Spectral stability for the wave equation with periodic forcing

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Abstract. We consider the spectral stability problem for Floquet-type systems such as the wave equation $v_{\tau\tau} = \gamma^2 v_{xx} - \psi v$ with periodic forcing ψ. Our approach is based on a comparison with finite-dimensional approximations. Specific results are obtained for a system where the forcing is due to a coupling between the wave equation and a time-period solution of a nonlinear beam equation. We prove (spectral) stability for some period and instability for another. The finite-dimensional approximations are controlled via computer-assisted estimates.

1. Introduction

The aim of this paper is to develop a nonperturbative method for analyzing the stability of certain periodically driven systems. We do this in the context of a wave equation

$$v_{\tau\tau}(\tau,x) = \gamma^2 v_{xx}(\tau,x) - \psi(\tau,x)v(\tau,x)\,, \qquad \tau\in\mathbb{R}\,, \qquad x\in(0,\pi)\,, \tag{1.1}$$

where ψ depends periodically on the time variable τ. For the function v we impose Dirichlet boundary conditions at $x=0$ and $x=\pi$. We consider a model where the coefficient ψ is determined canonically by the desired time-period T. In this case we prove spectral stability for some value of T and absence of spectral stability for another.

We say that the equation (1.1) is spectrally stable if the corresponding evolution operator $\Phi(T)$ has no spectrum outside the unit circle. To be more specific, let us write the second order equation (1.1) in the usual way as a pair of first order (in τ) equations: $v_\tau = \nu$ and $\nu_\tau = \gamma^2 v_{xx} - \psi v$. The solution depends linearly on the initial condition at time zero, and this defines the time-τ map $\Phi(\tau)$ via the equation

$$V(\tau) = \Phi(\tau)V(0)\,, \qquad V(\tau) = \begin{bmatrix} v(\tau,.) \\ \nu(\tau,.)\end{bmatrix}\,, \qquad \tau\in\mathbb{R}\,. \tag{1.2}$$

If ψ is time-periodic with period T, then the flow Φ satisfies $\Phi(\tau+T) = \Phi(\tau)\Phi(T)$. So the growth properties of Φ are determined by the properties of the linear operator $\Phi(T)$. We note that, formally, this operator is symplectic, and thus its spectrum is invariant under complex conjugation $z\mapsto\bar z$ and inversion $z\mapsto 1/z$.

Our spectral analysis of Φ involves a comparison principle for monotone families of Floquet systems. This allows us e.g. to bound the eigenvalues of $\Phi(T)$ on the unit circle from both sides by the eigenvalues obtained from certain finite-dimensional approximations. The finite-dimensional systems are still nontrivial, but we can estimate their Floquet spectrum by using computer-assisted techniques.

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Our analysis was motivated in part by numerical observations [6] on instabilities in a model of a suspension bridge. To be more precise, and to motivate our choice of the forcing ψ in (1.1), consider the following (Hamiltonian) system of partial differential equations:

$$\begin{aligned} u_{\tau\tau} &= -u_{xxxx} + \tfrac{1}{2}\bigl[f(u+v) + f(u-v)\bigr]\,, \\ v_{\tau\tau} &= \gamma^2 v_{xx} + \tfrac{1}{2}\bigl[f(u+v) - f(u-v)\bigr]\,. \end{aligned} \tag{1.3}$$

Here $u = u(\tau, x)$ and $v = v(\tau, x)$ are functions on $\mathbb{R} \times (0, \pi)$, satisfying Navier and Dirichlet boundary conditions, respectively, at $x = 0$ and $x = \pi$. The coupling function f is nonlinear and will be specified below.

The equations (1.3) are a simplified version of a model [6] for a suspension bridge. In this context, u describes the longitudinal modes of the bridge, and v describes the torsional modes. The function f models the force that the hangers apply to the deck; see also equation (4) and the ensuing discussion in [5]. Numerical studies on the model described in [6] indicate that there is a loss of stability in the torsional modes as the energy of the longitudinal modes exceeds a certain threshold. Since the torsional amplitudes are typically small, we will v-linearize the system (1.3) in the sense of dropping all terms of order v^2.

A reasonable choice for a simplified bridge model is $f(u) = -\kappa u - u^3$. With this choice of f, setting $v = 0$ in (1.3) reduces the system to a nonlinear beam equation for u. In order to show that this equation has a time-periodic solution with a given periods T, it is convenient to perform a change of variables $t = \alpha\tau$ with $\alpha = 2\pi/T$, so that T-periodicity in τ corresponds to 2π-periodicity in t. In these new variables, and for $f(u) = -\kappa u - u^3$, the system (1.3) becomes

$$\alpha^2 u_{tt} = -u_{xxxx} - \bigl(u^2 + \kappa\bigr)u\,, \tag{1.4}$$

$$\alpha^2 v_{tt} = \gamma^2 v_{xx} - \bigl(3u^2 + \kappa\bigr)v\,, \tag{1.5}$$

up to terms of order v^2.

Using the methods developed in [12], is it possible to find nontrivial solutions u of the beam equation (1.4) for many different values of the parameters. Some recent work that involves similar techniques can be found in [7-12]. Notice that the equation (1.5) for v is equivalent to (1.1), with $\psi = 3u^2 + \kappa$.

From now on, we restrict the values of the model parameters to

$$\gamma = \tfrac{7}{8}\,, \qquad \kappa = \tfrac{1}{2}\,. \tag{1.6}$$

We will comment on this choice later on. Denote by $\mathcal{B}$ the vector space of all real analytic functions u on $\mathbb{R}^2$ that are 2π-periodic in each argument and whose Fourier series is of the form $u(t, x) = \sum\limits_{n,k} u_{n,k} \cos(nt)\sin(kx)$, with $u_{n,k} = 0$ whenever nk is even.

Theorem 1.1. *For each $\alpha \in \bigl\{\frac{5}{4}, \frac{14}{11}\bigr\}$, the beam equation (1.4) has nontrivial solution $u \in \mathcal{B}$.*

Numerically we have computed solutions of (1.4) for many other rational values of α. They all lie on what looks like a branch of solutions, similar to the one shown in Fig. 4

(right) in [12]. Along the values of α considered, the solutions on this "branch" are of the form $u(t,x) = c_\alpha[\cos(t)\sin(t) + \text{small}]$, and the amplitude c_α is increasing with α, starting with $c_\alpha = 0$ for $\alpha = \sqrt{1+\kappa} = 1.22\ldots$. In a more realistic model of a suspension bridge, say with some damping included, we would expect to find a true branch of solutions, where the amplitude c_α of the longitudinal mode varies continuously with α.

Our proof Theorem 1.1 follows closely the proof of Theorem 1.3 in [12] on periodic solutions of the beam equation (1.4) with $\gamma = 1$ and $\kappa = 0$. But unlike in [12] we only consider rational values of α, since our analysis of (1.5) requires that $\beta = \gamma/\alpha$ be rational. Another restriction imposed by our analysis of (1.5) is that $p\beta \in \mathbb{Z}$ for some integer $p \geq 1$ that is not too large; otherwise our computer-assisted estimates take a prohibitive amount of time. This is our main reason for choosing relatively simple rational values for γ and α. In principle, our methods should apply to a large range of parameters with β and α rational. The only significant restriction is that the function $\psi = 3u^2 + \kappa$ be dominated by its average. (This holds in the cases considered here.) We have some idea on how to overcome this restriction, but this would complicate the analysis significantly.

Coming back to the problem mentioned at the beginning, we have the following result.

Theorem 1.2. *Consider the system (1.5) for both of the pairs (α, u) described in Theorem 1.1. If $\alpha = \frac{5}{4}$, then the system is spectrally stable. If $\alpha = \frac{14}{11}$ then the system is not spectrally stable.*

We have carried out purely numerical computations for a few more values of α. If we simply connect the dots, then our findings indicate that the torsional modes v are stable for small amplitudes of the longitudinal modes u, but that they become unstable as the amplitude of u increases past a value c_α where $\alpha \approx 1.27$.

In what follows, $\Phi(t)$ denotes the time-t map associated with the equation (1.5), where the forcing is 2π-periodic in t. Our strategy for estimating the spectrum of $\Phi(2\pi)$ is roughly as follows. For $\beta = q/p$ rational, the spectrum consists of isolated eigenvalues with finite multiplicities, together with a finite set of accumulation points consisting of p-th roots of unity. In order to estimate these eigenvalues, we consider one-parameter families of Floquet systems that have a certain monotonicity property. This property implies that the eigenvalues on the unit circle move either clockwise or counterclockwise as the parameter is increased, depending on the Krein signature of the eigenvalue. By choosing monotone families that start and end at two simple systems, this allows us to enclose eigenvalues of $\Phi(2\pi)$ between the corresponding eigenvalues of the two simple systems. Here "simple" means finite dimensional (modulo a trivial part) with sufficiently low dimension. The time-2π maps of these simple systems are matrices, which we estimate by integrating the first order equation at high accuracy. Then we estimate the eigenvalues of these matrices. (All this is done of course with rigorous error estimates.) In order to exclude bifurcations along the chosen families, we prove that the eigenvalues never cross certain "separating values".

The remaining part of this paper is organized as follows. In Section 2 we introduce some notation and review a few facts from Floquet-Krein theory that will be needed later on. One-parameter families of Floquet systems are discussed in Section 3. This includes the parameter-dependence of eigenvalues, monotone families, and bounds related to the above-mentioned separating values. All results given in the first two sections concern finite-dimensional systems only. But they apply to the problem at hand by taking limits, as will be shown in Section 5. In Section 4 we describe some general properties of the flow Φ. To be more precise, we first perform a symplectic change of variables U, and then work with $\tilde{\Phi} = U\Phi U^{-1}$. In Section 5 and Subsection 6.1 we give a proof of Theorem 1.2 and Theorem 1.1, respectively, based on three technical lemmas. Our proof of these lemmas is discussed in the remaining parts of Section 6 and described in detail in [13].

Figures 1 and 2 show the spectrum of $\tilde{\Phi}(2\pi)$ in the case $\alpha = \frac{5}{4}$ and $\alpha = \frac{14}{11}$, respectively. The eigenvalues on the left (right) correspond to eigenvectors that are even (odd) under translations $x \mapsto x+\pi$. Eigenvalues that are not marked with dots lie in the red and black arcs. The color indicates the Krein signature: red $\mapsto$ positive, black $\mapsto$ negative. Blue dashes mark the primary separating values (described later). The on-circle spectrum in these figures is accurate, but the values of the off-circle eigenvalues (blue dots) in Figure 2 are purely numerical.

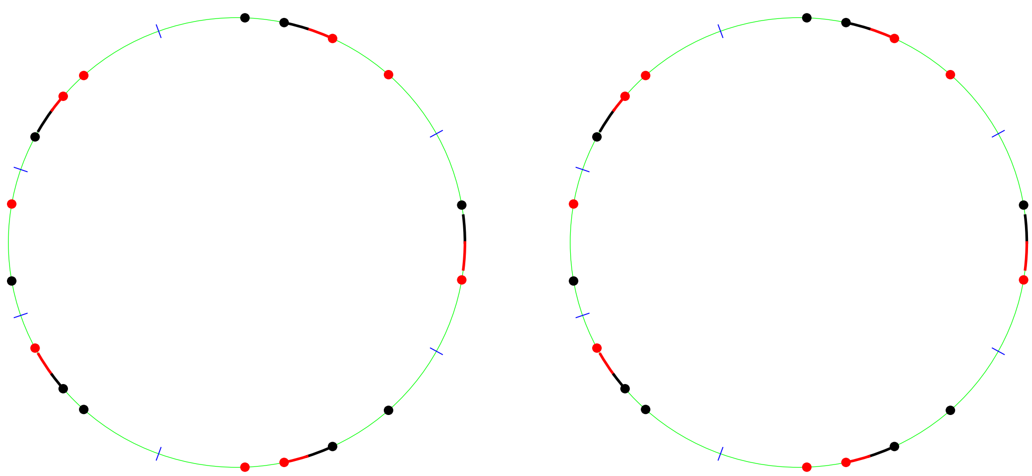

Figure 1. Spectrum for $\alpha = \frac{5}{4}$ in the even (left) and odd (right) subspace.

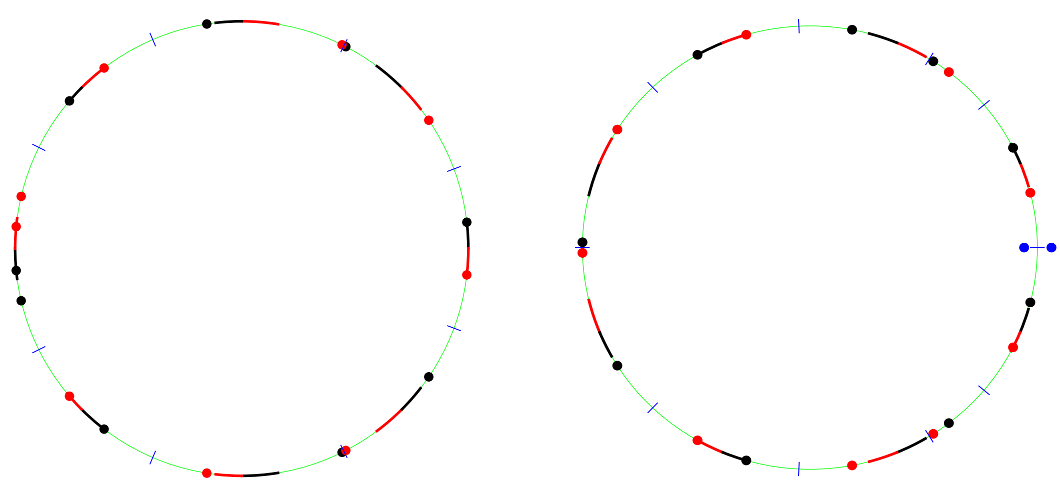

Figure 2. Spectrum for $\alpha = \frac{14}{11}$ in the even (left) and odd (right) subspace.

2. Floquet and Krein theory

In this section we cover some basic facts from Floquet-Krein theory [1,2] in finite dimensions. The same notation and concepts will be used later in our discussion of the system (1.5). The results themselves will be applied only to finite-dimensional approximations of this system. For other applications of the Krein signature to problems in partial differential equations, we refer to [3,4] and references therein.

2.1. The flow for periodic vector fields

Let $\mathfrak{B}$ be a Banach space. Unless specified otherwise, we assume that $\mathfrak{B}$ is finite-dimensional. Let $X : \mathbb{R} \to \mathcal{L}(\mathfrak{B})$ be a continuous curve in the space $\mathcal{L}(\mathfrak{B})$ of all linear operators on $\mathfrak{B}$. Floquet theory addresses the stability of the trivial solution of the evolution equation

$$\frac{d}{dt}V(t) = X(t)V(t)\,, \qquad V(t) \in \mathfrak{B}\,, \tag{2.1}$$

in the case where $X(t)$ depends periodically on time $t \in \mathbb{R}$. This can be done by considering the associated flow $\Phi : \mathbb{R} \to \mathcal{L}(\mathfrak{B})$, which is the solution of the initial value problem

$$\frac{d}{dt}\Phi(t) = X(t)\Phi(t)\,, \qquad \Phi(0) = \mathrm{I}\,. \tag{2.2}$$

The solution of (2.1) with initial condition $V(0) = V_0$ is then given by $V(t) = \Phi(t)V_0$.

A basic tool in Floquet theory is the following representation of the flow.

Lemma 2.1. *Let $T > 0$ be the fundamental period of X. Then there exists an operator $C \in \mathcal{L}(\mathfrak{B})$, and a periodic curve $P : \mathbb{R} \to \mathcal{L}(\mathfrak{B})$ with period T, such that*

$$\Phi(t) = P(t)e^{tC}\,, \qquad t \in \mathbb{R}\,. \tag{2.3}$$

Proof. First we note that $\Phi(t+T) = \Phi(t)\Phi(T)$, due to the periodicity of X. After choosing a branch of the logarithm whose domain of analyticity includes all eigenvalues of $\Phi(T)$, define $C = \frac{1}{2\pi}\log(\Phi(T))$. Clearly (2.3) holds if we set $P(t) = \Phi(t)e^{-tC}$. In addition,

$$P(t+T) = \Phi(t+T)e^{-(T+t)C} = \Phi(t)\Phi(T)e^{-TC}e^{-tC} = \Phi(t)e^{-tC} = P(T)\,, \tag{2.4}$$

as claimed. **QED**

The operator $\Phi(T)$ is called the *matrizant* of the Floquet system (2.2). Some other standard definitions are the following.

Definition 2.2. *The eigenvalues of $\Phi(T)$ are called the Floquet multipliers (of the given system). The eigenvalues of C are called Floquet exponents. Clearly, if r is a Floquet exponent then e^{Tr} is a Floquet multiplier. We note that Floquet exponents depend on the choice of C and are determined by $\Phi(T)$ only modulo $2\pi i/T$.*

Notice that $CV_0 = rV_0$ implies that $V(T) = e^{Tr}V_0$. Thus, every Floquet exponent r with positive (negative) real part gives rise to exponentially increasing (decreasing) solutions of (2.1). Suppose now that each Floquet exponent r is purely imaginary. If r is not semisimple, meaning that both $(C - r\mathrm{I})V_0 = 0$ and $(C - r\mathrm{I})V_1 = V_0$ admit nontrivial solutions, then $e^{tC}V_1$ grows linearly with t. Thus, we have the following.

Proposition 2.3. *The flow Φ is bounded if and only if every Floquet exponent has a nonpositive real part, and every Floquet exponent with zero real part is semisimple.*

Next we discuss Floquet systems that arise from second order equations. Let $\mathcal{H}$ be a finite-dimensional real Hilbert space. Consider the equation

$$\frac{d^2}{dt^2}v(t) = -\mathcal{X}(t)v(t)\,, \qquad v(t) \in \mathcal{H}\,, \tag{2.5}$$

with $\mathcal{X} : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ continuous and periodic with period $T > 0$. Introducing $\nu = \frac{d}{dt}v$ and

$$V = \begin{bmatrix} v \\ \nu \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & \mathrm{I} \\ -\mathcal{X} & 0 \end{bmatrix}, \tag{2.6}$$

the equation (2.5) reduces to an evolution equation (2.1) in the space $\mathfrak{B} = \mathcal{H}^2$. Assume now that $\mathcal{X}(t)$ is self-adjoint for all $t \in \mathbb{R}$. A straightforward computation shows that equation (2.5) arises from a Hamiltonian $H(t, v, \nu) = \frac{1}{2}\langle v, \mathcal{X}(t)v\rangle + \frac{1}{2}\langle \nu, \nu\rangle$, in the sense that the equation (2.1) can be written as

$$\frac{d}{dt}V(t) = \mathbb{J}\nabla H(t)\,, \qquad \mathbb{J} = \begin{bmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{bmatrix}. \tag{2.7}$$

For a proper discussion of Floquet exponents, we need to consider a complexification of $\mathcal{H}$. To simplify notation, the complexified space will again be denoted by $\mathcal{H}$. On the complex space $\mathcal{H}^2$ we consider the inner product

$$\langle V, V'\rangle = \langle v, v'\rangle + \langle \nu, \nu'\rangle\,, \qquad V = \begin{bmatrix} v \\ \nu \end{bmatrix} \in \mathcal{H}^2\,, \quad V' = \begin{bmatrix} v' \\ \nu' \end{bmatrix} \in \mathcal{H}^2\,, \tag{2.8}$$

where $\langle .\,, .\rangle$ is the inner product in $\mathcal{H}$. The convention adopted here is that $\langle .\,, .\rangle$ is antilinear in the first argument and linear in the second. We say that an operator on $\mathcal{H}^2$ is real if it leaves the real part of $\mathcal{H}^2$ invariant.

Definition 2.4. *A real operator $A \in \mathcal{L}(\mathcal{H}^2)$ is called symplectic if $A^*\mathbb{J}A = \mathbb{J}$.*

Denote by Φ the flow associated with the linear vector field X defined in (2.6). The operators $\mathcal{X}(t)$ are assumed to be real and self-adjoint.

Proposition 2.5. *The time-t maps $\Phi(t)$ are symplectic.*

Proof. Using that $\mathcal{X}^* = \mathcal{X}$ we have $X^*\mathbb{J} + \mathbb{J}X = 0$, and thus

$$\tfrac{d}{dt}\Phi(t)^*\mathbb{J}\Phi(t) = \Phi(t)^*X(t)^*\mathbb{J}\Phi(t) + \Phi(t)^*\mathbb{J}X(t)\Phi(t) = 0\,. \tag{2.9}$$

Given that $\Phi(0) = \mathrm{I}$, the assertion follows. **QED**

2.2. Eigenvalues and Krein signature

The main tool of Krein theory is the quadratic form

$$G(V,V') = \langle V, (i\mathbb{J})V'\rangle = i\langle v,\nu'\rangle - i\langle \nu, v'\rangle\,. \tag{2.10}$$

We remark that G is nondegenerate on $\mathcal{H}^2$, since $G(V, i\mathbb{J}V) = \langle V,V\rangle$. Notice also that $G(V,V)$ is real for all V, since $i\mathbb{J}$ is Hermitian. Another immediate consequence of the definitions is the following.

Proposition 2.6. *A real operator $A \in \mathcal{L}(\mathcal{H}^2)$ is symplectic if and only if $G(AV,AV') = G(V,V')$ for all $V,V' \in \mathcal{H}^2$.*

We conclude this section by mentioning some spectral properties of symplectic linear maps that will be needed later on.

First, consider an arbitrary bounded linear operator A on some (not necessarily finite-dimensional) Banach space $\mathfrak{B}$. Let Λ and Λ' be two disjoint compact sets in $\mathbb{C}$ whose union includes the spectrum of A. Then the Riesz projection $P(\Lambda, A)$ associated with the spectrum in Λ admits a representation

$$P(\Lambda, A) = P_\gamma(A) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_\gamma (z\mathrm{I} - A)^{-1}\,dz\,, \tag{2.11}$$

where γ is any regular cycle in the complement of $\Lambda\cup\Lambda'$ that has winding number 1 relative to every point in Λ and winding number 0 relative to every point in Λ'. If A is fixed, then the spectral subspace $P(\{\lambda\},A)\mathfrak{B}$ associated with an isolated eigenvalue λ of A will also be denoted by $E(\lambda)$. If λ has a finite algebraic multiplicity m, then $E(\lambda)$ is the null space of $(\lambda\mathrm{I} - A)^m$.

Assume now that A is a symplectic operator on $\mathcal{H}^2$. It follows readily from Definition 2.4 that A is nonsingular, and that the spectrum of A is invariant under complex conjugation $\lambda \mapsto \bar\lambda$ and inversion $\lambda \mapsto \lambda^{-1}$.

Lemma 2.7. *Let λ and λ' be eigenvalues of a symplectic A. If $\bar\lambda\lambda' \neq 1$ then the spectral subspaces $E(\lambda)$ and $E(\lambda')$ are G-orthogonal; that is, $G(V,V') = 0$ for all $V \in E(\lambda)$ and all $V' \in E(\lambda')$*

Proof. Straightforward algebraic manipulations yield the identity

$$G\big((A-\lambda\mathrm{I})^m V, A^m V'\big) = (-\bar\lambda)^m G\big(V, (A - \bar\lambda^{-1}\mathrm{I})^m V'\big) \tag{2.12}$$

for $m = 1$. This extends to $m > 1$ by induction. Taking m to be the algebraic multiplicity of λ, this shows that $E(\lambda)$ is G-orthogonal to the range of $\big(A - \bar\lambda^{-1}\mathrm{I}\big)^m$.

Clearly $E(\lambda')$ is invariant under any function of A, including $A - \bar\lambda^{-1}\mathrm{I}$. Assuming that $\lambda' \neq \bar\lambda^{-1}$, the restriction of $A - \bar\lambda^{-1}\mathrm{I}$ to $E(\lambda')$ is nonsingular. Thus, $E(\lambda')$ lies in the range of $(A - \bar\lambda^{-1}\mathrm{I})^m$, which is G-orthogonal to $E(\lambda)$. **QED**

Definition 2.8. *A subspace E of $\mathcal{H}^2$ is said to be Krein definite if $G(V,V) \neq 0$ for every nonzero $V \in E$. If $G(V,V)$ is positive (negative) for every nonzero $V \in E$ then we say that*

E has a positive (negative) Krein signature. An eigenvalue λ of a linear operator is said to have positive (negative) Krein signature if the corresponding spectral subspace $E(\lambda)$ has positive (negative) Krein signature.

We remark that an eigenvalue λ of a symplectic operator A can be Krein definite only if λ is semisimple and $|\lambda| = 1$. This follows e.g. from (2.12) with $m = 1$, which implies that the null space of $A - \lambda\mathrm{I}$ is G-orthogonal to the range of $A - \bar{\lambda}^{-1}\mathrm{I}$.

The following proposition illustrates the usefulness of the Krein signature for questions of stability and bifurcations.

Proposition 2.9. *Let A be a symplectic operator on $\mathcal{H}^2$. Let Λ be a closed arc on the unit circle whose endpoints are not eigenvalues of A. Let γ be a cycle in $\mathbb{C}$ such that $P(\Lambda, A) = P_\gamma(A)$, as described after (2.11). Assume that, within any given positive distance from A, there exists a symplectic operator $A' \in \mathcal{L}(\mathcal{H}^2)$ with the property that $P_\gamma(A')\mathcal{H}^2$ is Krein definite. Then every symplectic operator $A' \in \mathcal{L}(\mathcal{H}^2)$ within some positive distance of A has this property, and $P(\Lambda, A')\mathcal{H}^2$ has the same dimension as $P(\Lambda, A)\mathcal{H}^2$.*

Proof. It suffices to consider the positive signature case. Clearly the map $A' \mapsto P_\gamma(A')$ is continuous near A. Define $E_\gamma(A') = P_\gamma(A')\mathcal{H}^2$. In what follows, A' and A_n always denote symplectic operators on $\mathcal{H}^2$.

Under the given assumptions, there exists a sequence $A_n \to A$ such that G is positive definite when restricted to $E_\gamma(A_n)$. Thus, G is positive semidefinite on $E_\gamma(A)$. Assume for contradiction that $G(V, V) = 0$ for some nonzero $V \in E_\gamma(A)$. Using the Cauchy-Schwarz inequality we find that V is G-orthogonal to every vector in $E_\gamma(A)$. When combined with Lemma 2.7, this implies that V is G-orthogonal to every vector in $\mathcal{H}^2$. But this contradicts the fact that G is nondegenerate. Thus, G is positive definite on $E_\gamma(A)$.

If A' is sufficiently close to A, then G is positive definite on $E_\gamma(A')$ as well, and $E_\gamma(A')$ has the same dimension as $E_\gamma(A)$. As a result of positivity, $E_\gamma(A')$ cannot include an eigenvector V of A with an eigenvalue off the unit circle, since this would imply $G(V, V) = 0$ by Lemma 2.7. Thus, $P(\Lambda, A') = P_\gamma(A')$. **QED**

3. Parametrized families of Floquet systems

3.1. Separating sets

We will need to control eigenvalues on the unit circle $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ for continuous families of symplectic operators. The goal is to partition the circle $\mathbb{S}$ into arcs and then apply Proposition 2.9.

Definition 3.1. *Let A be a symplectic operator whose eigenvalues on $\mathbb{S}$ are all Krein definite. A finite set $Z \subset \mathbb{S}$ partitions $\mathbb{S} \setminus Z$ into arcs, and we say that Z is a separating set for A if all eigenvalues of A that lie on the same arc have the same Krein signature. We also assume that Z does not include any eigenvalues of A.*

Let now $s \mapsto A_s$ be a continuous family of symplectic operators on $\mathcal{H}^2$, where s ranges over some interval I. As a consequence of Proposition 2.9, we have the

Corollary 3.2. *Let Z be a finite subset of $\mathbb{S}$ that does not include any eigenvalues of A_s for any s. Assume that one of the operators A_s has the following property: Z is a separating set for A_s, and all eigenvalues of A lie on $\mathbb{S}$. Then each of the operators A_s has this property.*

Definition 3.3. *We say that $Z \subset \mathbb{S}$ is a separating set for the family $s \mapsto A_s$ if Z is a separating set for each A_s with $s \in I$.*

In the remaining part of this section we restrict our analysis to symplectic operators $\Phi(t)$ that are the time-t maps associated with a second order equation of the type (2.5). To simplify notation, let us assume from now on that $\mathcal{X}$ is periodic with period $T = 2\pi$. First we need to introduce some notation and basic facts.

Consider a solution curve V for the evolution equation $\frac{d}{dt}V(t) = X(t)V(t)$, where X is given by (2.6). Assume that $V(2\pi) = e^{2\pi r}V(0)$ for some complex number r. In other words, $V(0)$ is an eigenvector with eigenvalue $e^{2\pi r}$ of the matrizant $\Phi(2\pi)$ associated with the given system. If we set $W(t) = e^{-rt}V(t)$, then the curve W is 2π-periodic. A straightforward computation shows that the first component $w(t) = e^{-rt}v(t)$ of $W(t)$ satisfies the equation

$$\left(\tfrac{d}{dt} + r\right)^2 w(t) = -\mathcal{X}(t)w(t)\,. \tag{3.1}$$

Conversely, if this equation has a nonzero 2π-periodic solution w, and if we set $W = \left[\begin{smallmatrix} w \\ \omega \end{smallmatrix}\right]$, where $\omega = \left(\frac{d}{dt} + r\right)w$, then $W(0)$ is an eigenvector of $\Phi(2\pi)$ with eigenvalue $e^{2\pi r}$.

It is useful to write the equation (3.1) in the form $M(\eta)w = 0$, where $\eta = -ir$ and

$$M(\eta) = (\boldsymbol{n} + \eta)^2 - \mathcal{X}\,, \qquad \boldsymbol{n} = -i\tfrac{d}{dt}\,. \tag{3.2}$$

For a proper discussion of the operator $M(\eta)$, consider the vector space $\mathcal{C}$ of all continuous 2π-periodic curves $w : \mathbb{R} \to \mathcal{H}$, equipped with the inner product

$$\langle\!\langle w, w' \rangle\!\rangle = \frac{1}{\pi}\int_0^{2\pi} \big\langle w(t), w'(t) \big\rangle\, dt\,, \tag{3.3}$$

and define $\mathfrak{H}$ to be the completion of this space $\mathcal{C}$. A natural domain for the operator $M(\eta)$ is the space $\mathcal{D}$ of all curves $w \in \mathfrak{H}$ with the property that $\boldsymbol{n}^2 w$ belongs to $\mathfrak{H}$. Clearly $M(\eta)$ is a relatively compact perturbation of $\boldsymbol{n}^2$, so the spectrum of $M(\eta, s)$ consists of isolated eigenvalues of finite multiplicity. We assume from now on that $\mathcal{X}(t)$ is self-adjoint for each t. In this case, $M(\eta)$ is self-adjoint as well if η is real.

Proposition 3.4. *Assume that $t \mapsto \mathcal{X}(t)$ is real analytic. $V(0)$ is an eigenvector of $\Phi(2\pi)$ with eigenvalue $e^{2\pi i\eta}$ if and only if the equation $M(\eta)w = 0$ admits a nonzero solution $w \in \mathcal{D}$. In this case, $t \mapsto w(t)$ is analytic in a complex open neighborhood of $\mathbb{R}$.*

The proof of this proposition is straightforward; see Lemma 4.5 for a similar result in the case of the periodically forced wave equation.

Consider now two orbits $\frac{d}{dt}V = XV$ and $\frac{d}{dt}V' = XV'$. Assume that $V(0)$ and $V'(0)$ are eigenvectors of $\Phi(2\pi)$ with eigenvalues $e^{2\pi r}$ and $e^{2\pi r'}$, respectively. Let $w(t)$ and $w'(t)$

be the first components of $W(t) = e^{-rt}V(t)$ and $W'(t) = e^{-r't}V'(t)$, respectively. A straightforward computation shows that

$$G\bigl(V(t),V'(t)\bigr) = ie^{(\bar{r}+r')t}\Bigl[\bigl\langle w(t),\bigl(\tfrac{d}{dt}+r'\bigr)w'(t)\bigr\rangle - \bigl\langle\bigl(\tfrac{d}{dt}+r\bigr)w(t),w'(t)\bigr\rangle\Bigr]. \tag{3.4}$$

Since $\Phi(t)$ is symplectic, the left hand side of this equation does not depend on t. Assume now that $\eta = -ir$ is real. Setting $V' = V$ in (3.4) and taking the average over $t \in [0, 2\pi]$, we find that

$$G(V,V) = G_\eta(w) \stackrel{\text{def}}{=} -\langle\!\langle w, (\boldsymbol{n}+\eta)w\rangle\!\rangle\,. \tag{3.5}$$

Definition 3.5. *Let A and B be a bounded self-adjoint linear operator on a Hilbert space H. We say that A is strongly positive, in symbols $A \gg 0$, if there exists $a > 0$ such that $\langle h, Ah\rangle \geq a\langle h,h\rangle$ for all $h \in H$. If $B - A \gg 0$, then we also write $B \gg A$ or $A \ll B$.*

Coming back to parametrized systems, we consider the equation $M_s(\eta)w_s = 0$ associated with a curve $t \mapsto \mathcal{X}_s(t)$, all depending on a parameter s. For simplicity, we restrict now to affine curves $s \mapsto \mathcal{X}_s$. In this case we have

$$M_s(\eta) = (\boldsymbol{n}+\eta)^2 - \mathcal{X}_s\,, \qquad \mathcal{X}_s = \mathcal{X}_0 + sD\,. \tag{3.6}$$

We assume that $\mathcal{X}_0$ and D are self-adjoint bounded linear operators on $\mathfrak{H}$.

Lemma 3.6. *Assume that D is strongly positive. Assume that $M_{s_0}(\eta_0)$ has an eigenvalue zero for some $\eta_0, s_0 \in \mathbb{R}$. Then there exist real analytic functions $\tilde{s} : \mathbb{R} \to \mathbb{R}$ and $\tilde{w} : \mathbb{R} \to \mathfrak{H}$ nonzero, such that $\tilde{w}(\eta) \in \mathcal{D}$ and $M_{\tilde{s}(\eta)}(\eta)\tilde{w}(\eta) = 0$ for all $\eta \in \mathbb{R}$. Furthermore,*

$$\frac{d\tilde{s}}{d\eta} = -\frac{2\,G_\eta(\tilde{w}(\eta))}{\bigl\langle\!\bigl\langle \tilde{w}(\eta), D\tilde{w}(\eta)\bigr\rangle\!\bigr\rangle}\,, \qquad \tilde{s}(\eta_0) = s_0\,. \tag{3.7}$$

Moreover, if the eigenvalue zero of $M_{s_0}(\eta_0)$ is simple, then $\tilde{s}$ is unique, and $\tilde{w}$ is unique up to a constant factor.

Proof. The equation $M_s(\eta)w = 0$ is equivalent to the eigenvalue problem

$$\Bigl(D^{-1/2}\bigl[(\boldsymbol{n}+\eta)^2 - \mathcal{X}_0\bigr]D^{-1/2}\Bigr)g = sg\,. \tag{3.8}$$

where $g = D^{1/2}w$. The existence of the real analytic functions $\tilde{s}$ and $\tilde{w}$ now follows from Theorem 7.3.9 in [14]. Uniqueness in case of a simple eigenvalue zero follows from the fact that an analytic function, such as $\tilde{s}$ or $\langle w', \tilde{w}(.)\rangle$ for any $w' \in \mathfrak{H}$, has only isolated zeros, unless it is identically zero.

Differentiating the identity $M_{\tilde{s}}(\eta)\tilde{w} = 0$ with respect to η, we obtain

$$\left[2(\boldsymbol{n}+\eta) - \frac{d\tilde{s}}{d\eta}D\right]\tilde{w} + \Bigl[(\boldsymbol{n}+\eta)^2 - \mathcal{X}_0 - \tilde{s}D\Bigr]\frac{d\tilde{w}}{d\eta} = 0\,. \tag{3.9}$$

After taking the inner product with $\tilde{w}$, this yields

$$2\langle\!\langle \tilde{w}, (\boldsymbol{n}+\eta), \tilde{w} \rangle\!\rangle - \langle\!\langle \tilde{w}, D\tilde{w} \rangle\!\rangle \frac{d\tilde{s}}{d\eta} = 0\,. \tag{3.10}$$

Now we solve for $\frac{d\tilde{s}}{d\eta}$ and use (3.5) to get (3.7). **QED**

We will use this lemma in situations where the signature $G_\eta(\tilde{w}(\eta))$ cannot vanish. In this case, $\tilde{s}$ is strictly monotone and thus has an inverse $s \mapsto \eta_s$. From Lemma 3.6 we then get

$$\frac{d}{ds}\eta_s = -\frac{\langle\!\langle w_s, (\frac{d}{ds}\mathcal{X}_s) w_s \rangle\!\rangle}{2\, G_{\eta_s}(w_s)}\,, \qquad M_s(\eta_s) w_s = 0\,, \tag{3.11}$$

where $w_s = \tilde{w}(\eta_s)$ and $\frac{d}{ds}\mathcal{X}_s = D$. Notice that then Floquet exponents $e^{2\pi i \eta_s}$ with positive (negative) signature move (counter)clockwise on the unit circle as the parameter s is increased. Presumably (3.11) holds for more general curves $s \mapsto \mathcal{X}_s$. But we do not need such a generalization in this paper.

In the following two corollaries, we continue assuming that $s \mapsto \mathcal{X}_s$ is an affine family of bounded self-adjoint linear operators on $\mathfrak{H}$. But the parameter s is now restricted to $[-1, 1]$. Let $\mathcal{X}$ be an arbitrary bounded self-adjoint linear operator on $\mathfrak{H}$. The flows associated with $\mathcal{X}_s$ and $\mathcal{X}$ are denoted by Φ_s and Φ, respectively.

Corollary 3.7. *Assume that $\mathcal{X}_{-1} \lll \mathcal{X} \lll \mathcal{X}_1$. Let Z be a finite subset of the unit circle $\mathbb{S}$ that includes no eigenvalues of $\Phi_s(2\pi)$ for any s. Assume that for some s, the operator $A = \Phi_s(2\pi)$ has the following property: Z is a separating set for A, and all eigenvalues of A lie on $\mathbb{S}$. Then $A = \Phi(2\pi)$ has the same property.*

Proof. By Corollary 3.2, Z is a separating set for the family $s \mapsto \Phi_s(2\pi)$, and all eigenvalues of $\Phi_s(2\pi)$ for all s lie on the unit circle. Let $\mathcal{X}^r = (1-r)\mathcal{X}_0 + r\mathcal{X}$ for $r \in [0, 1]$. To each of these operators $\mathcal{X}^r$ we associate a family $s \mapsto \mathcal{X}^r_s$ by setting

$$\mathcal{X}^r_s = \mathcal{X}^r + \begin{cases} s(\mathcal{X}^r - \mathcal{X}_{-1}) & \text{if } s \in [-1, 0], \\ s(\mathcal{X}_1 - \mathcal{X}^r) & \text{if } s \in [0, 1]. \end{cases} \tag{3.12}$$

Here, and in what follows, we always assume that (r, s) belongs to $R = [0, 1] \times [-1, 1]$. Notice that $\mathcal{X}^0_s = \mathcal{X}_s$ and $\mathcal{X}^r_0 = \mathcal{X}^r$. Denote by Φ^r_s the flow associated with $\mathcal{X}^r_s$. Define $f(r, s)$ to be the distance (on $\mathbb{S}$) between the set Z and the eigenvalue of $\Phi^r_s(2\pi)$ on $\mathbb{S}$ that is closest to Z. Clearly f is continuous on R. Furthermore, f does not vanish on the segment $r = 0$. Thus, by compactness, Z is a separating set for $s \mapsto \Phi^r_s(2\pi)$, if $r > 0$ is sufficiently small. Denote by r_0 the largest value in $[0, 1]$ such that $s \mapsto \Phi^r_s(2\pi)$ has Z as a separating set for all $r < r_0$.

Consider $r < r_0$. By using (3.11) for the two affine segments of the curve $s \mapsto \mathcal{X}^r_s$, we see that the eigenvalues of $\Phi^r_s(2\pi)$ move monotonely along $\mathbb{S}$ as s increases from -1 to 1, covering an arc that starts at an eigenvalue of $\Phi_{-1}(2\pi)$ and ends at an eigenvalue of $\Phi_1(2\pi)$, without crossing the set Z. Thus, $f(r, .)$ is bounded from below by $\min(f(0, -1), f(0, 1))$, which is positive. This extends by continuity to $r = r_0$. Using Proposition 2.9, we conclude that Z is a separating set for $\Phi_s^{r_0}(2\pi)$, and that all eigenvalues of $\Phi_s^{r_0}(2\pi)$ lie on $\mathbb{S}$. If we assume that $r_0 < 1$, then $f^{-1}((0,\infty))$ includes a strip $(r_0-\varepsilon, r_0+\varepsilon)\times[-1,1]$ for some $\varepsilon > 0$, contradicting the defining property of r_0. Thus, $r_0 = 1$, and Corollary 3.7 is proved. **QED**

It is possible to deal with situations where not all eigenvalues lie on the unit circle, but additional conditions are required to avoid off-circle eigenvalues from interfering. The following covers a special case, where it suffices to include the point 1 in Z.

Corollary 3.8. *Assume that $\mathcal{X}_{-1} \lessdot \mathcal{X} \lessdot \mathcal{X}_1$. Let $Z_0 \ni 1$ be a finite subset of the unit circle $\mathbb{S}$ that includes no eigenvalues of $\Phi_s(2\pi)$ for any s. Assume that for some s, the operator $A = \Phi_s(2\pi)$ has the following property: Z_0 is a separating set for A, and all eigenvalues of A lie on the lie on $\mathbb{S}$, except for two simple real eigenvalues in $(0,1)\cup(1,\infty)$. Then $A = \Phi(2\pi)$ has the same property.*

Proof. Consider $Z = Z_0 \cup \{z^-, z^+\}$ where $z^\pm = \cos(\varepsilon) \pm i\sin(\varepsilon)$. By choosing $\varepsilon > 0$ sufficiently small, the assumptions of Corollary 3.8 are still satisfied if Z_0 is replaced by Z. Now we follow the proof of Corollary 3.7, but with the following changes.

It may happen that the eigenvalues of $\Phi_0^r(2\pi)$ in $(0,1)\cup(1,\infty)$ approach 1 as r is increased. With r_0 defined as before, consider the possibility that r_0 is the largest value in $[0,1]$ such that none of the operators $\Phi_s^r(2\pi)$ with $r < r_0$ has an eigenvalue 1. All eigenvalues of $\Phi_s^{r_0}(2\pi)$ in $\mathbb{S}\setminus\{1\}$ can be bounded away from Z as before. So $\Phi_0^{r_0}(2\pi)$ must have an eigenvalue 1.

Now consider perturbations $\hat{\mathcal{X}} = \mathcal{X} + t\mathrm{I}$ of $\mathcal{X}$, with $|t| > 0$ sufficiently small so that $\mathcal{X}_{-1} \lessdot \hat{\mathcal{X}} \lessdot \mathcal{X}_1$. Define operators $\hat{\mathcal{X}}_s^r$ and $\hat{\Phi}_s^r(2\pi)$ analogous to $\mathcal{X}_s^r$ and $\Phi_s^r(2\pi)$. By arguments analogous to those used in the proof of Proposition 4.6, we can find a sequence $n \mapsto t_n \neq 0$ converging to zero, such that the operator $\hat{\Phi}_0^{r_0}(2\pi)$ defined with $t = t_n$ has no eigenvalue 1 but two eigenvalues on $\mathbb{S}\setminus\{1\}$ close to 1. Let $r < r_0$. If we choose $t = t_n$ with n sufficiently large, then Z_0 is a separating set for the operators $\hat{\Phi}_s^r(2\pi)$. Thus, all eigenvalues of $\hat{\Phi}_0^r(2\pi)$ are bounded away from Z_0, uniformly in $r < r_0$ and large n. Taking $n \to \infty$ and then $r \to r_0$, we see that $\Phi_0^{r_0}(2\pi)$ cannot have an eigenvalue 1. Now we proceed as in the proof of Corollary 3.7 to show that $r_0 = 1$.

For reference later on, we remark that $\Phi(2\pi)$ has no eigenvalues in the arc bounded by $z^\pm$ that includes 1, since Z is a separating set for the family $s \mapsto \Phi_s^1(2\pi)$. **QED**

Remark 1. Under the assumption of Corollary 3.7 or Corollary 3.8, if $s \mapsto \lambda_s \in \mathbb{S}$ is a continuous curve of simple eigenvalues for the family $s \mapsto \Phi_s(2\pi)$, that is isolated from any other eigenvalue for that family, then one of the eigenvalues λ_s is also an eigenvalue of $\Phi(2\pi)$. This follows by including extra points in the separating sets.

3.2. Verifying separation

Let $\mathcal{X}_0$ be bounded self-adjoint linear operator on $\mathfrak{H}$. Let D be a positive self-adjoint linear operator on $\mathcal{H}$. We extend D to $\mathfrak{H}$ by setting $(Dw)(t) = Dw(t)$. Then we define $\mathcal{X}_s = \mathcal{X}_0 + sD$ for all $s \in [-1,1]$.

Our goal is to show that for some given $\eta \in \mathbb{R}$, none of the operators $M_s(\eta)$ with $s \in [-1,1]$ has an eigenvalue zero. To allow for better approximations, we replace the operator $M_s(\eta)$ by

$$\hat{M}_s(\eta) = \theta M_s(\eta)\theta = \theta(\boldsymbol{n}+\eta)^2\theta - \hat{\mathcal{X}}_0 - s\hat{D}\,, \tag{3.13}$$

where θ is some strictly positive operator on $\mathfrak{H}$. Here $\hat{\mathcal{X}}_0 = \theta\mathcal{X}_0\theta$ and $\hat{D} = \theta D\theta$. For concreteness let us assume that θ^{-1} is a bounded perturbation of $|\boldsymbol{n}|$. Then it is clear that $M_s(\eta)$ has an eigenvalue zero if and only if $\hat{M}_s(\eta)$ has an eigenvalue zero.

The idea is to approximate $\hat{\mathcal{X}}_0$ by a self-adjoint linear operator $\check{\mathcal{X}}_0$ on $\mathfrak{H}$ for which

$$\check{M}_s(\eta) = \theta(\boldsymbol{n}+\eta)^2\theta - \check{\mathcal{X}}_s \tag{3.14}$$

is easier to analyze. Here $\check{\mathcal{X}}_s = \check{\mathcal{X}}_0 + s\hat{D}$. In our application, $\check{M}_s(\eta)$ acts trivially outside some finite dimensional subspace of $\mathfrak{H}$, so $\check{M}$ is essentially a family of symmetric matrices. But here we make no such assumption. The first step is to prove a bound

$$\bigl\|\hat{\mathcal{X}}_0 - \check{\mathcal{X}}_0\bigr\| < C\,, \tag{3.15}$$

for some (small) positive constant $C > 0$. Then we choose appropriate parameter values $-1 \le s_0 < s_1 < \ldots < s_m = 1$ and verify the hypotheses of the following lemma.

Lemma 3.9. *With $C > 0$ satisfying (3.15), assume that $\check{M}_{s_j}(\eta)$ has no eigenvalue in $[-C,C]$, and that*

$$(s_j - s_{j-1})\bigl\|\hat{D}\bigr\| < 2C\,, \tag{3.16}$$

for $j = 1,2,\ldots,m$. Then none of the operators $\hat{M}_s(\eta)$ with $s \in [-1,1]$ has an eigenvalue zero.

Proof. First we note that the eigenvalues of $\check{M}_s(\eta)$ can be organized into a sequence $k \mapsto \check{\mu}_k(s)$ with each $\check{\mu}_k$ being real analytic on $[-1,1]$. Similarly for the corresponding eigenvectors $\check{w}_k(s)$. This follows e.g. from Theorem 7.3.9 in [14]. A computation analogous to (3.11) shows that

$$\frac{d}{ds}\check{\mu}_k(s) = -\frac{\bigl\langle \check{w}_k(s), \hat{D}\check{w}_k(s)\bigr\rangle}{\langle \check{w}_k(s), \check{w}_k(s)\rangle}\,. \tag{3.17}$$

Thus, the functions $\check{\mu}_k$ are decreasing, and

$$\bigl|\check{\mu}_k(s_j) - \check{\mu}_k(s_{j-1})\bigr| \le (s_j - s_{j-1})\bigl\|\hat{D}\bigr\| < 2C\,. \tag{3.18}$$

This shows that none of the curves $\check{\mu}_k$ can cross the interval $[-C,C]$, as s increases from s_{j-1} to s_j. Thus, $\check{M}(\eta,s)$ has no eigenvalue in $[-C,C]$ for all $s \in [-1,1]$.

Now we interpolate linearly between $\hat{\mathcal{X}}_s$ and $\check{\mathcal{X}}_s$ for a fixed but arbitrary $s \in [-1,1]$. By arguments analogous to the ones used above for the interpolation between $\check{\mathcal{X}}_{s_{j-1}}$ and $\check{\mathcal{X}}_{s_j}$, we find that

$$\bigl|\hat{\mu}_k(s) - \check{\mu}_k(s)\bigr| \le \bigl\|\hat{\mathcal{X}}_s - \check{\mathcal{X}}_s\bigr\| = \bigl\|\hat{\mathcal{X}}_0 - \check{\mathcal{X}}_0\bigr\| < C\,. \tag{3.19}$$

Such a bound holds for every eigenvalue $\hat{\mu}_k(s)$ of $\hat{M}_s(\eta)$. Combining this bound with the fact that $|\check{\mu}_k(s)| > C$, we find that

$$\bigl|\hat{\mu}(s)\bigr| \geq \bigl|\check{\mu}(s)\bigr| - \bigl|\hat{\mu}(s) - \check{\mu}(s)\bigr| > C - C = 0\,. \tag{3.20}$$

This shows that $\hat{M}_s(\eta)$ has no eigenvalue zero, for every $s \in [-1,1]$. **QED**

4. The wave equation with periodic forcing

4.1. The flow

In order to motivate our choices, consider a pair (α, u) as described in Theorem 1.1 and define $h = \alpha^{-2}\bigl(3u^2 + \kappa\bigr) - c$ for some constant $c \geq 0$. Using the notation $v(t) = v(t, \boldsymbol{.})$, the equation (1.5) can be written as

$$\frac{d^2}{dt^2}v(t) = -\beta^2\boldsymbol{k}^2 v(t) - cv(t) - H(t)v(t)\,, \qquad t \in \mathbb{R}\,, \tag{4.1}$$

where $\boldsymbol{k}^2 = -\partial_x^2$ with Dirichlet boundary conditions at $x = 0$ and $x = \pi$, and where $H(t)$ denotes multiplication by $h(t, \boldsymbol{.})$. This is an infinite-dimensional version of (2.5), with $\mathcal{X} = \beta^2\boldsymbol{k}^2 + c + H$. The vector field $X = X_H$ in the corresponding evolution equation (2.1) is given formally by

$$X_H(t) = \begin{bmatrix} 0 & \mathrm{I} \\ -\boldsymbol{y}^2 - H(t) & 0 \end{bmatrix}, \qquad \boldsymbol{y} = \bigl(\beta^2\boldsymbol{k}^2 + c\bigr)^{1/2}. \tag{4.2}$$

In this section, we consider the flow Φ_H associated with vector fields of this type, where $H(t)$ need not be a multiplication operator.

We start by choosing appropriate spaces for the initial conditions at $t = 0$. Given any real number $\rho \geq 0$, denote by $\mathcal{H}_\rho$ the Hilbert space of functions v on $\mathbb{R}$,

$$v(x) = \sum_{k=1}^{\infty} v_k \sin(kx)\,, \qquad \sum_{k=1}^{\infty} e^{2\rho k}|v_k|^2 < \infty\,, \tag{4.3}$$

equipped with the inner product

$$\langle v, v'\rangle_\rho = \sum_{k=1}^{\infty} e^{2\rho k}\,\overline{v_k}\,v'_k\,, \qquad v, v' \in \mathcal{H}_\rho\,. \tag{4.4}$$

On the space $\mathcal{H}_\rho^2$ of all pairs $V = \left[\begin{smallmatrix} v \\ \nu \end{smallmatrix}\right]$ with components in $v, \nu \in \mathcal{H}_\rho$ we use an inner product analogous to (2.8). Notice that, if ρ is positive, then the functions in $\mathcal{H}_\rho$ extend to analytic functions in the strip $|\mathrm{Im}\,(x)| < \rho$.

For the system considered in Theorem 1.2, the operators $H(t)$ are as described at the beginning of this section. The function $h = \alpha^{-2}\bigl(3u^2+\kappa\bigr) - c$ is 2π-periodic, even, and real analytic in both variables. So in this case, there exists $\varrho > 0$ such that the following holds whenever $0 \le \rho < \varrho$.

Property 4.1. *$t \mapsto H(t)$ is a 2π-periodic real analytic family of bounded self-adjoint linear operator on $\mathcal{H}_0$. If $0 < \rho < \varrho$ then the restriction to $\mathcal{H}_\rho$ defines a real analytic family of bounded linear operators on $\mathcal{H}_\rho$.*

In what follows, we allow $t \mapsto H(t)$ to be any family with this property. But we always assume that $0 \le \rho < \varrho$.

It is convenient to perform a (formally symplectic) change of variables

$$U = \begin{bmatrix} \boldsymbol{y}^{1/2} & 0 \\ 0 & \boldsymbol{y}^{-1/2} \end{bmatrix}. \tag{4.5}$$

The vector field in the new coordinates is given by

$$\tilde{X}_H = U X_H U^{-1} = \begin{bmatrix} 0 & \boldsymbol{y} \\ -\boldsymbol{y} - \tilde{H} & 0 \end{bmatrix}, \qquad \tilde{H} = \boldsymbol{y}^{-1/2} H \boldsymbol{y}^{-1/2}. \tag{4.6}$$

The natural domain for the operators $\tilde{X}_H(t)$ is $(\boldsymbol{y}^{-1}\mathcal{H}_\rho)^2$.

Proposition 4.2. *Let $0 \le \rho \le \rho' < \varrho$. For each $t \in \mathbb{R}$, the time-t map (with initial time zero) for the vector field $\tilde{X}_H$ defines a bounded linear operator $\tilde{\Phi}_X(t)$ from $\mathcal{H}^2_{\rho'}$ to $\mathcal{H}^2_\rho$. The operator norm of $\tilde{\Phi}_X(t)$ grows at most exponentially with $|t|$. If $\rho < \rho'$ then the flow $t \mapsto \tilde{\Phi}_X(t)$ defines a C^∞ function from $\mathbb{R}$ to the space $\mathcal{L}\bigl(\mathcal{H}^2_{\rho'}, \mathcal{H}^2_\rho\bigr)$ of bounded linear operators from $\mathcal{H}^2_{\rho'}$ to $\mathcal{H}^2_\rho$.*

Proof. To simplify notation we first consider $\rho' = \rho$. The vector field $\tilde{X}_H$ for $H = 0$ generates the continuous group of unitary operators

$$\tilde{\Phi}_0(t) = e^{t\tilde{X}_0} = \begin{bmatrix} \cos(t\boldsymbol{y}) & \sin(t\boldsymbol{y}) \\ -\sin(t\boldsymbol{y}) & \cos(t\boldsymbol{y}) \end{bmatrix} \tag{4.7}$$

on the space $\mathcal{H}^2_\rho$. The time-t maps $\tilde{\Phi}_H(t)$ for the vector field $\tilde{X}_H$, with initial time 0, can be obtained by solving the Duhamel equation

$$\tilde{\Phi}_H(t) = \tilde{\Phi}_0(t) + \int_0^t \tilde{\Phi}_0(t-s)\bigl[\tilde{X}_H(s) - \tilde{X}_0\bigr]\tilde{\Phi}_H(s)\,ds. \tag{4.8}$$

Using the operators $P_1 = [1\ 0]$ and $P_2 = [0\ 1]$ from $\mathcal{H}^2_{\rho'}$ to $\mathcal{H}_\rho$, this equation can also be written as

$$\tilde{\Phi}_H(t) = \tilde{\Phi}_0(t) - \int_0^t \tilde{\Phi}_0(t-s) P_2^* \tilde{H}(s) P_1 \tilde{\Phi}_H(s)\,ds. \tag{4.9}$$

Multiplying both sides of this equation from the left by $\tilde{\Phi}_0(-t)$, we obtain

$$A(t) = -\int_0^t B(s)\bigl[\mathrm{I} + A(s)\bigr]\, ds\,, \tag{4.10}$$

where

$$A(t) = \tilde{\Phi}_0(-t)\tilde{\Phi}_H(t) - \mathrm{I}\,, \tag{4.11}$$

and

$$\begin{aligned} B(s) &= \tilde{\Phi}_0(-s)P_2^*\tilde{H}(s)P_1\tilde{\Phi}_0(s) \\ &= \begin{bmatrix} -\sin(s\boldsymbol{y})\tilde{H}(s)\cos(s\boldsymbol{y}) & -\sin(s\boldsymbol{y})\tilde{H}(s)\sin(s\boldsymbol{y}) \\ \cos(s\boldsymbol{y})\tilde{H}(s)\cos(s\boldsymbol{y}) & \cos(s\boldsymbol{y})\tilde{H}(s)\sin(s\boldsymbol{y}) \end{bmatrix}. \end{aligned} \tag{4.12}$$

Notice that each $B(s)$ is a bounded linear operator on $\mathcal{H}^2_\rho$. Their operator norms satisfy a bound $\|B(s)\|_\rho \le b$ that is independent of s. Thus, the equation (4.10) for $A(\boldsymbol{.})$ can be solved by iteration,

$$A(t_0) = \sum_{n=1}^{\infty}(-1)^n \int_0^{t_0} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n B(t_1)B(t_2)\cdots B(t_n)\,, \tag{4.13}$$

and the solution satisfies a bound $\|A(t)\|_\rho \le e^{b|t|} - 1$. When combined with (4.11), this shows that, if $\rho' = \rho$, then $\Phi_H(t)$ belongs to $\mathcal{L}\bigl(\mathcal{H}^2_{\rho'}, \mathcal{H}^2_\rho\bigr)$ and the operator norm of $\Phi_H(t)$ grows at most exponentially with $|t|$. The same holds if $\rho' > \rho$ since $\mathcal{H}^2_{\rho'}$ is continuously embedded in $\mathcal{H}^2_\rho$, with $\|V\|_\rho \le \|V\|_{\rho'}$ for every $V \in \mathcal{H}^2_{\rho'}$.

Assume now that $\rho' > \rho$. From the explicit expressions (4.7) and (4.12) it is clear that $\tilde{\Phi}_0$ and B are of class C^∞ as curves in $\mathcal{L}\bigl(\mathcal{H}^2_{\rho_1}, \mathcal{H}^2_{\rho_2}\bigr)$, whenever $\rho' \ge \rho_1 > \rho_2 \ge \rho$. Using the equation (4.10) and the product rule of differentiation, we see that the same holds for the curve A, as well as for the curve $t \mapsto \tilde{\Phi}_H(t) = \tilde{\Phi}_0(t)[\mathrm{I} + A(t)]$. This completes the proof of Proposition 4.2. **QED**

4.2. The spectrum

Notice that $\tilde{H}(s)$ is compact for each s, due to the factors $\boldsymbol{y}^{-1/2}$ in (4.6). Thus, $B(s)$ is compact for each s. Using the equation (4.10) and Theorem 1.3 in [15] about strong integrals of compact operators, we find that $A(t)$ is compact for all t. This in turn implies that $\tilde{\Phi}_H(t) - \tilde{\Phi}_0(t) = \tilde{\Phi}_0(t)A(t)$ is compact for all t. In particular, the essential spectrum of $\tilde{\Phi}_H(2\pi)$ agrees with the essential spectrum Σ_e of $\tilde{\Phi}_0(2\pi)$. This follows e.g. from Theorem 4.5.35 in [14]. Notice that Σ_e is the set of all accumulation points of the sequence $k \mapsto e^{2\pi i y_k}$ for $k \in \mathbb{Z}$.

We are interested in the spectral stability of $\tilde{\Phi}_H(2\pi)$. Since Σ_e is included in the unit circle, it suffices to consider points λ in the spectrum of $\tilde{\Phi}_H(2\pi)$ that do not belong to Σ_e. The question is whether the results depend on the choice of the domain parameter ρ. The following can be applied e.g. to the operator $\tilde{\Phi}_H(2\pi)$ acting on $X = \mathcal{H}_\rho$ and $X' = \mathcal{H}_{\rho'}$ with $0 \le \rho \le \rho' < \varrho$.

Proposition 4.3. *Let X' and X be Banach spaces, with X' being continuously embedded and dense in X. Let L be a bounded linear operator on X that leaves X' invariant and defines a bounded linear operator L' on X'. Let λ be a point in the spectrum of L. Assume that the essential spectra of both L and L' are countable and do not include λ. Then λ is an eigenvalue for both L and L'. The corresponding spectral subspaces $E_\lambda \subset X$ and $E'_\lambda \subset X'$ are finite-dimensional and they agree.*

Proof. Under the given assumption, λ is an isolated eigenvalue of L with finite (algebraic) multiplicity. This follows e.g. from Theorem 4.5.33 in [14]. Similarly, if λ belongs to the spectrum of L', then λ is an isolated eigenvalue of L'. Denote by P_λ the Riesz projection in X onto the spectral subspace E_λ associated with the eigenvalue λ of L. It admits a representation of the form (2.11), and the integration contour γ can be chosen in the resolvent set of both L and L'. Thus, the restriction of P_λ to X' defines a bounded projection on X'. Given a fixed but arbitrary nonzero $y \in E_\lambda$, pick a sequence $n \mapsto x_n \in X'$ such that $x_n \to y$ in X. Let $y_n = P_\lambda x_n$. Then $y_n \to y$ for the norm in X. But y_n belongs to $E' = P_\lambda X'$ for all n, and since all norms on E' are equivalent, the sequence $n \mapsto y_n$ converges for the norm in X', and the limit has to be y. Since $y \in E_\lambda$ was arbitrary, this show that $E_\lambda \subset E'$. In particular, λ is an eigenvalue of L'. Since X' is continuously embedded in X we have $E'_\lambda = E' \subset E_\lambda$. Thus, $E'_\lambda = E_\lambda$ as claimed. **QED**

Assume that $0 \leq \rho \leq \rho' < \varrho$. Applying Proposition 4.3 to the operator $\tilde{\Phi}_H(2\pi)$ acting on $X = \mathcal{H}_\rho$ and on $X' = \mathcal{H}_{\rho'}$, we obtain the following.

Lemma 4.4. *The essential spectrum of $\tilde{\Phi}_H(2\pi)$ is the set Σ_e of all accumulation points of the sequence $k \mapsto e^{2\pi i y_k}$ for $k \in \mathbb{Z}$. The spectrum of $\tilde{\Phi}_H(2\pi)$ outside Σ_e consists of eigenvalues with finite multiplicities, and the corresponding spectral subspaces are independent of ρ.*

4.3. Borderline stability

The main result of this subsection concerns a borderline situation between (spectral) stability and instability. This situation has to be excluded in the case $\alpha = \frac{14}{11}$ described in Theorem 1.2.

Formally, we have a one-to-one correspondence between nonzero solutions w of the equation

$$\Big[\big(\tfrac{d}{dt} + r\big)^2 + \beta^2 \boldsymbol{k}^2 + c + H(t)\Big] w(t) = 0\,, \tag{4.14}$$

and eigenvectors V_0 of $\Phi_H(2\pi)$ with eigenvalue $e^{2\pi r}$. The maps $V_0 \mapsto w$ and $w \mapsto V_0$ are described before and after (3.1). In order to make this correspondence more precise, we need to discuss regularity properties.

To any curve $t \mapsto w(t)$ in $\mathcal{H}_0$ we can associate a function $w^\flat$ of two variables by setting $w^\flat(t,\boldsymbol{.}) = w(t)$ for all t. If $t \mapsto H(t)$ is a family of linear operators on $\mathcal{H}_0$ then we define $H^\flat w^\flat = (Hw)^\flat$. In what follows we will identify $w^\flat$ with w and $H^\flat$ with H.

Consider the space $\mathfrak{H} = \mathrm{L}^2\bigl([0,2\pi]\times[0,\pi]\bigr)$. A function w belongs to $\mathfrak{H}$ if and only if it admits a Fourier series

$$w = \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty} w_{n,k}E_{n,k}\,, \qquad E_{n,k}(t,x) = e^{int}\sin(kx)\,, \tag{4.15}$$

that converges in L^2. Given any nonnegative integer m, we defined $\mathfrak{H}'_m$ to be the space of all function $w \in \mathfrak{H}$ with the property that the function $(|\boldsymbol{n}|+|\boldsymbol{k}|)^m w$ with Fourier coefficients $(|n|+k)^m w_{n,k}$ belongs to $\mathfrak{H}$.

If we set $F = c + H$ and $\eta = -ir$, then the equation (4.14) can be written as

$$M_F(\eta)w = 0\,, \qquad M_F(\eta) = (\boldsymbol{n}+\eta)^2 - \beta^2\boldsymbol{k}^2 - F\,. \tag{4.16}$$

Below we will also consider the operator $M_0(\eta)$, which is defined by setting $F=0$ in the above equation. But, unless stated otherwise, we always assume that $F = c+H$.

We consider M_F as a densely defined linear operator on $\mathfrak{H}$. Its domain $\mathcal{D}_\eta$ is the set of all functions $w \in \mathfrak{H}$ with the property that $M_F(\eta)w$ belongs to $\mathfrak{H}$. A more explicit description will be given below.

Assume now that H satisfies Property 4.1, and that $H\mathfrak{H}'_m \subset \mathfrak{H}'_m$ for every $m \ge 0$.

Lemma 4.5. *Let β be a positive rational number. Then Σ_e is a finite set. Let η be a real number such that $\lambda = e^{2\pi i\eta}$ does not belong to Σ_e. If λ is an eigenvalue of $\tilde{\Phi}(2\pi)$ with eigenvector $\tilde{V}_0 \in \mathcal{H}_0^2$, then the corresponding function $w = w(t,x)$ belongs to $\mathcal{D}_\eta$ and satisfies $M_F(\eta)w = 0$. Conversely, if $w \in \mathcal{D}_\eta$ satisfies the equation $M_F(\eta)w = 0$, then the corresponding function $\tilde{V}_0$ belongs to $\mathcal{H}_0^2$ and is an eigenvector of $\tilde{\Phi}(2\pi)$ with eigenvalue λ.*

Proof. Write $\beta = q/p$ with q and p coprime positive integers. Then it is clear that Σ_e is a finite set, consisting of p-th roots of unity.

Assume that λ is an eigenvalue of $\tilde{\Phi}(2\pi)$ with eigenvector $\tilde{V}_0 \in \mathcal{H}_0^2$. By Lemma 4.4, this eigenvector belongs to $\mathcal{H}_\rho^2$ for every positive $\rho < \varrho$. And Proposition 4.2 implies that $t \mapsto \tilde{V}(t)$ is a C^∞ curve of function in $\mathcal{H}_\rho^2$. So it is clear that w belongs to $\mathcal{D}_\eta$ and satisfies the equation (4.16).

Before proving the converse, we note that $w \in \mathfrak{H}$ belongs to the domain $\mathcal{D}_\eta$ of $M_F(\eta)$ is and only if the series

$$M_0(\eta)w = \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty} w_{n,k}\mu_{n,k}(\eta)E_{n,k}\,, \qquad \mu_{n,k}(\eta) = (n+\eta)^2 - \beta^2k^2\,, \tag{4.17}$$

converges in $\mathfrak{H}$, where $w_{n,k}$ are the Fourier coefficients of w. Notice that each $E_{n,k}$ is an eigenfunction of $M_0(\eta)$ and that $\mu_{n,k}(\eta)$ is the corresponding eigenvalue. Under the given assumption on η, these eigenvalues satisfy a bound

$$|\mu_{n,k}(\eta)| = p^{-2}\bigl|(pn+p\eta+qk)(pn+p\eta-qk)\bigr| \ge C_\eta\bigl(|n|+k\bigr)\,, \tag{4.18}$$

with $C_\eta > 0$. So in particular, $M_0(\eta)$ has a compact inverse.

Assume now that the equation $M_F(\eta)w = 0$ has a nonzero solution $w \in \mathcal{D}_\eta$. Then $M_0(\eta)w - Fw = 0$ and thus

$$w = M_0(\eta)^{-1} F w\,. \tag{4.19}$$

The bound (4.18) implies that the right hand side of (4.19) belongs to $\mathfrak{H}'_{m+1}$ whenever $w \in \mathfrak{H}'_m$. Thus, w belongs to $\mathfrak{H}'_m$ for all $m \geq 0$. So w is of class C^∞. It is now clear that the corresponding function $\tilde{V}_0$ belongs to $\mathcal{H}_0^2$ and is an eigenvector of $\tilde{\Phi}_H(2\pi)$. **QED**

Proposition 4.6. *Let β be positive and rational. Assume that 1 is an isolated eigenvalue of $\tilde{\Phi}_H(\pi)$. Then for $\eta > 0$ sufficiently close to zero there exists $s = s(\eta)$ such that $\tilde{\Phi}_{H+s}(2\pi)$ has an eigenvalue $e^{2\pi i\eta}$. Furthermore $s(\eta) \to 0$ as $\eta \to 0$.*

Proof. We use the same notation as in the proof of Lemma 4.5.

Since the eigenvalue 1 is isolated, there exists some open interval J containing zero such that $e^{2\pi i\eta} \notin \Sigma_e$ whenever $\eta \in J$. From (4.18) we see that that if $\eta, \eta' \in J$ then

$$|\mu_{n,k}(\eta') - \mu_{n,k}(\eta)| = |(2n + \eta' + \eta)(\eta' - \eta)| \leq C'_\eta |\mu_{n,k}(\eta)|\,, \tag{4.20}$$

for some $C'_\eta > 0$. This shows that $\mathcal{D}_\eta = \mathcal{D}_0$ for all $\eta \in J$. This extends trivially to all η in some complex open neighborhood of J. Clearly $M_0(\eta) : \mathcal{D}_\eta \to \mathfrak{H}$ is a closed Fredholm operator. Since $F = c + H$ is bounded and compact relative to $M_0(\eta)$, the operator $M_F(\eta) : \mathcal{D}_\eta \to \mathfrak{H}$ is closed and Fredholm as well. Thus, $M_F(\eta)$ has compact resolvents. Furthermore, the family $\eta \mapsto M_F(\eta)w$ for any $w \in \mathcal{D}_0$ is analytic in an open neighborhood of J. Notice also that $M_F(\eta)$ is Hermitian for $\eta \in \mathbb{R}$. Thus, given that $M_F(0)$ has an eigenvalue zero, there exists a real analytic function s on J, with $s(0) = 0$, such that $s(\eta)$ is an eigenvalue of $M_F(\eta)$ for each $\eta \in J$. This follows e.g from Theorem 7.3.9 in [14]. Since an eigenvector w of $M_F(\eta)$ with eigenvalue s satisfies $M_{F+s}(\eta)w = 0$, the assertion follow. **QED**

5. Reduction to finite dimensions

5.1. Finite-dimensional approximations

Here we approximate the operators $H(t)$ by "truncated" operators $H_\ell(t)$ that act trivially outside an ℓ-dimensional subspace of $\mathcal{H}_0$. Suppose that we want to prove spectral stability for Φ_H. Our strategy is to look for a family of the form $\mathcal{X}_s = \beta^2 \boldsymbol{k}^2 + c + H_K + sD$ that can be shown to have a separating partition, and which satisfies $H_K - D \lll H_\ell \lll H_K + D$ for all $\ell > K$. Then we take $\ell \to \infty$.

For $\ell \geq 1$ consider the orthogonal projections P_ℓ on $\mathcal{H}_0$ defined by the equation

$$(P_\ell v)(x) = \sum_{k=1}^{\ell} v_k \sin(kx)\,, \qquad v \in \mathcal{H}_0\,, \quad x \in \mathbb{R}\,, \tag{5.1}$$

where $v_1, v_2, \ldots$ are the coefficients in the sine series (4.3) of v.

To a family $t \mapsto H(t)$ of bounded self-adjoint operators on $\mathcal{H}_0$ we associate a family $t \mapsto H_\ell(t)$ by setting $H_\ell(t) = P_\ell H(t) P_\ell$. Assume now that H has the Property 4.1.

Proposition 5.1. *Let $0 \le \rho < \varrho$. Then $\tilde{\Phi}_{H_\ell}(t) \to \tilde{\Phi}_H(t)$ in $\mathcal{L}(\mathcal{H}_\rho)$, uniformly in t on bounded subsets of $\mathbb{R}$.*

Proof. We adopt the notation from the previous section but include a subscript ℓ whenever we use the family H_ℓ in place of H.

Clearly, each family H_ℓ has the Property 4.1. Furthermore, $\tilde{H}_\ell(s) \to \tilde{H}(s)$ in norm, uniformly in $s \in \mathbb{R}$. Then $B_\ell(s) \to B(s)$ in norm, uniformly in $s \in \mathbb{R}$. Using (4.13), we find that $A_\ell(t) \to A(t)$ in norm, uniformly on bounded subsets of $\mathbb{R}$. Combining this with (4.11) proves the claim. **QED**

Using Proposition 5.1 together with the contour-integral formula (2.11) for spectral projections, we obtain the following.

Corollary 5.2. *Let λ be an eigenvalue $\tilde{\Phi}_H(2\pi)$ that lies outside Σ_e. Then for every ℓ there exists an eigenvalue λ_ℓ of $\tilde{\Phi}_{H_\ell}(2\pi)$ such that $\lambda_\ell \to \lambda$ as $\ell \to \infty$.*

Another consequence of Proposition 5.1 is that $\tilde{\Phi}_H(t)$ is symplectic for each t, as an operator on $\mathcal{H}_0$. It should be clear what symplecticity means in the current setting.

Notice that the restriction of $H_\ell(t)$ to $(\mathrm{I} - P_\ell)\mathcal{H}_0$ is identically zero. Correspondingly, the operator $\tilde{\Phi}_{H_\ell}(2\pi)$ has a set of trivial eigenvalues λ_k and $\overline{\lambda_k}$ that lie on the unit circle and are given by

$$\lambda_k = e^{2\pi i y_k}\,, \qquad y_k = \sqrt{\beta^2 k^2 + c}\,, \tag{5.2}$$

for $k > \ell$. The corresponding eigenvectors are $V_k = \left[\begin{smallmatrix}1\\ i\end{smallmatrix}\right] \sin(k\boldsymbol{.})$ and $\overline{V_k}$. The set of accumulation points for these eigenvalues is precisely the essential spectrum Σ_e of the operator $\tilde{\Phi}_H(2\pi)$. Thus, we have the following.

Corollary 5.3. *For each ℓ denote by Φ_{H_ℓ} the flow for the vector field X_{H_ℓ} restricted to $(P_\ell\mathcal{H}_0)^2$. Let λ be an eigenvalue $\tilde{\Phi}_H(2\pi)$ that lies outside Σ_e. Then for every ℓ there exists an eigenvalue λ_ℓ of $\Phi_{H_\ell}(2\pi)$ such that $\lambda_\ell \to \lambda$ as $\ell \to \infty$.*

In what follows we ignore the trivial action on $(\mathrm{I} - P_\ell)\mathcal{H}_0$ and regard H_ℓ as a family of operators on $P_\ell\mathcal{H}_0$. The corresponding space of 2π-periodic curves $w : \mathbb{R} \to P_\ell\mathcal{H}_0$, with the inner product (3.3), will be denoted by $\mathfrak{H}_\ell$.

Consider the basis in $P_\ell\mathcal{H}_0$ consisting of the vectors $e_k = \sin(k\boldsymbol{.})$ for $k = 1, 2, \ldots, \ell$. In this basis, a linear operator T on $\mathcal{H}_\ell$ is represented by a matrix $T_{k,j} = (Te_j)_k$. As a measure for the size of the k-th row of this matrix we define

$$\mathcal{R}_k(T) = \sum_{j=1}^{\ell} |T_{k,j}|\,. \tag{5.3}$$

Let now $K \ge 1$ be fixed. In what follows, we always assume that $\ell > K$. Our goal is to estimate the difference $H_\ell - H_K$ in a way that is uniform in ℓ.

To this end, assume that there exist positive constants $\delta, d_1, d_2, d_3, \ldots$ such that

$$d_k > \mathcal{R}_k\big(H_\ell(t) - H_{\scriptscriptstyle K}(t)\big) + \delta\,, \tag{5.4}$$

for all $\ell \geq k$. Let D be the linear operator on $P_\ell\mathcal{H}_0$ whose matrix is diagonal, with entries $D_{k,k} = d_k$.

Proposition 5.4. *As operators on $\mathfrak{H}_\ell$ we have $H_{\scriptscriptstyle K} - D \lll H_\ell \lll H_{\scriptscriptstyle K} + D$.*

Proof. Let $d_k' = d_k - \delta$, and let D' be the diagonal operator on $P_\ell\mathcal{H}_0$ whose matrix is diagonal, with entries $D'_{k,k} = d'_k$. Given any $t \in \mathbb{R}$, define $T = H_\ell(t) - H_{\scriptscriptstyle K}(t)$. By (5.4) we have

$$(D' \pm T)_{k,k} = d'_k \pm T_{k,k} \geq d'_k - |T_{k,k}| > \sum_{j\neq k} |T_{k,j}| = \sum_{j\neq k} |(D' \pm T)_{k,j}|\,. \tag{5.5}$$

This means that the matrix for $D' \pm T$ is diagonally dominated, with positive diagonal entries. By the Levy-Desplanques theorem, this implies that the operator $D' \pm T$ is positive definite. Thus,

$$H_{\scriptscriptstyle K}(t) - D' < H_\ell(t) < H_{\scriptscriptstyle K}(t) + D'\,. \tag{5.6}$$

Since $t \in \mathbb{R}$ was arbitrary, we have

$$\langle\!\langle w, (H_{\scriptscriptstyle K} - D')w\rangle\!\rangle < \langle\!\langle w, H_\ell w\rangle\!\rangle < \langle\!\langle w, (H_{\scriptscriptstyle K} + D')w\rangle\!\rangle\,, \tag{5.7}$$

for every $w \in \mathfrak{H}_\ell$. This proves the claim. **QED**

For reference later on let us compute the trivial Floquet multipliers associated with the operator $H_{K,s} = H_{\scriptscriptstyle K} + sD$. For $s = 0$ the eigenvalues of $\Phi_{H_{K,s}}(2\pi)$ are given by (5.2) for $k > K$. For arbitrary $s \in [-1, 1]$ the trivial eigenvalues are $\lambda_{k,s}$ and $\overline{\lambda_{k,s}}$, where

$$\lambda_{k,s} = e^{2\pi i\eta_{k,s}}\,, \qquad \eta_{k,s} = -n + \sqrt{\beta^2k^2 + c + sd_k}\,. \tag{5.8}$$

Here, n can be any integer. Recall that $k > K$, and $k \leq \ell$ if we restrict the flow to $(P_\ell\mathcal{H}_0)^2$. Choosing for n the integer part of βk and denoting by $\lfloor\beta k\rfloor$ the fractional part of βk, we have

$$\eta_{k,s} = \lfloor\beta k\rfloor + \frac{c + sd_k}{2\beta k}\big(1 + \epsilon_k(c + sd_k)\big)\,, \qquad \epsilon_k(x) = \mathcal{O}\big(k^{-2}x\big)\,. \tag{5.9}$$

An explicit computation of $G(V_k, V_k)$ shows that the Krein signature of $\lambda_{k,s}$ is negative, while the Krein signature of $\overline{\lambda_{k,s}}$ is positive. Alternatively, this can be seen from the dependence of these eigenvalues on the parameter s, using (3.11).

5.2. Application to the wave equation

Here we will state two technical lemmas that, together with Theorem 1.1, are shown to imply Theorem 1.2. The proof of Theorem 1.1 will be described in Section 6.

First a definition: for any real analytic function h of the form

$$h(t, x) = \sum_{n,k} h_{n,k}\cos(nt)\cos(kx)\,, \tag{5.10}$$

we define $\|h\| = \sum_{n,k} |h_{n,k}|$. This applies in particular to any product of two functions in $\mathcal{B}$; and in this case, we have $h_{n,k} = 0$ unless n and k are both even.

Now back to the setting of Theorem 1.2. To a given value $\alpha \in \left\{\frac{5}{4}, \frac{14}{11}\right\}$ we associate several quantities as follows. Denote by u the solution of the beam equation (1.4) described in Theorem 1.1. Our goal is to apply the steps described so far with $h = \alpha^{-2}\bigl(3u^2 + \kappa\bigr) - c$. But it is convenient to first consider a simpler function h. To be more precise, we now choose a triple (c, h, δ) where c and δ are positive real numbers, and where h is a Fourier polynomial of the form (5.10) that satisfies a bound

$$\bigl\|h - \bigl[\alpha^{-2}\bigl(3u^2 + \kappa\bigr) - c\bigr]\bigr\| < \delta\,. \tag{5.11}$$

In addition, we require that $h_{n,k} = 0$ unless n and k are both even. Of course we choose c close to the average value of $\alpha^{-2}\bigl(3u^2 + \kappa\bigr)$ and h close to $\alpha^{-2}\bigl(3u^2 + \kappa\bigr) - c$. So δ is in fact very small.

Define H to be the operator "pointwise multiplication by h" on $\mathfrak{H}$. The operator "pointwise multiplication by $h(t,\boldsymbol{.})$" on $\mathcal{H}_0$ is denoted by $H(t)$. Here, $\mathcal{H}_0$ and $\mathfrak{H}$ are the Hilbert spaces defined in Section 4. To H we associate a sequence of approximants H_ℓ as described in Section 5.

Besides (c, h, δ) we also choose a positive integer K_α. In what follows, ℓ always denotes a fixed but arbitrary integer larger than K_α. We regard $H_{K_\alpha}(t)$ and H_{K_α} as linear operators on $P_\ell\mathcal{H}_0$ and $\mathfrak{H}_\ell$, respectively.

Lemma 5.5. *If $\alpha = \frac{5}{4}$, then the operator $\Phi_{H_{K_\alpha}}(2\pi)$ has no eigenvalue off the unit circle. If $\alpha = \frac{14}{11}$, then the operator $\Phi_{H_{K_\alpha}}(2\pi)$ has exactly two real positive eigenvalues off the unit circle. Furthermore, all eigenvalues of all these operators are semisimple.*

We remark that the two off-circle eigenvalues for $\alpha = \frac{14}{11}$ appear in the odd subspace, as defined below.

Recall that $h_{n,k} = 0$ unless n and k are both even. This simplifies our task in the following way. For $\sigma \in \{0, 1\}$ denote by $\mathcal{H}_0^\sigma$ the subspace of $\mathcal{H}_0$ consisting of all functions $v \in \mathcal{H}_0$ whose coefficients v_k in the sine series (4.3) vanish unless $k \equiv \sigma \pmod 2$. Similarly, denote by $\mathfrak{H}^\sigma$ the subspace of $\mathfrak{H}$ consisting of all functions $w \in \mathfrak{H}$ whose coefficients $w_{n,k}$ in the series (4.15) vanish unless $k \equiv \sigma \pmod 2$. Due to the above-mentioned property of h, the subspaces $\mathcal{H}_0^\sigma$ and $\mathfrak{H}^\sigma$ are invariant under the operators $H(t)$ and H, respectively. So in particular, the time-t maps $\tilde{\Phi}_H(t)$ leave the spaces $(\mathcal{H}_0^\sigma)^2$ invariant. As a result, the spectral analysis for $\tilde{\Phi}_H(t)$ splits into two separate and independent tasks: in one step we can restrict all operators to the even ($\sigma = 0$) subspaces, and in the other step we can restrict to the odd ($\sigma = 1$) subspaces.

The following lemma refers to some quantities that were introduced in Subsection 3.2. Here we consider $\mathcal{X}_0 = \beta^2\boldsymbol{k}^2 + c + H_{K_\alpha}$. For the operator θ that appears in (3.14) we choose $\theta = (|\boldsymbol{n}| + |\boldsymbol{k}|)^{-1}$.

Lemma 5.6. *Let $\alpha \in \left\{\frac{5}{4}, \frac{14}{11}\right\}$ and $\sigma \in \{0, 1\}$ be fixed but arbitrary. Consider the restriction to the subspaces of parity σ, as described above. Let (c, h, δ) be as described earlier. Then there exists a sequence of real numbers $d_k < c$ satisfying (5.4), a constant $C > 0$, a self-adjoint linear operator $\check{\mathcal{X}}_0$ on $\mathfrak{H}_{K_\alpha}$ satisfying (3.15), and a separating set Z*

for $\Phi_{H_{K_\alpha}-D}(2\pi)$ that includes 1, such that the following holds. For each value $\eta \in [0,1)$ defining a primary (as described below) point $e^{2\pi i\eta}$ in Z, there exist $m > 0$ and real numbers $-1 = s_0 < s_1 < \ldots < s_m = 1$, such that the operator $\check{M}_{s_j}(\eta)$ has no eigenvalue in $[-C, C]$, and such that (3.16) holds, for $j = 1, 2, \ldots, m$.

Our proof of Lemma 5.5 and Lemma 5.6 is computer-assisted and will be described in Section 6.

Remark 2. Lemma 5.5 is formulated for a "computationally minimal" version of our proof, where the claims of that lemma are verified not for H_{K_α} directly, but for $H_{K_\alpha} - D$. Then Lemma 5.6 implies that the same holds for H_{K_α}. If Lemma 5.5 is verified directly, then one finds that all eigenvalues of $\Phi_{H_{K_\alpha}}(2\pi)$ are simple, and that the off-circle eigenvalues for $\alpha = \frac{14}{11}$ are $\lambda = 1.06\ldots$ and $1/\lambda$. In a "computationally extended" version of our proof, we verify an analogue of Lemma 5.5 for both $H_{K_\alpha} \pm D$. This yields accurate bounds on many on-circle eigenvalues of $\tilde{\Phi}(2\pi)$ via the method described in Remark 1. The eigenvalues of $\Phi_{H_{K_\alpha}+D}(2\pi)$ are close enough to those of $\Phi_{H_{K_\alpha}-D}(2\pi)$ that the difference is barely noticeable at the resolution of Figures 1 and 2. We do not know, however, whether the off-circle eigenvalues of $\tilde{\Phi}(2\pi)$ for $\alpha = \frac{14}{11}$ lie between the corresponding eigenvalues of $\Phi_{H_{K_\alpha}\pm D}(2\pi)$.

By "primary" points in Z we mean the following. Z defines a partition of the unit circle into arcs. By assumption, Z is a separating set for the operator $A = \Phi_{H_{K_\alpha}-D}(2\pi)$, so each arc Λ can be assigned an signature: the Krein signature of the eigenvalues of A that lie in Λ. We say that a point in Z is primary (for A) if the adjacent arc in the (counter)clockwise direction has a negative (positive) signature.

For a given pair (α, σ) as described in Lemma 5.6, the set of non-primary point in Z includes all values $\lfloor \beta k \rfloor$, and all values $1 - \lfloor \beta k \rfloor$ different from 1, as k ranges over the integers $k > K$ with parity $k \equiv \sigma \,(\mathrm{mod}\, 2)$. These are the accumulation points (as $k \to \infty$ with $\ell \geq k$) of the trivial eigenvalues $e^{2\pi i\eta_{k,s}}$ and $e^{2\pi i(1-\eta_{k,s})}$ described in (5.8) and (5.9). Since $d_k < c$ by assumption, the eigenvalues of opposite signature accumulate at a given point from opposite sides, independently of the parameter value $s \in [-1, 1]$.

In this context, let us add that 1 is not an accumulation point for the trivial eigenvalues when $\alpha = \frac{14}{11}$ and $\sigma = 1$. In this case $\beta = \frac{11}{16}$, and no odd multiple of β can be an integer.

Based on the two lemmas above, and on Theorem 1.1, we can now give a

Proof of Theorem 1.2. Consider a fixed $\alpha \in \left\{\frac{5}{4}, \frac{14}{11}\right\}$ and $\sigma \in \{0, 1\}$. Let $Z \subset \mathbb{S}$ be the set described in Lemma 5.6. The connected components of $\mathbb{S} \setminus Z$ will be referred to as arcs. Let $K = K_\alpha$. Define $\mathcal{X}_s = \beta^2 \boldsymbol{k}^2 + c + H_K + sD$, and denote by Φ_s the corresponding flow. Here, and in what follows, we always assume that $s \in [-1, 1]$.

Let $z = e^{2\pi iy}$ be a point in Z, with $0 \leq y < 1$. If z is a primary point, then by Lemma 5.6 and Lemma 3.9, none of the operators $M_s(y)$ has an eigenvalue zero. So in this case, none of the operators $\Phi_s(2\pi)$ has z as an eigenvalue. This follows from Proposition 3.4. Consider now an eigenvalue curve $s \mapsto \lambda_s = e^{2\pi i\eta_s}$ for the family $s \mapsto \Phi_s(2\pi)$, obtained by integrating (3.11), starting at $s = -1$. By Lemma 3.6, the curve $s \mapsto \eta_s$ is real analytic as long as $G_{\eta_s}(w_s)$ does not vanish, and by Proposition 2.9, this holds as long as λ_s stays in a single arc. Clearly λ_s never meets a primary point. And by the definition of a primary point, it is impossible to meet a non-primary point before meeting a primary point, as s is increased. Thus, the curve stays in a single arc. This shows that Z is a separating set for the entire family $s \mapsto \Phi_s(2\pi)$.

We first prove the claim in Theorem 1.2 for the flow defined by h instead of $\alpha^{-2}\bigl(3u^2 + \kappa\bigr) - c$. By Proposition 5.4 we have $H_K - D \lessdot H_\ell \lessdot H_K + D$ for all $\ell > K$. Thus, we can use Corollary 3.7 or Corollary 3.8 to conclude that that Z is a separating set for $\Phi_{H_\ell}(2\pi)$. Furthermore, the eigenvalues of $\Phi_{H_\ell}(2\pi)$ are all on the unit circle, except for two simple real eigenvalues in $(0,1) \cup (1,\infty)$ in the case $(\alpha,\sigma) = \bigl(\frac{14}{11},1\bigr)$. Now we take the limit $\ell \to \infty$ and use Corollary 5.3 to conclude that all eigenvalues of $\tilde{\Phi}_H(2\pi)$ lie on the unit circle, except in the following case.

Consider $(\alpha,\sigma) = \bigl(\frac{14}{11},1\bigr)$. In this case, all but two eigenvalues (counting multiplicities) of $\tilde{\Phi}_H(2\pi)$ must lie on the unit circle. These eigenvalues are all bounded away from 1. This follows e.g. from the fact (see the remark at the end of the proof of Corollary 3.8) that the operators $\Phi_{H_\ell}(2\pi)$ have no eigenvalues in some fixed open neighborhood of 1 in $\mathbb{S}$. The remaining two eigenvalues lie on $(0,\infty)$. Assume for contradiction that 1 is an eigenvalue of $\tilde{\Phi}_H(2\pi)$.

Consider small perturbation $H^s = H + s\mathrm{I}$ of H and the corresponding approximants H^s_ℓ. If $s \neq 0$ is chosen sufficiently close to zero, then $H_K - D \lessdot H^s_\ell \lessdot H_K + D$ for all $\ell > K$. This follows from the fact that Proposition 5.4 holds for any choice of $\delta > 0$ in (5.4). By Proposition 4.6 we can find a sequence $n \mapsto s_n \neq 0$ converging to zero, such that the operator $\tilde{\Phi}_{H^s}$ defined with $s = s_n$ has all of its eigenvalues in $\mathbb{S} \setminus \{1\}$. Here we have used that the eigenvalue 1 of $\tilde{\Phi}_H(2\pi)$ has multiplicity 2. Now we can choose n sufficiently large such that the inequality $H_K - D \lessdot H^s_\ell \lessdot H_K + D$ holds, and such that all eigenvalues of $\Phi_{H^s_\ell}(2\pi)$ lie in $\mathbb{S}\setminus\{1\}$, if $\ell > K$ is sufficiently large. But by Corollary 3.8 this is impossible, given that $\Phi_{H_K}(2\pi)$ has two simple eigenvalues in $(0,1) \cup (1,\infty)$, as stated in Lemma 5.5. So $\tilde{\Phi}_H(2\pi)$ must have two eigenvalues in $(0,1) \cup (1,\infty)$. This proves the analogue of Theorem 1.2 for the flow defined by h.

Consider now the function $\hat{h} = \alpha^{-2}\bigl(3u^2 + \kappa\bigr) - c$. First we note that $u \in \mathcal{B}$ is real analytic on $\mathbb{R}^2$. Thus, the family $t \mapsto \hat{H}(t)$ has the Property 4.1. The condition $\hat{H}\mathfrak{H}'_m \subset \mathfrak{H}'_m$ required by (and described before) Lemma 4.5 is clearly satisfied as well, for any $m \geq 0$.

Let $\ell \mapsto \hat{H}_\ell$ the sequence of approximants associated with $\hat{H}$. Below we will show that

$$H_K - D \lessdot \hat{H}_\ell \lessdot H_K + D\,, \tag{5.12}$$

for all $\ell > K$. This bound allows us to repeat the arguments above, with $\hat{H}_\ell$ in place of H_ℓ, and the proof of Theorem 1.2 will be complete.

By (5.11) we can find a real number r such that $\|\hat{h} - h\| < r < \delta$. Consider the space of all real analytic functions (5.10), equipped with the norm $\|.\|$ defined after (5.10). Denote by $\mathfrak{b}$ the completion of this space. Then $\mathfrak{b}$ is a Banach algebra. So the operator norm of $\hat{H} - H : \mathfrak{b} \to \mathfrak{b}$ is less than r. Regarding $\hat{H} - H$ as an infinite matrix, it is not hard to see that the operator norm of $\hat{H}_\ell - H_\ell : \mathfrak{b} \to \mathfrak{b}$ is less than r as well. Thus, the spectral radius of $\hat{H}_\ell - H_\ell$ is less than r, implying that

$$-r\mathrm{I} \lessdot \hat{H}_\ell - H_\ell \lessdot r\mathrm{I}\,, \tag{5.13}$$

as operators on $\mathfrak{H}_\ell$. Now we use again the fact that Proposition 5.4 holds for any choice of $\delta > 0$ in (5.4). This allows us to replace D by $D - r\mathrm{I}$ and conclude that

$$H_{\kappa} - (D - r\mathrm{I}) \leqslant H_\ell \leqslant H_{\kappa} + (D - r\mathrm{I})\,. \tag{5.14}$$

Combining this bound with (5.13) yields (5.12), as claimed. This completes the proof of Theorem 1.2. **QED**

6. The remaining proofs

6.1. The beam equation

In order to solve the equation (1.4) we first rewrite it as a fixed point problem:

$$u = \mathcal{F}_\alpha(u) \stackrel{\text{def}}{=} -L_\alpha^{-1} u^3\,, \qquad L_\alpha = \alpha^2\partial_t^2 + \partial_x^4 + \kappa\,. \tag{6.1}$$

For the domain of $\mathcal{F}_\alpha$ we choose one of the following spaces $\mathcal{B}_\rho$. Denote by $\mathcal{P}$ the span of all functions $(t,z) \mapsto \cos(nt)\sin(kx)$ on $\mathbb{R}^2$, with $n \geq 0$ and $k \geq 1$. Given a pair $\rho = (\rho_1, \rho_2)$ of positive real numbers, denote by $\mathcal{A}_\rho^\circ$ the closure of $\mathcal{P}$ with respect to the norm

$$\|u\|_\rho = \sum_{n,k} |u_{n,k}|(1+\rho_1)^n(1+\rho_2)^k\,, \qquad u(t,x) = \sum_{n=0}^\infty\sum_{k=1}^\infty u_{n,k}\cos(nt)\sin(kx)\,. \tag{6.2}$$

The functions in $\mathcal{A}_\rho^\circ$ extend analytically to the complex domain $|\mathrm{Im}\, t| < \log(1+\rho_1)$ and $|\mathrm{Im}\, x| < \log(1+\rho_2)$. The subspace of all functions $u \in \mathcal{A}_\rho^\circ$ whose Fourier coefficients $u_{n,k}$ vanish whenever nk is even will be denoted by $\mathcal{B}_\rho$. Clearly u^3 belongs to $\mathcal{B}_\rho$ whenever u does. And we will see below that L_α has a bounded inverse $L_\alpha^{-1} : \mathcal{B}_\rho \to \mathcal{B}_\rho$ for the values of α considered in Theorem 1.1.

In order to solve the fixed point problem for $\mathcal{F}_\alpha$, we first determine an approximate fixed point u_0 and write $u = u_0 + Ah$, where A is a suitable linear isomorphism of $\mathcal{B}_\rho$. Then u is a fixed point of $\mathcal{F}_\alpha$ if and only if h is a fixed point of the map $\mathcal{N}_\alpha$ defined by

$$\mathcal{N}_\alpha(h) = \mathcal{F}_\alpha(u_0 + Ah) - u_0 + (\mathrm{I} - A)h\,. \tag{6.3}$$

By choosing A to be an approximate inverse of $\mathrm{I} - D\mathcal{F}_\alpha(u_0)$ we can expect $\mathcal{N}_\alpha$ to be a contraction near u_0.

Given $r > 0$ and $u \in \mathcal{B}_\rho$, denote by $B_r(u)$ the open ball of radius r in $\mathcal{B}_\rho$, centered at u. Theorem 1.1 is proved by verifying the following bounds.

Lemma 6.1. *For each $\alpha \in \left\{\frac{5}{4}, \frac{14}{11}\right\}$ there exists a pair ρ of positive real numbers, a Fourier polynomial $u_0 \in \mathcal{B}_\rho$, a linear isomorphism $A : \mathcal{B}_\rho \to \mathcal{B}_\rho$, and positive constants K, δ, ε satisfying $\varepsilon + K\delta < \delta$, such that for every $\alpha \in R_\kappa$ the map $\mathcal{N}_\alpha$ defined by (6.1) and (6.3) is analytic on $B_\delta(0)$ and satisfies*

$$\|\mathcal{N}_\alpha(0)\|_\rho < \varepsilon\,, \qquad \|D\mathcal{N}_\alpha(h)\|_\rho < K\,, \quad h \in B_\delta(0)\,. \tag{6.4}$$

Based on this lemma we can now give a

Proof of Theorem 1.1. Lemma 6.1, together with the contraction mapping principle, implies that for each $\alpha \in \left\{\frac{5}{4}, \frac{14}{11}\right\}$ the map $\mathcal{N}_\alpha$ has a unique fixed point $h_* \in B_\delta(0)$. The corresponding function $u_* = u_0 + Ah_*$ is a fixed point of $\mathcal{F}_\alpha$ and thus solves the equation (1.4) as claimed. **QED**

Our proof of Lemma 6.1 is computer-assisted. It is essentially the same as the proof of Lemma 3.1 in [12], but simpler, since we only consider rational values of α here. The operator L_α used in [12] is the same as (6.1) but with $\kappa = 0$. The eigenvalues of L_α are given by $L_{n,k} = -\alpha^2 n^2 + k^4 + \kappa$. Since $-\alpha^2 n^2 + k^4$ takes on both positive and negative values, we cannot use the bounds given in [12]. Instead we use the following.

For $r, s \in \mathbb{R}$ define $r \vee s = \max(r, s)$ and $\|\lceil s \rfloor\| = \operatorname{dist}(s, \mathbb{Z})$.

Proposition 6.2. *Let $\alpha = p/q$, with p and q coprime positive integers that are not both odd. Assume that there exists $\delta > 0$ such that*

$$p\|\lceil \alpha^{-1} c_1 \rfloor\| \geq \delta\,, \qquad 1 - \tfrac{1}{18}\kappa q \geq \delta\,, \tag{6.5}$$

where $c_1 = \sqrt{1+\kappa}$. Then for all odd positive integers n and k,

$$|L_{n,k}| \geq \alpha^2 p^{-2}\bigl[(qk^2) \vee (pn) - \delta\bigr]\delta\,. \tag{6.6}$$

Proof. The eigenvalues of $\alpha^{-2} L_\alpha$ can be written as

$$\alpha^{-2} L_{n,k} = p^{-2}\bigl[q^2 k^4 c_k^2 - p^2 n^2\bigr]\,, \qquad c_k^2 = 1 + k^{-4}\kappa\,. \tag{6.7}$$

We will need the following two bounds. If $k = 1$ then

$$\bigl|qk^2 c_k - pn\bigr| = |qc_1 - pn| = p\bigl|\alpha^{-1} c_1 - n\bigr| \geq p\|\lceil \alpha^{-1} c_1 \rfloor\| \geq \delta\,. \tag{6.8}$$

And for $k \geq 3$ odd we have

$$\bigl|qk^2 c_k - pn\bigr| \geq \bigl|qk^2 - pn\bigr| - qk^2(c_k - 1) \geq 1 - \tfrac{1}{18}\kappa q \geq \delta\,. \tag{6.9}$$

Now we use the fact [12] that $\bigl|r^2 - s^2\bigr| \geq (2(r \vee s) - \delta)\delta$ whenever $|r - s| \geq \delta$ with $r, s > 0$. When applied to the difference of squares in (6.7), it yields

$$\alpha^{-2}|L_{n,k}| \geq p^{-2}\bigl[(qk^2 c_k) \vee (pn) - \delta\bigr]\delta\,, \tag{6.10}$$

which implies (6.6). **QED**

This estimate is used (in the procedure `Fouriers2.Beam.InvLinear`) to estimate truncation errors in the Fourier series for $L_\alpha^{-1} v$ with $v \in \mathcal{B}_\rho$. The remaining parts of the proof of Lemma 6.1 are essentially identical to those in [12]. We refer to [12] for a description and to [13] for the details of the proof. Some general aspects are described below.

6.2. Enclosures and bounds

Our proof of Lemmas 6.1, 5.5, and 5.6 consists in reducing these lemmas to successively simpler propositions, until the claims are trivial numerical statements that can be verified by a computer. This part of the proof is written in the programming language Ada [17] and can be found in [13]. Any description here is necessarily incomplete. The main steps will be discussed in the remaining part of this section. We start by describing some of the underlying principles and setting the context.

Consider a proposition "if $x \in X$ then $f(x) \in Y = \ldots$". In Ada this may be declared as a `procedure F(X: in Xtype; Y: out Ytype)`, and the definition of `F` provides a proof. Here X and Y have to be "representable" sets. In the notation described below, $X = \langle \mathtt{X}, \mathcal{X} \rangle$ belongs to $\langle \mathtt{Xtype}, \mathcal{X} \rangle$, and similarly for Y. We will refer to `F` as a bound on f.

Say we want to represent balls in a real Banach algebra $\mathcal{X}$ with unit $\mathbf{1}$. The type `Ball` used for this consists of pairs `S=(S.C,S.R)`, where `S.C` is a representable number (`Rep`) and `S.R` a nonnegative representable number (`Radius`). The corresponding ball in $\mathcal{X}$ is $\langle \mathtt{S}, \mathcal{X} \rangle = \{x \in \mathcal{X} : \|x - (\mathtt{S.C})\mathbf{1}\| \le \mathtt{S.R}\}$. The collection of all such balls is denoted by $\langle \mathtt{Ball}, \mathcal{X} \rangle$. Our bounds on some basic functions involving the type `Ball` can be found in the packages `Std_Balls` and `MPFR_Balls`.

An Ada `package` is simply a collection of definitions and procedures, centered around a few specific data types. Basic types such as `Ball` can be used to build more complex types. In fact, we build `Vector`, `Matrix`, `Taylor1`, `Fourier2`, ... from a generic type `Scalar`, which can be instantiated later to `Ball`, or to any type that provides a predefined set of `Scalar` operations.

The type `Fourier2` with `Scalar => Ball` and $\mathcal{X} = \mathbb{R}$ is used to compute enclosures for functions in a space $\mathcal{A}_\rho$ that includes the space $\mathcal{A}_\rho^\circ$ defined in Subsection 6.1. For definitions and basic bounds involving this type we refer to the package `Fouriers2`. A description of the type `Fourier2` can also be found in [7]. Bounds that are specific to the beam equation (1.4) are implemented in the child package `Fouriers2.Beam`. Other child packages will be mentioned in later subsections.

Since our integration procedure involves Taylor series, and to give a concrete example, let us describe here the type `Taylor1` in the case `Scalar => Ball` and $\mathcal{X} = \mathbb{R}$. Given a `Radius` ρ, consider the space $\mathcal{T}_\rho$ of all real analytic functions $g(t) = \sum_n g_n t^n$ on the interval $|t| < \rho$, obtained by completing the space of polynomials with respect to the norm $\|g\|_\rho = \sum_n |g_n|\rho^n$. Given a positive integer `D`, a `Taylor1` is a triple `P=(P.C,P.F,P.R)`, where `P.F` is a nonnegative integer, $\mathtt{P.R} = \rho$, and `P.C` is an `array(0..D) of Ball`. The corresponding set in $\langle \mathtt{Taylor1}, \mathcal{T}_\rho \rangle$ is defined as

$$\langle \mathtt{P}, \mathcal{T}_\rho \rangle = \sum_{n=0}^{m-1} \langle \mathtt{P.C(n)}, \mathbb{R} \rangle p_n + \sum_{n=m}^{D} \langle \mathtt{P.C(n)}, \mathcal{T}_\rho \rangle p_n \,, \qquad p_n(t) = t^n \,, \tag{6.11}$$

where $m = \min(\mathtt{P.F}, \mathtt{D}+1)$. For the operations that we need in our proof, this type of enclosure allows for simple and efficient bounds.

For reference later on, consider the problem of iterating a bound `F` on a continuous linear map $A : \mathcal{T}_\rho \to \mathcal{T}_\rho$. The representation (6.11) should make it clear that it is possible to use `F` to construct an enclosure for A in terms a $(\mathtt{D}+1) \times (\mathtt{D}+1)$ `Matrix` with entries of type `Ball`. Then 2^n iterations of `F` can be reduced to n iterations of a bound `Sqr(A1: in Matrix; A2: out Matrix)` on the map $A \mapsto A^2$.

6.3. Integration

Let us first comment on our choice of the triple (c, h, δ) satisfying (5.11). Given a pair (α, u) as described in Theorem 1.1, we first compute an enclosure on the function $\alpha^{-2}\bigl(3u^2 + \kappa\bigr)$. This enclosure consists of a Fourier polynomial g and some error terms. Then we choose $h = g - c$ where c is the average value of g. For the value of δ in (5.11) we choose a suboptimal upper bound on the norm on the left hand side of this equation.

Instead of $\Phi_{H_K}(2\pi)$ we compute the operator $\tilde{\Phi}_{H_K}(2\pi) = U\Phi_{H_K}(2\pi)U^{-1}$ which is somewhat better behaved. We regard $H_K(t)$ and $\tilde{\Phi}_{H_K}(t)$ as linear operators on the finite-dimensional spaces $\mathcal{H} = P_K\mathcal{H}_0$ and $\mathcal{H}^2$, respectively. So for all practical purposes these are matrices.

Our goal is to integrate the equation $\frac{d}{dt}\tilde{\Phi}_{H_K}(t) = \tilde{\mathcal{X}}_{H_K}(t)\tilde{\Phi}_{H_K}(t)$ with initial condition $\tilde{\Phi}_H(0) = \mathrm{I}$, where $\tilde{\mathcal{X}}_H$ is the vector field given by (4.6). To this end we choose intermediate times $0 = t_0 < t_1 < \ldots < t_m = 2\pi$ and integrate the equation in steps $j = 1, 2, \ldots, m$ from time t_{j-1} to time t_j. At step j of this procedure we use the Duhamel formula (4.8), adapted to initial time t_{j-1}. Defining a curve $\mathcal{V}_j$ by setting

$$\tilde{\Phi}_{H_K}(t) = \tilde{\Phi}_0(t - t_{j-1})\bigl[\tilde{\Phi}_{H_K}(t_{j-1}) + \mathcal{V}_j(t - t_{j-1})\bigr]\,, \qquad t_{j-1} \le t \le t_j\,, \tag{6.12}$$

where $\tilde{\Phi}_0$ is the trivial flow (4.7), the equation for $\mathcal{V}_j$ becomes

$$\mathcal{V}_j(\tau) = \int_0^\tau B_j(s)\bigl[\tilde{\Phi}_{H_K}(t_{j-1}) + \mathcal{V}_j(s)\bigr]\,ds\,. \tag{6.13}$$

Here

$$B_j(s) = \tilde{\Phi}_0(-s)\bigl[\tilde{X}_{H_K}(t_{j-1} + s) - \tilde{X}_0\bigr]\tilde{\Phi}_0(s)\,, \tag{6.14}$$

which is the operator (4.12) but with $\tilde{H}(s)$ replaced by $\tilde{H}_K(t_{j-1} + s)$.

Up to trivial factors $\sin(y_k s)$ and $\cos(y_k s)$, the matrix elements for $B_j(s)$ are linear combinations of two coefficients $h_k(t)$ from the cosine series $h(x,t) = \sum_k h_k(t)\cos(kx)$, where $t = t_{j-1} + s$. After expanding $B_j(s)$ into a Taylor series about $s = 0$, the equation (6.12) can be used to compute the Taylor series for $\mathcal{V}_j(s) = \mathcal{O}(s)$ recursively, order by order.

This is of course the well-known "Taylor method" of integration. In our programs, we compute a finite number of Taylor coefficients for $\mathcal{V}_j$ explicitly, and the remainder is estimated by using Lemma 5.1 in [10].

The data types used to represent matrices whose coefficients are Taylor series in t, or Taylor series in t whose coefficients are matrices, are defined in the package `MultiTaylors1`. The integration steps are organized in `MultiTaylors1.Phi`, with `LinFlow` being the main integration procedure.

6.4. Eigenvalues

Consider one of the symplectic operators $\tilde{\Phi}_{H_K}(2\pi)$ whose construction was described above. After restricting to one of the subspaces $(P_K\mathcal{H}_0^\sigma)^2$ of fixed parity σ and using the function $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\sin(k.)$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\sin(k.)$ as basis vectors, for $k \equiv \sigma\,(\mathrm{mod}\,2)$, we end up with a symplectic matrix A. To be more precise, we have an enclosure for A, meaning that each matrix element $A_{i,j}$ is in effect a `Ball` over $\mathbb{R}$. The procedures described below are implemented in the package `ScalVectors`, which uses `PolyRoots` for finding roots of polynomials.

As a first step we use QR factorization to obtain a matrix $B = QAQ^\top$ that is in lower Hessenberg form, where Q is a product of Householder reflections and thus orthogonal. It suffices to determine the eigenvalues and eigenvectors of B. The coefficients of the characteristic polynomial $p_0(z) = \det(z\mathrm{I} - B)$ for a lower Hessenberg $n \times n$ matrix B can be computed via a simple recursion relation. In fact, there are polynomials $p_1, p_2, \ldots, p_n$ that can be computed recursively as well [16], such that if all roots λ_i of p_0 are simple, then $p_j(\lambda_i)$ it the j-th component of the eigenvector associated with the eigenvalue λ_i.

So our task is reduced to finding the roots of p_0. For each root $\lambda \in (0,1)$ we divide p_0 by the polynomial $z \mapsto (z-\lambda)\big(z-\lambda^{-1}\big)$. The resulting polynomial p is of even degree $2m \le n$ and can be written as

$$p(z) = \sum_{k=-m}^{m} c_k z^{m+k} = c_0 z^m + z^m \sum_{k=1}^{m} c_k\big(z^k + z^{-k}\big)\,. \tag{6.15}$$

In the second equality we have used that $c_{-k} = c_k$ for all k, due to the fact that the spectrum of A is invariant under $z \mapsto z^{-1}$. Next we write

$$p(z) = z^m\left[c_0 + 2\sum_{k=1}^{m} c_k T_k(\omega)\right], \qquad \omega = \tfrac{1}{2}\big(z + z^{-1}\big), \tag{6.16}$$

where T_k is the k-th Chebyshev polynomial. The factor $[\cdots]$ in this equation is a polynomial P in ω. Its coefficients are computed using the Clenshaw algorithm.

In the cases considered, we find at most one root of p_0 in $(0,1)$, and m simple roots of P in $(-1,1)$. This is done first numerically, using bisection and then Newton's method. (This is the only step in our procedure that involves iteration.) Then the approximate zeros are verified rigorously using the contraction mapping principle. Notice that, if $\omega = \cos(\eta)$ is a zero of P, then $\lambda = e^{2\pi i\eta}$ is an eigenvalue of A. The corresponding eigenvector of A is used (only) to compute the Krein signature for λ.

6.5. Separation

The separating set Z for $\tilde{\Phi}_{H_K-D}(2\pi)$, and the primary points in Z, are determined in `ScalVectors.Phi.Separating_Eta`.

The numbers d_k in (5.4) are computed by estimating the right hand side of this equation and choosing d_k to be an upper bound, with $\delta > 0$ as described earlier. Recall that the argument of $\mathcal{R}_k$ in (5.4) is represented as a cosine series in t. We overestimate the value at t by replacing each term $c_k\cos(k.)$ by $|c_k|$. Doing this for H instead of H_ℓ

yields a bound (5.4) that holds for all t and all $\ell > K$. For details, we refer to the package `Fouriers2.FlokD`.

The main goal now is to implement the steps described in Subsection 3.2. The bounds used in this part can be found in the package `Fouriers2.FlokM`.

In the case considered here, $\hat{\mathcal{X}}_s = \theta(\beta^2\boldsymbol{k}^2 + c)\theta + \hat{H}_K + s\hat{D}$, where $\hat{H}_K = \theta H_K \theta$. First we choose a finite-dimensional subspace $\check{\mathfrak{H}}_K$ of $\mathfrak{H}_K$ by truncating the Fourier series in t to frequencies $|n| \le N$ for some $N > 0$. Then we define $\check{H}_K = P\hat{H}_K P$, where P is the orthogonal projection onto $\check{\mathfrak{H}}_K$. The corresponding operator $\check{\mathcal{X}}_0$ is defined in the obvious way, so that $\hat{\mathcal{X}}_0 - \check{\mathcal{X}}_0$ is equal to $A = \hat{H}_K - \check{H}_K$. The constant C in Lemma 5.6 is now obtained by computing an upper bound the operator norm of A. Since the Hilbert norm on $\mathfrak{H}_K$ is inconvenient for such estimates, we first compute the operator norm of A^m with respect to a different norm on (a dense subspace of) $\mathfrak{H}_K$ and then take the m-th root of the result. Here m is a power of 2. The powers of A are estimated by using a matrix enclosure (type `HMatrix`) on the operator A, of the type described at the end of Subsection 6.2.

The points $-1 = s_0 < s_1 < \ldots < s_m$ mentioned in Lemma 5.6 are chosen in such a way that (3.16) holds for all j. To prove that $\check{M}_{s_j}(\eta)$ has no eigenvalues in $[-C, C]$, we first compute $A = \check{M}_{s_j}(\eta)^{-1}$ and then verify that the operator norm of A is less than C^{-1}. This is done for each $\eta \in [0, 1)$ that corresponds to a primary value $e^{2\pi i\eta}$ in Z. To be more specific, the restriction of A to $\check{\mathfrak{H}}_K$ is given by a matrix (type `CMatrix`), so computing its norm is straightforward. The restriction of $\check{M}_s(\eta)$ to the orthogonal complement of $\check{\mathfrak{H}}_K$ is diagonal, with entries

$$b_{(n,k)} = \frac{(n+\eta)^2 - \beta^2 k^2 - c - s d_k}{(n+k)^2}\,, \qquad k \le K \quad n > N\,. \tag{6.17}$$

By our choice of $N > 0$, all these entries are positive. Using that $n \mapsto b_{n,k}$ is increasing, it suffices to check that $b_{N+1,k} > C$ for $s = 1$ and all $k \le K$.

6.6. Putting it all together

To see how the various steps are organized, it is best to look at the main program `Run_All`. This program calls five standalone procedures, each taking care of one or two of the main tasks described in the preceding subsections. The necessary parameters, such as the value of α, the parity σ, as well as several degrees and array dimensions, are passed to the standalone procedures via an argument of type `Pars`. These parameters are then used to instantiate the necessary packages. After that, the procedures defined in these packages are called to do the main work.

For the set of representable numbers (`Rep`) we choose either standard [19] extended floating-point numbers (type `LLFloat`) or high precision [20] floating-point numbers (type `MPFloat`), depending on the precision needed. Both types support controlled rounding. `Radius` is always a subtype of `LLFloat`. Our programs were run successfully on a standard desktop machine, using a public version of the gcc/gnat compiler [18].

By default, our programs run the "computationally minimal" version of the proof, as described in Remark 2. Instructions on how to run the other versions mentioned in Remark 2 can be found in the file `README`. For further details, we refer to [13].

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