On the existence in the sense of sequences of stationary solutions for some systems of non-Fredholm integro-differential equations

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Abstract. We prove the existence in the sense of sequences of stationary solutions for some systems of reaction-diffusion type equations in the appropriate H^2 spaces. It is established that under reasonable technical conditions the convergence in L^1 of the integral kernels yields the existence and the convergence in H^2 of the solutions. The nonlocal elliptic problems contain second order differential operators with and without Fredholm property.

1   Introduction

Let us recall that a linear operator L acting from a Banach space E into another Banach space F has the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. As a consequence, the problem $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. Such properties of Fredholm operators are broadly used in different methods of linear and nonlinear analysis.

Elliptic problems in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Lopatinskii conditions are fulfilled (see e.g. [1], [10], [12]). This is the main result of the theory of linear elliptic problems. When domains are not bounded, such conditions may be not sufficient and the Fredholm property may not be satisfied. For example, Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ does not satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^\alpha(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Linear elliptic problems in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions stated above, the limiting operators are invertible (see [13]). In certain simple cases, the limiting operators can be explicitly constructed. For example, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R}$$

with the coefficients of such operator having limits at infinity,

$$a_\pm = \lim_{x\to\pm\infty} a(x), \quad b_\pm = \lim_{x\to\pm\infty} b(x), \quad c_\pm = \lim_{x\to\pm\infty} c(x),$$

the limiting operators are given by:

$$L_\pm u = a_\pm u'' + b_\pm u' + c_\pm u.$$

Since the coefficients are constants, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ does not satisfy the Fredholm property, can be explicitly calculated by virtue of the Fourier transform:

$$\lambda_\pm(\xi) = -a_\pm \xi^2 + b_\pm i\xi + c_\pm, \quad \xi \in \mathbb{R}.$$

Invertibility of limiting operators is equivalent to the condition that the essential spectrum does not contain the origin.

In the case of general elliptic problems, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, such conditions may not be explicitly written.

When the operators fail to satisfy the Fredholm property, the standard solvability conditions may not be applicable and solvability relations are, in general, not known. There are some classes of operators for which solvability conditions are obtained. We illustrate them with the following example. Consider the equation

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where $a > 0$ is a constant. The operator L here coincides with its limiting operators. The homogeneous equation has a nonzero bounded solution. Hence the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can use the Fourier transform and find the solution explicitly. Solvability relations can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exists a solution of this equation in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}} \right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [23]). Here and further down S^d_r denotes the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Thus, though the operator is non Fredholm, solvability relations are formulated analogously. However, such similarity is only formal because the range of the operator is not closed.

In the case of the operator with a potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

the Fourier transform cannot help. Nevertheless, solvability relations in $\mathbb{R}^3$ can be obtained by a rather sophisticated application of the theory of self-adjoint operators (see [15]). As

before, solvability conditions are formulated in terms of orthogonality to solutions of the homogeneous adjoint problem. There are several other examples of linear elliptic operators without Fredholm property for which solvability relations can be derived (see [13]-[23]).

Solvability conditions play an important role in the analysis of nonlinear elliptic problems. When the operators do not satisfy the Fredholm property, in spite of some progress in understanding of linear problems, there exist only few examples where nonlinear non-Fredholm operators are studied (see [5]-[7], [22], [23], [26]). In the present article we deal with the nonlinear system, for which the Fredholm property may not be satisfied:

$$\frac{\partial u_k}{\partial t} = \Delta u_k + a_k u_k + \int_\Omega G_k(x-y) F_k(u_1(y,t), u_2(y,t), ..., u_N(y,t), y) dy, \quad 1 \le k \le N. \quad (1.2)$$

Here all $a_k \ge 0$ and Ω is a domain in $\mathbb{R}^d$, $d = 1, 2, 3$, the more physically interesting dimensions. Problems of that kind appear in cell population dynamics. The space variable x here corresponds to the cell genotype, $u_k(x,t)$ stand for the cell densities for various groups of cells as functions of their genotype and time. The right side of system (1.2) describes the evolution of cell densities via cell proliferation and mutations. Here the diffusion terms correspond to the change of genotype via small random mutations, and the nonlocal terms describe large mutations. In this context, $F_k(u_1, u_2, ..., u_N, x)$ are the rates of cell birth which depend on the vector function

$$u := (u_1, u_2, ..., u_N) \in \mathbb{R}^N \quad (1.3)$$

and x (density dependent proliferation), and the functions $G_k(x-y)$ show the proportion of newly born cells changing their genotype from y to x. Let us assume that they depend on the distance between the genotypes. In population dynamics the integro-differential equations describe models with intra-specific competition and nonlocal consumption of resources (see e.g. [2], [3], [8]). The existence of stationary solutions of (1.2) was studied in [22] via the fixed point technique. Related to problem (1.2), we consider the sequence of iterated systems of equations with $m \in \mathbb{N}$ and $1 \le k \le N$

$$\frac{\partial u_k}{\partial t} = \Delta u_k + a_k u_k + \int_\Omega G_{k,m}(x-y) F_k(u_1(y,t), u_2(y,t), ..., u_N(y,t), y) dy, \quad a_k \ge 0. \quad (1.4)$$

For $1 \le k \le N$, each sequence of kernels $G_{k,m}(x) \to G_k(x)$ as $m \to \infty$ in the appropriate function spaces discussed below. Let us prove that under the appropriate technical conditions each of systems (1.4) admits a unique stationary solution vector-function $u^{(m)}(x) \in H^2$, the limiting system (1.2) will possess a unique stationary solution $u(x) \in H^2$ and $u^{(m)}(x) \to u(x)$ in H^2 as $m \to \infty$, which is a so-called *existence of stationary solutions in the sense of sequences*. The similar ideas in the sense of standard Schrödinger type operators were exploited in [24] and [25]. The non Fredholm operators arise also when studying the so-called embedded solitons (see e.g. [11]).

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2 Formulation of the results

The nonlinear part of systems (1.2) and (1.4) will satisfy the regularity conditions analogous to the ones of [22].

Assumption 1. *Functions* $F_k(u,x) : \mathbb{R}^N \times \Omega \to \mathbb{R}, \quad 1 \le k \le N$ *are such that*

$$\sqrt{\sum_{k=1}^{N} F_k^2(u,x)} \le K|u|_{\mathbb{R}^N} + h(x) \quad for \quad u \in \mathbb{R}^N, \ x \in \Omega, \tag{2.1}$$

with a constant $K > 0$ *and* $h(x) : \Omega \to \mathbb{R}^+, \quad h(x) \in L^2(\Omega)$. *Moreover, they are Lipschitz continuous functions, such that for any* $u^{(1),(2)} \in \mathbb{R}^N, \ x \in \Omega$,

$$\sqrt{\sum_{k=1}^{N}(F_k(u^{(1)},x) - F_k(u^{(2)},x))^2} \le L|u^{(1)} - u^{(2)}|_{\mathbb{R}^N}, \tag{2.2}$$

where a constant $L > 0$.

Here and further down the norm of a vector given by (1.3) is

$$|u|_{\mathbb{R}^N} := \sqrt{\sum_{k=1}^{N} u_k^2}.$$

Evidently, the stationary solutions of (1.2) and (1.4), which exist under the appropriate technical conditions, will satisfy the system of nonlocal elliptic equations

$$\Delta u_k + \int_{\Omega} G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy + a_k u_k = 0, \quad a_k \ge 0, \quad 1 \le k \le N. \tag{2.3}$$

and for $m \in \mathbb{N}, \ 1 \le k \le N$

$$\Delta u_k^{(m)} + \int_{\Omega} G_{k,m}(x-y)F_k(u_1^{(m)}(y), u_2^{(m)}(y), ..., u_N^{(m)}(y), y)dy + a_k u_k^{(m)} = 0, \ a_k \ge 0. \tag{2.4}$$

We denote

$$(f_1(x), f_2(x))_{L^2(\Omega)} := \int_{\Omega} f_1(x)\bar{f}_2(x)dx,$$

with a slight abuse of notations when these functions are not square integrable, like for instance those used in the orthogonality conditions of Assumption 2 below. Indeed, if $f_1(x) \in L^1(\Omega)$ and $f_2(x)$ is bounded in Ω, then the integral in the right side of the definition above makes sense. In the first part of the work we consider the case of $\Omega = \mathbb{R}^d$, such that the appropriate Sobolev space is equipped with the norm

$$\|u\|_{H^2(\mathbb{R}^d, \ \mathbb{R}^N)}^2 := \sum_{k=1}^{N} \|u_k\|_{H^2(\mathbb{R}^d)}^2 = \sum_{k=1}^{N} \{\|u_k\|_{L^2(\mathbb{R}^d)}^2 + \|\Delta u_k\|_{L^2(\mathbb{R}^d)}^2\}. \tag{2.5}$$

We will also use the norm

$$\|u\|_{L^2(\mathbb{R}^d,\ \mathbb{R}^N)}^2 := \sum_{k=1}^{N} \|u_k\|_{L^2(\mathbb{R}^d)}^2.$$

The main issue for systems (2.3) and (2.4) above is that the operators $-\Delta - a_k : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $a_k \geq 0$ do not satisfy the Fredholm property, which is the obstacle when solving such systems. The similar situations arising in linear and nonlinear equations, both self- adjoint and non self-adjoint involving non Fredholm second or fourth order differential operators or systems of equations with non Fredholm operators have been studied extensively in recent years (see [15]-[27]). Let us make the following assumption on the integral kernels involved in the nonlocal parts of (2.4).

Assumption 2. *Let* $m \in \mathbb{N}$, $G_{k,m}(x) : \mathbb{R}^d \to \mathbb{R}$, *such that* $G_{k,m}(x) \in L^1(\mathbb{R}^d)$ *and* $G_{k,m}(x) \to G_k(x)$ *in* $L^1(\mathbb{R}^d)$ *as* $m \to \infty$ *with* $1 \leq k \leq N$, $1 \leq d \leq 3$ *and* $1 \leq l \leq N-1$, $l \in \mathbb{N}$ *with* $N \geq 2$.
I) Let $a_k > 0$, $1 \leq k \leq l$, *assume that* $xG_{k,m}(x) \in L^1(\mathbb{R}^d)$, *such that* $xG_{k,m}(x) \to xG_k(x)$ *in* $L^1(\mathbb{R}^d)$ *as* $m \to \infty$ *and for all* $m \in \mathbb{N}$

$$\left(G_{k,m}(x), \frac{e^{\pm i\sqrt{a_k}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0, \quad d = 1. \tag{2.6}$$

$$\left(G_{k,m}(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a_k}}, \quad d = 2, 3. \tag{2.7}$$

II) Let $a_k = 0$, $l+1 \leq k \leq N$, *assume that* $x^2 G_{k,m}(x) \in L^1(\mathbb{R}^d)$, *such that* $x^2 G_{k,m}(x) \to x^2 G_k(x)$ *in* $L^1(\mathbb{R}^d)$ *as* $m \to \infty$ *and for all* $m \in \mathbb{N}$

$$(G_{k,m}(x), 1)_{L^2(\mathbb{R}^d)} = 0 \quad and \quad (G_{k,m}(x), x_s)_{L^2(\mathbb{R}^d)} = 0, \quad 1 \leq s \leq d. \tag{2.8}$$

Let us use the hat symbol throughout the work to designate the standard Fourier transform, such that

$$\widehat{G_k}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} G_k(x)e^{-ipx}dx, \quad p \in \mathbb{R}^d. \tag{2.9}$$

Thus

$$\|\widehat{G_k}(p)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|G_k(x)\|_{L^1(\mathbb{R}^d)}. \tag{2.10}$$

Let us define the following auxiliary quantities for $m \in \mathbb{N}$

$$M_{k,m} := \max\left\{\left\|\frac{\widehat{G_{k,m}}(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}, \left\|\frac{p^2\widehat{G_{k,m}}(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}\right\}, \quad 1 \leq k \leq l. \tag{2.11}$$

$$M_{k,m} := \max\left\{\left\|\frac{\widehat{G_{k,m}}(p)}{p^2}\right\|_{L^\infty(\mathbb{R}^d)}, \left\|\widehat{G_{k,m}}(p)\right\|_{L^\infty(\mathbb{R}^d)}\right\}, \quad l+1 \leq k \leq N. \tag{2.12}$$

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Similarly, in the limiting case we have

$$M_k := \max\left\{\left\|\frac{\widehat{G_k}(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}, \quad \left\|\frac{p^2\widehat{G_k}(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}\right\}, \quad 1 \le k \le l. \tag{2.13}$$

$$M_k := \max\left\{\left\|\frac{\widehat{G_k}(p)}{p^2}\right\|_{L^\infty(\mathbb{R}^d)}, \quad \left\|\widehat{G_k}(p)\right\|_{L^\infty(\mathbb{R}^d)}\right\}, \quad l+1 \le k \le N. \tag{2.14}$$

Clearly, expressions (2.11) and (2.12) are finite due to Lemma A1 in one dimension and Lemma A2 for $d = 2, 3$ of [21] under Assumption 2 above. This enables us to define for each $m \in \mathbb{N}$

$$M_m := \max M_{k,m}, \quad 1 \le k \le N \tag{2.15}$$

with $M_{k,m}$ given by (2.11) and (2.12). Analogously, for the limiting case due to Lemmas A1 and A2 of the Appendix of [27] we define

$$M := \max M_k, \quad 1 \le k \le N, \tag{2.16}$$

which is finite. Our first main statement is as follows.

Theorem 3. *Let $\Omega = \mathbb{R}^d$, $d = 1, 2, 3$, Assumptions 1 and 2 hold and for all $m \in \mathbb{N}$ we have $\sqrt{2}(2\pi)^{\frac{d}{2}} M_m L \le 1 - \varepsilon$ for some $0 < \varepsilon < 1$. Then each system of equations (2.4) admits a unique solution $u^{(m)}(x) \in H^2(\mathbb{R}^d, \mathbb{R}^N)$, the limiting system of equations (2.3) has a unique solution $u(x) \in H^2(\mathbb{R}^d, \mathbb{R}^N)$, such that $u^{(m)}(x) \to u(x)$ in $H^2(\mathbb{R}^d, \mathbb{R}^N)$ as $m \to \infty$.*

The unique solution of each system (2.4) $u^{(m)}(x)$ is nontrivial provided the intersection of supports of the Fourier transforms of functions $supp\widehat{F_k(0,x)} \cap supp\widehat{G}_{k,m}$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$ for some $1 \le k \le N$. Similarly, the unique solution of the limiting system (2.3) $u(x)$ does not vanish identically if $supp\widehat{F_k(0,x)} \cap supp\widehat{G}_k$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$ for a certain $1 \le k \le N$.

The second part of the present work deals with the studies of the analogous system on the finite interval with periodic boundary conditions for the solution vector function and its first derivative, namely on $\Omega = I := [0, 2\pi]$. We assume the following about the integral kernels present in the nonlocal parts of system (2.4) in such case.

Assumption 4. *Let $\Omega = I$, $m \in \mathbb{N}$, $G_{k,m}(x) : I \to \mathbb{R}$, $G_{k,m}(x) \in L^\infty(I)$, such that $G_{k,m}(x) \to G_k(x)$ in $L^\infty(I)$ as $m \to \infty$ with $G_{k,m}(0) = G_{k,m}(2\pi)$, $1 \le k \le N$, where $N \ge 3$ and $1 \le l < q \le N - 1$, $l, q \in \mathbb{N}$.*
I) Let $a_k > 0$ and $a_k \ne n^2$, $n \in \mathbb{Z}$ for $1 \le k \le l$.
II) Let $a_k = n_k^2$, $n_k \in \mathbb{N}$ and

$$\left(G_{k,m}(x), \frac{e^{\pm i n_k x}}{\sqrt{2\pi}}\right)_{L^2(I)} = 0, \quad l+1 \le k \le q. \tag{2.17}$$

III) Let $a_k = 0$ and

$$(G_{k,m}(x), 1)_{L^2(I)} = 0, \quad q + 1 \le k \le N. \tag{2.18}$$

Let $F_k(u, 0) = F_k(u, 2\pi)$ for $u \in \mathbb{R}^N$ and $k = 1, ..., N$.

We introduce the Fourier transform for functions on the $[0, 2\pi]$ interval as

$$G_{k,n} := \int_0^{2\pi} G_k(x) \frac{e^{-inx}}{\sqrt{2\pi}} dx, \quad n \in \mathbb{Z} \tag{2.19}$$

and define the following auxiliary expressions for $m \in \mathbb{N}$

$$P_{k,m} := \max\left\{ \left\| \frac{G_{k,m,n}}{n^2 - a_k} \right\|_{l^\infty}, \left\| \frac{n^2 G_{k,m,n}}{n^2 - a_k} \right\|_{l^\infty} \right\}, \quad 1 \le k \le l. \tag{2.20}$$

$$P_{k,m} := \max\left\{ \left\| \frac{G_{k,m,n}}{n^2 - n_k^2} \right\|_{l^\infty}, \left\| \frac{n^2 G_{k,m,n}}{n^2 - n_k^2} \right\|_{l^\infty} \right\}, \quad l + 1 \le k \le q. \tag{2.21}$$

$$P_{k,m} := \max\left\{ \left\| \frac{G_{k,m,n}}{n^2} \right\|_{l^\infty}, \left\| G_{k,m,n} \right\|_{l^\infty} \right\}, \quad q + 1 \le k \le N. \tag{2.22}$$

In the limiting case, we will use

$$P_k := \max\left\{ \left\| \frac{G_{k,n}}{n^2 - a_k} \right\|_{l^\infty}, \left\| \frac{n^2 G_{k,n}}{n^2 - a_k} \right\|_{l^\infty} \right\}, \quad 1 \le k \le l. \tag{2.23}$$

$$P_k := \max\left\{ \left\| \frac{G_{k,n}}{n^2 - n_k^2} \right\|_{l^\infty}, \left\| \frac{n^2 G_{k,n}}{n^2 - n_k^2} \right\|_{l^\infty} \right\}, \quad l + 1 \le k \le q. \tag{2.24}$$

$$P_k := \max\left\{ \left\| \frac{G_{k,n}}{n^2} \right\|_{l^\infty}, \left\| G_{k,n} \right\|_{l^\infty} \right\}, \quad q + 1 \le k \le N. \tag{2.25}$$

Evidently, expressions (2.20), (2.21) and (2.22) are finite by virtue of Lemma A3 of [21] under Assumption 4 above. This allows us to define for each $m \in \mathbb{N}$

$$P_m := \max P_{k,m}, \quad 1 \le k \le N \tag{2.26}$$

with $P_{k,m}$ given by (2.20), (2.21) and (2.22). Similarly, in the limiting case by means of Lemma A3 of the Appendix of [27] we define

$$P := \max P_k, \quad 1 \le k \le N, \tag{2.27}$$

which is finite. Let us use here the corresponding functional space

$$H^2(I) := \{v(x) : I \to \mathbb{R} \mid v(x), v''(x) \in L^2(I), \quad v(0) = v(2\pi), \quad v'(0) = v'(2\pi)\},$$

aiming at $u_k(x) \in H^2(I)$, $1 \le k \le l$. We introduce the following auxiliary constrained subspaces

$$H_k^2(I) := \left\{ v \in H^2(I) \mid \left(v(x), \frac{e^{\pm i n_k x}}{\sqrt{2\pi}} \right)_{L^2(I)} = 0 \right\}, \quad n_k \in \mathbb{N}, \quad l+1 \le k \le q,$$

with the goal of having $u_k(x) \in H_k^2(I)$, $l+1 \le k \le q$. Finally,

$$H_0^2(I) := \{ v \in H^2(I) \mid (v(x), 1)_{L^2(I)} = 0 \}, \quad q+1 \le k \le N.$$

Our aim is to have $u_k(x) \in H_0^2(I)$, $q+1 \le k \le N$. The constrained subspaces defined above are Hilbert spaces as well (see e.g. Chapter 2.1 of [9]). The resulting space used for proving the existence in the sense of sequences of solutions $u(x) : I \to \mathbb{R}^N$ of system (2.3) will be the direct sum of the spaces given above, such that

$$H_c^2(I, \ \mathbb{R}^N) := \oplus_{k=1}^l H^2(I) \oplus_{k=l+1}^q H_k^2(I) \oplus_{k=q+1}^N H_0^2(I).$$

The corresponding Sobolev norm is given by

$$\|u\|_{H_c^2(I, \ \mathbb{R}^N)}^2 := \sum_{k=1}^N \{ \|u_k\|_{L^2(I)}^2 + \|u_k''\|_{L^2(I)}^2 \}$$

with $u(x) : I \to \mathbb{R}^N$. Another useful norm here is

$$\|u\|_{L^2(I, \ \mathbb{R}^N)}^2 := \sum_{k=1}^N \|u_k\|_{L^2(I)}^2.$$

Our second main result is as follows.

Theorem 5. *Let $\Omega = I$, Assumptions 1 and 4 hold and for all $m \in \mathbb{N}$ we have $2\sqrt{\pi} P_m L \le 1 - \varepsilon$ with some $0 < \varepsilon < 1$. Then each system of equations (2.4) possesses a unique solution $u^{(m)}(x) \in H_c^2(I, \ \mathbb{R}^N)$, the limiting system of equations (2.3) admits a unique solution $u(x) \in H_c^2(I, \ \mathbb{R}^N)$, such that $u^{(m)}(x) \to u(x)$ in $H_c^2(I, \ \mathbb{R}^N)$ as $m \to \infty$.*

The unique solution of each system (2.4) $u^{(m)}(x)$ is nontrivial provided the Fourier coefficients $G_{k,m,n} F_k(0, x)_n \ne 0$ for some $k = 1, ..., N$ and some $n \in \mathbb{Z}$. Similarly, the unique solution of limiting system (2.3) $u(x)$ does not vanish identically if $G_{k,n} F_k(0, x)_n \ne 0$ for some $k = 1, ..., N$ and some $n \in \mathbb{Z}$.

Remark. *We use the constrained subspaces $H_k^2(I)$ and $H_0^2(I)$ involved in the direct sum of spaces $H_c^2(I, \ \mathbb{R}^N)$, such that the Fredholm operators $-\dfrac{d^2}{dx^2} - n_k^2 : H_k^2(I) \to L^2(I)$ and $-\dfrac{d^2}{dx^2} : H_0^2(I) \to L^2(I)$ have trivial kernels.*

We conclude the article with the studies of our system on the product of sets, where one is the finite interval I with periodic boundary conditions as before and another is the

whole space of dimension not exceeding two, such that in our notations $\Omega = I \times \mathbb{R}^d = [0, 2\pi] \times \mathbb{R}^d$, $d = 1, 2$ and $x = (x_1, x_\perp)$ with $x_1 \in I$ and $x_\perp \in \mathbb{R}^d$. The total Laplace operator in such context will be

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \sum_{s=1}^{d} \frac{\partial^2}{\partial x_{\perp,s}^2}.$$

The appropriate Sobolev space for the problem is $H^2(\Omega, \mathbb{R}^N)$ of vector functions $u(x) : \Omega \to \mathbb{R}^N$, such that for k=1,...,N

$$u_k(x), \ \Delta u_k(x) \in L^2(\Omega), \ u_k(0, x_\perp) = u_k(2\pi, x_\perp), \ \frac{\partial u_k}{\partial x_1}(0, x_\perp) = \frac{\partial u_k}{\partial x_1}(2\pi, x_\perp)$$

with $x_\perp \in \mathbb{R}^d$ a.e. It is equipped with the norm

$$\|u\|_{H^2(\Omega, \ \mathbb{R}^N)}^2 := \sum_{k=1}^{N} \{\|u_k\|_{L^2(\Omega)}^2 + \|\Delta u_k\|_{L^2(\Omega)}^2\}.$$

Another norm used here is given by

$$\|u\|_{L^2(\Omega, \ \mathbb{R}^N)}^2 := \sum_{k=1}^{N} \|u_k\|_{L^2(\Omega)}^2.$$

Similarly to the whole space case studied in Theorem 3, the operators $-\Delta - a_k : H^2(\Omega) \to L^2(\Omega)$, $\quad a_k \geq 0$ do not possess the Fredholm property.

Assumption 6. *Let $m \in \mathbb{N}$, $G_{k,m}(x) : \Omega \to \mathbb{R}$, $G_{k,m}(x) \in L^1(\Omega)$,*

$$G_{k,m}(x) \to G_k(x) \ in \ L^1(\Omega), \ m \to \infty,$$

for $x_\perp \in \mathbb{R}^d$ a.e. $G_{k,m}(0, x_\perp) = G_{k,m}(2\pi, x_\perp) \in L^\infty(\mathbb{R}^d)$, such that

$$G_{k,m}(0, x_\perp) \to G_k(0, x_\perp), \ G_{k,m}(2\pi, x_\perp) \to G_k(2\pi, x_\perp) \ in \ L^\infty(\mathbb{R}^d), \ m \to \infty$$

and $F_k(u, 0, x_\perp) = F_k(u, 2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e., $u \in \mathbb{R}^N$, $d = 1, 2$ and $k = 1, ..., N$. Let $N \geq 3$ and $1 \leq l < q \leq N - 1$ with $l, q \in \mathbb{N}$.

I) Assume for $1 \leq k \leq l$ we have $n_k^2 < a_k < (n_k + 1)^2$, $n_k \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$, $x_\perp G_{k,m}(x) \in L^1(\Omega)$, such that $x_\perp G_{k,m}(x) \to x_\perp G_k(x)$ in $L^1(\Omega)$ as $m \to \infty$ and

$$\left(G_{k,m}(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{\pm i\sqrt{a_k - n^2} x_\perp}}{\sqrt{2\pi}} \right)_{L^2(\Omega)} = 0, \quad |n| \leq n_k, \quad d = 1, \tag{2.28}$$

$$\left(G_{k,m}(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{ipx_\perp}}{2\pi} \right)_{L^2(\Omega)} = 0, \quad p \in S^2_{\sqrt{a_k - n^2}}, \quad |n| \leq n_k, \quad d = 2. \tag{2.29}$$

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II) Assume for $l+1 \le k \le q$ we have $a_k = n_k^2$, $n_k \in \mathbb{N}$, $x_\perp^2 G_{k,m}(x) \in L^1(\Omega)$, such that $x_\perp^2 G_{k,m}(x) \to x_\perp^2 G_k(x)$ in $L^1(\Omega)$ as $m \to \infty$ and

$$\left(G_{k,m}(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{\pm i\sqrt{n_k^2 - n^2} x_\perp}}{\sqrt{2\pi}} \right)_{L^2(\Omega)} = 0, \quad |n| \le n_k - 1, \quad d = 1, \qquad (2.30)$$

$$\left(G_{k,m}(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{ipx_\perp}}{2\pi} \right)_{L^2(\Omega)} = 0, \quad p \in S_{\sqrt{n_k^2 - n^2}}^2, \quad |n| \le n_k - 1, \quad d = 2, \qquad (2.31)$$

$$\left(G_{k,m}(x_1, x_\perp), \frac{e^{\pm in_k x_1}}{\sqrt{2\pi}} \right)_{L^2(\Omega)} = 0, \quad \left(G_{k,m}(x_1, x_\perp), \frac{e^{\pm in_k x_1}}{\sqrt{2\pi}} x_{\perp,s} \right)_{L^2(\Omega)} = 0, \qquad (2.32)$$

for $1 \le s \le d$.

III) Assume for $q+1 \le k \le N$ we have $a_k = 0$, $x_\perp^2 G_{k,m}(x) \in L^1(\Omega)$, such that $x_\perp^2 G_{k,m}(x) \to x_\perp^2 G_k(x)$ in $L^1(\Omega)$ as $m \to \infty$ and

$$(G_{k,m}(x), 1)_{L^2(\Omega)} = 0, \quad (G_{k,m}(x), x_{\perp,s})_{L^2(\Omega)} = 0, \quad 1 \le s \le d. \qquad (2.33)$$

Let us use the Fourier transform for functions on such a product of sets, namely for $d = 1, 2$ and $k = 1, ..., N$

$$\widehat{G}_{k,n}(p) := \frac{1}{(2\pi)^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} dx_\perp e^{-ipx_\perp} \int_0^{2\pi} G_k(x_1, x_\perp) e^{-inx_1} dx_1, \quad p \in \mathbb{R}^d, \quad n \in \mathbb{Z}. \qquad (2.34)$$

Thus

$$\|\widehat{G}_{k,n}(p)\|_{L_{n,p}^\infty} := \sup_{\{p \in \mathbb{R}^d, n \in \mathbb{Z}\}} |\widehat{G}_{k,n}(p)| \le \frac{1}{(2\pi)^{\frac{d+1}{2}}} \|G_k\|_{L^1(\Omega)}.$$

For the technical purposes, we define the following quantities for $m \in \mathbb{N}$

$$R_{k,m} := \max\left\{ \left\| \frac{\widehat{G}_{k,m,n}(p)}{p^2 + n^2 - a_k} \right\|_{L_{n,p}^\infty}, \left\| \frac{(p^2 + n^2)\widehat{G}_{k,m,n}(p)}{p^2 + n^2 - a_k} \right\|_{L_{n,p}^\infty} \right\}, \quad 1 \le k \le l. \qquad (2.35)$$

$$R_{k,m} := \max\left\{ \left\| \frac{\widehat{G}_{k,m,n}(p)}{p^2 + n^2 - n_k^2} \right\|_{L_{n,p}^\infty}, \left\| \frac{(p^2 + n^2)\widehat{G}_{k,m,n}(p)}{p^2 + n^2 - n_k^2} \right\|_{L_{n,p}^\infty} \right\}, \quad l+1 \le k \le q. \qquad (2.36)$$

$$R_{k,m} := \max\left\{ \left\| \frac{\widehat{G}_{k,m,n}(p)}{p^2 + n^2} \right\|_{L_{n,p}^\infty}, \left\| \widehat{G}_{k,m,n}(p) \right\|_{L_{n,p}^\infty} \right\}, \quad q+1 \le k \le N. \qquad (2.37)$$

In the limiting case, we have

$$R_k := \max\left\{ \left\| \frac{\widehat{G}_{k,n}(p)}{p^2 + n^2 - a_k} \right\|_{L_{n,p}^\infty}, \left\| \frac{(p^2 + n^2)\widehat{G}_{k,n}(p)}{p^2 + n^2 - a_k} \right\|_{L_{n,p}^\infty} \right\}, \quad 1 \le k \le l. \qquad (2.38)$$

$$R_k := \max\left\{ \left\| \frac{\widehat{G}_{k,n}(p)}{p^2 + n^2 - n_k^2} \right\|_{L_{n,p}^\infty}, \left\| \frac{(p^2 + n^2)\widehat{G}_{k,n}(p)}{p^2 + n^2 - n_k^2} \right\|_{L_{n,p}^\infty} \right\}, \quad l+1 \le k \le q. \qquad (2.39)$$

$$R_k := \max\left\{\left\|\frac{\widehat{G}_{k,n}(p)}{p^2 + n^2}\right\|_{L_{n,p}^\infty}, \quad \left\|\widehat{G}_{k,n}(p)\right\|_{L_{n,p}^\infty}\right\}, \quad q + 1 \leq k \leq N. \tag{2.40}$$

Assumption 6 along with Lemmas A6, A5 and A4 of [21] yield that the expressions given by (2.35), (2.36) and (2.37) are finite, which enables us to define for each $m \in \mathbb{N}$

$$R_m := \max R_{k,m}, \quad k = 1, ..., N$$

with $R_{k,m}$ given in (2.35), (2.36) and (2.37). Analogously, in the limiting case by virtue of Lemmas A6, A5 and A4 of the Appendix of [27], we define the finite quantity

$$R := \max R_k, \quad k = 1, ..., N.$$

The final statement of the work is as follows.

Theorem 7. *Let $\Omega = I \times \mathbb{R}^d$, $d = 1, 2$, Assumptions 1 and 6 hold and for all $m \in \mathbb{N}$ we have $\sqrt{2}(2\pi)^{\frac{d+1}{2}} R_m L \leq 1 - \varepsilon$ for some $0 < \varepsilon < 1$. Then each system of equations (2.4) admits a unique solution $u^{(m)}(x) \in H^2(\Omega, \mathbb{R}^N)$, the limiting system of equations (2.3) has a unique solution $u(x) \in H^2(\Omega, \mathbb{R}^N)$, such that $u^{(m)}(x) \to u(x)$ in $H^2(\Omega, \mathbb{R}^N)$ as $m \to \infty$.*
The unique solution of each system (2.4) $u^{(m)}(x)$ is nontrivial provided that the intersection of supports of the Fourier transforms of functions $supp\widehat{F_k(0,x)}_n(p) \cap supp\widehat{G}_{k,m,n}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$ for some $k = 1, ..., N$ and for some $n \in \mathbb{Z}$. Similarly, the unique solution of limiting system (2.3) $u(x)$ does not vanish identically if $supp\widehat{F_k(0,x)}_n(p) \cap supp\widehat{G}_{k,n}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$ for a certain $k = 1, ..., N$ and for some $n \in \mathbb{Z}$.

Remark. *Note that in the work we deal with real valued vector functions by means of the assumptions on $F_k(u, x)$, $G_{k,m}(x)$ and $G_k(x)$, $k = 1, ..., N$ involved in the nonlocal terms of systems (2.4) and (2.3).*

3 The Whole Space Case

Proof of Theorem 3. By virtue of Theorem 3 of [22], each system of equations (2.4) possesses a unique solution $u^{(m)}(x) \in H^2(\mathbb{R}^d, \mathbb{R}^N)$, $m \in \mathbb{N}$. System (2.3) has a unique solution $u(x) \in H^2(\mathbb{R}^d, \mathbb{R}^N)$ as a result of Lemmas A1 and A2 of the Appendix of [27] in dimensions $d = 1$ and $d = 2, 3$ respectively along with Theorem 3 of [22].

We apply the standard Fourier transform (2.9) to both sides of systems (2.3) and (2.4). This gives us for $k = 1, ..., N$ and $m \in \mathbb{N}$

$$\widehat{u}_k(p) = (2\pi)^{\frac{d}{2}}\frac{\widehat{G}_k(p)\widehat{f}_k(p)}{p^2 - a_k}, \quad \widehat{u_k^{(m)}}(p) = (2\pi)^{\frac{d}{2}}\frac{\widehat{G}_{k,m}(p)\widehat{f}_{k,m}(p)}{p^2 - a_k} \tag{3.1}$$

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with $\widehat{f}_k(p)$ and $\widehat{f}_{k,m}(p)$ denoting the Fourier transforms of $F_k(u(x),x)$ and $F_k(u^{(m)}(x),x)$ respectively. Evidently, we have the upper bound

$$|\widehat{u_k^{(m)}}(p) - \widehat{u}_k(p)| \leq (2\pi)^{\frac{d}{2}}\left\|\frac{\widehat{G}_{k,m}(p)}{p^2 - a_k} - \frac{\widehat{G}_k(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}|\widehat{f}_k(p)|+$$

$$+(2\pi)^{\frac{d}{2}}\left\|\frac{\widehat{G}_{k,m}(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}|\widehat{f}_{k,m}(p) - \widehat{f}_k(p)|.$$

Hence

$$\|u_k^{(m)} - u_k\|_{L^2(\mathbb{R}^d)} \leq (2\pi)^{\frac{d}{2}}\left\|\frac{\widehat{G}_{k,m}(p)}{p^2 - a_k} - \frac{\widehat{G}_k(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}\|F_k(u(x),x)\|_{L^2(\mathbb{R}^d)}+$$

$$+(2\pi)^{\frac{d}{2}}\left\|\frac{\widehat{G}_{k,m}(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}\|F_k(u^{(m)}(x),x) - F_k(u(x),x)\|_{L^2(\mathbb{R}^d)}.$$

By means of inequality (2.2) of Assumption 1, we have

$$\sqrt{\sum_{k=1}^N \|F_k(u^{(m)}(x),x) - F_k(u(x),x)\|_{L^2(\mathbb{R}^d)}^2} \leq L\|u^{(m)} - u\|_{L^2(\mathbb{R}^d,\ \mathbb{R}^N)}. \tag{3.2}$$

Note that $u_k^{(m)}(x), u_k(x) \in H^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d),\ k = 1, ..., N,\ d \leq 3$ by means of the Sobolev embedding. Hence, we obtain

$$\|u^{(m)} - u\|_{L^2(\mathbb{R}^d,\ \mathbb{R}^N)}^2 \leq 2(2\pi)^d \sum_{k=1}^N \left\|\frac{\widehat{G}_{k,m}(p)}{p^2 - a_k} - \frac{\widehat{G}_k(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}^2 \|F_k(u(x),x)\|_{L^2(\mathbb{R}^d)}^2+$$

$$+2(2\pi)^d M_m^2 L^2 \|u^{(m)} - u\|_{L^2(\mathbb{R}^d,\ \mathbb{R}^N)}^2,$$

such that

$$\|u^{(m)} - u\|_{L^2(\mathbb{R}^d,\ \mathbb{R}^N)}^2 \leq \frac{2(2\pi)^d}{\varepsilon(2-\varepsilon)} \sum_{k=1}^N \left\|\frac{\widehat{G}_{k,m}(p)}{p^2 - a_k} - \frac{\widehat{G}_k(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}^2 \|F_k(u(x),x)\|_{L^2(\mathbb{R}^d)}^2.$$

By virtue of inequality (2.1) of Assumption 1, we have $F_k(u(x),x) \in L^2(\mathbb{R}^d),\ k = 1, ..., N$ for $u(x) \in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$. Thus,

$$u^{(m)}(x) \rightarrow u(x), \quad m \rightarrow \infty \tag{3.3}$$

in $L^2(\mathbb{R}^d,\ \mathbb{R}^N)$ due to Lemmas A1 and A2 of the Appendix of [27] for $d = 1$ and $d = 2, 3$ respectively. Obviously, for $k = 1, ..., N$ and $m \in \mathbb{N}$

$$p^2\widehat{u}_k(p) = (2\pi)^{\frac{d}{2}}\frac{p^2\widehat{G}_k(p)\widehat{f}_k(p)}{p^2 - a_k}, \quad p^2\widehat{u_k^{(m)}}(p) = (2\pi)^{\frac{d}{2}}\frac{p^2\widehat{G}_{k,m}(p)\widehat{f}_{k,m}(p)}{p^2 - a_k}.$$

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Therefore,

$$|p^2\widehat{u_k^{(m)}}(p) - p^2\widehat{u}_k(p)| \leq (2\pi)^{\frac{d}{2}}\left\|\frac{p^2\widehat{G}_{k,m}(p)}{p^2 - a_k} - \frac{p^2\widehat{G}_k(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}|\widehat{f}_k(p)| +$$

$$+(2\pi)^{\frac{d}{2}}\left\|\frac{p^2\widehat{G}_{k,m}(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}|\widehat{f}_{k,m}(p) - \widehat{f}_k(p)|.$$

By means of (3.2) we obtain for $k = 1, ..., N$

$$\|\Delta u_k^{(m)} - \Delta u_k\|_{L^2(\mathbb{R}^d)} \leq (2\pi)^{\frac{d}{2}}\left\|\frac{p^2\widehat{G}_{k,m}(p)}{p^2 - a_k} - \frac{p^2\widehat{G}_k(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}\|F_k(u(x), x)\|_{L^2(\mathbb{R}^d)} +$$

$$+(2\pi)^{\frac{d}{2}}\left\|\frac{p^2\widehat{G}_{k,m}(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}L\|u^{(m)} - u\|_{L^2(\mathbb{R}^d,\ \mathbb{R}^N)}.$$

Therefore, by means of Lemmas A1 and A2 of the Appendix of [27] in $d = 1$ and for $d = 2, 3$ respectively along with (3.3), we arrive at $\Delta u^{(m)}(x) \to \Delta u(x)$ in $L^2(\mathbb{R}^d,\ \mathbb{R}^N)$ as $m \to \infty$. Norm definition (2.5) yields that $u^{(m)}(x) \to u(x)$ in $H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ as $m \to \infty$.

Suppose the solution $u^{(m)}(x)$ of system (2.4) discussed above vanishes a.e. in $\mathbb{R}^d$ for some $m \in \mathbb{N}$. This will contradict to the assumption that the Fourier images of $G_{k,m}(x)$ and $F_k(0, x)$ do not vanish on a set of nonzero Lebesgue measure in $\mathbb{R}^d$ for a certain $1 \leq k \leq N$. The similar reasoning holds for the solution $u(x)$ of the limiting system of equations (2.3) discussed above. ∎

4   The Problem on the Finite Interval

Proof of Theorem 5. Evidently, for $1 \leq k \leq N$, we can estimate $|G_k(0) - G_k(2\pi)|$ from above by

$$|G_k(0) - G_{k,m}(0)| + |G_{k,m}(2\pi) - G_k(2\pi)| \leq 2\|G_{k,m}(x) - G_k(x)\|_{L^\infty(I)} \to 0, \quad m \to \infty$$

as assumed, such that $G_k(0) = G_k(2\pi)$. Clearly, under the stated conditions we have $G_{k,m}(x) \in L^1(I), \ m \in \mathbb{N}$ and $G_{k,m}(x) \to G_k(x)$ in $L^1(I)$ for $k = 1, ..., N$ as $m \to \infty$. By means of Theorem 5 of [22], each system (2.4) admits a unique solution $u^{(m)}(x)$ belonging to $H_c^2(I,\ \mathbb{R}^N)$ with $m \in \mathbb{N}$. System (2.3) possesses a unique solution $u(x)$ belonging to $H_c^2(I,\ \mathbb{R}^N)$ due to Lemma A3 of the Appendix of [27] along with Theorem 5 of [22].

Let us apply Fourier transform (2.19) to both sides of systems (2.3) and (2.4). This gives us for $n \in \mathbb{Z}$ and $1 \leq k \leq N$

$$u_{k,n} = \sqrt{2\pi}\frac{G_{k,n}f_{k,n}}{n^2 - a_k}, \quad u_{k,n}^{(m)} = \sqrt{2\pi}\frac{G_{k,m,n}f_{k,m,n}}{n^2 - a_k}, \quad m \in \mathbb{N} \tag{4.1}$$

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with $f_{k,n}$ and $f_{k,m,n}$ standing for the Fourier images of $F_k(u(x), x)$ and $F_k(u^{(m)}(x), x)$ respectively under transform (2.19). This allows us to derive the upper bound

$$|u_{k,n}^{(m)} - u_{k,n}| \le \sqrt{2\pi} \left\| \frac{G_{k,m,n}}{n^2 - a_k} - \frac{G_{k,n}}{n^2 - a_k} \right\|_{l^\infty} |f_{k,n}| + \sqrt{2\pi} \left\| \frac{G_{k,m,n}}{n^2 - a_k} \right\|_{l^\infty} |f_{k,m,n} - f_{k,n}|.$$

Hence

$$\|u_k^{(m)}(x) - u_k(x)\|_{L^2(I)} \le \sqrt{2\pi} \left\| \frac{G_{k,m,n}}{n^2 - a_k} - \frac{G_{k,n}}{n^2 - a_k} \right\|_{l^\infty} \|F_k(u(x), x)\|_{L^2(I)} +$$

$$+ \sqrt{2\pi} \left\| \frac{G_{k,m,n}}{n^2 - a_k} \right\|_{l^\infty} \|F_k(u^{(m)}(x), x) - F_k(u(x), x)\|_{L^2(I)}.$$

Inequality (2.2) of Assumption 1 gives us

$$\sqrt{\sum_{k=1}^{N} \|F_k(u^{(m)}(x), x) - F_k(u(x), x)\|_{L^2(I)}^2} \le L \|u^{(m)} - u\|_{L^2(I, \ \mathbb{R}^N)}. \tag{4.2}$$

Note that by virtue of the Sobolev embedding we have $u_k^{(m)}(x), \ u_k(x) \in H^2(I) \subset L^\infty(I), \ 1 \le k \le N$. Clearly,

$$\|u^{(m)} - u\|_{L^2(I, \ \mathbb{R}^N)}^2 \le 4\pi \sum_{k=1}^{N} \left\| \frac{G_{k,m,n}}{n^2 - a_k} - \frac{G_{k,n}}{n^2 - a_k} \right\|_{l^\infty}^2 \|F_k(u(x), x)\|_{L^2(I)}^2 +$$

$$+ 4\pi P_m^2 L^2 \|u^{(m)} - u\|_{L^2(I, \ \mathbb{R}^N)}^2.$$

Thus

$$\|u^{(m)} - u\|_{L^2(I, \ \mathbb{R}^N)}^2 \le \frac{4\pi}{\varepsilon(2 - \varepsilon)} \sum_{k=1}^{N} \left\| \frac{G_{k,m,n}}{n^2 - a_k} - \frac{G_{k,n}}{n^2 - a_k} \right\|_{l^\infty}^2 \|F_k(u(x), x)\|_{L^2(I)}^2.$$

Obviously, $F_k(u(x), x) \in L^2(I), \ k = 1, ..., N$ for $u(x) \in H_c^2(I, \ \mathbb{R}^N)$ via inequality (2.1) of Assumption 1. Then by virtue of the result of Lemma A3 of the Appendix of [27], we arrive at

$$u^{(m)}(x) \to u(x), \quad m \to \infty \tag{4.3}$$

in $L^2(I, \ \mathbb{R}^N)$. Apparently, for $n \in \mathbb{Z}, \ m \in \mathbb{N}, \ 1 \le k \le N$

$$n^2 u_{k,n} = \sqrt{2\pi} \frac{n^2 G_{k,n} f_{k,n}}{n^2 - a_k}, \quad n^2 u_{k,n}^{(m)} = \sqrt{2\pi} \frac{n^2 G_{k,m,n} f_{k,m,n}}{n^2 - a_k}.$$

Hence,

$$|n^2 u_{k,n}^{(m)} - n^2 u_{k,n}| \le \sqrt{2\pi} \left\| \frac{n^2 G_{k,m,n}}{n^2 - a_k} - \frac{n^2 G_{k,n}}{n^2 - a_k} \right\|_{l^\infty} |f_{k,n}| + \sqrt{2\pi} \left\| \frac{n^2 G_{k,m,n}}{n^2 - a_k} \right\|_{l^\infty} |f_{k,m,n} - f_{k,n}|,$$

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such that via (4.2)

$$\left\|\frac{d^2}{dx^2}u_k^{(m)} - \frac{d^2}{dx^2}u_k\right\|_{L^2(I)} \leq \sqrt{2\pi}\left\|\frac{n^2 G_{k,m,n}}{n^2 - a_k} - \frac{n^2 G_{k,n}}{n^2 - a_k}\right\|_{l^\infty}\|F_k(u(x), x)\|_{L^2(I)}+$$

$$+\sqrt{2\pi}\left\|\frac{n^2 G_{k,m,n}}{n^2 - a_k}\right\|_{l^\infty} L\|u^{(m)} - u\|_{L^2(I,\ \mathbb{R}^N)}.$$

By means of the result of Lemma A3 of the Appendix of [27] along with (4.3), we obtain

$$\frac{d^2}{dx^2}u^{(m)}(x) \to \frac{d^2}{dx^2}u(x), \quad m \to \infty$$

in $L^2(I,\ \mathbb{R}^N)$. Thus, $u^{(m)}(x) \to u(x)$ in the $H_c^2(I,\ \mathbb{R}^N)$ norm as $m \to \infty$.

Suppose $u^{(m)}(x)$ vanishes a.e. in I for some $m \in \mathbb{N}$. Then we will arrive at the contradiction to the assumption that the Fourier coefficients $G_{k,m,n}F_k(0, x)_n \neq 0$ for some $k = 1, ..., N$ and a certain $n \in \mathbb{Z}$. The similar argument holds for the solution $u(x)$ of the limiting system (2.3) discussed above. ∎

5 The Problem on the Product of Sets

Proof of Theorem 7. Clearly, for $1 \leq k \leq N$, the norm $\|G_k(0, x_\perp) - G_k(2\pi, x_\perp)\|_{L^\infty(\mathbb{R}^d)}$ can be bounded from above by

$$\|G_k(0, x_\perp) - G_{k,m}(0, x_\perp)\|_{L^\infty(\mathbb{R}^d)} + \|G_{k,m}(2\pi, x_\perp) - G_k(2\pi, x_\perp)\|_{L^\infty(\mathbb{R}^d)} \to 0, \quad m \to \infty$$

as assumed, such that $G_k(0, x_\perp) = G_k(2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e.. By virtue of Theorem 7 of [22], each system (2.4) possesses a unique solution $u^{(m)}(x) \in H^2(\Omega,\ \mathbb{R}^N)$, $m \in \mathbb{N}$. System (2.3) admits a unique solution $u(x) \in H^2(\Omega,\ \mathbb{R}^N)$ as a result of Lemmas A6, A5 and A4 of the Appendix of [27] along with Theorem 7 of [22].

Let us apply Fourier transform (2.34) to both sides of systems (2.3) and (2.4). This gives us for $k = 1, ..., N,\ n \in \mathbb{Z},\ p \in \mathbb{R}^d,\ d = 1, 2,\ m \in \mathbb{N}$

$$\widehat{u}_{k,n}(p) = (2\pi)^{\frac{d+1}{2}}\frac{\widehat{G}_{k,n}(p)\widehat{f}_{k,n}(p)}{p^2 + n^2 - a_k}, \quad \widehat{u}_{k,m,n}(p) = (2\pi)^{\frac{d+1}{2}}\frac{\widehat{G}_{k,m,n}(p)\widehat{f}_{k,m,n}(p)}{p^2 + n^2 - a_k}, \quad (5.1)$$

with $\widehat{f}_{k,n}(p)$ and $\widehat{f}_{k,m,n}(p)$ denoting the Fourier images of $F_k(u(x), x)$ and $F_k(u^{(m)}(x), x)$ respectively for $k = 1, ..., N$ under transform (2.34). This enables us to derive the upper bound

$$|\widehat{u}_{k,m,n}(p) - \widehat{u}_{k,n}(p)| \leq (2\pi)^{\frac{d+1}{2}}\left\|\frac{\widehat{G}_{k,m,n}(p)}{p^2 + n^2 - a_k} - \frac{\widehat{G}_{k,n}(p)}{p^2 + n^2 - a_k}\right\|_{L^\infty_{n,p}}|\widehat{f}_{k,n}(p)|+$$

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$$+(2\pi)^{\frac{d+1}{2}}\left\|\frac{\widehat{G}_{k,m,n}(p)}{p^2+n^2-a_k}\right\|_{L^\infty_{n,p}}|\widehat{f}_{k,m,n}(p)-\widehat{f}_{k,n}(p)|.$$

Hence for $k=1,...,N$

$$\|u_k^{(m)}(x)-u_k(x)\|_{L^2(\Omega)}\leq(2\pi)^{\frac{d+1}{2}}\left\|\frac{\widehat{G}_{k,m,n}(p)}{p^2+n^2-a_k}-\frac{\widehat{G}_{k,n}(p)}{p^2+n^2-a_k}\right\|_{L^\infty_{n,p}}\|F_k(u(x),x)\|_{L^2(\Omega)}+$$

$$+(2\pi)^{\frac{d+1}{2}}\left\|\frac{\widehat{G}_{k,m,n}(p)}{p^2+n^2-a_k}\right\|_{L^\infty_{n,p}}\|F_k(u^{(m)}(x),x)-F_k(u(x),x)\|_{L^2(\Omega)}.$$

Inequality (2.2) of Assumption 1 yields

$$\sqrt{\sum_{k=1}^N\|F_k(u^{(m)}(x),x)-F_k(u(x),x)\|^2_{L^2(\Omega)}}\leq L\|u^{(m)}(x)-u(x)\|_{L^2(\Omega,\ \mathbb{R}^N)}.\qquad(5.2)$$

Note that due to the Sobolev embedding we have $u_k^{(m)}(x),u_k(x)\in H^2(\Omega)\subset L^\infty(\Omega),\ k=1,...,N$. Evidently

$$\|u^{(m)}-u\|^2_{L^2(\Omega,\ \mathbb{R}^N)}\leq2(2\pi)^{d+1}\sum_{k=1}^N\left\|\frac{\widehat{G}_{k,m,n}(p)}{p^2+n^2-a_k}-\frac{\widehat{G}_{k,n}(p)}{p^2+n^2-a_k}\right\|^2_{L^\infty_{n,p}}\|F_k(u(x),x)\|^2_{L^2(\Omega)}+$$

$$+2(2\pi)^{d+1}R_m^2L^2\|u^{(m)}-u\|^2_{L^2(\Omega,\ \mathbb{R}^N)}.$$

Therefore, we obtain

$$\|u^{(m)}-u\|^2_{L^2(\Omega,\ \mathbb{R}^N)}\leq\frac{2(2\pi)^{d+1}}{\varepsilon(2-\varepsilon)}\sum_{k=1}^N\left\|\frac{\widehat{G}_{k,m,n}(p)}{p^2+n^2-a_k}-\frac{\widehat{G}_{k,n}(p)}{p^2+n^2-a_k}\right\|^2_{L^\infty_{n,p}}\|F_k(u(x),x)\|^2_{L^2(\Omega)}.$$

Evidently, $F_k(u(x),x)\in L^2(\Omega),\ k=1,...,N$ for $u(x)\in H^2(\Omega,\ \mathbb{R}^N)$ due to inequality (2.1) of Assumption 1. By virtue of the results of Lemmas A6, A5 and A4 of the Appendix of [27], we derive

$$u^{(m)}(x)\to u(x),\quad m\to\infty\qquad(5.3)$$

in $L^2(\Omega,\ \mathbb{R}^N)$. Obviously,

$$|(p^2+n^2)\widehat{u}_{k,m,n}(p)-(p^2+n^2)\widehat{u}_{k,n}(p)|\leq$$

$$\leq(2\pi)^{\frac{d+1}{2}}\left\|\frac{(p^2+n^2)\widehat{G}_{k,m,n}(p)}{p^2+n^2-a_k}-\frac{(p^2+n^2)\widehat{G}_{k,n}(p)}{p^2+n^2-a_k}\right\|_{L^\infty_{n,p}}|\widehat{f}_{k,n}(p)|+$$

$$+(2\pi)^{\frac{d+1}{2}}\left\|\frac{(p^2+n^2)\widehat{G}_{k,m,n}(p)}{p^2+n^2-a_k}\right\|_{L^\infty_{n,p}}|\widehat{f}_{k,m,n}(p)-\widehat{f}_{k,n}(p)|.$$

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Via (5.2), this gives us

$$\|\Delta u_k^{(m)}(x) - \Delta u_k(x)\|_{L^2(\Omega)} \le$$

$$\le (2\pi)^{\frac{d+1}{2}} \left\| \frac{(p^2+n^2)\widehat{G}_{k,m,n}(p)}{p^2+n^2-a_k} - \frac{(p^2+n^2)\widehat{G}_{k,n}(p)}{p^2+n^2-a_k} \right\|_{L^\infty_{n,p}} \|F_k(u(x),x)\|_{L^2(\Omega)}+$$

$$+(2\pi)^{\frac{d+1}{2}} \left\| \frac{(p^2+n^2)\widehat{G}_{k,m,n}(p)}{p^2+n^2-a_k} \right\|_{L^\infty_{n,p}} L \|u^{(m)}(x) - u(x)\|_{L^2(\Omega,\ \mathbb{R}^N)}.$$

By virtue of (5.3) along with the results of Lemmas A6, A5 and A4 of the Appendix of [27], we obtain

$$\Delta u^{(m)}(x) \to \Delta u(x), \quad m \to \infty$$

in $L^2(\Omega,\ \mathbb{R}^N)$. Thus, we arrive at

$$u^{(m)}(x) \to u(x), \quad m \to \infty$$

in $H^2(\Omega,\ \mathbb{R}^N)$. Suppose $u^{(m)}(x)$ vanishes a.e. in Ω for some $m \in \mathbb{N}$. This gives us the contradiction to the assumption that there exist $k = 1, ..., N$ and $n \in \mathbb{Z}$, such that $supp\widehat{G}_{k,m,n}(p) \cap supp\widehat{F_k(0,x)}_n(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$. The similar argument is valid for the solution $u(x)$ of limiting system (2.3) considered above. ∎

6   Discussion

Let us conclude the work with a short discussion of biological interpretations of our results. All tissues and organs in a biological organism are characterized by the cell distribution with respect to their genotype. Without mutations all cells would possess the same genotype. Due to mutations, the genotype changes and represents a certain distribution around its principal value. The stationary solutions of such systems yield a stationary cell distribution with respect to the genotype. Existence of these stationary distributions is an important property of biological organisms which allows their existence as steady state systems. We prove the existence of stationary solutions in the space of integrable functions decaying at infinity. Biologically this implies that the cell distribution with respect to the genotype decays as the distance from the main genotype increases. Our results show what conditions should be imposed on cell proliferation, mutations and influx to arrive at such distributions. In the context of population dynamics, our results apply also to biological species where individuals are distributed around some average genotype. In this case, the existence of stationary solutions corresponds to the existence of biological species (see [4]).

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