CHAIN ON CONE

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ABSTRACT. We consider a loop of a chain thrown like a lasso on a fixed right circular cone. The system is in the standard homogeneous gravity field. The axis of the cone is vertical. It is shown that under certain vertex angles chain's loop has an oblique equilibrium.

1. Statement of the Problem and Main Theorem

In inheritance from the classical epoch we have obtained two most famous variational problems.

The first one is the catenary problem was formulated and solved by Johann Bernoulli in 1690.

The brachistochrone curve problem was independently considered and solved also approximately about 1690 by Johann Bernoulli, Christiaan Huygens and Gottfried Wilhelm Leibniz.

These both variational problems are simultaneously beautiful non trivial and can be solved by pure analytical means.

This is absolutely exclusive situation: there are very few variational problems that are not artificially composed but shows up from mechanics and can be solved by hands.

In this short note we propose one of such problems.

Ends of thin homogeneous chain are connected to obtain a loop. The mass of the chain is m and its length is l. This loop is putted on a right cone with vertex angle 2α, $\alpha \in (0, \pi/2)$; see Fig. 1. The surface of the cone is smooth. The axis of the cone is vertical. The system is in the standard $mg-$gravity field.

Find all equilibriums of chain's loop on the cone.

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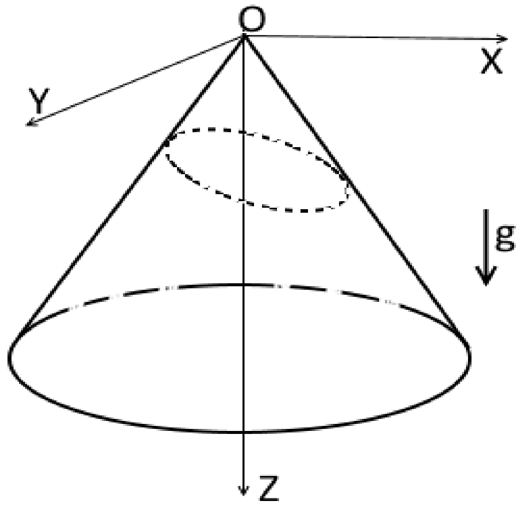

FIGURE 1.

This problem has trivial solution: the loop forms a circle lying in horizontal plane. But do another equilibriums exist?

Theorem 1. *If $\pi/6 < \alpha < \pi/4$, then there is a unique (up to rotations about cone's axis) oblique equilibrium (see Fig. 1).*

For all another α the chain has only trivial equilibrium.

Remark 1. *Numerical simulations show that at this oblique equilibrium the chain is not contained in a plane.*

2. PROOF

Introduce a Cartesian frame $Oxyz$ such that the axis Z is directed down along the cone axis and the origin O is cone's vertex. Introduce also a cylindrical coordinate frame such that

$$(z, r, \psi), \quad x = r\cos\psi, \quad y = r\sin\psi.$$

The cone is given by the equation

$$z = ar, \quad a = \cot\alpha > 0.$$

Let

$$r = r(\psi), \quad r(\psi + 2\pi) = r(\psi)$$

be the equation of the curve described by the chain. Infinitesimal arclength element of such a curve is as follows

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{(1 + a^2)(r')^2 + r^2}d\psi, \quad r' = \frac{d}{d\psi}r(\psi).$$

Then $Z-$coordinate of the centre of mass of the chain is expressed by the formula.

$$Z[r(\cdot)] = \frac{a}{l}\int_0^{2\pi} r(\psi)ds.$$

So we are looking for conditional extremals of the functional $r(\cdot) \mapsto Z[r(\cdot)]$ in class of $2\pi-$periodic functions $r(\psi)$ under the condition

$$\int_0^{2\pi} ds = l. \tag{2.1}$$

The corresponding Lagrange function is

$$L(r, r') = \Big(\frac{a}{l}r + \lambda\Big)\sqrt{(1 + a^2)(r')^2 + r^2},$$

with a Lagrange multiplier λ [1].

Since the Lagrangian L does not depend on ψ we have the following integral of the Lagrange equations

$$h = r'\frac{\partial L}{\partial r'} - L,$$

or

$$-\Big(\frac{a}{l}r + \lambda\Big)r^2 = h\sqrt{(1 + a^2)(r')^2 + r^2}. \tag{2.2}$$

So that the constant h and the expression $-\Big(\frac{a}{l}r + \lambda\Big)$ must have the same sign for all $\psi \in \mathbb{R}$ i.e.

$$\operatorname{sgn} h = -\operatorname{sgn}\Big(\frac{a}{l}r + \lambda\Big). \tag{2.3}$$

Under this assumption (we justify this assumption in the sequel) take square from both sides of (2.2)

$$(1 + a^2)(r')^2 + r^2 - \frac{1}{h^2}\Big(\frac{a}{l}r + \lambda\Big)^2 r^4 = 0,$$

and perform a change of variables $r \mapsto \rho$ by the formula

$$r = \frac{\lambda l}{a}\rho, \quad \rho = \rho(\psi) \tag{2.4}$$

to obtain

$$(1 + a^2)(\rho')^2 + \rho^2 - u^2(\rho + 1)^2\rho^4 = 0, \tag{2.5}$$

where

$$u^2 = \frac{\lambda^4 l^2}{h^2 a^2}. \tag{2.6}$$

The condition (2.1) takes the form

$$\int_0^{2\pi} \sqrt{(1 + a^2)(\rho')^2 + \rho^2}d\psi = \frac{a}{|\lambda|}. \tag{2.7}$$

Our plan is as follows. Choosing a constant u^2 we will find $2\pi-$solution $\rho(\psi)$ of equation (2.5) and then take $|\lambda|$ to satisfy (2.7).By reason that will be clear in the sequel we assume that

$$\lambda < 0. \tag{2.8}$$

Then already known u^2 and $|\lambda|$ we will substitute to (2.6) and find a constant h^2.

The solution ρ must also be such that the expression

$$\Big(\frac{a}{l}r+\lambda\Big)=\lambda(\rho+1)$$

is sign-definite for all ψ. If the solution $\rho(\psi)$ provides this condition then (2.3) is satisfied by choosing the sign of h.

Introducing new variable $t=\psi/\sqrt{a^2+1}$, rewrite equation (2.5) as follows

$$\frac{1}{(\rho+1)^2\rho^4}\dot\rho^2+\frac{1}{\rho^2(\rho+1)^2}=u^2,\quad \dot\rho=\frac{d\rho}{dt}. \tag{2.9}$$

Now we are looking for the solution $\rho(t)$ of equation (2.9) that has period

$$\frac{2\pi}{\sqrt{a^2+1}}.$$

Equation (2.9) has the form of energy integral of a classical mechanical system with kinetic energy

$$E=\frac{1}{(\rho+1)^2\rho^4}\dot\rho^2$$

and a potential energy

$$V(\rho)=\frac{1}{\rho^2(\rho+1)^2}.$$

Use this observation to analyze system (2.9).

Having a graph of the function V we see that the point $C=(-1/2,0)$ on the phase plane $(\rho,\dot\rho)$ is an equilibrium of the type "centre". This equilibrium is surrounded with closed curves that are squeezed between vertical lines $\rho=-1$ and $\rho=0$. These trajectories are marked with parameter u^2. There are no periodic orbits in other domains of the phase space.

Therefore all the periodic solutions satisfy the condition

$$\rho(t)\in(-1,0),\quad t\in\mathbb{R}$$

consequently one has $\rho(t)+1>0$. By virtue of formulas (2.8) and (2.4) it follows that $r>0$.

The equilibrium C corresponds to the value $u^2=16$ and gives the trivial equilibrium of the chain.

Separating variables in (2.9), find period of the trajectory $\rho(t)$ by the formula:

$$T(u)=-2\int_{\rho_-}^{\rho_+}\frac{d\rho}{\rho\sqrt{u^2(\rho+1)^2\rho^2-1}},\quad \rho_\pm=\frac{-1\pm\sqrt{1-4/u}}{2}.$$

The function $T(u)$ is defined for $u>4$. It is not hard to show that the function T is continuous and decreased.

Thus each root u of the equation

$$T(u)=\frac{2\pi}{\sqrt{1+a^2}}$$

corresponds to the solution of (2.9) with desired period. This solution gives the oblique equilibrium.

The proof of the theorem is concluded by a lemma.

Lemma 1. *The following formulas hold*

$$\lim_{u\to 4+} T(u) = \sqrt{2}\pi, \quad \lim_{u\to\infty} T(u) = \pi. \tag{2.10}$$

Indeed, the oblique equilibrium shows up iff

$$\pi < \frac{2\pi}{\sqrt{1+a^2}} < \sqrt{2}\pi,$$

or

$$\pi/6 < \alpha < \pi/4.$$

The theorem is proved.

2.1. **Proof of the Lemma.** The first one of formulas (2.10) is most simple; we prove it by using little bit informal mechanical argument. Rigorous proof of this formula follows by the same manner as we employ below to prove the second formula of (2.10).

Up to third order terms in the neighbourhood of the point C equation (2.9) has the form

$$\dot{\xi}^2 + 2\xi^2 = const, \quad \rho = -\frac{1}{2} + \xi.$$

So that the period of small oscillations is equal to $2\pi/\sqrt{2}$ or

$$\lim_{u\to 4+} T(u) = \sqrt{2}\pi.$$

Let us check the second formula. Observe that

$$T(u) = -\frac{2}{u}\int_{\rho_-}^{\rho_+} \frac{d\rho}{\rho\sqrt{(\rho-\rho_-)(\rho-\rho_+)(\rho-\tilde{\rho}_-)(\rho-\tilde{\rho}_+)}},$$

where

$$\tilde{\rho}_\pm = \frac{-1\pm\sqrt{1+4/u}}{2}.$$

Introducing a small parameter $\epsilon = 1/u \to 0$ as $u \to \infty$, we get

$$\begin{aligned} \rho_+ &= -\epsilon + O(\epsilon^2), & \rho_- &= -1 + \epsilon + O(\epsilon^2), \\ \tilde{\rho}_+ &= \epsilon + O(\epsilon^2), & \tilde{\rho}_- &= -1 - \epsilon + O(\epsilon^2). \end{aligned}$$

For small ϵ the following inequalities hold

$$\rho_+ < -\epsilon, \quad \rho_- > -1 + \epsilon.$$

It is easy to see that

$$T(u) = -2\epsilon\int_{-\frac{1}{2}}^{\rho_+} \frac{d\rho}{\rho\sqrt{(\rho-\rho_-)(\rho-\rho_+)(\rho-\tilde{\rho}_-)(\rho-\tilde{\rho}_+)}} + O(\sqrt{\epsilon}).$$

Transform this integral as follows:

$$\int_{-\frac{1}{2}}^{\rho_+}\frac{d\rho}{\rho\sqrt{(\rho-\rho_-)(\rho-\rho_+)(\rho-\tilde\rho_-)(\rho-\tilde\rho_+)}}$$
$$=\int_{-\frac{1}{2}}^{\rho_+}\frac{(1+O(\epsilon))\,d\rho}{\rho(\rho+1)\sqrt{(\rho-\rho_+)(\rho+\rho_+-\rho_+-\tilde\rho_+)}}.$$

Sinse $-\rho_+-\tilde\rho_+=O(\epsilon^2)$ the last integral equals

$$\int_{-\frac{1}{2}}^{\rho_+}\frac{(1+O(\epsilon))\,d\rho}{\rho(\rho+1)\sqrt{\rho^2-\rho_+^2}}.$$

The integral

$$\int_{-\frac{1}{2}}^{\rho_+}\frac{d\rho}{\rho(\rho+1)\sqrt{\rho^2-\rho_+^2}}$$

is computed explicitly, nevertheless the formula is very large and we do not bring it; write down the asymptotic

$$\int_{-\frac{1}{2}}^{\rho_+}\frac{d\rho}{\rho(\rho+1)\sqrt{\rho^2-\rho_+^2}}=-\frac{\pi}{2\epsilon}+O\Big(\ln\frac{1}{\epsilon}\Big).$$

Gathering all these formulas we yield $T(u)=\pi+O(\sqrt{\epsilon})$.

The Lemma is proved.

References

[1] Bruce van Brunt: The Calculus of Variations, Springer, New York, 2004.