Decay of Complex-time Determinantal Correlation Functionals in Lattices

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Abstract

We supplement the determinantal bounds of [4] for many-body localization of free fermions, by considering the high dimensional case and complex-time correlations. Our proof uses the analyticity of correlation functions via the Hadamard three-line theorem. We show that the dynamical localization for the one-particle system yields the dynamical localization for the many-point fermionic correlation functions, with respect to the Hausdorff distance. In [4], a stronger notion of decay for many-particle configurations was used but only at dimension one and for real times. Considering determinantal correlation functionals for complex times is important in the study of weakly interacting fermions.

Keywords: disordered fermion system, many-body localization, determinant bound

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1 Introduction

Since a few years, the problem of (Anderson) localization in many-body systems is garnering attention. The mathematical understanding of this phenomenon for *interacting* quantum particles, as adressed in 2006 by [1] for weakly interacting fermions at small densities, is a long-term goal. In 2009, [2, 3] contributed first rigorous results. In 2016, [4] proved an exponential decay of many-particle correlations at any temperature for free fermions in *one-dimensional* lattices with disorder. Via the Jordan-Wigner transformation, this includes the celebrated disordered XY spin chains. This paper has attracted much attention and it has already been cited many times in one and a half year. See, e.g., [5–13].

As pointed out in [4], it is an interesting open question (a) whether the main results [4, Theorems 1.1 and 1.2] can be generalized to higher dimensions. Another open question (b) is their generalization for *complex*-time correlation functions. This last point is relevant because such correlation functions (of free fermions) can be useful to study localization of *weakly interacting* fermion systems on lattices. In fact, (free) complex-time correlation functions appear in the perturbative expansion of (full) correlations for weakly interacting systems. See, for instance, [14, Section 5.4.1].

By considering the many-body localization in the sense of the Hausdorff distance, like in [3], we propose an answer to both questions (a) and (b), using the Hadamard three-line theorem (Section 4). See Corollary 2.3, which, together with Theorem 2.2, is the main result of the current paper.

2 Setup of the Problem and Main Results

(i): Let $d \in \mathbb{N}$. For a fixed parameter $\epsilon \in (0,1]$, we define

$$\mathfrak{d}_\epsilon(\mathcal{X}_1, \mathcal{X}_2) \doteq \max\left\{ \max_{x_1 \in \mathcal{X}_1} \min_{x_2 \in \mathcal{X}_2} |x_1 - x_2|^\epsilon , \max_{x_2 \in \mathcal{X}_2} \min_{x_1 \in \mathcal{X}_1} |x_1 - x_2|^\epsilon \right\}, \qquad \mathcal{X}_1, \mathcal{X}_2 \subset \mathbb{Z}^d, \tag{1}$$

which is the well-known Hausdorff distance between the two sets, associated with the metric $(x_1, x_2) \mapsto |x_1 - x_2|^\epsilon$ on $\mathbb{Z}^d$.

(ii): We consider (non-relativistic) fermions in the lattice $\mathbb{Z}^d$ with arbitrary finite spin set S. Thus, we define the one-particle Hilbert space to be $\mathfrak{h} \doteq \ell^2(\mathbb{Z}^d; \mathbb{C}^{\mathrm{S}})$, the canonical orthonormal basis $\{\mathfrak{e}_{x,\sigma}\}_{(x,\sigma)\in\mathbb{Z}^d\times\mathrm{S}}$ of which is

$$\mathfrak{e}_{x_0,\sigma_0}(x,\sigma) \doteq \delta_{x,x_0}\delta_{\sigma,\sigma_0}, \qquad x, x_0 \in \mathbb{Z}^d,\ \sigma, \sigma_0 \in \mathrm{S}. \tag{2}$$

(iii): Let $(\Omega, \mathfrak{F}, \mathfrak{a})$ be a standard[1] probability space. As is usual, $\mathbb{E}[\,\cdot\,]$ denotes the expectation value associated with the probability measure $\mathfrak{a}$. We consider $\mathfrak{F}$-measurable families $\{H_\omega\}_{\omega\in\Omega} \subset \mathcal{B}(\mathfrak{h})$ of bounded one-particle Hamiltonians satisfying the following (one-body localization) assumption, at fixed $\beta \in \mathbb{R}^+$:

Condition 2.1
There is a Borel set $I \subset \mathbb{R}$ as well as constants $\epsilon \in (0,1]$, D and $\mu \in \mathbb{R}^+$ such that, for all $x_1 \in \mathbb{Z}^d$ and $R > 0$,

$$\sum_{x_2 \in \mathbb{Z}^d : |x_1 - x_2|^\epsilon \geq R} \mathbb{E}\left[\sup_{z \in \mathbb{S}_\beta} \max_{\sigma_1,\sigma_2 \in \mathrm{S}} \left| \left\langle \mathfrak{e}_{x_1,\sigma_1}, \frac{\mathrm{e}^{izH_\omega}\chi_I(H_\omega)}{1+\mathrm{e}^{\beta H_\omega}} \mathfrak{e}_{x_2,\sigma_2} \right\rangle_{\mathfrak{h}} \right| \right] \leq D\,\mathrm{e}^{-\mu R}, \tag{3}$$

where

$$\mathbb{S}_\beta \doteq \mathbb{R} - i\,[0,\beta], \qquad \beta \in \mathbb{R}^+, \tag{4}$$

χ_I is the characteristic function of the set I, and $|x_1 - x_2|$ the euclidean distance between the lattice points $x_1, x_2 \in \mathbb{Z}^d$.

This assumption is similar to the so-called strong exponential dynamical localization in I, see, e.g., [15, Definition 7.1]. Note that, for $\epsilon \in (0,1]$, $(x_1, x_2) \mapsto |x_1 - x_2|^\epsilon$ defines a translation invariant metric on the lattice $\mathbb{Z}^d$. Observe also that, for all $\beta \in \mathbb{R}^+$ and $z \in \mathbb{S}_\beta$, the function $\lambda \mapsto |\mathrm{e}^{z\lambda}(1+\mathrm{e}^{\beta\lambda})^{-1}|$ on $\mathbb{R}$ is bounded by 1. In particular, the left-hand side of (3) is bounded by the eigenfunction correlator [15, Eq. (7.1)]. Condition 2.1 replaces [4, Eq. (1.19)], noting that

$$\rho(s,t) = \frac{\mathrm{e}^{i(t-s)H_\omega}}{1+\mathrm{e}^{\beta H_\omega}}, \qquad s,t \in \mathbb{R}, \tag{5}$$

is the main example they have in mind [4, Eq. (2.37)].

(iv): Let $\mathrm{CAR}(\mathfrak{h})$ be the CAR C^*–algebra generated by the unit $\mathbf{1}$ and $\{a(\varphi)\}_{\varphi\in\mathfrak{h}}$. For any $A_1, A_2 \in \mathrm{CAR}(\mathfrak{h})$ and any $z_1, z_2 \in \mathbb{C}$, we define

$$\mathbb{O}_{z_1,z_2}(A_1, A_2) \doteq \begin{cases} A_1A_2 & \text{if} \quad \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2), \\ -A_2A_1 & \text{if} \quad \mathrm{Im}(z_1) > \mathrm{Im}(z_2). \end{cases}$$

[1] I.e., $\mathfrak{F}$ is the Borel σ-algebra of a Polish space Ω.

(v): For any $\beta \in \mathbb{R}^+$ and $\omega \in \Omega$, we define the quasi-free state $\rho_\omega \equiv \rho_{\beta,\omega}$ by the condition

$$\rho_\omega \left(a(\varphi_1)^* a(\varphi_2) \right) = \left\langle \varphi_2, \frac{1}{1+\mathrm{e}^{\beta H_\omega}} \varphi_1 \right\rangle_{\mathfrak{h}} , \qquad \varphi_1, \varphi_2 \in \mathfrak{h}. \tag{6}$$

This state is the unique KMS state at inverse temperature $\beta \in \mathbb{R}^+$ associated with the unique strongly continuous group $\{\tau_t^{(\omega)}\}_{t\in\mathbb{R}}$ of (Bogoliubov) automorphisms of $\mathrm{CAR}(\mathfrak{h})$ satisfying

$$\tau_t^{(\omega)} \left(a\left(\varphi\right) \right) = a(\mathrm{e}^{itH_\omega}\varphi) \, , \qquad t \in \mathbb{R}, \ \varphi \in \mathfrak{h}. \tag{7}$$

Note that, for all $\varphi \in \mathfrak{h}$, the maps

$$t \mapsto \tau_t^{(\omega)} \left(a\left(\varphi\right) \right) \quad \text{and} \quad t \mapsto \tau_t^{(\omega)} \left(a\left(\varphi\right)^* \right)$$

on $\mathbb{R}$ uniquely extend to entire functions on the whole complex plane $\mathbb{C}$: For any $z \in \mathbb{C}$ and $\varphi \in \mathfrak{h}$,

$$\tau_z^{(\omega)} \left(a\left(\varphi\right)^* \right) \doteq a(\mathrm{e}^{izH_\omega}\varphi)^* \quad \text{and} \quad \tau_z^{(\omega)} \left(a\left(\varphi\right) \right) \doteq a(\mathrm{e}^{i\bar{z}H_\omega}\varphi). \tag{8}$$

Observe additionally that, for any $z_1, z_2 \in \mathbb{C}$ and $\varphi_1, \varphi_2 \in \mathfrak{h}$,

$$\begin{aligned} & \rho_\omega \left(\mathbb{O}_{z_1,z_2} \left(\tau_{z_1}^{(\omega)}(a(\varphi_1)^*), \tau_{z_2}^{(\omega)}(a(\varphi_2)) \right) \right) \\ = & \left\{ \begin{array}{ll} \left\langle \varphi_2, \frac{\mathrm{e}^{i(z_1-z_2)H_\omega}}{1+\mathrm{e}^{\beta H_\omega}} \varphi_1 \right\rangle_{\mathfrak{h}} & \text{if} \quad \mathrm{Im}\left(z_1\right) \leq \mathrm{Im}\left(z_2\right), \\ -\left\langle \varphi_2, \frac{\mathrm{e}^{(\beta+i(z_1-z_2))H_\omega}}{1+\mathrm{e}^{\beta H_\omega}} \varphi_1 \right\rangle_{\mathfrak{h}} & \text{if} \quad \mathrm{Im}\left(z_1\right) > \mathrm{Im}\left(z_2\right). \end{array} \right. \end{aligned} \tag{9}$$

The aim of the current paper is to show that strong one-body localization, in the sense of Condition 2.1, yields the corresponding many-body localization for the quasi-free state ρ_ω, in the sense of the Hausdorff distance, as stated in Corollary 2.3. This is achieved by estimating, in Theorem 4.1, determinants of the form

$$\det \left[G_\omega \left(\left(\varphi_k, z_k\right), \left(\varphi_{N+l}, z_{N+l}\right) \right) \right]_{k,l=1}^N \tag{10}$$

in terms of the entries of one single row or column. In (10), $\beta \in \mathbb{R}^+$, $N \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_{2N} \in \mathfrak{h}$ are normalized vectors, $z_1, \ldots, z_{2N} \in \mathbb{S}_\beta$ and

$$G_\omega \left(\left(\varphi_k, z_k\right), \left(\varphi_{N+l}, z_{N+l}\right) \right) \doteq \rho_\omega \left(\mathbb{O}_{z_k,z_{N+l}} \left(\tau_{z_k}^{(\omega)}(a(\varphi_k)^*), \tau_{z_{N+l}}^{(\omega)}(a(\varphi_{N+l})) \right) \right)$$

is the two-point, complex-time-ordered correlation function associated with the quasi-free state ρ_ω.

Theorem 2.2
Let $\{H_\omega\}_{\omega\in\Omega} \subset \mathcal{B}(\mathfrak{h})$ be a family of bounded Hamiltonians. For all $\omega \in \Omega$, $\beta \in \mathbb{R}^+$, $N \in \mathbb{N}$, norm-one vectors $\varphi_1, \ldots, \varphi_{2N} \in \mathfrak{h}$, and $z_1, \ldots, z_{2N} \in \mathbb{S}_\beta$ (see (4))

$$\begin{aligned} & \left| \det \left[G_\omega \left(\left(\varphi_k, z_k\right), \left(\varphi_{N+l}, z_{N+l}\right) \right) \right]_{k,l=1}^N \right| \\ \leq \ & \min \left\{ \min_{k\in\{1,\ldots,N\}} \sum_{l=1}^N \left| G_\omega \left(\left(\varphi_k, z_k\right), \left(\varphi_{N+l}, z_{N+l}\right) \right) \right| , \min_{l\in\{1,\ldots,N\}} \sum_{k=1}^N \left| G_\omega \left(\left(\varphi_k, z_k\right), \left(\varphi_{N+l}, z_{N+l}\right) \right) \right| \right\} . \end{aligned}$$

Proof. Fix all parameters of the theorem. By expanding the determinant along a fixed row or column, for any $m \in \{1, \ldots, N\}$,

$$\begin{aligned}
&\det \left[G_\omega \left((\varphi_k, z_k), (\varphi_{N+l}, z_{N+l})\right)\right]_{k,l=1}^N \\
&= \sum_{n=1}^{N} (-1)^{m+n} G_\omega \left((\varphi_m, z_m), (\varphi_{N+n}, z_{N+n})\right) \\
&\times \det \left[G_\omega \left((\varphi_k, z_k), (\varphi_{N+l}, z_{N+l})\right)\right]_{\substack{k\in\{1,\ldots,N\}\backslash\{m\} \\ l\in\{1,\ldots,N\}\backslash\{n\}}} \\
&= \sum_{n=1}^{N} (-1)^{m+n} G_\omega \left((\varphi_n, z_n), (\varphi_{N+m}, z_{N+m})\right) \\
&\times \det \left[G_\omega \left((\varphi_k, z_k), (\varphi_{N+l}, z_{N+l})\right)\right]_{\substack{k\in\{1,\ldots,N\}\backslash\{n\} \\ l\in\{1,\ldots,N\}\backslash\{m\}}} .
\end{aligned}$$

Then, the assertion directly follows from Lemma 3.2. ■

Corollary 2.3
If Condition 2.1 holds true then, for all $\beta \in \mathbb{R}^+$, $N \in \mathbb{N}$, $\mathcal{X}_1 = \{x_1, \ldots, x_N\}$, $\mathcal{X}_2 = \{x_{N+1}, \ldots, x_{2N}\} \subset \mathbb{Z}^d$ such that $|\mathcal{X}_1| = |\mathcal{X}_2| = N$, and $z_1, \ldots, z_{2N} \in \mathbb{S}_\beta$,

$$\mathbb{E}\left[\max_{\sigma_1,\ldots,\sigma_{2N}} \left|\det\left[G_\omega\left((\chi_I(H_\omega)\mathfrak{e}_{x_k,\sigma_k}, z_k), (\chi_I(H_\omega)\mathfrak{e}_{x_{N+l},\sigma_{N+l}}, z_{N+l})\right)\right]_{k,l=1}^N\right|\right] \leq D\,\mathrm{e}^{-\mu \mathfrak{d}_\epsilon(\mathcal{X}_1,\mathcal{X}_2)},$$

where $\mathfrak{d}_\epsilon(\mathcal{X}_1, \mathcal{X}_2)$ is the Hausdorff distance (1) between the N-particle configurations $\mathcal{X}_1$ and $\mathcal{X}_2$. Recall that χ_I is the characteristic function of the Borel set I and note that the constants ϵ, D and μ are exactly the same as in Condition 2.1.

Proof. Combine Condition 2.1 and Theorem 2.2 with Equations (8) and (9). ■

A similar estimate can be obtained for Pfaffians of the two-point correlation functions, by the same methods, because they also can be seen, like in the proof of Lemma 3.2, as many-point correlation functions of free fermions. See, e.g., [16, Equations (6.6.9) and (6.6.10)]. We omit the details.

The analogue of [4, Theorem 1.1], i.e., an estimate like Corollary 2.3 for the many-point correlation functions at fixed $\omega \in \Omega$, instead of an estimate for their expectation values, easily follows by replacing Condition 2.1 with a similar bound for a fixed $\omega \in \Omega$. We also omit the details.

The estimate obtained here is a version of [4, Theorem 1.2] which holds at any dimension $d \in \mathbb{N}$ and for any complex times within the strip $\mathbb{S}_\beta$. However, two observations in relation with [4] are important to mention:

- Since, for any $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2 \subset \mathbb{Z}^d$,

  $$\mathfrak{d}_\epsilon(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2) \leq \max\left\{\mathfrak{d}_\epsilon(\mathcal{X}_1, \mathcal{Y}_1), \mathfrak{d}_\epsilon(\mathcal{X}_2, \mathcal{Y}_2)\right\},$$

  we have

  $$\mathfrak{d}_\epsilon(\mathcal{X}, \mathcal{Y}) \leq \mathfrak{d}_\epsilon^{(S)}(\mathcal{X}, \mathcal{Y}) \doteq \min_{\pi\in\mathcal{S}_N} \max_{j\in\{1,\ldots,N\}} \left|x_j - y_{\pi(j)}\right|^\epsilon$$

  for any set $\mathcal{X} = \{x_1, \ldots, x_N\} \subset \mathbb{Z}^d$ and $\mathcal{Y} = \{y_1, \ldots, y_N\} \subset \mathbb{Z}^d$ of $N \in \mathbb{N}$ (different) lattice points. Here, $\mathcal{S}_N$ is the set of all permutations π of N elements. The distance we use, i.e., the Hausdorff distance (1), is therefore weaker than the symmetrized configuration distance $\mathfrak{d}_\epsilon^{(S)}$ [4, Equation (1.13) and remarks below]. Nevertheless, Corollary 2.3 yields the main features of localization. Whether Corollary 2.3 holds true, at any dimension, when $\mathfrak{d}_\epsilon$ is replaced with $\mathfrak{d}_\epsilon^{(S)}$ is an open question. See also discussions of [3, Section 1.3].

- The proofs of [4, Theorems 1.1 and 1.2] use that, for all $N \in \mathbb{N}$, $x_1, \ldots, x_{2N} \in \mathbb{Z}^d$, $\sigma_1, \ldots, \sigma_{2N} \in \mathrm{S}$, and $t_1, \ldots, t_{2N} \in \mathbb{R}$, the $N \times N$ matrix

$$\mathbf{M} \doteq \left[\left\langle \mathfrak{e}_{x_{N+l},\sigma_{N+l}}, \rho\left(t_{N+l}, t_k\right) \mathfrak{e}_{x_k,\sigma_k} \right\rangle_{\mathfrak{h}} \right]_{k,l=1}^{N}$$

(cf. (5)) defines an operator on $\mathbb{C}^N$ of norm at most 1. This is true even for complex times, provided that

$$z_1 = \cdots = z_N \in \mathbb{S}_\beta , \quad z_{N+1} = \cdots = z_{2N} \in \mathbb{S}_\beta , \quad \mathrm{Im}\left(z_N\right) \leq \mathrm{Im}\left(z_{N+1}\right) . \tag{11}$$

However, this is generally not true when $z_1, \ldots, z_{2N} \in \mathbb{S}_\beta$ are different from each other. For this reason, instead of a bound on the norm of $\mathbf{M}$, our proof uses (in an essential way) the analyticity of correlation functions with respect to complex times.

The results of this paper are also reminiscent of [3, Theorem 1.1] where a bound like Corollary 2.3, with the Hausdorff distance but for complex times satisfying (11), can be found for n-particle correlation functions. Note, additionally, that in [3] a particle interaction is included, but *no* particle statistics is taken into account: The n-particle Hilbert space is the full space $\ell^2\left(\mathbb{Z}^{nd}\right)$. By contrast, we consider many-*fermion* systems, which would correspond in [3, Theorem 1.1] to restrict $\ell^2\left(\mathbb{Z}^{nd}\right)$ to its subspace of antisymmetric functions. In this situation, the one-particle localization theory cannot be directly used, even in the free fermion case. Moreover, we do not fix the particle number, by using the grand-canonical setting.

Finally, observe that *free*, complex-time-ordered, many-point correlations appear in the perturbative expansion of *interacting* correlation functions. See, e.g., [14, Section 5.4.1]. Therefore, as a first step towards the proof of localization in fully interacting fermion systems, it is important to establish localization for these correlations, as stated in Corollary 2.3. For instance, by combining Corollary 2.3 with [14, Theorem 5.4.4], one can show that a local, weak interaction cannot destroy the (static) localization of the thermal, many-point correlation functions of free fermions in lattices.

3 Universal Bounds on Determinants from the Hadamard Three-line Theorem

For any permutation π of $n \in \mathbb{N}$ elements with sign $(-1)^\pi$, we define the monomial $\mathbb{O}_\pi(A_1, \ldots, A_n) \in \mathrm{CAR}(\mathfrak{h})$ in $A_1, \ldots, A_n \in \mathrm{CAR}(\mathfrak{h})$ by the product

$$\mathbb{O}_\pi\left(A_1, \ldots, A_n\right) \doteq (-1)^\pi A_{\pi^{-1}(1)} \cdots A_{\pi^{-1}(n)} . \tag{12}$$

In other words, $\mathbb{O}_\pi$ places the operator A_k at the $\pi(k)$th position in the monomial $(-1)^\pi A_{\pi^{-1}(1)} \cdots A_{\pi^{-1}(n)}$. Further, for all $k, l \in \{1, \ldots, n\}$, $k \neq l$,

$$\pi_{k,l} : \{1, 2\} \to \{1, 2\} \tag{13}$$

is the identity function if $\pi(k) < \pi(l)$, otherwise $\pi_{k,l}$ interchanges 1 and 2. Then, the following identities holds true for quasi–free states:

Lemma 3.1
Let ρ be a quasi–free state on $\mathrm{CAR}(\mathfrak{h})$. For any $N \in \mathbb{N}$, all permutations π of $2N$ elements and $\varphi_1, \ldots, \varphi_{2N} \in \mathfrak{h}$,

$$\begin{aligned} &\det\left[\rho\Big(\mathbb{O}_{\pi_{k,N+l}}\left(a(\varphi_k)^*, a(\varphi_{N+l})\right)\Big)\right]_{k,l=1}^{N} \\ &= \rho\Big(\mathbb{O}_\pi\left(a(\varphi_1)^*, \ldots, a(\varphi_N)^*, a(\varphi_{2N}), \ldots, a(\varphi_{N+1})\right)\Big). \end{aligned} \tag{14}$$

Proof. See [17, Lemma 3.1]. ■

Using Lemma 3.1 and the Hadamard three-line theorem (via Corollary 4.2), we obtain a universal bound on determinants of the form (10):

Lemma 3.2
Fix $H = H^* \in \mathcal{B}(\mathfrak{h})$. Let the quasi–free state ρ on $\mathrm{CAR}(\mathfrak{h})$ be the unique KMS state at inverse temperature $\beta \in \mathbb{R}^+$ associated with the unique strongly continuous group $\{\tau_t\}_{t\in\mathbb{R}}$ of automorphisms of $\mathrm{CAR}(\mathfrak{h})$ satisfying (7)-(8) for $H_\omega = H$. Then, for any $N \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_{2N} \in \mathfrak{h}$ and $z_1, \ldots, z_{2N} \in \mathbb{S}_\beta$ (see (4)),

$$\left|\det\left[\rho\Big(\mathbb{O}_{z_k,z_{N+l}}\left(\tau_{z_k}(a(\varphi_k)^*), \tau_{z_{N+l}}(a(\varphi_{N+l}))\right)\Big)\right]_{k,l=1}^N\right| \leq \prod_{k=1}^{2N} \|\varphi_k\|_{\mathfrak{h}} .$$

Proof. Fix all parameters of the lemma and choose any permutation π of $2N$ elements such that, for all $k, l \in \{1, \ldots, N\}$,

$$\mathrm{Im}(z_k) \leq \mathrm{Im}(z_{N+l}) \Leftrightarrow \pi(k) < \pi(N+l) \ . \tag{15}$$

Then, by Lemma 3.1,

$$\begin{aligned}
&\det\left[\rho\Big(\mathbb{O}_{z_k,z_{N+l}}\left(\tau_{z_k}(a(\varphi_k)^*), \tau_{z_{N+l}}(a(\varphi_{N+l}))\right)\Big)\right]_{k,l=1}^N \qquad (16)\\
= \ &\rho\Big(\mathbb{O}_\pi\left(\tau_{z_1}(a(\varphi_1)^*), \ldots, \tau_{z_N}(a(\varphi_N)^*), \tau_{z_{2N}}(a(\varphi_{2N})), \ldots, \tau_{z_{N+1}}(a(\varphi_{N+1}))\right)\Big).
\end{aligned}$$

Define the entire analytic map Υ from $\mathbb{C}^{2N}$ to $\mathbb{C}$ by

$$\begin{aligned}
\Upsilon(\xi_1, \ldots, \xi_{2N}) \doteq \rho\Big(\mathbb{O}_\pi\Big(&\tau_{\xi_1+\cdots+\xi_{2N-\pi(1)+1}}(a(\varphi_1)^*), \ldots, \tau_{\xi_1+\cdots+\xi_{2N-\pi(N)+1}}(a(\varphi_N)^*),\\
&\tau_{\xi_1+\cdots+\xi_{2N-\pi(2N)+1}}(a(\varphi_{2N})), \ldots, \tau_{\xi_1+\cdots+\xi_{2N-\pi(N+1)+1}}(a(\varphi_{N+1}))\Big)\Big). \qquad (17)
\end{aligned}$$

Now, impose additionally that the permutation π of $2N$ elements used in (16)-(17) satisfies, for any $k, l \in \{1, \ldots, N\}$, $k \neq l$, the conditions

$$\mathrm{Im}(z_k) < \mathrm{Im}(z_l) \Leftrightarrow \pi(k) < \pi(l) \, ; \ \mathrm{Im}(z_{2N-k}) < \mathrm{Im}(z_{2N-l}) \Leftrightarrow \pi(2N-k) < \pi(2N-l) \, .$$

Ergo, by (15),

$$\mathrm{Im}(z_{\pi^{-1}(1)}) \leq \cdots \leq \mathrm{Im}(z_{\pi^{-1}(N)}) \leq \mathrm{Im}(z_{\pi^{-1}(2N)}) \leq \cdots \leq \mathrm{Im}(z_{\pi^{-1}(N+1)})$$

and, by (16)-(17), the assertion follows if we can bound the function Υ on the tube $\mathfrak{T}_{2N}$ defined below by (19) for $n = 2N$. Since Υ is uniformally bounded on $\mathfrak{T}_{2N}$, it suffices to bound the function Υ on the boundary

$$\partial\mathfrak{T}_{2N} \doteq \left\{(\xi_1, \ldots, \xi_{2N}) \in \mathbb{C}^{2N} : \forall j \in \{1, \ldots, 2N\}, \ \mathrm{Im}(\xi_j) \in \{-\beta, 0\} \, , \ \sum_{j=1}^{2N} \mathrm{Im}(\xi_j) \in \{-\beta, 0\}\right\},$$

by Corollary 4.2. By the KMS property [14, Section 5.3.1], note that, for all $t_1, \ldots, t_{2N} \in \mathbb{R}$ and $k \in \{1, \ldots, 2N\}$,

$$\Upsilon(t_1, \ldots, t_{k-1}, t_k - i\beta, t_{k+1}, \ldots, t_{2N}) = \Upsilon(t_{k+1}, \ldots, t_{2N}, t_1, \ldots, t_k)$$

while

$$\sup|\Upsilon|\left(\mathbb{R}^{2N}\right) \leq \prod_{k=1}^{2N} \|\varphi_k\|_{\mathfrak{h}} .$$

As a consequence,

$$\sup |\Upsilon| (\mathfrak{T}_{2N}) = \sup |\Upsilon| (\partial \mathfrak{T}_{2N}) \leq \prod_{k=1}^{2N} \|\varphi_k\|_{\mathfrak{h}} \tag{18}$$

and the assertion follows from (16), (17) and (19). ■

Observe that estimates like (18) are related to the generalization of the Hölder inequality to non–commutative L^p–spaces. See, e.g., [18].

4 Appendix: Log convexity of Multivariable Analytic Functions on Tubes

Fix $\beta \in \mathbb{R}^+$. Let

$$\mathfrak{T}_1 \doteq \{\xi \in \mathbb{C} : \mathrm{Im}\{\xi\} \in [-\beta, 0]\} = \mathbb{S}_\beta,$$

(see (4)) and $f : \mathfrak{T}_1 \to \mathbb{C}$ be a bounded continuous function. Define the map $B_f : [-\beta, 0] \to [-\infty, \infty)$ by

$$B_f^{(1)}(s) \doteq \ln \left(\sup_{t\in\mathbb{R}} |f(t + is)| \right).$$

We use the convention $\ln 0 \doteq -\infty$ and $0 \cdot (-\infty) \doteq -\infty$. Then, the Hadamard three-line theorem [19, Theorem 12.3] states:

Theorem 4.1
Let $\beta \in \mathbb{R}^+$ and $f : \mathfrak{T}_1 \to \mathbb{C}$ be a bounded continuous function. If f is holomorphic in the interior of $\mathfrak{T}_1$ then $B_f^{(1)}$ is a convex function.

This theorem has the following generalization to holomorphic functions in several variables: For all $n \in \mathbb{N}$, let $K_n \subset \mathbb{R}^n$ be the simplex

$$K_n \doteq \{(s_1, \ldots, s_n) : s_1, \ldots, s_n \in [-\beta, 0]\,,\ s_1 + \cdots + s_n \geq -\beta\}$$

and define, for all $n \in \mathbb{N}$, the "tube"

$$\mathfrak{T}_n \doteq \{(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n : (\mathrm{Im}\{\xi_1\}, \ldots, \mathrm{Im}\{\xi_n\}) \in K_n\}. \tag{19}$$

Define further the map $B_f^{(n)} : K_n \to [-\infty, \infty)$ by

$$B_f^{(n)}(s_1, \ldots, s_n) \doteq \ln \left(\sup_{(t_1,\ldots,t_n)\in\mathbb{R}^n} |f(t_1 + is_1, \ldots, t_n + is_n)| \right)$$

with $f : \mathfrak{T}_n \to \mathbb{C}$ being a bounded continuous function. Then, we obtain the following corollary:

Corollary 4.2
Let $\beta \in \mathbb{R}^+$, $n \in \mathbb{N}$ and $f : \mathfrak{T}_n \to \mathbb{C}$ be a bounded continuous function. If f is holomorphic in the interior of $\mathfrak{T}_n$ then $B_f^{(n)}$ is a convex function.

Proof. Fix all parameters of the corollary and assume that f is holomorphic in the interior of $\mathfrak{T}_n$. Take $(s_1, \ldots, s_n) \in K_n$ and $(s_1', \ldots, s_n') \in K_n$. For all $(t_1, \ldots, t_n) \in \mathbb{R}^n$, define the function $F_{(t_1,\ldots,t_n)} : \mathfrak{T}_1 \to \mathbb{C}$ by

$$F_{(t_1,\ldots,t_n)}(\xi) \doteq f\left(t_1 + i(s_1(1 + \xi\beta^{-1}) - s_1'\xi\beta^{-1}), \ldots, t_n + i(s_n(1 + \xi\beta^{-1}) - s_n'\xi\beta^{-1})\right).$$

For all $\xi \in \mathfrak{T}_1$, note that

$$\left(t_1 + i(s_1(1+\xi\beta^{-1}) - s_1'\xi\beta^{-1}), \ldots, t_n + i(s_n(1+\xi\beta^{-1}) - s_n'\xi\beta^{-1})\right) \in \mathfrak{T}_n \ ,$$

by convexity of K_n. This function is bounded and continuous on $\mathfrak{T}_1$, and holomorphic in the interior of $\mathfrak{T}_1$. Hence, by Theorem 4.1, for all $\alpha \in [0,1]$,

$$\begin{aligned} \ln\left(\sup_{t\in\mathbb{R}} \left|F_{(t_1,\ldots,t_n)}(t - i\alpha\beta)\right|\right) \ \leq\ & \alpha \ln\left(\sup_{t\in\mathbb{R}} \left|F_{(t_1,\ldots,t_n)}(t - i\beta)\right|\right) \\ & + (1-\alpha)\ln\left(\sup_{t\in\mathbb{R}} \left|F_{(t_1,\ldots,t_n)}(t)\right|\right) \ . \end{aligned} \tag{20}$$

Since $\ln$ is a monotonically increasing, continuous function, for all $\alpha \in [0,1]$,

$$B_f^{(n)}(\alpha s_1' + (1-\alpha)s_1, \ldots, \alpha s_n' + (1-\alpha)s_n) = \sup_{(t_1,\ldots,t_n)\in\mathbb{R}^n} \ln\left(\sup_{t\in\mathbb{R}} \left|F_{(t_1,\ldots,t_n)}(t - i\alpha\beta)\right|\right) \ ,$$

which, by (20), in turn implies that

$$B_f^{(n)}(\alpha s_1' + (1-\alpha)s_1, \ldots, \alpha s_n' + (1-\alpha)s_n) \leq (1-\alpha)\, B_f^{(n)}(s_1, \ldots, s_n) + \alpha B_f^{(n)}(s_1', \ldots, s_n')$$

for all $\alpha \in [0,1]$. ■

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