OPTIMIZATION OF THE LOWEST EIGENVALUE FOR LEAKY STAR GRAPHS

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Abstract. We consider the problem of geometric optimization for the lowest eigenvalue of the two-dimensional Schrödinger operator with an attractive δ-interaction of a fixed strength the support of which is a star graph with finitely many edges of an equal length $L \in (0, \infty]$. Under the constraint of fixed number of the edges and fixed length of them, we prove that the lowest eigenvalue is maximized by the fully symmetric star graph. The proof relies on the Birman-Schwinger principle, properties of the Macdonald function, and on a geometric inequality for polygons circumscribed into the unit circle.

1. Introduction

The topic addressed in this note is the spectral optimization problem for the lowest eigenvalue of the two-dimensional Schrödinger operator with a δ-interaction supported by a star graph. This problem can be regarded as a two-dimensional counterpart of our recent analysis [EL16] of δ-interactions supported by conical surfaces in $\mathbb{R}^3$. As in the said paper, in order to obtain the main result we have to combine the Birman-Schwinger principle with a certain geometric inequality. To advocate the use of this principle as a powerful tool of proving spectral optimization results is an additional motivation of our present considerations.

To describe our main result we need to introduce some notations. In what follows we consider a star graph $\Sigma_N = \Sigma_N(L) \subset \mathbb{R}^2$, which has $N \geq 2$ edges of length $L \in (0, \infty]$ each. Being enumerated in the clockwise manner, these edges are, up to an overall rotation, characterized by the angles $\phi = \phi(\Sigma_N) = \{\phi_1, \phi_2, \ldots, \phi_N\}$ between the neighboring edges. We also adopt the natural assumptions $\phi_n \in (0, 2\pi)$ for all $n \in \{1, \ldots, N\}$ and $\sum_{n=1}^{N} \phi_n = 2\pi$; when convenient we say that ϕ is the vector of angles for the star graph Σ_N. Furthermore, by Γ_N we denote the star graph with maximum symmetry, having the same number $N \geq 2$ of edges of the same length $L \in (0, \infty]$ each, and whose vector of angles is given by $\phi = \phi(\Gamma_N) = \left\{ \frac{2\pi}{N}, \frac{2\pi}{N}, \ldots, \frac{2\pi}{N} \right\}$. For brevity we refer to Γ_N as to the symmetric star graph. Examples of star graphs Σ_N and Γ_N for $N = 5$ are plotted in Figure 1.1.

Given a real number $\alpha > 0$, we consider spectral problem for the self-adjoint operator H_{α, Σ_N} corresponding via the first representation theorem to the closed, densely defined, symmetric,
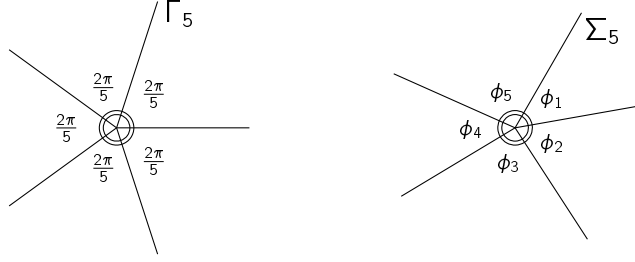

Figure 1.1. The star graphs Γ_5 and Σ_5 with $N = 5$ and $L < \infty$.

and semi-bounded quadratic form in $L^2(\mathbb{R}^2)$,

$$(1.1) \qquad \mathfrak{h}_{\alpha,\Sigma_N}[u] := \|\nabla u\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} - \alpha \|u|_{\Sigma_N}\|^2_{L^2(\Sigma_N)}, \qquad \mathrm{dom}\, \mathfrak{h}_{\alpha,\Sigma_N} := H^1(\mathbb{R}^2);$$

here $u|_{\Sigma_N}$ denotes the usual trace of $u \in H^1(\mathbb{R}^2)$ onto Σ_N; cf. [BEKŠ94, Sec. 2] and [BEL14, Sec. 3.2]. The operator H_{α,Σ_N} is usually called Schrödinger operator with the δ-interaction of strength α supported by Σ_N. In recent years the investigation of Schrödinger operators with singular interactions supported by sets of lower dimensionality became a topic of permanent interest – see, *e.g.* , [BEL14, BLL13, BEKŠ94, BEW09, DR14, EN03, EV16, LO16, P15], the monograph [EK], and the references therein. The typical physical use of the Hamiltonian H_{α,Σ_N} is to model electron behavior of dilute electron gas in quantum wire networks, that is, very thin structures constructed from semiconductor or other materials, the confinement being realized by potential jump between different material types. With an acceptable idealization we can neglect the 'wire' width and suppose that the interaction is supported by the network skeleton, for instance, by the graph Σ_N. One usually employs the name *'leaky star graph'* for such models [E08] to underline that, in contrast to the standard quantum graphs models [BK], they do not neglect quantum tunnelling. Another physical use of the Hamiltonian H_{α,Σ_N} can be found in the few-body quantum mechanics with zero-range interactions – see, *e.g.* , [BK13, BD06, CDR08, HKPC17, LL63].

Recall that the essential spectrum of H_{α,Σ_N} coincides with the set $[0,\infty)$ if $L < \infty$ and with the set $[-\frac{1}{4}\alpha^2, +\infty)$ if $L = \infty$. The negative discrete spectrum is known to be non-empty and finite unless we have simultaneously $L = \infty$, $N = 2$, and $\phi = \{\pi, \pi\}$; cf. Section 2 below for details. By $\lambda_1^\alpha(\Sigma_N)$ we denote the spectral threshold of H_{α,Σ_N} which, except for the mentioned trivial case, is an isolated eigenvalue.

The aim of this paper is to demonstrate that $\lambda_1^\alpha(\Sigma_N)$ is maximized by the symmetric star graph Γ_N. A precise formulation of this claim is the contents of the following theorem.

Theorem 1.1. *For all $\alpha > 0$ we have*

$$(1.2) \qquad \max_{\Sigma_N(L)} \lambda_1^\alpha\left(\Sigma_N(L)\right) = \lambda_1^\alpha\left(\Gamma_N(L)\right),$$

where the maximum is taken over all star graphs with $N \geq 2$ edges of a given length $L \in (0, \infty]$. In the case $L < \infty$ the equality in (1.2) is achieved if and only if Σ_N and Γ_N are congruent.

Our method of the proof of Theorem 1.1 relies on the Birman-Schwinger principle for H_{α,Σ_N} and on the trick proposed in [E05a, EHL06] and further applied and developed in [BFK$^+$16, EL16, L16]. The main geometric ingredient in the proof of Theorem 1.1 is an inequality for the lengths of the diagonals for polygons circumscribed into the unit circle. This inequality can be viewed as a discrete version of the mean chord-length inequality known for smooth arcs [ACF$^+$03, EHL06, Lu66]. Finally, we mention that a slightly different discrete version of the mean chord-length inequality is used in [E05b, E06] to prove a spectral isoperimetric inequality for Schrödinger operators with point interactions.

Organisation of the paper. In Section 2 we recall the known spectral properties of H_{α,Σ_N} that are needed in this paper. Section 3 is devoted to the Birman-Schwinger principle for H_{α,Σ_N} and its consequences. Theorem 1.1 is proven in Section 4. The paper is concluded by Section 5 containing a discussion of the obtained results and their possible extensions and generalizations.

2. **The spectral problem for δ-interactions supported by star graphs**

Throughout this section, Σ_N is an arbitrary star graph in $\mathbb{R}^2$ with $N \geq 2$ edges of length $L \in (0, \infty]$ each; *cf.* Figure 1.1 for the case $N = 5$. The edges of Σ_N are labeled by $\sigma_1, \sigma_2, \ldots, \sigma_N$ being enumerated in the clockwise way. For convenience we extend the sequence of edges in a periodic manner by letting $\sigma_{kN+n} := \sigma_n$ for all $k \in \mathbb{N}$ and $n \in \{1, 2, \ldots, N\}$. By $\phi_n \in (0, 2\pi)$ with $n \in \mathbb{N}$ we denote the magnitude of the angle between the edges σ_n and σ_{n+1}, measured again in the clockwise direction. The vector of angles $\phi = \phi(\Sigma_N) = \{\phi_1, \phi_2, \ldots, \phi_N\}$ satisfies $\phi_n \in (0, 2\pi)$ for all $n \in \{1, \ldots, N\}$ and $\sum_{n=1}^{N} \phi_n = 2\pi$. At the same time, the (attractive) coupling strength α is a fixed positive number.

We are interested in spectral properties of the self-adjoint operator H_{α,Σ_N} in $L^2(\mathbb{R}^2)$ introduced via the first representation theorem [K, Thm. VI 2.1] as associated with the closed, densely defined, symmetric, and semi-bounded quadratic form $\mathfrak{h}_{\alpha,\Sigma_N}$ defined in (1.1); see [BEKŠ94, Sec. 2] and also [BEL14, Prop. 3.1].

For the sake of completeness, let us add a few words about the operator domain of H_{α,Σ_N}. Let $\widetilde{\Sigma}_N = \widetilde{\Sigma}_N(\infty) \subset \mathbb{R}^2$ be the natural extension of Σ_N up to the star graph with semi-infinite edges $\{\widetilde{\sigma}_n\}_{n=1}^{N}$; in the case $L = \infty$ it is, of course, trivial, we clearly have $\widetilde{\Sigma}_N = \Sigma_N$. The graph $\widetilde{\Sigma}_N$ obviously splits $\mathbb{R}^2$ into N wedge-type domains $\{\Omega_n\}_{n=1}^{N}$ with the angles $\{\phi_n\}_{n=1}^{N}$ at their corners. Again for the sake of convenience we extend the sequences of edges for $\widetilde{\Sigma}_N$ and of respective wedges putting $\widetilde{\sigma}_{kN+n} := \widetilde{\sigma}_n$ and $\Omega_{kN+n} := \Omega_n$ for $k \in \mathbb{N}$ and $n \in \{1, \ldots, N\}$. For any $u \in L^2(\mathbb{R}^2)$ we introduce the notation $u_n := u|_{\Omega_n}$, $n \in \mathbb{N}$. Then the operator domain of H_{α,Σ_N} consists of functions $u \in H^1(\mathbb{R}^2)$, which satisfy $\Delta u_n \in L^2(\Omega_n)$, $n \in \mathbb{N}$, in the distributional sense and the δ-type boundary condition on each edge,

$$\partial_{\nu_n} u_n|_{\widetilde{\sigma}_{n+1}} + \partial_{\nu_{n+1}} u_{n+1}|_{\widetilde{\sigma}_{n+1}} = \alpha \chi_{\sigma_n} u|_{\widetilde{\sigma}_{n+1}}, \qquad n \in \mathbb{N},$$

in the sense of traces, where $\chi_{\sigma_n} \colon \widetilde{\sigma}_n \to \{0,1\}$, $n \in \mathbb{N}$, is the characteristic function of the set $\sigma_n \subset \widetilde{\sigma}_n$, and where $\partial_{\nu_n} u_n|_{\widetilde{\sigma}_n}$ and $\partial_{\nu_n} u_n|_{\widetilde{\sigma}_{n+1}}$ denote the traces of normal derivatives of u_n onto the edges $\widetilde{\sigma}_n$ and $\widetilde{\sigma}_{n+1}$, respectively, with the normal vector at the boundary of Ω_n pointing outward. Moreover, for any $u \in \operatorname{dom} H_{\alpha,\Sigma_N}$ we have $H_{\alpha,\Sigma_N} u = -\oplus_{n=1}^{N} \Delta u_n$. The reader may consult [BEL14, Sec. 3.2] for a more precise description of $\operatorname{dom} H_{\alpha,\Sigma_N}$.

It is not surprising that the operator H_{α,Σ_N} has a non-empty essential spectrum. More specifically, we have the following statement.

Proposition 2.1. *Let $\Sigma_N = \Sigma_N(L) \subset \mathbb{R}^2$ be a star graph with $N \geq 2$ edges of length $L \in (0,\infty]$. Then for all $\alpha > 0$ the essential spectrum of H_{α,Σ_N} is characterized as follows:*

(i) $\sigma_{\mathrm{ess}}(H_{\alpha,\Sigma_N}) = [0,\infty)$ *if $L < \infty$;*

(ii) $\sigma_{\mathrm{ess}}(H_{\alpha,\Sigma_N}) = [-\tfrac{1}{4}\alpha^2, \infty)$ *if $L = \infty$.*

The claim (i) of this proposition is easy to verify, because the essential spectrum of the Laplacian in the whole space $\mathbb{R}^2$ equals $[0,\infty)$ and introducing a δ-interaction supported by Σ_N with $L < \infty$ leads to a compact perturbation in the sense of resolvent differences; we refer to [BEKŠ94, Thm. 3.1] or to [BEL14, Thm. 4.2]. For a proof of Proposition 2.1 (ii) see [EN03, Prop. 5.4].

As for the discrete spectrum of H_{α,Σ_N}, various properties of it are investigated in or follow from [BEW09, CDR08, DR14, EI01, EN03, KP16, KL14, P15]. For our purposes we need the following statement.

Proposition 2.2. *Let $\Sigma_N = \Sigma_N(L) \subset \mathbb{R}^2$ be a star graph with $N \geq 2$ edges of length $L \in (0,\infty]$. If $L = \infty$ and $N = 2$ hold simultaneously, we additionally assume that $\phi(\Sigma_2) \neq \{\pi,\pi\}$. Then $\#\sigma_{\mathrm{dis}}(H_{\alpha,\Sigma_N}) \geq 1$ holds for all $\alpha > 0$[1].*

Nontriviality of $\sigma_{\mathrm{dis}}(H_{\alpha,\Sigma_N})$ for $L < \infty$ follows from [KL14, Thm. 3.1]. It can be alternatively proven via the test function argument as in the proof of [KL16, Prop. 2]. The fact that $\sigma_{\mathrm{dis}}(H_{\alpha,\Sigma_N})$ is non-void for $L = \infty$ is shown in [EN03, Thm. 5.7]; note that by [KL16, Thm. E.1] this discrete spectrum is finite.

Summarizing, the essential spectrum of H_{α,Σ_N} coincides with the interval $[0,\infty)$ if $L < \infty$ and with the interval $[-\tfrac{1}{4}\alpha^2, \infty)$ if $L = \infty$, and there is at least one isolated eigenvalue below the threshold of the essential spectrum, unless simultaneously $L = \infty$, $N = 2$, and $\phi(\Sigma_2) = \{\pi,\pi\}$ hold.

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[1]Given a self-adjoint operator T we denote by $\#\sigma_{\mathrm{dis}}(T)$ the cardinality of the discrete spectrum with the multiplicities of the eigenvalues taken into account.

3. Birman-Schwinger principle

In this section we formulate a version of the Birman-Schwinger-type principle for the operator H_{α,Σ_N} and derive a related characterization of its lowest eigenvalue $\lambda_1^\alpha(\Sigma_N)$. A standing assumption throughout this section is $L < \infty$, although most of the results hold or can be reformulated for $L = \infty$ as well.

First of all, we parametrize each edge σ_n of Σ_N by the unit-speed mapping $\sigma_n \colon \mathcal{I} \to \mathbb{R}^2$ with $\mathcal{I} := [0, L]$, i.e. $|\sigma_n'(s)| = 1$ holds for all $s \in \mathcal{I}$. We choose this parametrisation in such a way that $\sigma_1(0) = \sigma_2(0) = \cdots = \sigma_N(0)$. Clearly, the Hilbert spaces $L^2(\Sigma_N)$ and $\mathcal{H} := \oplus_{n=1}^N L^2(\mathcal{I})$ can be identified. For any $\psi \in \mathcal{H}$ we denote by $[\psi]_n \in L^2(\mathcal{I})$ (with $n \in \{1, 2, \ldots, N\}$) its n-th component in the orthogonal decomposition of $\mathcal{H}$.

Furthermore, we define a weakly singular integral operator $Q_{\Sigma_N}(\kappa) \colon \mathcal{H} \to \mathcal{H}$ for $\kappa > 0$ by

$$(3.1) \qquad [Q_{\Sigma_N}(\kappa)\psi]_n(s) := \sum_{m=1}^N \frac{1}{2\pi} \int_0^L K_0\left(\kappa|\sigma_n(s) - \sigma_m(t)|\right) [\psi]_m(t)\mathrm{d}t,$$

where $n \in \{1, 2, \ldots, N\}$ and $K_0(\cdot)$ is the modified Bessel function of the second kind having the order $\nu = 0$; cf. [AS64, §9.6]. In the next proposition we state basic properties of this integral operator.

Proposition 3.1. *Let* $\Sigma_N = \Sigma_N(L) \subset \mathbb{R}^2$ *be a star graph with* $N \geq 2$ *edges of length* $L \in (0, \infty)$. *Then the operator* $Q_{\Sigma_N}(\kappa)$ *in* (3.1) *is self-adjoint, compact, and non-negative for all* $\kappa > 0$.

Proof. Self-adjointness and non-negativity of $Q_{\Sigma_N}(\kappa)$ directly follow from more general results in [B95]. Compactness of $Q_{\Sigma_N}(\kappa)$ is a consequence of [BEKŠ94, Lem. 3.2]. $\qquad\square$

Now we have all the tools to formulate our Birman-Schwinger-type condition for H_{α,Σ_N}.

Theorem 3.2. *Let* $\Sigma_N = \Sigma_N(L) \subset \mathbb{R}^2$ *be a star graph with* $N \geq 2$ *edges of length* $L \in (0, \infty)$. *Let the self-adjoint operator* H_{α,Σ_N} *in* $L^2(\mathbb{R}^2)$ *represent the quadratic form in* (1.1) *and let the operator-valued function* $(0, \infty) \ni \kappa \mapsto Q_{\Sigma_N}(\kappa)$ *be as in* (3.1). *Then the following claims hold:*

(i) $\dim \ker(H_{\alpha,\Sigma_N} + \kappa^2) = \dim \ker(I - \alpha Q_{\Sigma_N}(\kappa))$ *for all* $\kappa > 0$.

(ii) *The mapping* $u \mapsto u|_{\Sigma_N}$ *is a bijection between* $\ker(H_{\alpha,\Sigma_N} + \kappa^2)$ *and* $\ker(I - \alpha Q_{\Sigma_N}(\kappa))$.

Proof. The claim (i) is a particular case of [BEKŠ94, Lem. 2.3 (iv)]. The claim (ii) follows from an abstract statement in [B95, Lem. 1]. $\qquad\square$

We conclude this section by two corollaries of Theorem 3.2.

Corollary 3.3. *Let the assumptions be as in Theorem 3.2, and moreover, let* $\kappa > 0$ *be such that* $\lambda_1^\alpha(\Sigma_N) = -\kappa^2$. *Then the following claims hold:*

(i) $\ker(I - \alpha Q_{\Sigma_N}(\kappa)) = \mathrm{span}\{\psi_\star\}$ where $\psi_\star \in \mathcal{H}$ is a positive function.

(ii) For the symmetric star graph Γ_N the corresponding function $\psi_\star$ depends only on the distance from the vertex of Γ_N; i.e. $[\psi_\star]_1 = [\psi_\star]_2 = \cdots = [\psi_\star]_N$.

Proof. The proof of (i) is completely analogous to the proofs of [EL16, Prop. 2.3] and of [L16, Cor. 3.3], hence we skip it to avoid self-repetition.

In order to prove (ii) we define the subspace

$$\mathcal{H}_{\mathrm{sym}} = \{\psi \oplus \psi \oplus \cdots \oplus \psi \colon \psi \in L^2(\mathcal{I})\}$$

of the Hilbert space $\mathcal{H}$. It is not difficult to check that the operator $Q_{\Gamma_N}(\kappa)$ can be rewritten as an orthogonal sum $Q_{\Gamma_N}^{\|}(\kappa) \oplus Q_{\Gamma_N}^{\perp}(\kappa)$ with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_{\mathrm{sym}} \oplus \mathcal{H}_{\mathrm{sym}}^{\perp}$. Thus, the function $\psi_\star$ in (i) is an eigenfunction of at least one of the operators $Q_{\Gamma_N}^{\|}(\kappa)$ and $Q_{\Gamma_N}^{\perp}(\kappa)$.

It is clear that $\psi_\star$, being a positive function, can not belong to the space $\mathcal{H}_{\mathrm{sym}}^{\perp}$. This allows us to conclude $\psi_\star \in \mathcal{H}_{\mathrm{sym}}$ by which the claim of (ii) is proven. $\square$

Next we provide the second consequence of Theorem 3.2. The proof of this corollary is analogous to the one of [EL16, Prop. 2.3] or of [L16, Cor. 3.4] which is why we can skip it.

Corollary 3.4. *Let the assumptions be as in Theorem 3.2. Let $\kappa > 0$ be fixed. Then the following claims hold:*

(i) $\sup \sigma(\alpha Q_{\Sigma_N}(\kappa)) \geq 1$ if and only if $\lambda_1^\alpha(\Sigma_N) \leq -\kappa^2$.

(ii) $\sup \sigma(\alpha Q_{\Sigma_N}(\kappa)) = 1$ if and only if $\lambda_1^\alpha(\Sigma_N) = -\kappa^2$.

4. **Proof of Theorem 1.1**

Now we are finally in position to establish Theorem 1.1. Throughout this section, we assume that $\Sigma_N = \Sigma_N(L) \subset \mathbb{R}^2$ is a star graph with $N \geq 2$ edges of length $L \in (0, \infty]$.

First of all, we suppose that the star graph has edges of finite length $L < \infty$, and that they are parametrized as in Section 3 via the mappings $\{\sigma_n\}_{n=1}^N$. Let $\Gamma_N \subset \mathbb{R}^2$ be the symmetric star graph having the same length $L \in (0, \infty)$ of the edges, which are also parametrized by the unit-speed mappings $\{\gamma_n\}_{n=1}^N$ in the same way. In addition, we assume that Σ_N is not congruent to Γ_N, in other words, the corresponding vectors of angles $\phi(\Sigma_N)$ and $\phi(\Gamma_N)$ do not coincide. Without loss of generality we may assume that the vertices of Σ_N and of Γ_N are both located at the origin.

Recall that $\lambda_1^\alpha(\Sigma_N)$ and $\lambda_1^\alpha(\Gamma_N)$ denote the lowest (negative) eigenvalues of H_{α,Σ_N} and of H_{α,Γ_N}, respectively. Furthermore, we fix $\kappa > 0$ such that $\lambda_1^\alpha(\Gamma_N) = -\kappa^2$. By Corollary 3.3 (i) we have $\ker(I - \alpha Q_{\Gamma_N}(\kappa)) = \mathrm{span}\{\psi_\star\}$, where $\psi_\star \in \mathcal{H} \cong L^2(\Gamma_N)$ is a positive function, which

according to Corollary 3.3 (ii) depends on the distance from the origin only. We normalise $\psi_\star$ so that $\|\psi_\star\|_{L^2(\Gamma_N)} = 1$. Note that by Corollary 3.4 (ii) we also have $\sup \sigma(\alpha Q_{\Gamma_N}(\kappa)) = 1$.

Let $\mathbb{S}^1 \subset \mathbb{R}^2$ be the unit circle centred at the origin. In what follows we fix the points on $\mathbb{S}^1$ by $y_n := L^{-1}\sigma_n(L)$ and $x_n := L^{-1}\gamma_n(L)$ for all $n \in \{1, \ldots, N\}$. The sequences of points $\{y_n\}_{n=1}^N$ and $\{x_n\}_{n=1}^N$ can be interpreted as vertices of polygons circumscribed into $\mathbb{S}^1$. Next, we extend these sequences to all $n \in \mathbb{N}$ in a natural periodic manner with the period N. We also adopt the notation $\angle\sigma_n\sigma_m \in (0, 2\pi)$ for the magnitude of the angle between the edges σ_m and σ_n, measured from σ_m to σ_n in the clockwise direction. Using the cosine law for triangles and Jensen's inequality applied for the concave function $(0, \pi) \ni x \mapsto \sin^2(x)$ we find

$$
\begin{aligned}
\sum_{n=1}^N |y_{n+m} - y_n|^2 &= 4\sum_{n=1}^N \sin^2\left(\frac{\angle\sigma_{n+m}\sigma_n}{2}\right) \leq 4N\sin^2\left(\sum_{n=1}^N \frac{\angle\sigma_{n+m}\sigma_n}{2N}\right) \\
&= 4N\sin^2\left(\sum_{k=0}^{m-1}\sum_{n=1}^N \frac{\angle\sigma_{n+k+1}\sigma_{n+k}}{2N}\right) \\
&= 4N\sin^2\left(\frac{\pi m}{N}\right) = \sum_{n=1}^N |x_{n+m} - x_n|^2,
\end{aligned}
$$

(4.1)

for any $m \in \mathbb{N}_0$. Note that the sequence $\{\angle\sigma_{n+1}\sigma_n\}_{n=1}^N$ does not consist of equal numbers, since the star graph Σ_N is not congruent to the symmetric star graph Γ_N. Thus, the inequality in (4.1) is strict at least for $m = 1$, because the function $(0, \pi) \ni x \mapsto \sin^2(x)$ is strictly concave. Furthermore, the cosine law for triangles implies that for any $s, t \in \mathcal{I}$ and $n, m \in \mathbb{N}$ one has the identities

$$
\begin{aligned}
|\sigma_n(s) - \sigma_m(t)|^2 &= (s - t)^2 + st|y_n - y_m|^2, \\
|\gamma_n(s) - \gamma_m(t)|^2 &= (s - t)^2 + st|x_n - x_m|^2.
\end{aligned}
$$

Next we introduce an auxiliary function by

$$
\mathbb{R}_+ \ni x \mapsto F_{s,t}(x) := K_0\left(\kappa\sqrt{(s - t)^2 + stx}\right), \qquad \text{for } s, t \in \mathcal{I}, \ \kappa > 0.
$$

Differentiating it with respect to x twice, we get

$$
\begin{aligned}
F'_{s,t}(x) &= -\frac{\kappa st K_1(\kappa\sqrt{(s - t)^2 + stx})}{2\sqrt{(s - t)^2 + stx}} < 0, \\
F''_{s,t}(x) &= \frac{\kappa s^2 t^2 K_1(\kappa\sqrt{(s - t)^2 + stx})}{4((s - t)^2 + stx)^{3/2}} - \frac{\kappa^2 s^2 t^2 K'_1(\kappa\sqrt{(s - t)^2 + stx})}{4((s - t)^2 + stx)} > 0,
\end{aligned}
$$

where we have employed that the Bessel function $K_1(\cdot)$ is monotonously decreasing and positive [AS64, §9.6]. Thus, we conclude that $F_{s,t}$ is a strictly decreasing and strictly convex

function. Using the min-max principle we can further infer that

$$2\pi \sup \sigma(Q_{\Sigma_N}(\kappa)) \geq \int_0^L \int_0^L \left(\sum_{n=1}^N \sum_{m=1}^N K_0\left(\kappa|\sigma_n(s) - \sigma_m(t)|\right) \right) \psi_\star(s)\psi_\star(t)\mathrm{d}s\mathrm{d}t$$

(4.2)
$$= \sum_{m=0}^{N-1} \int_0^L \int_0^L \left(\sum_{n=1}^N F_{s,t}\left(|y_{n+m} - y_n|^2\right) \right) \psi_\star(s)\psi_\star(t)\mathrm{d}s\mathrm{d}t.$$

Plugging now the geometric inequality (4.1) into (4.2) and using the positivity of $\psi_\star$, properties of the function $F_{s,t}(\cdot)$, and Jensen's inequality, we obtain

$$2\pi \sup \sigma(Q_{\Sigma_N}(\kappa)) \geq N \sum_{m=0}^{N-1} \int_0^L \int_0^L \left(F_{s,t}\left(\sum_{n=1}^N \frac{|y_{n+m} - y_n|^2}{N} \right) \right) \psi_\star(s)\psi_\star(t)\mathrm{d}s\mathrm{d}t$$

$$> N \sum_{m=0}^{N-1} \int_0^L \int_0^L F_{s,t}\left(|x_{n+m} - x_n|^2\right) \psi_\star(s)\psi_\star(t)\mathrm{d}s\mathrm{d}t$$

$$= \sum_{m=0}^{N-1} \int_0^L \int_0^L \left(\sum_{n=1}^N F_{s,t}\left(|x_{n+m} - x_n|^2\right) \right) \psi_\star(s)\psi_\star(t)\mathrm{d}s\mathrm{d}t$$

$$= \int_0^L \int_0^L \left(\sum_{n=1}^N \sum_{m=1}^N K_0\left(\kappa|\gamma_n(s) - \gamma_m(t)|\right) \right) \psi_\star(s)\psi_\star(t)\mathrm{d}s\mathrm{d}t$$

$$= 2\pi \sup \sigma(Q_{\Gamma_N}(\kappa)) = \frac{2\pi}{\alpha}.$$

This means that $\sup \sigma(\alpha Q_{\Sigma_N}(\kappa)) > 1$ and using Corollary 3.4 we get the inequality

$$\lambda_1^\alpha(\Sigma_N) < -\kappa^2 = \lambda_1^\alpha(\Gamma_N)$$

which concludes the proof for $L < \infty$.

In the case of semi-infinite edges we first show that $H_{\alpha,\Sigma_N(L)}$ and $H_{\alpha,\Gamma_N(L)}$ converge to $H_{\alpha,\Sigma_N(\infty)}$ and $H_{\alpha,\Gamma_N(\infty)}$ in the strong resolvent sense as $L \to \infty$. Indeed, the sequences of quadratic forms $\mathfrak{h}_{\alpha,\Sigma_N(L)}$ and $\mathfrak{h}_{\alpha,\Gamma_N(L)}$ are decreasing in L in the sense of form ordering and for any $u \in H^1(\mathbb{R}^2)$ the dominated convergence theorem implies $\lim_{L\to\infty} \mathfrak{h}_{\alpha,\Sigma_N(L)}[u] = \mathfrak{h}_{\alpha,\Sigma_N(\infty)}[u]$ and $\lim_{L\to\infty} \mathfrak{h}_{\alpha,\Gamma_N(L)}[u] = \mathfrak{h}_{\alpha,\Gamma_N(\infty)}[u]$. Hence, the desired strong resolvent convergence follows from [RS-I, Thm. S.16]. By [W, Satz 9.26 (b)] we then get

$$\lim_{L\to\infty} \lambda_1^\alpha(\Sigma_N(L)) = \lambda_1^\alpha(\Sigma_N(\infty)) \quad \text{and} \quad \lim_{L\to\infty} \lambda_1^\alpha(\Gamma_N(L)) = \lambda_1^\alpha(\Gamma_N(\infty)).$$

Now the claim for infinite star graphs follows by passing to the limit $L \to \infty$ in the inequality $\lambda_1^\alpha(\Sigma_N(L)) \leq \lambda_1^\alpha(\Gamma_N(L))$ demonstrated above. $\square$

Remark 4.1. Note that for $L = \infty$ the equality in Theorem 1.1 is apparently achieved *if and only if* $\Sigma_N(\infty) \cong \Gamma_N(\infty)$, however, the used method of the proof through the spectral convergence is not refined enough to make this conclusion. We do not pursue this question further, instead we review below several less trivial extensions of the above considerations.

5. **Discussion**

The present analysis adds one more item to the long list of various 'isoperimetric' results having in mathematical physics a tradition almost a century old starting from the papers [F23, K24]. The symmetry embedded in the result comes naturally from the symmetry of the interaction. This would change when the latter is violated, for instance, by choosing the edges of the graph having different lengths, or alternatively by adding a potential bias in the spirit of [EV16] to the Schrödinger operator in question. There is no need to stress that the problem then becomes more complicated. Moreover, the solution may not be obvious also in situations where the support does exhibit a symmetry but the interaction is modified; as examples one can mention star graphs of the type considered here supporting a δ'-interaction as in [BEL14], or a three-dimensional 'star' with edges supporting a δ-interaction of co-dimension two as in [BFK$^+$16, EK02, EK08, EK16]. In both these cases one naturally expects that the extremal configuration(s) could depend on the number N of the star rays.

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