ASYMPTOTICS OF THE BOUND STATE INDUCED BY δ-INTERACTION SUPPORTED ON A WEAKLY DEFORMED PLANE

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Abstract. In this paper we consider the three-dimensional Schrödinger operator with a δ-interaction of strength $\alpha > 0$ supported on an unbounded surface parametrized by the mapping $\mathbb{R}^2 \ni x \mapsto (x, \beta f(x))$, where $\beta \in [0, \infty)$ and $f: \mathbb{R}^2 \to \mathbb{R}$, $f \not\equiv 0$, is a C^2-smooth, compactly supported function. The surface supporting the interaction can be viewed as a local deformation of the plane. It is known that the essential spectrum of this Schrödinger operator coincides with $[-\frac{1}{4}\alpha^2, +\infty)$. We prove that for all sufficiently small $\beta > 0$ its discrete spectrum is non-empty and consists of a unique simple eigenvalue. Moreover, we obtain an asymptotic expansion of this eigenvalue in the limit $\beta \to 0+$. On a qualitative level this eigenvalue tends to $-\frac{1}{4}\alpha^2$ exponentially fast as $\beta \to 0+$.

1. Introduction

1.1. **Motivation.** Various physical systems can be effectively described by Schrödinger operators with δ-interactions supported on sets of zero Lebesgue measure. To mention just a few, these operators are used:

– in mesoscopic physics in the model of leaky quantum graphs [EK, Chap. 10];

– for the description of atoms in strong magnetic fields [BD06];

– in the theory of semiconductors as a model for excitons [HKPC17];

– for the analysis of high contrast photonic crystals [FK96, HL17].

One can expect that this list will keep expanding, in particular, with the simplicity and versatility of the model in mind. This is certainly a motivation to investigate its properties by rigorous mathematical means.

One of the most traditional problems concerns the relation between the geometry of the support of the δ-interaction and the spectrum of the corresponding Schrödinger operator; see

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the review [E08], the monograph [EK], and the references therein. A prominent particular question, addressed in numerous papers (see *e.g.* [BEL14a, EI01, EK03, EL17, OP16, P15]), is to analyze whether bound states below the threshold of the essential spectrum are induced by an attractive δ-interaction supported on an unbounded, asymptotically flat hypersurface.

In the two-dimensional setting, this question is answered affirmatively in [EI01], provided that the asymptotically straight curve is not a straight line. In the space dimension $d \geq 4$, a circular conical surface is a non-trivial example [LO16] of an asymptotically flat hypersurface such that an attractive δ-interaction of any strength, supported on it, induces no bound states. Apparently, the three-dimensional case happens to be the most subtle. In this space dimension, existence of bound states (in fact, infinitely many of them) is shown in [BEL14a, LO16, OP16] for all interaction strengths in the geometric setting of conical surfaces, which is a special class of asymptotically flat surfaces. On the other hand, for the most natural geometric setting of locally deformed planes, existence of at least one bound state below the threshold is proven in [EK03] only in the strong-coupling regime. For the same geometry, the question of existence of bound states below the threshold for an arbitrary strength of an attractive δ-interaction still remains open and challenging.

The aim of this paper is to make one more step towards the complete answer to this open question. Specifically, we prove the existence of bound states induced by δ-interactions supported on locally deformed planes, in the small deformation limit. As a by-product of the proof we obtain that for sufficiently small deformation the discrete spectrum consists of unique simple eigenvalue. Moreover, we derive an asymptotic expansion of this eigenvalue in terms of the profile of the deformation.

Notations. Throughout the paper $g(\beta,\delta) = o_{\rm u}(h(\beta))$ and $g(\beta,\delta) = \mathcal{O}_{\rm u}(h(\beta))$ are standard asymptotic notations in the limit $\beta \to 0+$, which are additionally uniform in $\delta \in [0,1]$. For a Hilbert space $\mathcal{H}$ we denote by $\mathcal{B}(\mathcal{H})$ the space of bounded, everywhere defined linear operators in $\mathcal{H}$. We denote by $(L^2(\mathbb{R}^d), (\cdot,\cdot)_{L^2(\mathbb{R}^d)})$ (respectively, by $(L^2(\mathbb{R}^d;\mathbb{C}^d), (\cdot,\cdot)_{L^2(\mathbb{R}^d;\mathbb{C}^d)})$) the usual L^2-spaces over $\mathbb{R}^d$, $d \in \mathbb{N}$, of scalar-valued (respectively, vector-valued) functions. By $\mathcal{F}\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ we abbreviate the unitary Fourier-Plancherel operator; with a slight abuse of terminology we will refer to it as to Fourier transformation in $\mathbb{R}^2$. In the same vein, for any $\psi \in L^2(\mathbb{R}^2)$ its Fourier transform $\mathcal{F}\psi$ will be denoted by $\widehat{\psi} \in L^2(\mathbb{R}^2)$. By $H^1(\mathbb{R}^d)$ we denote the first order L^2-based Sobolev space over $\mathbb{R}^d$, $d \in \mathbb{N}$. For a C^2-smooth surface $\Gamma \subset \mathbb{R}^3$, $(L^2(\Gamma), (\cdot,\cdot)_{L^2(\Gamma)})$ is the usual L^2-space over Γ, where the inner product $(\cdot,\cdot)_{L^2(\Gamma)}$ is introduced via the canonical Hausdorff measure $\sigma(\cdot)$ on Γ; *cf.* [Le, App. C.8].

1.2. **The spectral problem for δ-interaction supported on a locally deformed plane.** Let $\Gamma = \Gamma_\beta(f) \subset \mathbb{R}^3$, with $\beta \in [0,\infty)$, be an unbounded surface given by

$$\Gamma := \Big\{(x_1,x_2,x_3) \in \mathbb{R}^3 \colon x_3 = \beta f(x_1,x_2)\Big\} \subset \mathbb{R}^3\,, \tag{1.1}$$

where $f\colon \mathbb{R}^2 \to \mathbb{R}$ ($f \not\equiv 0$) is a C^2-smooth, compactly supported function. The surface Γ can be viewed as a local deformation of the plane $\mathbb{R}^2 \times \{0\}$. We also point out that in view of the identity $\Gamma_{-\beta}(f) = \Gamma_\beta(-f)$ it is enough to consider non-negative values β only. In what follows we set $\mathcal{S} := \operatorname{supp} f$ and denote by $L_f > 0$ the Lipschitz constant of f; *i.e.* the minimal positive number such that $|f(x) - f(y)| \le L_f|x-y|$ holds for all $x, y \in \mathbb{R}^2$. By the mean-value theorem we infer that the inequality $|\nabla f| \le L_f$ holds pointwise. Taking the smoothness of Γ into account, it is not difficult to check that the mapping $\Omega \mapsto \sigma(\Omega \cap \Gamma)$ defines a measure on $\mathbb{R}^3$, which belongs to the *generalized Kato class*; *cf.* [BEKŠ94, Sec. 2].

Let a constant $\alpha > 0$ be fixed. According to [BEKŠ94, Sec. 2] and also to [BEL14b, Prop 3.1], the symmetric quadratic form

$$H^1(\mathbb{R}^3) \ni u \mapsto \mathfrak{h}_{\alpha,\beta}[u] := \|\nabla u\|^2_{L^2(\mathbb{R}^3;\mathbb{C}^3)} - \alpha\|u|_\Gamma\|^2_{L^2(\Gamma)}\,, \tag{1.2}$$

is closed, densely defined, symmetric, and semi-bounded in $L^2(\mathbb{R}^3)$; here $u|_\Gamma$ denotes the trace of u onto Γ. Recall that the trace map $H^1(\mathbb{R}^3) \ni u \to u|_\Gamma \in L^2(\Gamma)$ is well defined and continuous [McL, Thm. 3.38]. Now we are in position to define the Hamiltonian with δ-interaction supported on Γ, the main object of the present paper.

Definition 1.1. *The self-adjoint Schrödinger operator $\mathsf{H}_{\alpha,\beta}$ in $L^2(\mathbb{R}^3)$ corresponding to the formal differential expression $-\Delta - \alpha\,\delta(x-\Gamma)$, $\alpha > 0$, is defined via the first representation theorem* [Ka, Thm. VI 2.1] *as associated with the quadratic form $\mathfrak{h}_{\alpha,\beta}$ in* (1.2).

The surface Γ is referred to as the support of the δ-interaction and the constant $\alpha > 0$ is usually called the strength of this interaction. Schrödinger operators with δ-interactions supported on locally deformed planes were first investigated in [EK03] and then subsequently in [BEHL17, BEL14b, E17]. In the following theorem we collect some previously known fundamental spectral properties of $\mathsf{H}_{\alpha,\beta}$.

Theorem 1.1. *The spectrum of the self-adjoint operator $\mathsf{H}_{\alpha,\beta}$ introduced in Definition 1.1 is characterised as follows.*

(i) $\sigma_{\rm ess}(\mathsf{H}_{\alpha,\beta}) = \big[-\frac14\alpha^2, +\infty\big)$.

(ii) $\sigma_{\rm d}(\mathsf{H}_{\alpha,\beta}) \ne \varnothing$ *for all $\beta > 0$ and all $\alpha > 0$ large enough.*

(iii) $\sigma_{\rm d}(\mathsf{H}_{\alpha,0}) = \varnothing$ *for $\beta = 0$ and all $\alpha > 0$.*

For a proof of item (i) see [EK03, Thm. 4.1, Rem. 4.2] and [BEL14b, Thm. 4.10]. A proof of item (ii) can be found in [EK03, Thm. 4.3]. The claim of (iii) easily follows via separation of variables. Our considerations are inspired by the open question, whether $\sigma_{\mathrm{d}}(\mathsf{H}_{\alpha,\beta}) \neq \varnothing$ holds for all $\alpha, \beta > 0$; *cf.* [E08, Problem 7.5].

1.3. **Main result.** Informally speaking, the main result of this paper says that the discrete spectrum of $\mathsf{H}_{\alpha,\beta}$ consists of exactly one simple eigenvalue for all sufficiently small $\beta > 0$. Moreover, an asymptotic expansion of this eigenvalue in terms of α, β and of the function f is found. In order to formulate this result precisely, we denote by $\lambda_1^\alpha(\beta)$ the lowest spectral point of $\mathsf{H}_{\alpha,\beta}$.

Theorem 1.2. *Let $\alpha > 0$ be fixed and let the self-adjoint operator $\mathsf{H}_{\alpha,\beta}$ be as in Definition 1.1. Set*

$$\mathcal{D}_{\alpha,f} := \int_{\mathbb{R}^2} |p|^2 \left(\alpha^2 - \frac{2\alpha^3}{\sqrt{4|p|^2 + \alpha^2} + \alpha} \right) |\widehat{f}(p)|^2 \mathrm{d}p > 0,$$

where $\widehat{f}$ is the Fourier transform of f. Then $\#\sigma_{\mathrm{d}}(\mathsf{H}_{\alpha,\beta}) = 1$ holds for all sufficiently small $\beta > 0$ and, moreover, $\lambda_1^\alpha(\beta)$ admits the asymptotic expansion

$$\lambda_1^\alpha(\beta) = -\frac{\alpha^2}{4} - \exp\left(-\frac{16\pi}{\mathcal{D}_{\alpha,f}\beta^2} \right) \big(1 + o(1)\big), \qquad \beta \to 0+ \,. \tag{1.3}$$

The proof of this result relies on the Birman-Schwinger principle [BEKŠ94] for $\mathsf{H}_{\alpha,\beta}$. Inspired by the technique developed in [BFK$^+$17, CK11, EK02, EK08, EK15], we take the advantage of rewriting the Birman-Schwinger condition in the perturbative form, in which the resolvent of the two-dimensional free Laplacian appears. A technically demanding step is to expand this new condition with respect to the small parameter β. Following the strategy similar in spirit to the one used in [S76], we derive from this condition an implicit scalar equation on the principal eigenvalue of $\mathsf{H}_{\alpha,\beta}$. Careful inspection of this equation yields the existence and uniqueness of its solution for all sufficiently small $\beta > 0$, as well as the expansion of this unique solution in the asymptotic regime $\beta \to 0+$. Surprisingly, an integral representation of the relativistic Schrödinger operator [IT93] arises in this asymptotic analysis. The obtained implicit equation seems to be of an independent interest, because it allows to extract more terms in the asymptotic expansion for $\lambda_1^\alpha(\beta)$. However, we will not elaborate on this point here.

Organisation of the paper. In Section 2 we recall the standard formulation of the Birman-Schwinger principle for the Hamiltonian $\mathsf{H}_{\alpha,\beta}$ and employ it to obtain a useful lower bound on $\lambda_1^\alpha(\beta)$. Furthermore, we derive a perturbative reformulation of the Birman-Schwinger principle and expand the new Birman-Schwinger condition with respect to the small parameter β. In Section 3 we prove our main result, formulated in Theorem 1.2. We conclude the paper by Section 4 containing a discussion on possible generalizations of the obtained results.

2. Birman-Schwinger principle

2.1. **Standard formulation.** Birman-Schwinger principle (BS-principle in what follows) is a powerful tool for the spectral analysis of Schrödinger operators. Its generalization, which covers δ-interactions supported on hypersurfaces, is derived in [BEKŠ94]; see [BLL13, B95, Po01] for some refinements.

In what follows, let $\lambda < 0$ and set $\kappa := \sqrt{-\lambda}$. Green's function corresponding to the differential expression $-\Delta + \kappa^2$ in $\mathbb{R}^3$ has the following well known form

$$G_\kappa(x-y) = \frac{e^{-\kappa|x-y|}}{4\pi|x-y|}.$$

Let the surface $\Gamma = \Gamma_\beta(f) \subset \mathbb{R}^3$ be as in (1.1). Parametrizing Γ by the mapping

$$r_\beta \colon \mathbb{R}^2 \to \mathbb{R}^3, \qquad r_\beta(x) := (x, \beta f(x)), \tag{2.1}$$

we can naturally express the surface measure on Γ through the Lebesgue measure on $\mathbb{R}^2$ via the relation $d\sigma(x) = g_\beta(x)dx$, where the Jacobian g_β is explicitly given by

$$g_\beta(x) = \left(1 + \beta^2|\nabla f(x)|^2\right)^{1/2}.$$

Next we introduce the *weakly singular integral operator* $\mathsf{Q}_\beta(\kappa) \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $\kappa > 0$, acting as

$$(\mathsf{Q}_\beta(\kappa)\psi)(x) := \int_{\mathbb{R}^2} g_\beta(x)^{1/2} G_\kappa(r_\beta(x) - r_\beta(y))\, g_\beta(y)^{1/2}\psi(y)dy. \tag{2.2}$$

Note that the linear mapping $\mathsf{J}_\beta \colon L^2(\Gamma) \to L^2(\mathbb{R}^2)$, $(\mathsf{J}_\beta\psi)(x) = g_\beta(x)^{1/2}\psi(r_\beta(x))$, is an isometric isomorphism and it is not difficult to check that $\mathsf{Q}_\beta(\kappa) = \mathsf{J}_\beta R_{mm}(\mathsf{i}\kappa)\mathsf{J}_\beta^{-1}$, where the operator $R_{mm}(\mathsf{i}\kappa) \colon L^2(\Gamma) \to L^2(\Gamma)$ is defined as

$$(R_{mm}(\mathsf{i}\kappa)\psi)(x) := \int_\Gamma G_\kappa(x-y)\psi(y)d\sigma(y).$$

In fact, $R_{mm}(\mathsf{i}\kappa)$ is the Birman-Schwinger operator introduced in [BEKŠ94, Sec. 2], see also [B95]. In view of this identification, we get from [B95] that $\mathsf{Q}_\beta(\kappa)$ is a bounded, self-adjoint, non-negative operator in $L^2(\mathbb{R}^2)$. Next theorem contains a BS-principle for the Schrödinger operator $\mathsf{H}_{\alpha,\beta}$ in Definition 1.1. We remark that while this formulation of the BS-principle is not the same as in [BEKŠ94, Lem. 2.3 (iv)] and [B95, Lem. 1], it can be easily derived from those claims using the identity $\mathsf{Q}_\beta(\kappa) = \mathsf{J}_\beta R_{mm}(\mathsf{i}\kappa)\mathsf{J}_\beta^{-1}$.

Theorem 2.1. *Let the self-adjoint operator $\mathsf{H}_{\alpha,\beta}$ be as in Definition 1.1 and the operator-valued function $\mathbb{R}_+ \ni \kappa \mapsto \mathsf{Q}_\beta(\kappa)$ be as in (2.2). Then it holds that*

$$\forall\, \kappa > 0, \qquad \dim\ker\left(\mathsf{H}_{\alpha,\beta} + \kappa^2\right) = \dim\ker\left(\mathsf{I} - \alpha\mathsf{Q}_\beta(\kappa)\right).$$

In the following lemma we recall the properties of $\mathsf{Q}_0(\kappa)$ (*i.e.* for $\beta = 0$). Since these properties are easy to prove and difficult to find in the literature we provide a short argument.

Lemma 2.2. *The operator* $\mathsf{Q}_0(\kappa)$ *is unitarily equivalent via the Fourier transformation to the multiplication operator in* $\mathbb{R}^2$ *with the function*

$$\mathbb{R}^2 \ni p \mapsto \frac{1}{2\sqrt{|p|^2+\kappa^2}}.$$

In particular, $\sigma(\mathsf{Q}_0(\kappa)) = [0, \frac{1}{2\kappa}]$ *and the operator-valued function* $\mathbb{R}_+ \ni \kappa \mapsto \mathsf{Q}_0(\kappa)$ *is real-analytic.*

Proof. Recall that for the Fourier transform of the convolution of $\psi_1, \psi_2 \in L^2(\mathbb{R}^2)$ we have $\mathcal{F}(\psi_1 \star \psi_2) = \widehat{\psi}_1\widehat{\psi}_2$. Using this formula and the fact that $p \mapsto \frac{1}{2\sqrt{|p|^2+\kappa^2}}$ is the Fourier transform of $\mathbb{R}^2 \ni x \mapsto \frac{e^{-\kappa|x|}}{4\pi|x|}$ we get for any $\psi \in L^2(\mathbb{R}^2)$

$$\begin{aligned}\mathsf{Q}_0(\kappa)\psi &= \mathcal{F}^{-1}\mathcal{F}\int_{\mathbb{R}^2} G_\kappa(\cdot - y)\psi(y)\mathrm{d}y \\ &= \mathcal{F}^{-1}\left(\mathcal{F}\left(\frac{e^{-\kappa|\cdot|}}{4\pi|\cdot|}\right)\widehat{\psi}\right) = \mathcal{F}^{-1}\left(\frac{\widehat{\psi}}{2\sqrt{|\cdot|^2+\kappa^2}}\right),\end{aligned}$$

and the main claim of the lemma immediately follows. The analyticity of $\mathbb{R}_+ \ni \kappa \mapsto \mathsf{Q}_0(\kappa)$ is a consequence of the same property of the operator-valued function of multiplication by $\mathbb{R}_+ \ni \kappa \mapsto \frac{1}{2\sqrt{|p|^2+\kappa^2}}$. Moreover, we have

$$\sigma(\mathsf{Q}_0(\kappa)) = \overline{\left\{\lambda \in \mathbb{R} \colon \lambda = \tfrac{1}{2\sqrt{|p|^2+\kappa^2}} \text{ for some } p \in \mathbb{R}^2\right\}} = \left[0, \tfrac{1}{2\kappa}\right]. \qquad \square$$

By means of the BS-principle in Theorem 2.1 we obtain a useful lower bound on the lowest spectral point $\lambda_1^\alpha(\beta)$ of $\mathsf{H}_{\alpha,\beta}$.

Proposition 2.1. *Let* $\lambda_1^\alpha(\beta)$ *be the lowest spectral point of the self-adjoint operator* $\mathsf{H}_{\alpha,\beta}$ *introduced in Definition 1.1. Then the following lower bound*

$$\lambda_1^\alpha(\beta) \geq -\frac{\alpha^2}{4}\left(1 + \beta^2 L_f^2\right)$$

holds for all $\alpha, \beta > 0$*. In particular,* $\lambda_1^\alpha(\beta) \to -\frac{1}{4}\alpha^2-$ *as* $\beta \to 0+$*.*

Proof. In view of Theorem 1.1 (i) we clearly have $\lambda_1^\alpha(\beta) \leq -\frac{1}{4}\alpha^2$ and if $\lambda_1^\alpha(\beta) < -\frac{1}{4}\alpha^2$, then necessarily $\lambda_1^\alpha(\beta) \in \sigma_{\mathrm{d}}(\mathsf{H}_{\alpha,\beta})$ holds. Applying the Schur test [Te, Lem. 0.32] for the operator $\alpha\mathsf{Q}_\beta(\kappa)$ we get, using monotonicity of $G_\kappa(\cdot)$ in combination with the inequalities

$|r_\beta(x) - r_\beta(y)| \geq |x-y|$ and $|\nabla f| \leq L_f$, the following bound

$$\begin{aligned}\|\alpha \mathsf{Q}_\beta(\kappa)\| &\leq \alpha(1+\beta^2 L_f^2)^{1/2} \sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^2} G_\kappa(r_\beta(x) - r_\beta(y))\mathrm{d}y \\ &\leq \alpha(1+\beta^2 L_f^2)^{1/2} \sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\mathrm{e}^{-\kappa|x-y|}}{4\pi|x-y|}\mathrm{d}y \\ &= \alpha(1+\beta^2 L_f^2)^{1/2} \int_{\mathbb{R}^2} \frac{\mathrm{e}^{-\kappa|y|}}{4\pi|y|}\mathrm{d}y \\ &= \alpha(1+\beta^2 L_f^2)^{1/2} \frac{1}{2}\int_0^\infty e^{-\kappa r}\mathrm{d}r = \frac{\alpha}{2\kappa}(1+\beta^2 L_f^2)^{1/2}.\end{aligned}$$

Consequently, for $\kappa > \frac{\alpha}{2}(1+\beta^2 L_f^2)^{1/2}$ holds $\|\alpha \mathsf{Q}_\beta(\kappa)\| < 1$ and by the BS-principle in Theorem 2.1 we get $-\kappa^2 \notin \sigma_{\mathrm{d}}(\mathsf{H}_{\alpha,\beta})$. Finally, we conclude that

$$\lambda_1^\alpha(\beta) \geq -\frac{\alpha^2}{4}(1+\beta^2 L_f^2). \qquad \square$$

2.2. **Perturbative reformulation.** In our considerations it is convenient to deal with a perturbative reformulation of the BS-principle. This technique has already been successfully applied in [BFK+17, CK11, EK02, EK08, EK15] for the case of interactions supported on curves. To this aim, for $\kappa \geq \frac{1}{2}\alpha$ we set $\delta := \sqrt{\kappa^2 - \frac{1}{4}\alpha^2}$ and define the operator-valued function

$$\mathsf{D}_\beta(\delta) := \mathsf{Q}_\beta(\kappa) - \mathsf{Q}_0(\kappa),$$

which is real-analytic in $\delta \in (0,\infty)$ by [BLLR]; see also [BLLR17, Prop. 3.2 (i)] and [BL12, Props. 6.13 and 6.14]. Moreover, the expression for g_β yields that $\mathsf{D}_\beta(\delta)$ is real-analytic in $\beta \in (0,\infty)$ as well. Next, for $\kappa > \frac{1}{2}\alpha$ we define

$$\mathsf{B}_\alpha(\delta) := \Big(\mathsf{I} - \alpha\mathsf{Q}_0(\kappa)\Big)^{-1}, \tag{2.3}$$

where existence and boundedness of the inverse for $\mathsf{I} - \alpha\mathsf{Q}_0(\kappa)$ are guaranteed by Lemma 2.2.

With these auxiliary operators in hands, we have

$$\dim\ker(\mathsf{I} - \alpha\mathsf{Q}_\beta(\kappa)) = \dim\ker\Big(\mathsf{I} - \alpha\mathsf{Q}_0(\kappa) - \alpha\mathsf{D}_\beta(\delta)\Big) = \dim\ker\Big(\mathsf{I} - \alpha\mathsf{B}_\alpha(\delta)\mathsf{D}_\beta(\delta)\Big).$$

Thus, the BS-principle formulated in Theorem 2.1 yields

$$\forall\, \kappa > \frac{\alpha}{2}, \qquad \dim\ker\Big(\mathsf{H}_{\alpha,\beta} + \kappa^2\Big) = \dim\ker\Big(\mathsf{I} - \alpha\mathsf{B}_\alpha(\delta)\mathsf{D}_\beta(\delta)\Big). \tag{2.4}$$

In the next lemma we collect the properties of the operator family $\mathsf{B}_\alpha(\delta)$. In the following, we denote by $-\Delta_{\mathbb{R}^2}$ the usual self-adjoint free Laplacian in $L^2(\mathbb{R}^2)$, whose resolvent is abbreviated by $\mathsf{R}(z) := (-\Delta_{\mathbb{R}^2} + z)^{-1}$ for $z > 0$.

Lemma 2.3. *The operator* $\mathsf{B}_\alpha(\delta)$, $\delta > 0$, *in* (2.3) *admits the representation:*

$$\mathsf{B}_\alpha(\delta) = \frac{\alpha^2}{2}\mathsf{R}(\delta^2) + \mathsf{N}_\alpha(\delta) \tag{2.5}$$

with

$$\mathsf{N}_\alpha(\delta) := 1 + \alpha\mathsf{R}\Big(\delta^2 + \tfrac{1}{4}\alpha^2\Big)^{1/2}\Big(\alpha\mathsf{R}\Big(\delta^2 + \tfrac{1}{4}\alpha^2\Big)^{1/2} + 2\Big)^{-1}, \qquad \delta \geq 0.$$

Moreover, the operator-valued function $\mathsf{N}_\alpha(\delta)$ *satisfies the following properties.*

(i) *The estimate* $\|\mathsf{N}_\alpha(\delta)\| \leq \frac{3}{2}$ *is valid for all* $\delta \geq 0$.

(ii) *The convergence* $\mathsf{N}_\alpha(\delta) \to \mathsf{N}_\alpha(0)$ *holds in the operator norm as* $\delta \to 0+$.

(iii) $(0,\infty) \ni \delta \mapsto \mathsf{N}_\alpha(\delta)$ *is real-analytic.*

(iv) *The estimate*[1] $\|\partial_\delta \mathsf{N}_\alpha(\delta)\| \leq \frac{\delta}{\alpha^2}$ *is valid for all* $\delta \geq 0$.

In particular, representation (2.5) *yields real-analyticity of* $\mathsf{B}_\alpha(\delta)$ *with respect to* $\delta \in (0,\infty)$.

Proof. By Lemma 2.2, the operator $\mathsf{B}_\alpha(\delta)$ is unitarily equivalent (via the Fourier transformation) to the operator of multiplication with the function

$$f_{\alpha,\delta}(p) := \left(1 - \frac{\alpha}{2\tau_{\alpha,\delta}(p)}\right)^{-1},$$

where $\tau_{\alpha,\delta}(p) := \sqrt{|p|^2 + \delta^2 + \frac{1}{4}\alpha^2}$. Note that the function $f_{\alpha,\delta}$ can be decomposed as $f_{\alpha,\delta}(p) = m_{\alpha,\delta}(p) + n_{\alpha,\delta}(p)$ with

$$m_{\alpha,\delta}(p) := \frac{\alpha^2}{2(|p|^2+\delta^2)} \qquad \text{and} \qquad n_{\alpha,\delta}(p) := 1 + \frac{\alpha}{2\tau_{\alpha,\delta}(p) + \alpha}.$$

Observe that we have

$$n_{\alpha,\delta}(p) \leq n_{\alpha,\delta}(0) = 1 + \frac{\alpha}{2\tau_{\alpha,\delta}(0) + \alpha} \leq 1 + \frac{\alpha}{2\tau_{\alpha,0}(0) + \alpha} = 1 + \frac{1}{2} = \frac{3}{2}. \tag{2.6}$$

Clearly, the operators of multiplication with $m_{\alpha,\delta}$ and with $n_{\alpha,\delta}$ are unitarily equivalent via the inverse Fourier transformation to $\frac{\alpha^2}{2}\mathsf{R}(\delta^2)$ and to $\mathsf{N}_\alpha(\delta)$, respectively. In particular, an upper bound in (i) holds, thanks to (2.6).

The estimate

$$\begin{aligned}\|\mathsf{N}_\alpha(\delta) - \mathsf{N}_\alpha(0)\| &= \sup_{p\in\mathbb{R}^2}\left|\frac{\alpha}{2\tau_{\alpha,\delta}(p)+\alpha} - \frac{\alpha}{2\tau_{\alpha,0}(p)+\alpha}\right| \\ &\leq \frac{1}{2\alpha}\sup_{p\in\mathbb{R}^2}|\tau_{\alpha,0}(p) - \tau_{\alpha,\delta}(p)| \leq \frac{\delta^2}{2\alpha}\sup_{p\in\mathbb{R}^2}\frac{1}{\tau_{\alpha,0}(p)+\tau_{\alpha,\delta}(p)} \leq \frac{\delta^2}{2\alpha^2},\end{aligned}$$

[1] Here and in the following we define the derivative of an operator-valued function $\mathbb{R}_+ \ni \delta \mapsto \mathsf{A}(\delta)$ as the limit in the operator-norm of the fraction $\frac{\mathsf{A}(\delta')-\mathsf{A}(\delta)}{\delta'-\delta}$ as $\delta' \to \delta$.

implies the convergence in (ii). Analyticity of $(0,\infty)\ni\delta\mapsto \mathsf{R}(\frac{1}{4}\alpha^2+\delta^2)$ yields the claim of (iii).

Define $\partial_\delta \mathsf{N}_\alpha(\delta)\colon L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ as the operator being unitarily equivalent via the Fourier transformation to the multiplication with the function

$$\partial_\delta n_{\alpha,\delta}(p) = -\frac{2\alpha\delta}{\left(2\tau_{\alpha,\delta}(p)+\alpha\right)^2\tau_{\alpha,\delta}(p)}.$$

Next, we show that the operator $\partial_\delta \mathsf{N}_\alpha(\delta)$ defined as above satisfies

$$\lim_{\delta'\to\delta}\left\|\frac{\mathsf{N}_\alpha(\delta')-\mathsf{N}_\alpha(\delta)}{\delta'-\delta}-\partial_\delta\mathsf{N}_\alpha(\delta)\right\| = 0\,. \tag{2.7}$$

Applying the mean-value theorem we obtain

$$\left|\frac{n_{\alpha,\delta'}(p)-n_{\alpha,\delta}(p)}{\delta'-\delta}-\partial_\delta n_{\alpha,\delta}(p)\right| = \left|\partial^2_{\delta\delta}n_{\alpha,\delta_\star}(p)(\delta'-\delta)\right|,$$

where $\delta_\star\in(0,\infty)$ lies between δ and δ'. A straightforward calculations shows

$$\partial^2_{\delta\delta}n_{\alpha,\delta}(p) = -\frac{2\alpha}{(2\tau_{\alpha,\delta}(p)+\alpha)^2\tau_{\alpha,\delta}(p)}+\frac{8\alpha\delta^2}{(2\tau_{\alpha,\delta}(p)+\alpha)^3\tau_{\alpha,\delta}(p)^2}+\frac{2\alpha\delta^2}{(2\tau_{\alpha,\delta}(p)+\alpha)^2\tau_{\alpha,\delta}(p)^3},$$

which implies

$$\left|\partial^2_{\delta\delta}n_{\alpha,\delta}(p)\right|\le\frac{3}{\alpha^2},$$

and hence

$$\sup_{p\in\mathbb{R}^2}\left|\frac{n_{\alpha,\delta'}(p)-n_{\alpha,\delta}(p)}{\delta'-\delta}-\partial_\delta n_{\alpha,\delta}(p)\right| = \frac{3}{\alpha^2}\left|\delta'-\delta\right|.$$

This completes the verification of (2.7).

Finally, we get

$$\begin{aligned}\|\partial_\delta\mathsf{N}_\alpha(\delta)\| &= \sup_{p\in\mathbb{R}^2}\frac{2\alpha\delta}{\left(2\tau_{\alpha,\delta}(p)+\alpha\right)^2\tau_{\alpha,\delta}(p)}\\ &= \frac{2\alpha\delta}{\left(2\tau_{\alpha,\delta}(0)+\alpha\right)^2\tau_{\alpha,\delta}(0)}\le\frac{2\alpha\delta}{\left(2\tau_{\alpha,0}(0)+\alpha\right)^2\tau_{\alpha,0}(0)}=\frac{\delta}{\alpha^2},\end{aligned}$$

and the claim of (iv) is proven. □

In what follows, we identify $x\in\mathbb{R}^2$ with $(x,0)\in\mathbb{R}^3$. For a given $V\colon\mathbb{R}^2\to\mathbb{R}_+$ we introduce the integral kernels

$$\mathsf{D}_{\beta,V}(\delta)(x,y) := V(x)\mathsf{D}_\beta(\delta)(x,y)V(y) \tag{2.8a}$$

$$\mathsf{D}^{(1)}_V(\delta)(x,y) := V(x)G_\kappa(x-y)V(y)E(x,y), \tag{2.8b}$$

where

$$E(x,y) := \frac{|\nabla f(x)|^2+|\nabla f(y)|^2}{4}-\frac{|f(x)-f(y)|^2(\kappa|x-y|+1)}{2|x-y|^2}.$$

Furthermore, we work out a representation for the operator-valued function $\mathsf{D}_{\beta,V}(\delta)$ associated with the kernel in (2.8a) under certain limitation on the growth for V.

Proposition 2.2. *Let $V\colon \mathbb{R}^2 \to \mathbb{R}_+$ satisfy $V(x) \le c\exp\left(\frac{\alpha}{4}|x|\right)$ for all $x \in \mathbb{R}^2$ with some constant $c > 0$. Let the integral kernels $\mathsf{D}_{\beta,V}(\delta)$ and $\mathsf{D}_V^{(1)}(\delta)$, $\delta \in [0,1]$, $\beta \in (0,1]$, be as in (2.8). Then there exist constants $C_j = C_j(\alpha, f, c) > 0$, $j = 1,2,3$, such that the following claims hold.*

(i) *For all $x, y \in \mathbb{R}^2$, the pointwise bound*

$$|\mathsf{D}_V^{(1)}(\delta)(x,y)| \le C_1 G_{\frac{\alpha}{4}}(x-y)\left[1 + \frac{1}{2}\kappa|x-y|\right], \tag{2.9}$$

holds, the kernel $\mathsf{D}_V^{(1)}(\delta)(x,y)$ defines the self-adjoint operator $\mathsf{D}_V^{(1)}(\delta) \in \mathcal{B}(L^2(\mathbb{R}^2))$, and, in addition, $\|\mathsf{D}_V^{(1)}(\delta)\| \le C_2$.

(ii) *For all $x, y \in \mathbb{R}^2$, the decomposition*

$$\mathsf{D}_{\beta,V}(\delta)(x,y) = \mathsf{D}_V^{(1)}(\delta)(x,y)\beta^2 + \mathsf{D}_{\beta,V}^{(2)}(\delta)(x,y)\beta^4 \tag{2.10}$$

holds, the kernel $\mathsf{D}_{\beta,V}^{(2)}(\delta)(x,y)$ defines the self-adjoint operator $\mathsf{D}_{\beta,V}^{(2)}(\delta) \in \mathcal{B}(L^2(\mathbb{R}^2))$, and, in addition, $\|\mathsf{D}_{\beta,V}^{(2)}(\delta)\| \le C_3$.

In particular, the kernel $\mathsf{D}_{\beta,V}(\delta)(x,y)$ induces the self-adjoint operator

$$\mathsf{D}_{\beta,V}(\delta) = \mathsf{D}_V^{(1)}(\delta)\beta^2 + \mathsf{D}_{\beta,V}^{(2)}(\delta)\beta^4 \in \mathcal{B}(L^2(\mathbb{R}^2)).$$

Proof. (i) Recall that $\mathcal{S} = \operatorname{supp} f$ and let $\mathcal{B}_R \subset \mathbb{R}^2$ be an open ball of the radius $R > 0$ centred at the origin such that the inclusion $\mathcal{S} \subset \mathcal{B}_R$ holds. The subset of $\mathbb{R}^2 \times \mathbb{R}^2$, where the factor $E(x,y)$ in the expression (2.8b) for $\mathsf{D}_V^{(1)}(\delta)(x,y)$ is not equal to zero can be covered by two (intersecting) sets

$$\mathcal{U} := \mathcal{B}_R \times \mathcal{B}_R \quad \text{and} \quad \mathcal{V} := (\mathcal{S} \times \mathcal{B}_R^{\mathrm{c}}) \cup (\mathcal{B}_R^{\mathrm{c}} \times \mathcal{S}), \tag{2.11}$$

where $\mathcal{B}_R^{\mathrm{c}} := \mathbb{R}^2 \setminus \overline{\mathcal{B}_R}$. Applying the bound $\frac{1}{4}|x-y| > \frac{1}{4}(|x| + |y|) - \frac{1}{2}R$ (valid for all $(x,y) \in \mathcal{V}$) we get

$$V(x)G_\kappa(x-y)V(y) \le c^2 \exp\left(\frac{\alpha R}{2}\right) G_{\frac{\alpha}{4}}(x-y), \qquad \forall (x,y) \in \mathcal{U} \cup \mathcal{V}, \tag{2.12}$$

where we also used monotonicity of Green's function with respect to κ.

Employing the inequality $|\nabla f| \le L_f$ we can pointwise estimate the factor E by

$$|E(x,y)| \le L_f^2 \left[1 + \frac{1}{2}\kappa|x-y|\right]. \tag{2.13}$$

Combining (2.12) and (2.13) we get the bound (2.9) with

$$C_1 := c^2 L_f^2 \exp\left(\frac{\alpha R}{2}\right). \tag{2.14}$$

Taking into account that the integral kernel of $\mathsf{D}_V^{(1)}(\delta)$ is symmetric, we obtain from (2.9) using the Schur test that

$$\begin{aligned}\|\mathsf{D}_V^{(1)}(\delta)\| &\le C_1 \int_{\mathbb{R}^2} G_{\frac{\alpha}{4}}(x)\mathrm{d}x + C_1\frac{\kappa}{2}\int_{\mathbb{R}^2} G_{\frac{\alpha}{4}}(x)|x|\mathrm{d}x \\ &= C_1\frac{1}{2}\int_0^\infty e^{-\frac{\alpha}{4}r}\mathrm{d}r + C_1\frac{\kappa}{4}\int_0^\infty e^{-\frac{\alpha}{4}r} r\mathrm{d}r \\ &= C_1\left(\frac{2}{\alpha}+\frac{4\kappa}{\alpha}\right) \le C_1\left(\frac{6}{\alpha}+2\right) =: C_2,\end{aligned}$$

in the last step of the above estimates we used that $\kappa = \sqrt{\frac{1}{4}\alpha^2+\delta^2} \le \frac{1}{2}\alpha + \delta \le \frac{1}{2}\alpha + 1$ for all $\delta \in [0,1]$. Thus, the kernel $\mathsf{D}_V^{(1)}(\delta)(x,y)$ defines the operator $\mathsf{D}_V^{(1)}(\delta) \in \mathcal{B}(L^2(\mathbb{R}^2))$. Self-adjointness of $\mathsf{D}_V^{(1)}(\delta)$ is a consequence of the identity $\mathsf{D}_V^{(1)}(\delta)(x,y) = \mathsf{D}_V^{(1)}(\delta)(y,x)$.

(ii) For $x,y \in \mathbb{R}^2$, we introduce $\rho_\beta(x,y) := |r_\beta(x) - r_\beta(y)|^2$, where the mapping $r_\beta\colon \mathbb{R}^2 \to \mathbb{R}^3$ is as in (2.1). A simple computation yields

$$\rho_\beta(x,y) = |x-y|^2 + |f(x)-f(y)|^2\beta^2.$$

Furthermore, we define the function $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ by $F(s) := \frac{\mathrm{e}^{-\kappa\sqrt{s}}}{4\pi\sqrt{s}}$ and compute its first and second derivatives

$$\begin{aligned}F'(s) &= -\frac{\mathrm{e}^{-\kappa\sqrt{s}}\left(\kappa s^{1/2}+1\right)}{8\pi s^{3/2}} = -F(s)\frac{\kappa s^{1/2}+1}{2s}, \\ F''(s) &= \frac{\mathrm{e}^{-\kappa\sqrt{s}}\left[\kappa^2 s + 3\kappa s^{1/2}+3\right]}{16\pi s^{5/2}} = F(s)\frac{\kappa^2 s+3\kappa s^{1/2}+3}{4s^2}.\end{aligned}$$

Taylor expansion of $F(\cdot)$ in the vicinity of $s \in (0,\infty)$ with the remainder in the Lagrange form reads as follows

$$F(t) = F(s) + F'(s)(t-s) + F''(s+\theta\cdot(t-s))\frac{(t-s)^2}{2}, \qquad \theta = \theta(s,t) \in (0,1).$$

For $x,y \in \mathbb{R}^2$ we define an auxiliary function $\mu\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}_+$ by

$$\mu(x,y) := \rho_0(x,y) + \theta(\rho_0(x,y),\rho_\beta(x,y))|f(x)-f(y)|^2\beta^2.$$

For the sake of brevity we denote

$$H(x,y) := \frac{|f(x)-f(y)|^2(\kappa|x-y|+1)}{2|x-y|^2}, \quad K(x,y) := \frac{|\nabla f(x)|^2+|\nabla f(y)|^2}{4},$$

and

$$K_1(x,y) := (K(x,y) - H(x,y))\,\beta^2 = E(x,y)\beta^2, \tag{2.15a}$$

$$K_2(x,y) := (g_\beta(x)g_\beta(y))^{1/2} - 1 - \beta^2 K(x,y), \tag{2.15b}$$

$$K_3(x,y) := H(x,y)\left(1 - (g_\beta(x)g_\beta(y))^{1/2}\right)\beta^2, \tag{2.15c}$$

$$K_4(x,y) := g_\beta(x)^{1/2} F''(\mu(x,y)) g_\beta(y)^{1/2} \frac{|f(x) - f(y)|^4}{2}\beta^4. \tag{2.15d}$$

Dependence of the above kernels on β and f is not indicated in the notations as no confusion can arise. Thus, the integral kernel $\mathsf{D}_{\beta,V}(\delta)(x,y)$ can be decomposed as

$$\begin{aligned}\mathsf{D}_{\beta,V}(\delta)(x,y) &= V(x)\,[\mathsf{Q}_\beta(\delta)(x,y) - \mathsf{Q}_0(\delta)(x,y)]\,V(y)\\ &= V(x)\left(G_\kappa(x-y)\sum_{j=1}^{3} K_j(x,y) + K_4(x,y)\right)V(y).\end{aligned}$$

Hence, the expansion (2.10) holds with the integral kernel of $\mathsf{D}^{(2)}_{\beta,V}(\delta)$ given by

$$\mathsf{D}^{(2)}_{\beta,V}(\delta)(x,y) = \frac{V(x)V(y)}{\beta^4}\Big[G_\kappa(x-y)\,(K_2(x,y) + K_3(x,y)) + K_4(x,y)\Big]. \tag{2.16}$$

With the aid of the definitions (2.15b), (2.15c) for the kernels $K_j(\cdot,\cdot)$, $j = 2,3$, one obtains using $\beta \in (0,1]$ and $|\nabla f| \le L_f$ that

$$|K_2(x,y)| \le C_{2,f}\beta^4 \qquad \text{and} \qquad |K_3(x,y)| \le C_{3,f}\,(\kappa|x-y| + 1)\,\beta^4, \tag{2.17}$$

with some constants $C_{2,f}, C_{3,f} > 0$. Taking into account that F'' is a decreasing function and using that $\beta \in (0,1]$ we estimate K_4 in (2.15d) as

$$|K_4(x,y)| \le C_{4,f} G_\kappa(x-y)\left(\kappa^2|x-y|^2 + 3\kappa|x-y| + 3\right)\beta^4. \tag{2.18}$$

with some constant $C_{4,f} > 0$. Finally, combining the estimates (2.17), (2.18) and the expression for $\mathsf{D}^{(2)}_{\beta,V}(\delta)(\cdot,\cdot)$ in (2.16) we end up with

$$|\mathsf{D}^{(2)}_{\beta,V}(\delta)(x,y)| \le C_1 C_3' G_{\frac{\alpha}{4}}(x-y)\left[5 + 4\kappa|x-y| + \kappa^2|x-y|^2\right],$$

where $C_3' := \max\{C_{2,f}, C_{3,f}, C_{4,f}\}$ and C_1 is as in (2.14). Applying the Schur test once again we get

$$\begin{aligned}\|\mathsf{D}^{(2)}_{\beta,V}(\delta)\| &\le C_1 C_3' \int_{\mathbb{R}^2} G_{\frac{\alpha}{4}}(x)\left[5 + 4\kappa|x| + \kappa^2|x|^2\right]\mathrm{d}x\\ &= \frac{1}{2} C_1 C_3' \int_0^\infty e^{-\frac{\alpha}{4}r}\left[5 + 4\kappa r + \kappa^2 r^2\right]\mathrm{d}r = C_1 C_3'\left[\frac{10}{\alpha} + \frac{32}{\alpha^2}\kappa + \frac{64}{\alpha^3}\kappa^2\right]\\ &\le C_1 C_3'\left[\frac{10}{\alpha} + \frac{32}{\alpha^2}\left(\frac{\alpha}{2} + 1\right) + \frac{64}{\alpha^3}\left(\frac{\alpha^2}{4} + 1\right)\right] =: C_3,\end{aligned}$$

where we used the bounds $\kappa^2 \le \frac{1}{4}\alpha^2 + 1$ and $\kappa \le \frac{1}{2}\alpha + 1$. Thus, we have shown $\mathsf{D}^{(2)}_{\beta,V}(\delta) \in \mathcal{B}(L^2(\mathbb{R}^2))$. Self-adjointness of $\mathsf{D}^{(2)}_{\beta,V}(\delta)$ follows from $\mathsf{D}^{(2)}_{\beta,V}(\delta)(x,y) = \mathsf{D}^{(2)}_{\beta,V}(\delta)(y,x)$. ∎

In the next proposition we show real analyticity of $\mathsf{D}_{\beta,V}(\delta)$ with respect to δ and β. Furthermore, we estimate the norm of $\partial_\delta \mathsf{D}_{\beta,V}(\delta)$.

Proposition 2.3. *Let the assumptions be as in Proposition 2.2. Then the following claims hold.*

(i) *The operator-valued function $(0,1)^2 \ni (\delta,\beta) \mapsto \mathsf{D}_{\beta,V}(\delta)$ is real-analytic in both arguments.*

(ii) $\|\partial_\delta \mathsf{D}_{\beta,V}(\delta)\| = \mathcal{O}_{\mathrm{u}}(1)$ *as* $\beta \to 0+$.

Proof. (i) Combining [Ka, Thm. III 3.12] (and the discussion following it) with the claims of Proposition 2.2 we conclude that it suffices to check real-analyticity with respect to $\delta,\beta \in (0,1)$ of the scalar-valued functions

$$(0,1)^2 \ni (\delta,\beta) \mapsto (\mathsf{D}_{\beta,V}(\delta)h_1, h_2)_{L^2(\mathbb{R}^2)}\,,$$

where $h_1, h_2 \in C_0^\infty(\mathbb{R}^2)$. The latter follows from real analyticity of $(0,1)^2 \ni (\delta,\beta) \mapsto \mathsf{D}_\beta(\delta)$ in δ and β, because the function V is locally bounded.

(ii) Differentiating the integral kernel $\mathsf{D}_{\beta,V}(\delta)(x,y)$ with respect to δ we find

$$\begin{aligned}
\partial_\delta \mathsf{D}_{\beta,V}(\delta)(x,y) &= V(x)\partial_\delta(\mathsf{D}_\beta(\delta)(x,y))V(y)\\
&= \frac{\delta}{\kappa}V(x)\left[g_\beta(x)^{1/2}\partial_\kappa\Big(G_\kappa(r_\beta(x)-r_\beta(y))\Big)g_\beta(y)^{1/2} - \partial_\kappa\Big(G_\kappa(x-y)\Big)\right]V(y)\\
&= \frac{\delta}{4\pi\kappa}V(x)\left[e^{-\kappa|x-y|} - g_\beta(x)^{1/2}e^{-\kappa|r_\beta(x)-r_\beta(y)|}g_\beta(y)^{1/2}\right]V(y).
\end{aligned}$$

Next, we show that the integral operator $\partial_\delta \mathsf{D}_{\beta,V}(\delta)\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ associated with the above kernel satisfies

$$\lim_{\delta'\to\delta}\left\|\frac{\mathsf{D}_{\beta,V}(\delta') - \mathsf{D}_{\beta,V}(\delta)}{\delta'-\delta} - \partial_\delta \mathsf{D}_{\beta,V}(\delta)\right\| = 0. \tag{2.19}$$

Applying the mean-value theorem for the integral kernels, we get

$$\left|\frac{\mathsf{D}_{\beta,V}(\delta')(x,y) - \mathsf{D}_{\beta,V}(\delta)(x,y)}{\delta'-\delta} - \partial_\delta \mathsf{D}_{\beta,V}(\delta)(x,y)\right| = |\partial^2_{\delta\delta}\mathsf{D}_{\beta,V}(\delta_\star)(x,y)(\delta'-\delta)|\,,$$

where $\delta_\star$ lies between δ' and δ. Standard calculations yield

$$\begin{aligned}
\partial^2_{\delta\delta}\mathsf{D}_{\beta,V}(\delta)(x,y) = \frac{1}{4\pi}V(x)\Bigg(&\Big(g_\beta(x)g_\beta(y)\Big)^{1/2}\left(\frac{\delta^2}{\kappa^3} - \frac{1}{\kappa} + \frac{\delta^2|r_\beta(x)-r_\beta(y)|}{\kappa^2}\right)e^{-\kappa|r_\beta(x)-r_\beta(y)|}\\
&- \left(\frac{\delta^2}{\kappa^3} - \frac{1}{\kappa} + \frac{\delta^2|x-y|}{\kappa^2}\right)e^{-\kappa|x-y|}\Bigg)V(y).
\end{aligned}$$

Using the inequality $|r_\beta(x)-r_\beta(y)|\le 2|x-y|$ which holds for $L_f\beta\le\sqrt{3}$ together with the estimates $\|g_\beta\|_\infty\le 1+L_f$ and $\kappa>\alpha/2$ we get

$$|\partial^2_{\delta\delta}\mathsf{D}_{\beta,V}(\delta)(x,y)|\le\frac{1}{4\pi}\Big(1+L_f\Big)V(x)\left(\frac{4}{\alpha}+\frac{16\delta^2}{\alpha^3}+\frac{12\delta^2}{\alpha^2}|x-y|\right)e^{-\frac{\alpha}{2}|x-y|}V(y).$$

Performing the analysis as in the *Step 1* of the proof for Proposition 2.2 we get

$$|\partial^2_{\delta\delta}\mathsf{D}_{\beta,V}(\delta)(x,y)|\le\frac{C}{4\pi}\Big(1+|x-y|\Big)e^{-\frac{\alpha}{4}|x-y|},$$

with some constant $C=C(\alpha,f)>0$. By the Schur test we obtain

$$\begin{aligned}\left\|\frac{\mathsf{D}_{\beta,V}(\delta')-\mathsf{D}_{\beta,V}(\delta)}{\delta'-\delta}-\partial_\delta\mathsf{D}_{\beta,V}(\delta)\right\|&\le|\delta-\delta'|\frac{C}{4\pi}\int_{\mathbb{R}^2}(1+|x|)e^{-\frac{\alpha}{4}|x|}\mathrm{d}x\\&=|\delta-\delta'|C\left(\frac{2}{\alpha}+\frac{8}{\alpha^2}\right).\end{aligned}$$

Therefore, the convergence (2.19) is verified.

Furthermore, the subset of $\mathbb{R}^2\times\mathbb{R}^2$, where $\partial_\delta\mathsf{D}_{\beta,V}(\delta)(x,y)\neq 0$, can be covered by two (intersecting) sets $\mathcal{U}=\mathcal{B}_R\times\mathcal{B}_R$ and $\mathcal{V}=(\mathcal{S}\times\mathcal{B}_R^{\rm c})\cup(\mathcal{B}_R^{\rm c}\times\mathcal{S})$ defined as in (2.11). For any $(x,y)\in\mathcal{U}$ we get

$$|\partial_\delta\mathsf{D}_{\beta,V}(\delta)(x,y)|\le\frac{\delta}{2\pi\alpha}e^{\alpha R}\left[(1+\beta^2L_f^2)^{1/2}+1\right]e^{-\frac{\alpha}{4}(|x|+|y|)}.\tag{2.20}$$

Using the inequalities $|r_\beta(x)-r_\beta(y)|\ge|x-y|$ and $\frac14|x-y|\ge\frac{|x|+|y|}{4}-\frac12R$ we get for any $(x,y)\in\mathcal{V}$ the estimate

$$|\partial_\delta\mathsf{D}_{\beta,V}(\delta)(x,y)|\le\frac{\delta}{2\pi\alpha}e^{\alpha R}\left[(1+\beta^2L_f^2)^{1/2}+1\right]e^{-\frac{\alpha}{4}(|x|+|y|)}.\tag{2.21}$$

Combining the bounds (2.20) and (2.21) we obtain

$$|\partial_\delta\mathsf{D}_{\beta,V}(\delta)(x,y)|\le\frac{e^{\alpha R}\delta}{2\pi\alpha}\left[2+L_f\right]e^{-\frac{\alpha}{4}(|x|+|y|)}.$$

Hence, by the Schur test we find using that $\beta\in(0,1]$

$$\begin{aligned}\|\partial_\delta\mathsf{D}_{\beta,V}(\delta)\|&\le\frac{e^{\alpha R}\delta}{2\pi\alpha}\left[2+L_f\right]\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}e^{-\frac{\alpha}{4}(|x|+|y|)}\mathrm{d}y\\&=\frac{e^{\alpha R}\delta}{\alpha}\left[2+L_f\right]\int_0^\infty e^{-\frac{\alpha}{4}r}r\mathrm{d}r=\frac{16e^{\alpha R}\delta}{\alpha^3}\left[2+L_f\right],\end{aligned}$$

and the claim (ii) follows. ∎

In what follows we employ for $V\equiv 1$ the shorthand notation $\mathsf{D}^{(1)}(\delta):=\mathsf{D}^{(1)}_1(\delta)$.

Corollary 2.4. *The integral kernel* $\mathsf{D}^{(1)}(\delta)(\cdot,\cdot)$ *in* (2.8b) *with* $\delta \in [0,1]$ *and* $V \equiv 1$ *satisfies*

$$\mathcal{D}_{\alpha,f}(\delta) := 2\alpha^3 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathsf{D}^{(1)}(\delta)(x,y)\mathrm{d}x\mathrm{d}y < \infty. \tag{2.22}$$

In addition, the function $[0,1] \ni \delta \mapsto \mathcal{D}_{\alpha,f}(\delta)$ *is continuous.*

Proof. Note that there exists an integrable majorant for the integrand in (2.22) with $\delta \in [0,1]$ given by

$$\mathbb{R}^2 \times \mathbb{R}^2 \ni (x,y) \mapsto 2\alpha^3 C_1 G_{\frac{\alpha}{4}}(x-y)\left[1 + \left(\frac{\alpha}{4} + \frac{1}{2}\right)|x-y|\right]\chi_{\mathcal{T}}(x,y),$$

where C_1 is as in (2.9), $\mathcal{T} = (\mathcal{S}\times\mathbb{R}^2)\cup(\mathbb{R}^2\times\mathcal{S})$ and $\chi_{\mathcal{T}}\colon \mathbb{R}^2\times\mathbb{R}^2 \to \{0,1\}$ is the characteristic function of $\mathcal{T}$. Hence, finiteness of $\mathcal{D}_{\alpha,f}(\delta)$ directly follows. Furthermore, taking into account the pointwise continuity of the integrand in (2.22) with respect to δ, continuity of $[0,1] \ni \delta \mapsto \mathcal{D}_{\alpha,f}(\delta)$ is a consequence of the dominated convergence theorem. □

Finally, we obtain an alternative formula for $\mathcal{D}_{\alpha,f}(0)$ in terms of the Fourier transform of f. In the proof of this proposition we use an integral representation of the relativistic Schrödinger operator; see [IT93] and also [LL01, §7.12].

Proposition 2.5. *The value* $\mathcal{D}_{\alpha,f} := \mathcal{D}_{\alpha,f}(0)$ *in* (2.22) *with* $\delta = 0$ *can be represented as*

$$\mathcal{D}_{\alpha,f} = \int_{\mathbb{R}^2} |p|^2\left(\alpha^2 - \frac{2\alpha^3}{\sqrt{4|p|^2+\alpha^2}+\alpha}\right)|\widehat{f}(p)|^2\mathrm{d}p > 0.$$

Proof. First, we decompose $\mathcal{D}_{\alpha,f}$ as $\mathcal{D}_{\alpha,f} = \mathcal{D}^{(1)}_{\alpha,f} - \mathcal{D}^{(2)}_{\alpha,f}$ with

$$\mathcal{D}^{(1)}_{\alpha,f} := \frac{\alpha^3}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{-\frac{\alpha}{2}|x-y|}}{4\pi|x-y|}\left(|\nabla f(x)|^2 + |\nabla f(y)|^2\right)\mathrm{d}x\mathrm{d}y,$$

$$\mathcal{D}^{(2)}_{\alpha,f} := \frac{\alpha^3}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{-\frac{\alpha}{2}|x-y|}}{4\pi|x-y|}\frac{|f(x)-f(y)|^2}{|x-y|^2}\left(\alpha|x-y|+2\right)\mathrm{d}x\mathrm{d}y.$$

Then, we find by elementary computations

$$\begin{aligned}\mathcal{D}^{(1)}_{\alpha,f} &= \alpha^3\left(\int_{\mathbb{R}^2}|\nabla f(x)|^2\mathrm{d}x\right)\left(\int_{\mathbb{R}^2}\frac{e^{-\frac{\alpha}{2}|y|}}{4\pi|y|}\mathrm{d}y\right)\\ &= \frac{\alpha^3}{2}\left(\int_{\mathbb{R}^2}|p|^2|\widehat{f}(p)|^2\mathrm{d}p\right)\left(\int_{\mathbb{R}_+}e^{-\frac{\alpha}{2}r}\mathrm{d}r\right) = \alpha^2\int_{\mathbb{R}^2}|p|^2|\widehat{f}(p)|^2\mathrm{d}p.\end{aligned} \tag{2.23}$$

Next, using the identities [IT93, Eq. (2.2) and (2.4) for $d = 2$] we get

$$\begin{aligned}\int_{\mathbb{R}^2}\left(\sqrt{|p|^2+\tfrac{1}{4}\alpha^2}-\tfrac{1}{2}\alpha\right)|\widehat{f}(p)|^2\mathrm{d}p &= -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(f(x)f(y)-f(x)^2\right)n(x-y)\mathrm{d}x\mathrm{d}y\\ &= \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|f(x)-f(y)|^2\,n(x-y)\mathrm{d}x\mathrm{d}y,\end{aligned}$$

for $n(\cdot)$ given by

$$
\begin{aligned}
n(x) &= 2(2\pi)^{-3/2}\left(\frac{\alpha}{2}\right)^{3/2}|x|^{-3/2}K_{3/2}\left(\frac{\alpha}{2}|x|\right)\\
&= 2(2\pi)^{-3/2}\left(\frac{\alpha}{2}\right)^{3/2}|x|^{-3/2}\frac{\left(\frac{\pi}{2}\right)^{1/2}\exp\left(-\frac{\alpha}{2}|x|\right)\left(\frac{2}{\alpha|x|}+1\right)}{\left(\frac{\alpha}{2}|x|\right)^{1/2}}\\
&= \frac{\exp\left(-\frac{\alpha}{2}|x|\right)}{4\pi|x|}\frac{1}{|x|^2}\left(2+\alpha|x|\right),
\end{aligned}
$$

where in between we used the representation

$$
K_{3/2}(x) = \frac{\left(\frac{\pi}{2}\right)^{1/2}\exp(-x)\left(\frac{1}{x}+1\right)}{x^{1/2}}
$$

for the modified Bessel function $K_{3/2}(\cdot)$. Hence, we get

$$
\mathcal{D}^{(2)}_{\alpha,f} = \alpha^3\int_{\mathbb{R}^2}\left(\sqrt{|p|^2+\tfrac{1}{4}\alpha^2}-\tfrac{1}{2}\alpha\right)|\widehat{f}(p)|^2\mathrm{d}p = 2\alpha^3\int_{\mathbb{R}^2}\frac{|p|^2}{\sqrt{4|p|^2+\alpha^2}+\alpha}|\widehat{f}(p)|^2\mathrm{d}p. \qquad (2.24)
$$

Finally, combining (2.23) and (2.24) we obtain

$$
\mathcal{D}_{\alpha,f} = \mathcal{D}^{(1)}_{\alpha,f} - \mathcal{D}^{(2)}_{\alpha,f} = \int_{\mathbb{R}^2}|p|^2\left(\alpha^2 - \frac{2\alpha^3}{\sqrt{4|p|^2+\alpha^2}+\alpha}\right)|\widehat{f}(p)|^2\mathrm{d}p.
$$

In particular, $\mathcal{D}_{\alpha,f} > 0$ follows from positivity almost everywhere in $\mathbb{R}^2$ of the expression in the round brackets standing in the integrand in the above formula. □

3. Proof of Theorem 1.2

We split the proof of the main result into three steps.

Step 1: Spectral equation. In order to derive the spectral equation, we introduce an auxiliary function[2] $V(x) := \mathrm{e}^{\frac{\alpha}{4}|x|}$, $\alpha > 0$. Note that V satisfies the growth condition in Proposition 2.2 with $c = 1$ and moreover $V^{-1} \in L^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Furthermore, we associate with the kernel in (2.8a) the operator $\mathsf{D}_{\beta,V}(\delta) \in \mathcal{B}(L^2(\mathbb{R}^2))$ as in Proposition 2.2. Recall that the operator $\mathsf{D}_{\beta,V}(\delta)$ admits the representation

$$
\mathsf{D}_{\beta,V}(\delta) = \mathsf{D}^{(1)}_{V}(\delta)\beta^2 + \mathsf{D}^{(2)}_{\beta,V}(\delta)\beta^4, \qquad (3.1)
$$

where $\mathsf{D}^{(1)}_{V}(\delta), \mathsf{D}^{(2)}_{\beta,V}(\delta) \in \mathcal{B}(L^2(\mathbb{R}^2))$ and we also have $\|\mathsf{D}_{\beta,V}(\delta)\| = \mathcal{O}_{\mathrm{u}}(\beta^2)$ as $\beta\to 0+$. Next, we define the product

$$
\mathsf{B}_{\alpha,V}(\delta) := V^{-1}\mathsf{B}_\alpha(\delta)V^{-1}, \qquad \delta > 0,
$$

[2]Introducing V is a purely technical step, needed for a regularization. The final result does not depend on the particular choice of V.

where $\mathsf{B}_\alpha(\delta)$ is as in (2.3). The spectral condition (2.4) can be rewritten as

$$\forall\, \kappa > \frac{\alpha}{2}, \qquad \dim\ker\left(\mathsf{H}_{\alpha,\beta} + \kappa^2\right) = \dim\ker\left(\mathsf{I} - \alpha\mathsf{B}_{\alpha,V}(\delta)\mathsf{D}_{\beta,V}(\delta)\right).$$

To compute the dimension of $\ker(\mathsf{I} - \alpha\mathsf{B}_{\alpha,V}(\delta)\mathsf{D}_{\beta,V}(\delta))$ we investigate the asymptotic behaviour of $\mathsf{B}_{\alpha,V}(\delta)$ as $\delta \to 0+$. First, we observe that the decomposition in Lemma 2.3 yields

$$\mathsf{B}_{\alpha,V}(\delta) = \frac{\alpha^2}{2} V^{-1}\mathsf{R}(\delta^2)V^{-1} + \mathsf{N}_{\alpha,V}(\delta)\,,$$

where $\mathsf{N}_{\alpha,V}(\delta) := V^{-1}\mathsf{N}_\alpha(\delta)V^{-1}$. Lemma 2.3 (iii) implies that $\mathbb{R}_+ \ni \delta \mapsto \mathsf{N}_{\alpha,V}(\delta)$ is real analytic. Observe that $\mathsf{R}(\delta^2)$ is an integral operator with the kernel $\frac{1}{2\pi}K_0(\delta|x-y|)$, where $K_0(\cdot)$ is the modified Bessel function of the second kind and order zero; *cf.* [AS, §9.6]. The function K_0 admits an asymptotic expansion (see [AS, Eq. 9.6.13])

$$K_0(z) = -\ln\frac{z}{2} - \gamma + \mathcal{O}(z^2\ln z), \qquad z \to 0+\,, \tag{3.2}$$

where $\gamma \approx 0.577\ldots$ is the Euler-Mascheroni constant. In accordance to the asymptotics (3.2), the operator-valued function $\delta \mapsto \frac{1}{2}V^{-1}\mathsf{R}(\delta^2)V^{-1}$ can be decomposed as follows

$$\frac{1}{2}V^{-1}\mathsf{R}(\delta^2)V^{-1} = \mathsf{L}(\delta) + \mathsf{M}(\delta)\,,$$

where

$$\mathsf{L}(\delta) := -\frac{\ln\delta}{4\pi}\left(\cdot, V^{-1}\right)_{L^2(\mathbb{R}^2)} V^{-1}$$

and $\mathsf{M}(\delta)\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $\delta > 0$, is a bounded integral operator with the kernel

$$\mathsf{M}(\delta)(x,y) := \frac{1}{4\pi}V^{-1}(x)\left[K_0(\delta|x-y|) + \ln\delta\right]V^{-1}(y)\,.$$

Define also the bounded integral operator $\mathsf{M}(0)\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ with the kernel

$$\mathsf{M}(0)(x,y) := -\frac{1}{4\pi}V^{-1}(x)\left[\gamma + \ln\frac{|x-y|}{2}\right]V^{-1}(y)\,.$$

Mimicking the arguments from [S76, Prop. 3.2] we conclude that the operator-valued function $(0,\infty) \ni \delta \mapsto \mathsf{M}(\delta)$ is real analytic and that

$$\|\mathsf{M}(\delta) - \mathsf{M}(0)\| \to 0, \qquad \delta \to 0+\,.$$

We define the integral operator $\mathsf{M}'(\delta)\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ via the kernel

$$\mathsf{M}'(\delta)(x,y) := \frac{1}{4\pi\delta}V^{-1}(x)\Big(1 - K_1(\delta|x-y|)\delta|x-y|\Big)V^{-1}(y),$$

where $K_1(\cdot)$ is the modified Bessel function of the second kind and order $\nu = 1$; *cf.* [AS, §9.6]. Analogously, for the $\mathsf{M}(\delta)$ one check the following convergence

$$\lim_{\delta'\to\delta}\left\|\frac{\mathsf{M}(\delta') - \mathsf{M}(\delta)}{\delta' - \delta} - \mathsf{M}'(\delta)\right\| = 0.$$

Consequently, $\mathsf{M}'(\delta)$ can be identified with $\partial_\delta \mathsf{M}(\delta)$. Furthermore, using the inequality $1 - xK_1(x) < x$ we get by the Schur test

$$\begin{aligned} \|\partial_\delta \mathsf{M}(\delta)\| &\le \frac{1}{4\pi\delta} \sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\frac{\alpha}{4}|x|} e^{-\frac{\alpha}{4}|y|} \Big|1 - \delta|x-y| K_1(\delta|x-y|)\Big| \mathrm{d}y \\ &\le \frac{1}{4\pi} \sup_{x\in\mathbb{R}^2} \left(e^{-\frac{\alpha}{4}|x|} \int_{\mathbb{R}^2} e^{-\frac{\alpha}{4}|y|} (|x| + |y|) \mathrm{d}y \right) \\ &= \frac{1}{2} \left[\left(\sup_{x\in\mathbb{R}^2} |x| e^{-\frac{\alpha}{4}|x|} \right) \int_0^\infty e^{-\frac{\alpha}{4}r} r \mathrm{d}r + \int_0^\infty e^{-\frac{\alpha}{4}r} r^2 \mathrm{d}r \right] = \frac{32}{\alpha^3} \Big(e^{-1} + 2 \Big) \end{aligned} \tag{3.3}$$

Next, denote

$$\mathsf{G}_{\alpha,\beta}(\delta) := \Big(\alpha^2 \mathsf{M}(\delta) + \mathsf{N}_{\alpha,V}(\delta) \Big) \mathsf{D}_{\beta,V}(\delta).$$

Real analyticity of $\mathsf{D}_{\beta,V}(\delta)$, $\mathsf{N}_{\alpha,V}(\delta)$, and $\mathsf{M}(\delta)$ with respect to $\delta, \beta \in (0,1)$ implies that $\mathsf{G}_{\alpha,\beta}(\delta)$ is also real-analytic in $\delta, \beta \in (0,1)$. It follows from the expansion (3.1) and the above estimates that $\mathsf{G}_{\alpha,\beta}(\delta)$ is a bounded operator, whose norm behaves as $\|\mathsf{G}_{\alpha,\beta}(\delta)\| = \mathcal{O}_{\mathrm{u}}(\beta^2)$ as $\beta \to 0+$. Using Lemma 2.3 (iv), Propositions 2.2 and 2.3, and the estimate (3.3) we get applying the triangle inequality for the operator norm

$$\begin{aligned} \|\partial_\delta \mathsf{G}_{\alpha,\beta}(\delta)\| \le & \Big[\alpha^2 \|\partial_\delta \mathsf{M}(\delta)\| + \|\partial_\delta \mathsf{N}_{\alpha,V}(\delta)\| \Big] \|\mathsf{D}_{\beta,V}(\delta)\| \\ & + \Big[\alpha^2 \|\mathsf{M}(\delta)\| + \|\mathsf{N}_{\alpha,V}(\delta)\| \Big] \|\partial_\delta \mathsf{D}_{\beta,V}(\delta)\| = \mathcal{O}_{\mathrm{u}}(1), \qquad \beta \to 0+ . \end{aligned} \tag{3.4}$$

Next, for all sufficiently small $\beta > 0$, the operator $\mathsf{I} - \alpha \mathsf{G}_{\alpha,\beta}(\delta)$ is invertible and $\mathsf{I} - \alpha \mathsf{B}_{\alpha,V}(\delta) \mathsf{D}_{\beta,V}(\delta)$ can be factorized as

$$\mathsf{I} - \alpha \mathsf{B}_{\alpha,V}(\delta) \mathsf{D}_{\beta,V}(\delta) = (\mathsf{I} - \alpha \mathsf{G}_{\alpha,\beta}(\delta)) (\mathsf{I} - \mathsf{P}_{\alpha,\beta}(\delta)),$$

where $\mathsf{P}_{\alpha,\beta}(\delta)$ is the rank-one operator given by

$$\begin{aligned} \mathsf{P}_{\alpha,\beta}(\delta) &:= (\mathsf{I} - \alpha \mathsf{G}_{\alpha,\beta}(\delta))^{-1} \mathsf{L}(\delta) \alpha^3 \mathsf{D}_{\beta,V}(\delta) \\ &= -\alpha^3 \frac{\ln \delta}{4\pi} \Big(\cdot, \mathsf{D}_{\beta,V}(\delta) V^{-1} \Big)_{L^2(\mathbb{R}^2)} (\mathsf{I} - \alpha \mathsf{G}_{\alpha,\beta}(\delta))^{-1} V^{-1}. \end{aligned}$$

Thus, we get for all sufficiently small $\beta > 0$

$$\forall \delta > 0, \qquad \dim\ker (\mathsf{I} - \alpha \mathsf{B}_{\alpha,V}(\delta) \mathsf{D}_{\beta,V}(\delta)) = \dim\ker (\mathsf{I} - \mathsf{P}_{\alpha,\beta}(\delta)).$$

Observe that $\dim\ker (\mathsf{I} - \mathsf{P}_{\alpha,\beta}(\delta)) \in \{0,1\}$. Using the relation $\dim\ker(\mathsf{I} - \mathsf{P}) = 1$ *if, and only if,* $\operatorname{Tr} \mathsf{P} = 1$ (true for any rank-one operator P), we find that $\dim\ker(\mathsf{I} - \mathsf{P}_{\alpha,\beta}(\delta)) = 1$ *if, and only if,*

$$\boxed{4\pi + \alpha^3 \ln \delta \Big(\mathsf{D}_{\beta,V}(\delta) V^{-1}, (\mathsf{I} - \alpha \mathsf{G}_{\alpha,\beta}(\delta))^{-1} V^{-1} \Big)_{L^2(\mathbb{R}^2)} = 0 .} \tag{3.5}$$

In view of this reduction, for all sufficiently small $\beta > 0$, each solution $\delta > 0$ of the equation (3.5) corresponds to a simple eigenvalue $-\frac{\alpha^2}{4} - \delta^2$ of $\mathsf{H}_{\alpha,\beta}$.

Step 2: Existence and uniqueness of solution for (3.5). Define the function

$$\eta_\alpha(\beta,\delta) := 2\alpha^3 \left(\mathsf{D}_{\beta,V}(\delta)V^{-1}, (\mathsf{I}-\alpha\mathsf{G}_{\alpha,\beta}(\delta))^{-1}V^{-1}\right)_{L^2(\mathbb{R}^2)}.$$

We remark that the function $\eta_\alpha(\cdot,\cdot)$ is real-analytic in $\delta,\beta>0$ lying in a sufficiently small right neighbourhood of the origin, thanks to real-analyticity with respect to the same parameters of the operator-valued functions $\mathsf{D}_{\beta,V}(\delta)$ and $\mathsf{G}_{\alpha,\beta}(\delta)$; see Proposition 2.3 and the discussion in *Step 1*. The spectral condition (3.5) can be equivalently written as

$$\eta_\alpha(\beta,\delta) = -\frac{8\pi}{\ln\delta}. \tag{3.6}$$

Applying the Neumann series argument and using $\|\mathsf{G}_{\alpha,\beta}(\delta)\| = \mathcal{O}_{\mathrm{u}}(\beta^2)$ $(\beta\to 0+)$ we find

$$\|\,(\mathsf{I}-\alpha\mathsf{G}_{\alpha,\beta}(\delta))^{-1} - \mathsf{I}\| = o_{\mathrm{u}}(1), \qquad \beta\to 0+\,.$$

Hence, we conclude from $\|\mathsf{D}_{\beta,V}(\delta)\| = \mathcal{O}_{\mathrm{u}}(\beta^2)$ as $\beta\to 0+$ that $\eta_\alpha(\beta,\delta)=\mathcal{O}_{\mathrm{u}}(\beta^2)$ as $\beta\to 0+$. Combining the expansion (3.1) and Corollary 2.4 we arrive at

$$\begin{aligned}\eta_\alpha(\beta,\delta) &= 2\alpha^3\beta^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathsf{D}_V^{(1)}(\delta)(x,y)V^{-1}(x)V^{-1}(y)\mathrm{d}x\mathrm{d}y + \mathcal{O}_{\mathrm{u}}(\beta^4)\\ &= 2\alpha^3\beta^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathsf{D}^{(1)}(\delta)(x,y)\mathrm{d}x\mathrm{d}y + \mathcal{O}_{\mathrm{u}}(\beta^4)\\ &= \mathcal{D}_{\alpha,f}(\delta)\beta^2 + \mathcal{O}_{\mathrm{u}}(\beta^4), \qquad \beta\to 0+\,.\end{aligned} \tag{3.7}$$

Since $\mathcal{D}_{\alpha,f} = \mathcal{D}_{\alpha,f}(0) > 0$ by Proposition 2.5, Corollary 2.4 yields that $\eta_\alpha(\beta,\delta)>0$ holds for all sufficiently small $\delta,\beta>0$. The continuous function $(0,1)\ni\delta\mapsto -\frac{8\pi}{\ln\delta}$ vanishes as $\delta\to 0+$ and its range coincides with $(0,\infty)$. Hence, for all sufficiently small $\beta>0$ the equation (3.6) has at least one solution $\delta(\beta)>0$, which satisfies $\delta(\beta)\to 0+$ taking the lower bound in Proposition 2.1 into account. In particular, we proved that $\#\sigma_{\mathrm{d}}(\mathsf{H}_{\alpha,\beta})\geq 1$ holds for all sufficiently small $\beta>0$.

It remains to show that in fact for all sufficiently small $\beta>0$ holds $\#\sigma_{\mathrm{d}}(\mathsf{H}_{\alpha,\beta})=1$. Indeed, the equation (3.6) can be rewritten as

$$\widetilde{\eta}_\alpha(\beta,\delta)=0 \qquad \text{with } \widetilde{\eta}_\alpha(\beta,\delta) := \exp\left(-\frac{8\pi}{\eta_\alpha(\beta,\delta)}\right) - \delta.$$

Suppose that $\beta>0$ is small enough and that $\widetilde{\eta}_\alpha(\beta,\delta)=0$ has two solutions $\delta_1,\delta_2\in(0,1)$ such that $\delta_1<\delta_2$. By Rolle's theorem there exists a point $\delta_\star\in(\delta_1,\delta_2)$ such that

$$(\partial_\delta\widetilde{\eta}_\alpha)(\beta,\delta_\star) = 0. \tag{3.8}$$

Computing the partial derivative of $\widetilde{\eta}_\alpha$ with respect to δ we get

$$\partial_\delta\widetilde{\eta}_\alpha(\beta,\delta) = \frac{8\pi\partial_\delta\eta_\alpha(\beta,\delta)}{\eta_\alpha(\beta,\delta)^2}\exp\left(-\frac{8\pi}{\eta_\alpha(\beta,\delta)}\right) - 1. \tag{3.9}$$

Differentiating the operator-valued function $(\mathsf{I} - \mathsf{G}_\alpha(\delta))^{-1}$ with respect to δ we find

$$\begin{aligned}\lim_{\delta'\to\delta} & \frac{(\mathsf{I} - \alpha\mathsf{G}_{\alpha,\beta}(\delta'))^{-1} - (\mathsf{I} - \alpha\mathsf{G}_{\alpha,\beta}(\delta))^{-1}}{\delta' - \delta} \\ & = \alpha \lim_{\delta'\to\delta} (\mathsf{I} - \alpha\mathsf{G}_{\alpha,\beta}(\delta'))^{-1} \frac{\mathsf{G}_{\alpha,\beta}(\delta') - \mathsf{G}_{\alpha,\beta}(\delta)}{\delta' - \delta} (\mathsf{I} - \alpha\mathsf{G}_{\alpha,\beta}(\delta))^{-1} \\ & = \alpha(\mathsf{I} - \alpha\mathsf{G}_{\alpha,\beta}(\delta))^{-1} \partial_\delta \mathsf{G}_{\alpha,\beta}(\delta)(\mathsf{I} - \alpha\mathsf{G}_{\alpha,\beta}(\delta))^{-1}.\end{aligned}$$

Hence, differentiating the scalar function η_α with respect to δ and applying Proposition 2.2 and the estimate (3.4), we end up with

$$\begin{aligned}\partial_\delta \eta_\alpha(\beta,\delta) &= 2\alpha^3 \left(\partial_\delta \mathsf{D}_{\beta,V}(\delta) V^{-1}, (\mathsf{I} - \alpha\mathsf{G}_{\alpha,\beta}(\delta))^{-1} V^{-1} \right)_{L^2(\mathbb{R}^2)} \\ & \quad + 2\alpha^4 \left(\mathsf{D}_{\beta,V}(\delta) V^{-1}, (\mathsf{I} - \alpha\mathsf{G}_{\alpha,\beta}(\delta))^{-1} (\partial_\delta \mathsf{G}_{\alpha,\beta}(\delta)) (\mathsf{I} - \alpha\mathsf{G}_{\alpha,\beta}(\delta))^{-1} V^{-1} \right)_{L^2(\mathbb{R}^2)} \\ &= \mathcal{O}_\mathrm{u}(1), \quad \beta \to 0+ .\end{aligned}$$

Eventually, we derive from (3.9) that $\partial_\delta \widetilde{\eta}_\alpha(\beta,\delta) = -1 + o_\mathrm{u}(1)$ as $\beta \to 0+$, which contradicts to (3.8) for all sufficiently small $\beta > 0$.

Step 3: Asymptotic expansion. Let $\delta(\beta) > 0$ be the unique solution of (3.6) for sufficiently small $\beta > 0$. Substituting the expansion (3.7) into the spectral condition (3.6) and making an additional use of $\delta(\beta) = o(1)$ (as $\beta \to 0+$) we get

$$8\pi + \ln \delta(\beta) \beta^2 \mathcal{D}_{\alpha,f}(\delta(\beta)) + o(\ln \delta(\beta)\beta^2) = 0, \qquad \beta \to 0+ .$$

Applying Corollary 2.4 we obtain

$$8\pi + \ln \delta(\beta) \beta^2 \mathcal{D}_{\alpha,f} + o(\ln \delta(\beta)\beta^2) = 0, \qquad \beta \to 0+ .$$

Hence, we deduce

$$\delta(\beta) = \exp\left(-\frac{8\pi}{\mathcal{D}_{\alpha,f}\beta^2} \right) (1 + o(1)), \qquad \beta \to 0+ . \tag{3.10}$$

Finally, the asymptotic expansion of $\lambda_1^\alpha(\beta)$ in (1.3) follows from (3.10) and the identity $\lambda_1^\alpha(\beta) = -\frac{1}{4}\alpha^2 - \delta^2(\beta)$. □

4. Discussion

Apparently, a similar asymptotic analysis can be performed in space dimensions $d \geq 4$, where not much is known apart from the result in [LO16] mentioned above. We note that a convincing physical motivation is missing in this case, so far at least, and also one can expect here that for all sufficiently small $\beta > 0$ the discrete spectrum would be empty.

It is also worth noting that analogous spectral problem can be considered for the Robin Laplacian in a locally perturbed half-space. In view of [EM14] one may expect that the existence of the unique bound state for all sufficiently small $\beta > 0$ will depend on the function f, defining the profile of the deformation. However, the technique to deal with the asymptotic analysis should be different for the Robin spectral problem, because a Birman-Schwinger-type principle with an explicitly given integral operator is not available in this setting.

Finally, let us point out that in the present paper we have not touched the case where the interaction support is a topologically non-trivial surface which could be regarded as a certain analogue of spectral analysis in infinite, topologically nontrivial layers [CEK04]. It is not so clear to what extent the main result and the technique of the present paper can be generalized to include such more involved geometries.

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