{2}}_{0,A}(\Omega) \cap H^1_A(\Omega), \ \forall\, k \in \mathbb{N}, \ \|\Phi\|_{Z_{\Omega,A}} < \infty\}$$

respectively. Let us also introduce the generalized homogenous Sobolev norms

$$\|\Phi\|_{\dot{Y}_{\Omega,A}} := \Big(\sum_{k=1}^{\infty} \lambda_k \||-i\nabla + A|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \ \textit{and} \ \|\Phi\|_{\dot{Z}_{\Omega,A}} := \Big(\sum_{k=1}^{\infty} \lambda_k \|(-i\nabla + A)\phi_k\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}.$$

The notations $\|\Phi\|_{\dot{Y}_{\Omega}}, \|\Phi\|_{Y_{\Omega}}, \|\Phi\|_{\dot{Z}_{\Omega}}, \|\Phi\|_{Z_{\Omega}}$ will be used in the article when the magnetic vector potential $A(x)$ vanishes in Ω, analogously to Section 3 of [8]. Let us note the following equivalence of magnetic and non magnetic norms.

Lemma 2. *Assume that the vector potential $A(x) \in \mathbb{C}^1(\bar{\Omega}, \mathbb{R}^3)$ and the Coulomb gauge is chosen, such that $divA = 0$.*

a) Let $\Phi(x) \in Y_{\Omega,A}$. Then the norms $\|\Phi\|_{Y_{\Omega,A}}, \|\Phi\|_{\dot{Y}_{\Omega,A}}, \|\Phi\|_{Y_\Omega}$ and $\|\Phi\|_{\dot{Y}_\Omega}$ are equivalent.

b) Let $\Phi(x) \in Z_{\Omega,A}$. Then the norms $\|\Phi\|_{Z_{\Omega,A}}, \|\Phi\|_{\dot{Z}_{\Omega,A}}, \|\Phi\|_{Z_\Omega}$ and $\|\Phi\|_{\dot{Z}_\Omega}$ are equivalent.

Proof. Let us note that the statement b) of the lemma is the result of the part c) of Lemma A.2 of [6]. By means of the part a) of Lemma A.2 of [6], we have

$$C_1\|\nabla f\|_{L^2(\Omega)} \le \|(-i\nabla + A)f\|_{L^2(\Omega)} \le C_2\|\nabla f\|_{L^2(\Omega)}$$

for $f(x) \in H^1_{0,A}(\Omega)$, where $C_{1,2} > 0$ are constants. Hence, for the quadratic forms

$$C_1^2(-\Delta f, f)_{L^2(\Omega)} \le ((-i\nabla + A)^2 f, f)_{L^2(\Omega)} \le C_2^2(-\Delta f, f)_{L^2(\Omega)}.$$

This implies that for the square roots of these operators we have

$$C_1(|p|f, f)_{L^2(\Omega)} \le (|-i\nabla + A|f, f)_{L^2(\Omega)} \le C_2(|p|f, f)_{L^2(\Omega)},$$

where $|p|$ denotes the operator $\sqrt{-\Delta}$. Therefore, for the components of $\Phi = \{\phi_k\}_{k=1}^\infty$ we obtain

$$C_1\||p|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)} \le \||-i\nabla + A|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)} \le C_2\||p|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)}, \quad k \in \mathbb{N}. \tag{2.1}$$

Without loss of generality, we assume that in inequality (2.1) we have $C_1 \le 1$ and $C_2 \ge 1$. Then

$$C_1[\||p|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)} + \|\phi_k\|^2_{L^2(\Omega)}] \le \||-i\nabla + A|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)} + C_1\|\phi_k\|^2_{L^2(\Omega)}$$

yields

$$C_1\|\phi_k\|^2_{H^{\frac{1}{2}}(\Omega)} \le \|\phi_k\|^2_{H_A^{\frac{1}{2}}(\Omega)}, \quad k \in \mathbb{N}. \tag{2.2}$$

Similarly, (2.1) gives us

$$\||-i\nabla + A|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)} + \|\phi_k\|^2_{L^2(\Omega)} \le C_2\||p|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)} + \|\phi_k\|^2_{L^2(\Omega)}$$

and therefore

$$\|\phi_k\|^2_{H_A^{\frac{1}{2}}(\Omega)} \le C_2\|\phi_k\|^2_{H^{\frac{1}{2}}(\Omega)}, \quad k \in \mathbb{N}. \tag{2.3}$$

Let us multiply both sides of (2.2) and (2.3) by λ_k and sum over $k \in \mathbb{N}$. Thus, we arrive at

$$\sqrt{C_1}\|\Phi\|_{Y_\Omega} \le \|\Phi\|_{Y_{\Omega,A}} \le \sqrt{C_2}\|\Phi\|_{Y_\Omega},$$

such that the norms $\|.\|_{Y_{\Omega,A}}$ and $\|.\|_{Y_\Omega}$ are equivalent. The equivalence of $\|.\|_{Y_\Omega}$ and $\|.\|_{\dot{Y}_\Omega}$ norms was established in Lemma 2 of [1]. We multiply all sides of inequality (2.1) by λ_k and add up over $k \in \mathbb{N}$. This yields

$$\sqrt{C_1}\|\Phi\|_{\dot{Y}_\Omega} \le \|\Phi\|_{\dot{Y}_{\Omega,A}} \le \sqrt{C_2}\|\Phi\|_{\dot{Y}_\Omega},$$

such that the norms $\|.\|_{\dot{Y}_{\Omega,A}}$ and $\|.\|_{\dot{Y}_\Omega}$ are equivalent as well. $\square$

Let $\Psi = \{\psi_k\}_{k=1}^{\infty}$ be a wave function and the magnetic, relativistic kinetic energy operator acts on it $T_{A,m}\Psi = (\sqrt{(-i\nabla + A)^2 + m^2} - m)\Psi$ componentwise. We have the following two technical statements.

Lemma 3. *The domain of the magnetic, semi-relativistic kinetic energy operator is given by $D(T_{A,m}) = Z_{\Omega,A} \subseteq X_\Omega$.*

Proof. Let $\Psi \in Z_{\Omega,A}$. Hence

$$\|\Psi\|_{Z_{\Omega,A}} = (\sum_{k=1}^{\infty} \lambda_k [\|(-i\nabla + A)\psi_k\|^2_{L^2(\Omega)} + \|\psi_k\|^2_{L^2(\Omega)}])^{\frac{1}{2}} \geq (\sum_{k=1}^{\infty} \lambda_k \|\psi_k\|^2_{L^2(\Omega)})^{\frac{1}{2}} = \|\Psi\|_{X_\Omega},$$

such that $\Psi \in X_\Omega$. Let us estimate $\|T_{A,m}\psi_k\|^2_{L^2(\Omega)}$ as

$$([(-i\nabla + A)^2 + m^2]\psi_k, \psi_k)_{L^2(\Omega)} + m^2\|\psi_k\|^2_{L^2(\Omega)} - 2m(\sqrt{(-i\nabla + A)^2 + m^2}\psi_k, \psi_k)_{L^2(\Omega)}$$

$$\leq \|(-i\nabla + A)\psi_k\|^2_{L^2(\Omega)} + 2m^2\|\psi_k\|^2_{L^2(\Omega)} \leq c(m)\|\psi_k\|^2_{H^1_A(\Omega)},$$

where $c(m)$ is a mass dependent constant. Thus

$$\|T_{A,m}\Psi\|^2_{X_\Omega} = \sum_{k=1}^{\infty} \lambda_k \|T_{A,m}\psi_k\|^2_{L^2(\Omega)} \leq c(m)\|\Psi\|^2_{Z_{\Omega,A}} < \infty.$$

$\square$

Lemma 4. *The operator $T_{A,m}$ generates the group $e^{-iT_{A,m}t}$, $t \in \mathbb{R}$, of unitary operators on X_Ω.*

Let us rewrite our magnetic, semi-relativistic Schrödinger-Poisson system for $x \in \Omega$ into the form

$$\Psi_t = -iT_{A,m}\Psi + F[\Psi(x,t)],\ \textit{where}\ F[\Psi] := i^{-1}V[\Psi]\Psi, \tag{2.4}$$

$$-\Delta V[\Psi] = n[\Psi],\ \textit{where}\ V|_{\partial\Omega} = 0,$$

$$n[\Psi] = \sum_{k=1}^{\infty} \lambda_k |\psi_k|^2$$

and establish the following technical result.

Lemma 5. *The map defined in (2.4) $F : Z_{\Omega,A} \to Z_{\Omega,A}$ is locally Lipschitz continuous.*

Proof. Let $\Psi,\ \Phi \in Z_{\Omega,A}$ with $\Psi = \{\psi_k\}_{k=1}^{\infty},\ \Phi = \{\phi_k\}_{k=1}^{\infty}$ and $t \in [0,T)$. Then, according to the result of Lemma 5 of [1],

$$\|F[\Psi] - F[\Phi]\|_{Z_\Omega} \leq C(\|\Psi\|_{Z_\Omega}^2 + \|\Phi\|_{Z_\Omega}^2)\|\Psi - \Phi\|_{Z_\Omega}.$$

The result of the lemma follows from the equivalence of magnetic and non magnetic norms established in Lemma 2 above. $\square$

From standard arguments (see for instance Theorem 1.7 of [13]) and Lemma 5 it follows that our magnetic, semi-relativistic Schrödinger-Poisson system admits a unique mild solution Ψ belonging to $Z_{\Omega,A}$ on a time interval $[0,T)$, for a certain $T > 0$, satisfying the integral equation

$$\Psi(t) = e^{-iT_{A,m}t}\Psi(0) + \int_0^t e^{-iT_{A,m}(t-s)}F[\Psi(s)]ds \tag{2.5}$$

in $Z_{\Omega,A}$. Furthermore,

$$\lim_{t\nearrow T}\|\Psi(t)\|_{Z_{\Omega,A}} = \infty$$

in the case if T is finite. Let us also note that Ψ is a unique strong solution in X_Ω, such that $\Psi \in \mathbb{C}([0,T); Z_{\Omega,A}) \cap \mathbb{C}^1([0,T); X_\Omega)$. Below we will establish that this solution is in fact global in time. First let us prove the following proposition.

Lemma 6. *Suppose for the unique mild solution* (2.5) *of the magnetic, semi-relativistic Schrödinger-Poisson system (1.7)-(1.10) at $t = 0$ the functions $\{\psi_k(x,0)\}_{k=1}^{\infty}$ form a complete orthonormal system in $L^2(\Omega)$. Then, for any $t \in [0,T)$, the set $\{\psi_k(x,t)\}_{k=1}^{\infty}$ remains a complete orthonormal system in $L^2(\Omega)$. Moreover, the X_Ω-norm is preserved: $\|\Psi(x,t)\|_{X_\Omega} = \|\Psi(x,0)\|_{X_\Omega},\ t \in [0,T)$.*

Proof. By means of (1.7), we obtain

$$\frac{d}{dt}(\psi_k,\psi_l)_{L^2(\Omega)} = -i((T_{A,m} + V_\Psi)\psi_k,\psi_l)_{L^2(\Omega)} + i(\psi_k,(T_{A,m} + V_\Psi)\psi_l)_{L^2(\Omega)} = 0.$$

This gives us

$$(\psi_k(x,t),\psi_l(x,t))_{L^2(\Omega)} = (\psi_k(x,0),\psi_l(x,0))_{L^2(\Omega)} = \delta_{k,l}, \quad k,l \in \mathbb{N}.$$

Here $\delta_{k,l}$ denotes the Kronecker symbol. Thus, for $k \in \mathbb{N}$

$$\|\psi_k(.,t)\|_{L^2(\Omega)}^2 = \|\psi_k(.,0)\|_{L^2(\Omega)}^2.$$

Therefore, for $t \in [0,T)$, we have the preservation of the X_Ω-norm, namely

$$\|\Psi(.,t)\|_{X_\Omega} = \left(\sum_{k=1}^{\infty} \lambda_k \|\psi_k(x,t)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{\infty} \lambda_k \|\psi_k(x,0)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} = \|\Psi(.,0)\|_{X_\Omega}.$$

For the unique given solution $\Psi(t)$ of our magnetic, semi-relativistic Schrödinger-Poisson system on $[0,T)$, we arrive at the time-dependent one-particle Hamiltonian

$$H_{A,V_\Psi}(t) = T_{A,m} + V_\Psi(t,x),$$

where the potential V_Ψ solves $-\Delta V_\Psi(t,x) = n[\Psi(t)]$ with Dirichlet boundary conditions, see (1.2). The properties of V_Ψ are discussed in more detail in Lemma 8 below. Accordingly, the components of $\Psi(t)$ satisfy the *non-autonomous* magnetic, semi-relativistic Schrödinger equation $i\partial_t \psi_k(t,x) = H_{A,V_\Psi}(t)\psi_k(t,x)$, for $k \in \mathbb{N}$, on the time interval $[0,T)$. Therefore, by means of Theorem X.71 of [14], there exists a propagator, denoted as $e^{-i\int_0^t H_{A,V_\Psi}(\tau)d\tau}$ such that for $t \in [0,T)$,

$$\psi_k(x,t) = e^{-i\int_0^t H_{A,V_\Psi}(\tau)d\tau}\psi_k(x,0), \;\; k \in \mathbb{N}. \tag{2.6}$$

Consider an arbitrary function $f(x) \in L^2(\Omega)$. Obviously, we have the expansion

$$f(x) = \sum_{k=1}^{\infty} (f(y), \psi_k(y,0))_{L^2(\Omega)} \psi_k(x,0)$$

and analogously

$$e^{i\int_0^t H_{A,V_\Psi}(\tau)d\tau} f(x) = \sum_{k=1}^{\infty} (e^{i\int_0^t H_{A,V_\Psi}(\tau)d\tau} f(y), \psi_k(y,0))_{L^2(\Omega)} \psi_k(x,0).$$

Therefore, by virtue of (2.6) we derive the expansion

$$f(x) = \sum_{k=1}^{\infty} (f(y), \psi_k(y,t))_{L^2(\Omega)} \psi_k(x,t)$$

for $t \in [0,T)$. $\square$

Moreover, we establish the conservation of energy for the solutions to the magnetic, semi-relativistic Schrödinger-Poisson system in the following sense.

Lemma 7. *For the unique mild solution* (2.5) *of the Schrödinger-Poisson system (1.7)-(1.10) and for any value of time* $t \in [0,T)$ *we have the identity*

$$\sum_{k\in\mathbb{N}} \lambda_k \|T^{\frac{1}{2}}_{A,m}\psi_k(x,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla V[\Psi(x,t)]\|^2_{L^2(\Omega)}$$

$$= \sum_{k\in\mathbb{N}} \lambda_k \|T^{\frac{1}{2}}_{A,m}\psi_k(x,0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla V[\Psi(x,0)]\|^2_{L^2(\Omega)}.$$

Proof. The complex conjugation of the Schrödinger-Poisson system (1.7) gives us

$$-i\frac{\partial\bar{\psi}_k}{\partial t}=\overline{T_{A,m}\psi_k}+V[\psi]\bar{\psi}_k,\quad k\in\mathbb{N}. \tag{2.7}$$

We add the k-th equation of the original system (1.7) multiplied by $\frac{\partial\bar{\psi}_k}{\partial t}$, and the k-th equation of (2.7) multiplied by $\frac{\partial\psi_k}{\partial t}$ and arrive at

$$\frac{\partial}{\partial t}\|T_{A,m}^{\frac{1}{2}}\psi_k\|_{L^2(\Omega)}^2+\int_\Omega V[\psi]\frac{\partial}{\partial t}|\psi_k|^2dx=0,\quad k\in\mathbb{N}.$$

Hence, multiplying by λ_k, and summing over k, we obtain

$$\frac{\partial}{\partial t}\sum_{k\in\mathbb{N}}\lambda_k\|T_{A,m}^{\frac{1}{2}}\psi_k(x,t)\|_{L^2(\Omega)}^2+\int_\Omega V[\Psi(x,t)]\frac{\partial}{\partial t}n[\Psi(x,t)]dx=0. \tag{2.8}$$

It can be easily verified that

$$\frac{\partial}{\partial t}\|\nabla V[\Psi(x,t)]\|_{L^2(\Omega)}^2=2\int_\Omega V[\Psi(x,t)]\frac{\partial}{\partial t}n[\Psi(x,t)]dx.$$

By substituting this identity into (2.8), we complete the proof of the lemma. □

With the auxiliary statements proven above at our disposal, we may now prove our main result, Theorem 1.

Proof of Theorem 1. The proof follows from the blow-up alternative and the conservation laws. Let us recall that the mild solution of the Schrödinger-Poisson system (1.7)-(1.10) is given by (2.5). We apply the norm $\|.\|_{\dot{Z}_{\Omega,A}}$ to both sides of (2.5), which yields

$$\|\Psi(t)\|_{\dot{Z}_{\Omega,A}}\le\|e^{-iT_{A,m}t}\Psi(0)\|_{\dot{Z}_{\Omega,A}}+\int_0^t\|e^{-iT_{A,m}(t-s)}F[\Psi(s)]\|_{\dot{Z}_{\Omega,A}}ds.$$

Hence,

$$\|\Psi(t)\|_{\dot{Z}_{\Omega,A}}\le\|\Psi(0)\|_{\dot{Z}_{\Omega,A}}+\int_0^t\|F[\Psi(s)]\|_{\dot{Z}_{\Omega,A}}ds.$$

In the proof of Theorem 1 of [1], it was shown that

$$\|F[\Psi]\|_{\dot{Z}_\Omega}\le C\|\Psi\|_{\dot{Y}_\Omega}^2\|\Psi\|_{\dot{Z}_\Omega}.$$

By virtue of the equivalence of the magnetic and non magnetic norms established in our Lemma 2 above, we obtain

$$\|F[\Psi]\|_{\dot{Z}_{\Omega,A}}\le C\|\Psi\|_{\dot{Y}_{\Omega,A}}^2\|\Psi\|_{\dot{Z}_{\Omega,A}}. \tag{2.9}$$

The energy conservation established in Lemma 7 above gives us the boundedness of the quantity

$$\sum_{k\in\mathbb{N}} \lambda_k \|T_{A,m}^{\frac{1}{2}} \psi_k(x,t)\|_{L^2(\Omega)}^2, \quad t \in [0,T)$$

by the amount of the initial energy of our magnetic, semi-relativistic Schrödinger-Poisson system. Since the $\|.\|_{X_\Omega}$ norm is conserved for our unique mild solution according to Lemma 6, we have the boundedness of the quantity

$$\sum_{k\in\mathbb{N}} \lambda_k (\sqrt{(-i\nabla + A)^2 + m^2}\psi_k(x,t), \psi_k(x,t))_{L^2(\Omega)}$$

for $t \in [0,T)$. Because $\sqrt{(-i\nabla + A)^2 + m^2} \geq |-i\nabla + A|$ in the sense of the quadratic forms, we derive the boundedness of

$$\|\Psi\|_{\dot{Y}_{\Omega,A}}^2 = \sum_{k\in\mathbb{N}} \lambda_k \||-i\nabla + A|^{\frac{1}{2}}\psi_k\|_{L^2(\Omega)}^2, \quad t \in [0,T)$$

for our unique mild solution. Then (2.9) yields

$$\|F[\Psi]\|_{\dot{Z}_{\Omega,A}} \leq C_0 \|\Psi\|_{\dot{Z}_{\Omega,A}},$$

where C_0 is a positive constant obtained by virtue of our conservation laws discussed above. This gives us

$$\|\Psi(t)\|_{\dot{Z}_{\Omega,A}} \leq \|\Psi(0)\|_{\dot{Z}_{\Omega,A}} + C_0 \int_0^t \|\Psi(s)\|_{\dot{Z}_{\Omega,A}} ds.$$

By the Gronwall's lemma,

$$\|\Psi(t)\|_{\dot{Z}_{\Omega,A}} \leq \|\Psi(0)\|_{\dot{Z}_{\Omega,A}} e^{C_0 t}, \quad t \in [0,T).$$

Therefore, by virtue of the blow-up alternative, our magnetic, semi-relativistic Schrödinger-Poisson system is globally well-posed in $Z_{\Omega,A}$. □

We conclude the article with addressing the properties of the scalar potential function involved in our magnetic, semi-relativistic Schrödinger-Poisson system.

Lemma 8. *For* $(\Psi, \underline{\lambda}) \in \mathcal{L}$ *we have*

$$n_{\psi,\lambda} = \sum_{k=1}^{\infty} \lambda_k |\psi_k|^2 \in L^2(\Omega).$$

Let $V_{\psi,\lambda}$ *denote the Coulomb potential induced by* $n_{\psi,\lambda}$*, such that*

$$-\Delta V_{\psi,\lambda}(x) = n_{\psi,\lambda}(x), \quad x \in \Omega; \quad V_{\psi,\lambda}(x) = 0, \quad x \in \partial\Omega.$$

Then $V_{\psi,\lambda}(x) \in H_0^1(\Omega) \cap H^2(\Omega)$.

Proof. Clearly, by virtue of the Schwarz inequality we have the upper bound

$$\|n_{\psi,\lambda}\|^2_{L^2(\Omega)} = \sum_{k,s\in\mathbb{N}} \lambda_k\lambda_s \int_\Omega |\psi_k(x)|^2|\psi_s(x)|^2dx \leq \left(\sum_{k\in\mathbb{N}} \lambda_k \sqrt{\int_\Omega |\psi_k(x)|^4 dx}\right)^2.$$

By means of the Hölder's inequality,

$$\int_\Omega |\psi_k(x)|^4 dx \leq \left(\int_\Omega |\psi_k(x)|^6 dx\right)^{\frac{2}{3}} |\Omega|^{\frac{1}{3}}.$$

Let us use the standard Sobolev inequality (see e.g. p.186 of [10])

$$\|\nabla f\|_{L^2(\Omega)} \geq c_s\|f\|_{L^6(\Omega)},$$

where $c_s > 0$ is a constant. This along with the Diamagnetic inequality

$$\int_\Omega |(-i\nabla + A)f|^2 dx \geq \int_\Omega |\nabla|f||^2 dx$$

(see e.g. p.179 of [10]) gives us

$$\sqrt{\int_\Omega |\psi_k(x)|^4 dx} \leq \frac{|\Omega|^{\frac{1}{6}}}{c_s^2} \int_\Omega |\nabla|\psi_k||^2 dx \leq \frac{|\Omega|^{\frac{1}{6}}}{c_s^2} \|(-i\nabla + A)\psi_k\|^2_{L^2(\Omega)}.$$

Therefore,

$$\|n_{\psi,\lambda}\|^2_{L^2(\Omega)} \leq \frac{|\Omega|^{\frac{1}{3}}}{c_s^4}(\sum_{k\in\mathbb{N}} \lambda_k\|(-i\nabla + A)\psi_k\|^2_{L^2(\Omega)})^2 = \frac{|\Omega|^{\frac{1}{3}}}{c_s^4}\|\Psi\|^4_{\dot{Z}_{\Omega,A}} < \infty,$$

such that $n_{\psi,\lambda}(x) \in L^2(\Omega)$. Then, by means of our Poisson equation we have $\Delta V_{\psi,\lambda} \in L^2(\Omega)$.

Let $\{\mu_k^0\}_{k\in\mathbb{N}}$ denote the set of the Dirichlet eigenvalues for the negative Laplace operator on $L^2(\Omega)$, such that $\mu_k^0 > 0$, $k \in \mathbb{N}$ and μ_1^0 is the lowest eigenvalue. Thus, since $V_{\psi,\lambda} = (-\Delta)^{-1}n_{\psi,\lambda}$, we have the estimate

$$\|V_{\psi,\lambda}\|_{L^2(\Omega)} \leq \frac{1}{\mu_1^0}\|n_{\psi,\lambda}\|_{L^2(\Omega)} < \infty.$$

Furthermore, since $V_{\psi,\lambda}$ vanishes on the Lipschitz boundary of our bounded set Ω as assumed, $V_{\psi,\lambda}$ is a trace zero function in $H^1(\Omega)$. Let us also note that $V_{\psi,\lambda}(x) \in L^\infty(\Omega) \subset H^2(\Omega)$ by virtue of the Sobolev embedding. ∎

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