Existence in the sense of sequences of stationary solutions for some non-Fredholm integro-differential equations

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Abstract. We establish the existence in the sense of sequences of stationary solutions for some reaction-diffusion type equations in the appropriate H^2 spaces. It is shown that under reasonable technical conditions the convergence in L^1 of the integral kernels implies the existence and the convergence in H^2 of the solutions. The nonlocal elliptic equations involve second order differential operators with and without Fredholm property.

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1 Introduction

We recall that a linear operator L which acts from a Banach space E into another Banach space F possesses the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. Consequently, the problem $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. These properties of Fredholm operators are widely used in various approaches of linear and nonlinear analysis.

Elliptic equations in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Lopatinskii conditions are satisfied (see e.g. [1], [9], [11]). This is the main result of the theory of linear elliptic equations. When domains are unbounded, these conditions may be insufficient and the Fredholm property may not be satisfied. For instance, Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ fails to satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Linear elliptic equations in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions cited above, limiting operators are invertible (see [12]). In some simple cases, limiting operators can be explicitly constructed. For instance, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R}$$

with the coefficients of this operator having limits at infinity,

$$a_\pm = \lim_{x\to\pm\infty} a(x), \quad b_\pm = \lim_{x\to\pm\infty} b(x), \quad c_\pm = \lim_{x\to\pm\infty} c(x),$$

the limiting operators are given by:

$$L_\pm u = a_\pm u'' + b_\pm u' + c_\pm u.$$

Due to the fact that the coefficients are constant, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ fails to satisfy the Fredholm property, can be explicitly computed by means of the Fourier transform:

$$\lambda_\pm(\xi) = -a_\pm \xi^2 + b_\pm i\xi + c_\pm, \quad \xi \in \mathbb{R}.$$

Invertibility of limiting operators is equivalent to the condition that the essential spectrum does not contain the origin.

In the case of general elliptic equations, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, these conditions may not be explicitly written.

When the operators are non-Fredholm the usual solvability conditions may not be applicable and solvability conditions are, in general, not known. There are some classes of operators for which solvability conditions are derived. We illustrate them with the following example. Let us consider the problem

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where $a > 0$ is a constant. The operator L here coincides with its limiting operators. The homogeneous problem admits a nonzero bounded solution. Hence the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can use the Fourier transform and find the solution precisely. Solvability conditions can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exists a solution of this problem in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}} \right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [21]). Here and below S_r^d stands the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Hence, though the operator fails to satisfy the Fredholm property, solvability conditions are formulated similarly. However, this similarity is only formal due to the fact that the range of the operator is not closed.

In the case of the operator involving a potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

the Fourier transform is not of any help. Nevertheless, solvability conditions in $\mathbb{R}^3$ can be derived by a rather sophisticated application of the theory of self-adjoint operators (see [14]). As before, solvability relations are formulated in terms of orthogonality to solutions of the homogeneous adjoint problem. There are several other examples of linear elliptic operators without Fredholm property for which solvability conditions can be obtained (see [12]-[21]).

Solvability relations play a crucial role in the analysis of nonlinear elliptic equations. When the operators are non-Fredholm, in spite of some progress in understanding of linear equations, there exist only few examples where nonlinear non-Fredholm operators are analyzed (see [4]-[6], [20], [21], [24]). In the present work we consider the nonlinear problem, for which the Fredholm property may not be satisfied:

$$\frac{\partial u}{\partial t} = \Delta u + au + \int_\Omega G(x-y)F(u(y,t),y)dy, \quad a \geq 0. \tag{1.2}$$

Here Ω is a domain in $\mathbb{R}^d$, $\quad d = 1, 2, 3$, the more physically interesting dimensions. Equations of that kind appear in cell population dynamics. The space variable x here corresponds to the cell genotype, $u(x,t)$ denotes the cell density as a function of their genotype and time. The right side of problem (1.2) describes the evolution of cell density due to cell proliferation and mutations. Here the diffusion term corresponds to the change of genotype via small random mutations, and the nonlocal term describes large mutations. In this context, $F(u,x)$ is the rate of cell birth which depends on u and x (density dependent proliferation), and the function $G(x-y)$ shows the proportion of newly born cells which change their genotype from y to x. We assume that it depends on the distance between the genotypes. In population dynamics the integro-differential equations describe models with intra-specific competition and nonlocal consumption of resources (see e.g. [2], [3], [7]). The existence of stationary solutions of (1.2) was studied in [20] using the fixed point technique. Related to problem (1.2), we consider the sequence of iterated equations with $m \in \mathbb{N}$

$$\frac{\partial u}{\partial t} = \Delta u + au + \int_\Omega G_m(x-y)F(u(y,t),y)dy, \quad a \geq 0. \tag{1.3}$$

The sequence of kernels $G_m(x) \to G(x)$ as $m \to \infty$ in the appropriate function spaces discussed below. We will show that under the certain technical conditions each of equations (1.3) has a unique stationary solution $u_m(x) \in H^2$, the limiting problem (1.2) will have a unique stationary solution $u(x) \in H^2$ and $u_m(x) \to u(x)$ in H^2 as $m \to \infty$, which is a so-called *existence of stationary solutions in the sense of sequences.* The similar ideas in the sense of standard Schrödinger type operators were exploited in [22] and [23]. The operators without Fredholm property arise also when studying the so-called embedded solitons (see e.g. [10]).

2 Formulation of the results

The nonlinear part of problems (1.2) and (1.3) will satisfy the regularity conditions analogous to the ones of [20].

Assumption 1. *Function $F(u,x) : \mathbb{R} \times \Omega \to \mathbb{R}$ is such that*

$$|F(u,x)| \le k|u| + h(x) \quad for \quad u \in \mathbb{R}, \ x \in \Omega, \tag{2.1}$$

where a constant $k > 0$ and $h(x) : \Omega \to \mathbb{R}^+$, $\quad h(x) \in L^2(\Omega)$. Furthermore, it is a Lipschitz continuous function, such that

$$|F(u_1,x) - F(u_2,x)| \le l|u_1 - u_2| \quad for \quad any \quad u_{1,2} \in \mathbb{R}, \quad x \in \Omega \tag{2.2}$$

with a constant $l > 0$.

Obviously, the stationary solutions of (1.2) and (1.3), which exist under certain technical conditions, will satisfy the nonlocal elliptic equations

$$\Delta u + \int_\Omega G(x-y)F(u(y),y)dy + au = 0, \ a \ge 0 \tag{2.3}$$

and

$$\Delta u_m + \int_\Omega G_m(x-y)F(u_m(y),y)dy + au_m = 0, \ a \ge 0, \ m \in \mathbb{N}. \tag{2.4}$$

Let us denote

$$(f_1(x), f_2(x))_{L^2(\Omega)} := \int_\Omega f_1(x)\bar{f}_2(x)dx,$$

with a slight abuse of notations when these functions are not square integrable, like for instance those used in the orthogonality conditions of Theorem 1 below. Indeed, if $f_1(x) \in L^1(\Omega)$ and $f_2(x) \in L^\infty(\Omega)$, then the integral in the right side of the definition above makes sense. In the first part of the article we treat the case of $\Omega = \mathbb{R}^d$, such that the appropriate Sobolev space is equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta u\|^2_{L^2(\mathbb{R}^d)}. \tag{2.5}$$

The main issue for equations (2.3) and (2.4) above is that the operator $-\Delta - a : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \ a \ge 0$ fails to satisfy the Fredholm property, which is the obstacle when solving these equations. The similar situations arising in linear and nonlinear equations, both self- adjoint and non self-adjoint involving non Fredholm second or fourth order differential operators or even systems of equations with non Fredholm operators have been treated extensively in recent years (see [14]-[24]). Our first main result is as follows.

Theorem 1. *Let $\Omega = \mathbb{R}^d$, $m \in \mathbb{N}$, $G_m(x) : \mathbb{R}^d \to \mathbb{R}$, $G_m(x) \in L^1(\mathbb{R}^d)$, such that $G_m(x) \to G(x)$ in $L^1(\mathbb{R}^d)$ as $m \to \infty$ and Assumption 1 holds.*

I) When $a > 0$ we assume that $xG_m(x) \in L^1(\mathbb{R}^d)$, such that $xG_m(x) \to xG(x)$ in $L^1(\mathbb{R}^d)$ as $m \to \infty$, orthogonality relations (6.7) hold if $d = 1$ and (6.22) when $d = 2, 3$ and

$$\sqrt{2}(2\pi)^{\frac{d}{2}} N_{a, \ d, \ m} \ l \le 1 - \varepsilon \tag{2.6}$$

for all $m \in \mathbb{N}$ *with some* $0 < \varepsilon < 1$. *Then each of the equations (2.4) admits a unique solution* $u_m(x) \in H^2(\mathbb{R}^d)$ *and the limiting equation (2.3) has a unique solution* $u(x) \in H^2(\mathbb{R}^d)$.

II) When $a = 0$ *we assume that* $x^2 G_m(x) \in L^1(\mathbb{R}^d)$, *such that* $x^2 G_m(x) \to x^2 G(x)$ *in* $L^1(\mathbb{R}^d)$ *as* $m \to \infty$, *orthogonality conditions (6.26) hold,* $d = 1, 2, 3$ *and*

$$\sqrt{2}(2\pi)^{\frac{d}{2}} N_{0,\ d,\ m}\ l \le 1 - \varepsilon \tag{2.7}$$

for all $m \in \mathbb{N}$ *with some* $0 < \varepsilon < 1$. *Then each of the equations (2.4) possesses a unique solution* $u_m(x) \in H^2(\mathbb{R}^d)$ *and the limiting equation (2.3) admits a unique solution* $u(x) \in H^2(\mathbb{R}^d)$.

In both cases I) and II) we have $u_m(x) \to u(x)$ *in* $H^2(\mathbb{R}^d)$ *as* $m \to \infty$.

The unique solution of each problem (2.4) $u_m(x)$ *is nontrivial provided the intersection of supports of the Fourier transforms of functions* $supp\widehat{F(0,x)} \cap supp\widehat{G}_m$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}^d$. *Similarly, the unique solution of the limiting problem (2.3)* $u(x)$ *does not vanish identically if* $supp\widehat{F(0,x)} \cap supp\widehat{G}$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}^d$.

The second part of the article is devoted to the studies of the analogous problem on the finite interval with periodic boundary conditions, i.e. $\Omega = I := [0,\ 2\pi]$ and the appropriate functional space is

$$H^2(I) = \{u(x) : I \to \mathbb{R} \mid u(x), u''(x) \in L^2(I), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi)\}.$$

We define the following auxiliary constrained subspaces

$$H_0^2(I) := \{u \in H^2(I) \mid \Big(u(x), \frac{e^{\pm i n_0 x}}{\sqrt{2\pi}}\Big)_{L^2(I)} = 0\},\ n_0 \in \mathbb{N} \tag{2.8}$$

and

$$H_{0,\ 0}^2(I) := \{u \in H^2(I) \mid (u(x), 1)_{L^2(I)} = 0\}, \tag{2.9}$$

which are Hilbert spaces as well (see e.g. Chapter 2.1 of [8]). Our second main result is as follows.

Theorem 2. *Let* $\Omega = I,\ m \in \mathbb{N},\ G_m(x) : I \to \mathbb{R},\ G_m(x) \in L^\infty(I)$, *such that* $G_m(x) \to G(x)$ *in* $L^\infty(I),\ m \to \infty,\ G_m(0) = G_m(2\pi),\ F(u,0) = F(u,2\pi)$ *for* $u \in \mathbb{R}$ *and Assumption 1 holds.*

I) When $a > 0$ *and* $a \neq n^2,\ n \in \mathbb{Z}$, *we assume that*

$$2\sqrt{\pi}\mathcal{N}_{a,\ m} l \le 1 - \varepsilon \tag{2.10}$$

for all $m \in \mathbb{N}$ *with some* $0 < \varepsilon < 1$. *Then each of the equations (2.4) possesses a unique solution* $u_m(x) \in H^2(I)$ *and the limiting equation (2.3) admits a unique solution* $u(x) \in H^2(I)$.

II) If $a=n_0^2,\ n_0\in\mathbb{N}$, let us assume that orthogonality relations (6.42) hold and

$$2\sqrt{\pi}\mathcal{N}_{n_0^2,\ m}l\le 1-\varepsilon \tag{2.11}$$

for all $m\in\mathbb{N}$ with some $0<\varepsilon<1$. Then each of the equations (2.4) has a unique solution $u_m(x)\in H_0^2(I)$ and the limiting problem (2.3) admits a unique solution $u(x)\in H_0^2(I)$.

III) When $a=0$, assume that orthogonality relations (6.46) hold and

$$2\sqrt{\pi}\mathcal{N}_{0,\ m}l\le 1-\varepsilon \tag{2.12}$$

for all $m\in\mathbb{N}$ with some $0<\varepsilon<1$. Then each of the equations (2.4) admits a unique solution $u_m(x)\in H_{0,\ 0}^2(I)$ and the limiting equation (2.3) possesses a unique solution $u(x)\in H_{0,\ 0}^2(I)$.

In all the cases I), II and III) we have $u_m(x)\to u(x)$ as $m\to\infty$ in the norms of $H^2(I),\ H_0^2(I)$ and $H_{0,\ 0}^2(I)$ respectively.

The unique solution of each problem (2.4) $u_m(x)$ is nontrivial provided the Fourier coefficients $G_{m,n}F(0,x)_n\ne 0$ for some $n\in\mathbb{Z}$. Similarly, the unique solution of the limiting problem (2.3) $u(x)$ does not vanish identically if $G_nF(0,x)_n\ne 0$ for some $n\in\mathbb{Z}$.

Remark. *We use the constrained subspaces $H_0^2(I)$ and $H_{0,\ 0}^2(I)$ in cases II) and III) of the theorem above respectively, such that the Fredholm operators $-\dfrac{d^2}{dx^2}-n_0^2:H_0^2(I)\to L^2(I)$ and $-\dfrac{d^2}{dx^2}:H_{0,\ 0}^2(I)\to L^2(I)$ have empty kernels.*

Let us conclude the article with the studies of our problem on the product of spaces, where one is the finite interval with periodic boundary conditions as before and another is the whole space of dimension not exceeding two, such that in our notations $\Omega=I\times\mathbb{R}^d=[0,2\pi]\times\mathbb{R}^d,\ d=1,2$ and $x=(x_1,x_\perp)$ with $x_1\in I$ and $x_\perp\in\mathbb{R}^d$. The appropriate Sobolev space for the problem is $H^2(\Omega)$ defined as

$$\{u(x):\Omega\to\mathbb{R}\ |\ u(x),\Delta u(x)\in L^2(\Omega),\ u(0,x_\perp)=u(2\pi,x_\perp),\ u_{x_1}(0,x_\perp)=u_{x_1}(2\pi,x_\perp)\},$$

where $x_\perp\in\mathbb{R}^d$ *a.e.* and u_{x_1} stands for the derivative of $u(x)$ with respect to the first variable x_1. Analogously to the whole space case treated in Theorem 1, the operator $-\Delta-a:H^2(\Omega)\to L^2(\Omega),\quad a\ge 0$ fails to possess the Fredholm property. Our final main result is as follows.

Theorem 3. *Let $\Omega=I\times\mathbb{R}^d,\ d=1,2,\quad m\in\mathbb{N},\quad G_m(x):\Omega\to\mathbb{R}$, such that*

$$G_m(x)\in L^1(\Omega),\quad G_m(x)\to G(x)\quad in\quad L^1(\Omega)\quad as\quad m\to\infty,$$

for $x_\perp\in\mathbb{R}^d$ a.e.

$$G_m(0,x_\perp)=G_m(2\pi,x_\perp)\in L^\infty(\mathbb{R}^d),$$

$F(u,0,x_\perp)=F(u,2\pi,x_\perp),\quad u\in\mathbb{R}$ and Assumption 1 holds. Moreover, let us assume that

$$G_m(0,x_\perp)\to G(0,x_\perp),\quad G_m(2\pi,x_\perp)\to G(2\pi,x_\perp),\quad m\to\infty$$

in $L^\infty(\mathbb{R}^d)$.

I) If $n_0^2 < a < (n_0+1)^2$, $n_0 \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$, *let* $x_\perp G_m(x) \in L^1(\Omega)$, *such that* $x_\perp G_m(x) \to x_\perp G(x)$ *in* $L^1(\Omega)$ *as* $m \to \infty$, *condition (6.86) holds if dimension* $d = 1$ *and (6.87) if* $d = 2$ *and*

$$\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_{a,\ m} l \le 1 - \varepsilon \tag{2.13}$$

for all $m \in \mathbb{N}$ *with some* $0 < \varepsilon < 1$. *Then each of the equations (2.4) admits a unique solution* $u_m(x) \in H^2(\Omega)$ *and the limiting equation (2.3) has a unique solution* $u(x) \in H^2(\Omega)$.

II) When $a = n_0^2$, $n_0 \in \mathbb{N}$, *let* $x_\perp^2 G_m(x) \in L^1(\Omega)$, *such that* $x_\perp^2 G_m(x) \to x_\perp^2 G(x)$ *in* $L^1(\Omega)$ *as* $m \to \infty$, *conditions (6.69), (6.71) hold if dimension* $d = 1$ *and conditions (6.70), (6.71) hold if* $d = 2$ *and*

$$\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_{n_0^2,\ m} l \le 1 - \varepsilon \tag{2.14}$$

for all $m \in \mathbb{N}$ *with some* $0 < \varepsilon < 1$. *Then each of the equations (2.4) possesses a unique solution* $u_m(x) \in H^2(\Omega)$ *and the limiting equation (2.3) admits a unique solution* $u(x) \in H^2(\Omega)$.

III) When $a = 0$, *let* $x_\perp^2 G_m(x) \in L^1(\Omega)$, *such that* $x_\perp^2 G_m(x) \to x_\perp^2 G(x)$ *in* $L^1(\Omega)$ *as* $m \to \infty$, *conditions (6.62) hold and*

$$\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_{0,\ m} l \le 1 - \varepsilon \tag{2.15}$$

for all $m \in \mathbb{N}$ *with some* $0 < \varepsilon < 1$. *Then each of the equations (2.4) admits a unique solution* $u_m(x) \in H^2(\Omega)$ *and the limiting equation (2.3) has a unique solution* $u(x) \in H^2(\Omega)$.

In all the cases I), II and III) we have $u_m(x) \to u(x)$ *in* $H^2(\Omega)$ *as* $m \to \infty$.

The unique solution of each equation (2.4) $u_m(x)$ *is nontrivial provided that the intersection of supports of the Fourier transforms of functions* $supp\widehat{F(0,x)}_n \cap supp\widehat{G}_{m,n}$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}^d$ *for some* $n \in \mathbb{Z}$. *Similarly, the unique solution of the limiting equation (2.3)* $u(x)$ *does not vanish identically if* $supp\widehat{F(0,x)}_n \cap supp\widehat{G}_n$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}^d$ *for some* $n \in \mathbb{Z}$.

Remark. *Note that in the article we deal with real valued functions due to the assumptions on* $F(u,x)$, $G_m(x)$ *and* $G(x)$ *involved in the nonlocal terms of equations (2.3) and (2.4).*

3 The Whole Space Case

Proof of Theorem 1. By means of Theorem 1 of [20], each equation (2.4) admits a unique solution $u_m(x) \in H^2(\mathbb{R}^d)$, $m \in \mathbb{N}$. Equation (2.3) possesses a unique solution $u(x) \in H^2(\mathbb{R}^d)$ as a result of Lemma A1 of the Appendix in dimension $d = 1$ and via Lemma A2 when $d = 2, 3$ along with Theorem 1 of [20].

Let us apply the standard Fourier transform (6.1) to both sides of equations (2.3) and (2.4). This yields

$$\widehat{u}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G}(p)\widehat{f}(p)}{p^2 - a}, \quad \widehat{u}_m(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G}_m(p)\widehat{f}_m(p)}{p^2 - a}, \quad m \in \mathbb{N}, \tag{3.1}$$

where $\widehat{f}(p)$ and $\widehat{f}_m(p)$ stand for the Fourier transforms of $F(u(x), x)$ and $F(u_m(x), x)$ respectively. Obviously, we have the estimate from above

$$|\widehat{u}_m(p) - \widehat{u}(p)| \le (2\pi)^{\frac{d}{2}} \left\| \frac{\widehat{G}_m(p)}{p^2 - a} - \frac{\widehat{G}(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)} |\widehat{f}(p)| + (2\pi)^{\frac{d}{2}} \left\| \frac{\widehat{G}_m(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)} |\widehat{f}_m(p) - \widehat{f}(p)|,$$

such that

$$\|u_m - u\|_{L^2(\mathbb{R}^d)} \le (2\pi)^{\frac{d}{2}} \left\| \frac{\widehat{G}_m(p)}{p^2 - a} - \frac{\widehat{G}(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)} \|F(u(x), x)\|_{L^2(\mathbb{R}^d)} +$$

$$+ (2\pi)^{\frac{d}{2}} \left\| \frac{\widehat{G}_m(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)} \|F(u_m(x), x) - F(u(x), x)\|_{L^2(\mathbb{R}^d)}.$$

By means of inequality (2.2) of Assumption 1, we have

$$\|F(u_m(x), x) - F(u(x), x)\|_{L^2(\mathbb{R}^d)} \le l \|u_m - u\|_{L^2(\mathbb{R}^d)}. \tag{3.2}$$

Note that $u_m(x), u(x) \in H^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$, $d \le 3$ by virtue of the Sobolev embedding. Thus, we arrive at

$$\|u_m - u\|_{L^2(\mathbb{R}^d)} \left\{ 1 - (2\pi)^{\frac{d}{2}} l \left\| \frac{\widehat{G}_m(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)} \right\} \le$$

$$\le (2\pi)^{\frac{d}{2}} \left\| \frac{\widehat{G}_m(p)}{p^2 - a} - \frac{\widehat{G}(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)} \|F(u(x), x)\|_{L^2(\mathbb{R}^d)}.$$

By virtue of (2.6) if $a > 0$ and (2.7) in the case of $a = 0$, we arrive at

$$\|u_m - u\|_{L^2(\mathbb{R}^d)} \le \frac{(2\pi)^{\frac{d}{2}}}{\varepsilon} \left\| \frac{\widehat{G}_m(p)}{p^2 - a} - \frac{\widehat{G}(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)} \|F(u(x), x)\|_{L^2(\mathbb{R}^d)}.$$

By means of inequality (2.1) of Assumption 1, we have $F(u(x), x) \in L^2(\mathbb{R}^d)$ for $u(x) \in H^2(\mathbb{R}^d)$. Therefore,

$$u_m(x) \to u(x), \quad m \to \infty \tag{3.3}$$

in $L^2(\mathbb{R}^d)$ due to Lemma A1 of the Appendix for $d = 1$ and Lemma A2 when $d = 2, 3$. Clearly,

$$p^2 \widehat{u}(p) = (2\pi)^{\frac{d}{2}} \frac{p^2 \widehat{G}(p)\widehat{f}(p)}{p^2 - a}, \quad p^2 \widehat{u}_m(p) = (2\pi)^{\frac{d}{2}} \frac{p^2 \widehat{G}_m(p)\widehat{f}_m(p)}{p^2 - a}, \quad m \in \mathbb{N},$$

such that

$$|p^2\widehat{u}_m(p)-p^2\widehat{u}(p)|\le (2\pi)^{\frac{d}{2}}\left\|\frac{p^2\widehat{G}_m(p)}{p^2-a}-\frac{p^2\widehat{G}(p)}{p^2-a}\right\|_{L^\infty(\mathbb{R}^d)}|\widehat{f}(p)|+$$

$$+(2\pi)^{\frac{d}{2}}\left\|\frac{p^2\widehat{G}_m(p)}{p^2-a}\right\|_{L^\infty(\mathbb{R}^d)}|\widehat{f}_m(p)-\widehat{f}(p)|.$$

Thus, using (3.2) we arrive at

$$\|\Delta u_m-\Delta u\|_{L^2(\mathbb{R}^d)}\le (2\pi)^{\frac{d}{2}}\left\|\frac{p^2\widehat{G}_m(p)}{p^2-a}-\frac{p^2\widehat{G}(p)}{p^2-a}\right\|_{L^\infty(\mathbb{R}^d)}\|F(u(x),x)\|_{L^2(\mathbb{R}^d)}+$$

$$+(2\pi)^{\frac{d}{2}}\left\|\frac{p^2\widehat{G}_m(p)}{p^2-a}\right\|_{L^\infty(\mathbb{R}^d)} l\|u_m-u\|_{L^2(\mathbb{R}^d)}.$$

Hence, by virtue of Lemma A1 of the Appendix when $d=1$ and Lemma A2 for $d=2,3$ along with (3.3), we have $\Delta u_m(x)\to\Delta u(x)$ in $L^2(\mathbb{R}^d)$ as $m\to\infty$. Norm definition (2.5) implies that $u_m(x)\to u(x)$ in $H^2(\mathbb{R}^d)$ as $m\to\infty$.

Suppose the solution $u_m(x)$ of equation (2.4) discussed above vanishes a.e. in $\mathbb{R}^d$ for a certain $m\in\mathbb{N}$. This will contradict to our assumption that the Fourier images of $G_m(x)$ and $F(0,x)$ do not vanish on a set of nonzero Lebesgue measure in $\mathbb{R}^d$. The analogous reasoning holds for the solution $u(x)$ of the limiting problem (2.3) studied above. ■

4 The Problem on the Finite Interval

Proof of Theorem 2. Note that under the given conditions we have $G_m(x)\in L^1(I),\ m\in\mathbb{N}$ and $G_m(x)\to G(x)$ in $L^1(I)$ as $m\to\infty$. By virtue of Theorem 2 of [20], each equation (2.4) possesses a unique solution $u_m(x)$ belonging to $H^2(I)$ in case I) of the Theorem, to $H_0^2(I)$ in case II) and to $H^2_{0,\ 0}(I)$ in case III) with $m\in\mathbb{N}$. Equation (2.3) has a unique solution $u(x)$ belonging to $H^2(I)$ in case I), to $H_0^2(I)$ in case II) and to $H^2_{0,\ 0}(I)$ in case III) as a result of Lemma A3 of the Appendix along with Theorem 2 of [20].

Let us apply Fourier transform (6.33) to both sides of equations (2.3) and (2.4). This yields for $n\in\mathbb{Z}$

$$u_n=\sqrt{2\pi}\frac{G_nf_n}{n^2-a},\quad u_{m,n}=\sqrt{2\pi}\frac{G_{m,n}f_{m,n}}{n^2-a},\quad m\in\mathbb{N} \tag{4.1}$$

with f_n and $f_{m,n}$ denoting the Fourier images of $F(u(x),x)$ and $F(u_m(x),x)$ respectively under transform (6.33). This enables us to obtain the estimate from above

$$|u_{m,n}-u_n|\le\sqrt{2\pi}\left\|\frac{G_{m,n}}{n^2-a}-\frac{G_n}{n^2-a}\right\|_{l^\infty}|f_n|+\sqrt{2\pi}\left\|\frac{G_{m,n}}{n^2-a}\right\|_{l^\infty}|f_{m,n}-f_n|,$$

such that

$$\|u_m(x)-u(x)\|_{L^2(I)} \le \sqrt{2\pi}\left\|\frac{G_{m,n}}{n^2-a}-\frac{G_n}{n^2-a}\right\|_{l^\infty}\|F(u(x),x)\|_{L^2(I)}+$$

$$+\sqrt{2\pi}\left\|\frac{G_{m,n}}{n^2-a}\right\|_{l^\infty}\|F(u_m(x),x)-F(u(x),x)\|_{L^2(I)}.$$

Inequality (2.2) of Assumption 1 yields

$$\|F(u_m(x),x)-F(u(x),x)\|_{L^2(I)} \le l\|u_m(x)-u(x)\|_{L^2(I)}. \tag{4.2}$$

Note that by means of the Sobolev embedding we have $u_m(x), u(x) \in H^2(I) \subset L^\infty(I)$. Ediventy,

$$\|u_m(x)-u(x)\|_{L^2(I)}\left\{1-\sqrt{2\pi}l\left\|\frac{G_{m,n}}{n^2-a}\right\|_{l^\infty}\right\} \le \sqrt{2\pi}\left\|\frac{G_{m,n}}{n^2-a}-\frac{G_n}{n^2-a}\right\|_{l^\infty}\|F(u(x),x)\|_{L^2(I)}.$$

Let us use inequalities (2.10), (2.11) and (2.12) in cases I), II) and III) of the theorem respectively. Thus, we arrive at

$$\|u_m(x)-u(x)\|_{L^2(I)} \le \frac{\sqrt{2\pi}}{\varepsilon}\left\|\frac{G_{m,n}}{n^2-a}-\frac{G_n}{n^2-a}\right\|_{l^\infty}\|F(u(x),x)\|_{L^2(I)}.$$

Clearly, $F(u(x),x) \in L^2(I)$ for $u(x) \in H^2(I)$ due to bound (2.1) of Assumption 1. Then by means of the result of Lemma A3 of the Appendix, we have

$$u_m(x) \to u(x), \quad m \to \infty \tag{4.3}$$

in $L^2(I)$. Evidently,

$$|n^2u_{m,n}-n^2u_n| \le \sqrt{2\pi}\left\|\frac{n^2G_{m,n}}{n^2-a}-\frac{n^2G_n}{n^2-a}\right\|_{l^\infty}|f_n|+\sqrt{2\pi}\left\|\frac{n^2G_{m,n}}{n^2-a}\right\|_{l^\infty}|f_{m,n}-f_n|,$$

such that via (4.2)

$$\|u_m''(x)-u''(x)\|_{L^2(I)} \le \sqrt{2\pi}\left\|\frac{n^2G_{m,n}}{n^2-a}-\frac{n^2G_n}{n^2-a}\right\|_{l^\infty}\|F(u(x),x)\|_{L^2(I)}+$$

$$+\sqrt{2\pi}\left\|\frac{n^2G_{m,n}}{n^2-a}\right\|_{l^\infty}l\|u_m(x)-u(x)\|_{L^2(I)}.$$

By virtue of the result of Lemma A3 of the Appendix along with (4.3), we arrive at

$$u_m''(x) \to u''(x), \quad m \to \infty$$

in $L^2(I)$. Therefore, $u_m(x) \to u(x)$ in the $H^2(I)$ norm as $m \to \infty$.

Suppose $u_m(x) = 0$ a.e. in I for some $m \in \mathbb{N}$. Then we will obtain the contradiction to the assumption that the Fourier coefficients $G_{m,n}F(0,x)_n \neq 0$ for some $n \in \mathbb{Z}$. The analogous argument holds for the solution $u(x)$ of the limiting equation (2.3). ■

5 The Problem on the Product of Spaces

Proof of Theorem 3. By means of Theorem 3 of [20], each equation (2.4) admits a unique solution $u_m(x) \in H^2(\Omega),\ m \in \mathbb{N}$. Equation (2.3) possesses a unique solution $u(x) \in H^2(\Omega)$ as a result of Lemmas A6, A5 and A4 of the Appendix in cases I), II) and III) respectively along with Theorem 3 of [20].

We apply Fourier transform (6.54) to both sides of equations (2.3) and (2.4). This yields for $n \in \mathbb{Z},\ p \in \mathbb{R}^d,\ d = 1, 2,\ m \in \mathbb{N}$

$$\widehat{u}_n(p) = (2\pi)^{\frac{d+1}{2}} \xi_n^a(p) \widehat{f}_n(p), \quad \widehat{u}_{m,n}(p) = (2\pi)^{\frac{d+1}{2}} \xi_{m,n}^a(p) \widehat{f}_{m,n}(p) \tag{5.1}$$

with $\widehat{f}_n(p)$ and $\widehat{f}_{m,n}(p)$ standing for the Fourier images of $F(u(x), x)$ and $F(u_m(x), x)$ under transform (6.54) and $\xi_n^a(p),\ \xi_{m,n}^a(p)$ defined in (6.57). This allows us to obtain the upper bound

$$|\widehat{u}_{m,n}(p) - \widehat{u}_n(p)| \leq (2\pi)^{\frac{d+1}{2}} \|\xi_{m,n}^a(p) - \xi_n^a(p)\|_{L^\infty_{n,p}} |\widehat{f}_n(p)| + (2\pi)^{\frac{d+1}{2}} \|\xi_{m,n}^a(p)\|_{L^\infty_{n,p}} |\widehat{f}_{m,n}(p) - \widehat{f}_n(p)|,$$

such that

$$\|u_m(x) - u(x)\|_{L^2(\Omega)} \leq (2\pi)^{\frac{d+1}{2}} \|\xi_{m,n}^a(p) - \xi_n^a(p)\|_{L^\infty_{n,p}} \|F(u(x), x)\|_{L^2(\Omega)} +$$

$$+ (2\pi)^{\frac{d+1}{2}} \|\xi_{m,n}^a(p)\|_{L^\infty_{n,p}} \|F(u_m(x), x) - F(u(x), x)\|_{L^2(\Omega)}.$$

Bound (2.2) of Assumption 1 implies

$$\|F(u_m(x), x) - F(u(x), x)\|_{L^2(\Omega)} \leq l \|u_m(x) - u(x)\|_{L^2(\Omega)}. \tag{5.2}$$

Note that by virtue of the Sobolev embedding we have $u_m(x), u(x) \in H^2(\Omega) \subset L^\infty(\Omega)$. Apparently,

$$\|u_m(x) - u(x)\|_{L^2(\Omega)} \{1 - (2\pi)^{\frac{d+1}{2}} l \|\xi_{m,n}^a(p)\|_{L^\infty_{n,p}}\} \leq$$

$$\leq (2\pi)^{\frac{d+1}{2}} \|\xi_{m,n}^a(p) - \xi_n^a(p)\|_{L^\infty_{n,p}} \|F(u(x), x)\|_{L^2(\Omega)}.$$

We use bounds (2.13), (2.14) and (2.15) in cases I), II) and III) of the theorem respectively. Hence, we derive

$$\|u_m(x) - u(x)\|_{L^2(\Omega)} \leq \frac{(2\pi)^{\frac{d+1}{2}}}{\varepsilon} \|\xi_{m,n}^a(p) - \xi_n^a(p)\|_{L^\infty_{n,p}} \|F(u(x), x)\|_{L^2(\Omega)}.$$

Obviously, $F(u(x), x) \in L^2(\Omega)$ for $u(x) \in H^2(\Omega)$ via inequality (2.1) of Assumption 1. By means of the results of Lemmas A6, A5 and A4 of the Appendix in cases I), II) and III) of the theorem respectively, we obtain

$$u_m(x) \to u(x), \quad m \to \infty \tag{5.3}$$

in $L^2(\Omega)$. Apparently,

$$|(p^2+n^2)\widehat{u}_{m,n}(p)-(p^2+n^2)\widehat{u}_n(p)| \le (2\pi)^{\frac{d+1}{2}}\|(p^2+n^2)\xi^a_{m,n}(p)-(p^2+n^2)\xi^a_n(p)\|_{L^\infty_{n,p}}|\widehat{f}_n(p)|+$$

$$+(2\pi)^{\frac{d+1}{2}}\|(p^2+n^2)\xi^a_{m,n}(p)\|_{L^\infty_{n,p}}|\widehat{f}_{m,n}(p)-\widehat{f}_n(p)|,$$

which yields via (5.2)

$$\|\Delta u_m(x)-\Delta u(x)\|_{L^2(\Omega)} \le (2\pi)^{\frac{d+1}{2}}\|(p^2+n^2)\xi^a_{m,n}(p)-(p^2+n^2)\xi^a_n(p)\|_{L^\infty_{n,p}}\|F(u(x),x)\|_{L^2(\Omega)}+$$

$$+(2\pi)^{\frac{d+1}{2}}\|(p^2+n^2)\xi^a_{m,n}(p)\|_{L^\infty_{n,p}}l\|u_m(x)-u(x)\|_{L^2(\Omega)}.$$

By means of (5.3) along with the results of Lemmas A6, A5 and A4 of the Appendix in cases I), II) and III) of the theorem respectively, we arrive at

$$\Delta u_m(x) \to \Delta u(x), \quad m \to \infty$$

in $L^2(\Omega)$. This proves that

$$u_m(x) \to u(x), \quad m \to \infty$$

in $H^2(\Omega)$.

Suppose $u_m(x) = 0$ a.e. in Ω for some $m \in \mathbb{N}$. This yields the contradiction to the assumption that there exists $n \in \mathbb{Z}$ for which $supp\widehat{G}_{m,n} \cap supp\widehat{F(0,x)}_n$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$. The analogous reasoning is valid for the solution $u(x)$ of the limiting problem (2.3). ■

6 Appendix

Let $G(x)$ be a function, $G(x) : \mathbb{R}^d \to \mathbb{R}, \quad d \le 3$. We designate its standard Fourier transform via the hat symbol as

$$\widehat{G}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} G(x)e^{-ipx}dx, \ p \in \mathbb{R}^d. \tag{6.1}$$

Hence

$$\|\widehat{G}(p)\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{(2\pi)^{\frac{d}{2}}}\|G\|_{L^1(\mathbb{R}^d)} \tag{6.2}$$

and $G(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\widehat{G}(q)e^{iqx}dq, \ x \in \mathbb{R}^d$. We introduce the auxiliary quantities for $m \in \mathbb{N}$

$$N_{a,\ d,\ m} := max\Big\{\Big\|\frac{\widehat{G}_m(p)}{p^2-a}\Big\|_{L^\infty(\mathbb{R}^d)}, \quad \Big\|\frac{p^2\widehat{G}_m(p)}{p^2-a}\Big\|_{L^\infty(\mathbb{R}^d)}\Big\} \tag{6.3}$$

when $a>0$ and

$$N_{0,\ d,\ m} := max\Big\{\Big\|\frac{\widehat{G}_m(p)}{p^2}\Big\|_{L^\infty(\mathbb{R}^d)},\quad \Big\|\widehat{G}_m(p)\Big\|_{L^\infty(\mathbb{R}^d)}\Big\} \tag{6.4}$$

for $a=0$. Similarly, in the limiting case

$$N_{a,\ d} := max\Big\{\Big\|\frac{\widehat{G}(p)}{p^2-a}\Big\|_{L^\infty(\mathbb{R}^d)},\quad \Big\|\frac{p^2\widehat{G}(p)}{p^2-a}\Big\|_{L^\infty(\mathbb{R}^d)}\Big\},\quad a>0 \tag{6.5}$$

and

$$N_{0,\ d} := max\Big\{\Big\|\frac{\widehat{G}(p)}{p^2}\Big\|_{L^\infty(\mathbb{R}^d)},\quad \Big\|\widehat{G}(p)\Big\|_{L^\infty(\mathbb{R}^d)}\Big\},\quad a=0. \tag{6.6}$$

Lemma A1. *Let the assumptions of Theorem 1 hold in dimension $d=1$.*
a) If $a>0$, let

$$\Big(G_m(x), \frac{e^{\pm i\sqrt{a}x}}{\sqrt{2\pi}}\Big)_{L^2(\mathbb{R})} = 0, \quad m\in\mathbb{N}. \tag{6.7}$$

Then

$$\frac{\widehat{G}_m(p)}{p^2-a} \to \frac{\widehat{G}(p)}{p^2-a}, \quad \frac{p^2\widehat{G}_m(p)}{p^2-a} \to \frac{p^2\widehat{G}(p)}{p^2-a}, \quad m\to\infty \tag{6.8}$$

in $L^\infty(\mathbb{R})$, such that

$$\Big\|\frac{\widehat{G}_m(p)}{p^2-a}\Big\|_{L^\infty(\mathbb{R})} \to \Big\|\frac{\widehat{G}(p)}{p^2-a}\Big\|_{L^\infty(\mathbb{R})}, \quad \Big\|\frac{p^2\widehat{G}_m(p)}{p^2-a}\Big\|_{L^\infty(\mathbb{R})} \to \Big\|\frac{p^2\widehat{G}(p)}{p^2-a}\Big\|_{L^\infty(\mathbb{R})}, \quad m\to\infty. \tag{6.9}$$

Moreover,

$$\sqrt{2}(2\pi)^{\frac{1}{2}}N_{a,\ 1}l \le 1-\varepsilon \tag{6.10}$$

holds.
b) When $a=0$, let

$$(G_m(x),1)_{L^2(\mathbb{R})} = 0 \quad and \quad (G_m(x),x)_{L^2(\mathbb{R})} = 0, \quad m\in\mathbb{N}. \tag{6.11}$$

Then

$$\frac{\widehat{G}_m(p)}{p^2} \to \frac{\widehat{G}(p)}{p^2}, \quad \widehat{G}_m(p)\to\widehat{G}(p), \quad m\to\infty \tag{6.12}$$

in $L^\infty(\mathbb{R})$, such that

$$\Big\|\frac{\widehat{G}_m(p)}{p^2}\Big\|_{L^\infty(\mathbb{R})} \to \Big\|\frac{\widehat{G}(p)}{p^2}\Big\|_{L^\infty(\mathbb{R})}, \quad \|\widehat{G}_m(p)\|_{L^\infty(\mathbb{R})} \to \|\widehat{G}(p)\|_{L^\infty(\mathbb{R})}, \quad m\to\infty. \tag{6.13}$$

Furthermore,

$$\sqrt{2}(2\pi)^{\frac{1}{2}}N_{0,\ 1}l \le 1-\varepsilon \tag{6.14}$$

holds.

Proof. In both parts a) and b) of the lemma by means of (6.2)

$$\|\widehat{G}_m(p) - \widehat{G}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}}\|G_m(x) - G(x)\|_{L^1(\mathbb{R})} \to 0, \quad m \to \infty \tag{6.15}$$

as assumed. Let us first establish part a) of the lemma. Via the trivial limiting argument, we have

$$\left(G(x), \frac{e^{\pm i\sqrt{a}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0 \tag{6.16}$$

as well. Indeed, if $w(x) \in L^\infty(\mathbb{R})$ is the right side of the inner product in orthogonality condition (6.7), then

$$|(G(x), w(x))_{L^2(\mathbb{R})}| = |(G(x) - G_m(x), w(x))_{L^2(\mathbb{R})}| \leq \|w\|_{L^\infty(\mathbb{R})}\|G_m - G\|_{L^1(\mathbb{R})} \to 0$$

as $m \to \infty$ as assumed, which yields (6.16). Therefore, by means of the part a) of Lemma A1 of [20], we have

$$N_{a,\,1} < \infty.$$

We introduce the intervals on the real line

$$I_\delta^+ := [\sqrt{a} - \delta, \sqrt{a} + \delta], \quad I_\delta^- := [-\sqrt{a} - \delta, -\sqrt{a} + \delta], \quad \sqrt{a} > \delta > 0,$$

such that $I_\delta := I_\delta^+ \cup I_\delta^-$. This yields

$$\frac{\widehat{G}_m(p)}{p^2 - a} - \frac{\widehat{G}(p)}{p^2 - a} = \frac{\widehat{G}_m(p) - \widehat{G}(p)}{p^2 - a}\chi_{I_\delta^c} + \frac{\widehat{G}_m(p) - \widehat{G}(p)}{p^2 - a}\chi_{I_\delta^+} + \frac{\widehat{G}_m(p) - \widehat{G}(p)}{p^2 - a}\chi_{I_\delta^-}, \tag{6.17}$$

where χ_A here and further down denotes the characteristic function of a set A, A^c stands for its complement. The first term in the right side of (6.17) can be easily bounded in the norm from above using (6.2) as

$$\left\|\frac{\widehat{G}_m(p) - \widehat{G}(p)}{p^2 - a}\chi_{I_\delta^c}\right\|_{L^\infty(\mathbb{R})} \leq \frac{\|G_m(x) - G(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}\delta^2} \to 0$$

as $m \to \infty$ due to one of our assumptions. Clearly,

$$\widehat{G}(\pm\sqrt{a}) = 0, \quad \widehat{G}_m(\pm\sqrt{a}) = 0, \quad m \in \mathbb{N}$$

due to orthogonality relations (6.16) and (6.7). This enables us to use the representation formulas

$$\widehat{G}(p) = \int_{\sqrt{a}}^p \frac{d\widehat{G}(q)}{dq}dq, \quad \widehat{G}_m(p) = \int_{\sqrt{a}}^p \frac{d\widehat{G}_m(q)}{dq}dq, \quad m \in \mathbb{N}.$$

By means of the definition of the Fourier transform (6.1), we have

$$\left|\frac{d\widehat{G}_m(p)}{dp} - \frac{d\widehat{G}(p)}{dp}\right| \le \frac{1}{\sqrt{2\pi}}\|xG_m(x) - xG(x)\|_{L^1(\mathbb{R})}, \quad p \in \mathbb{R}.$$

This enables us to obtain the bound

$$\left\|\frac{\widehat{G}_m(p) - \widehat{G}(p)}{p^2 - a}\chi_{I_\delta^+}\right\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}}\frac{\|xG_m(x) - xG(x)\|_{L^1(\mathbb{R})}}{2\sqrt{a} - \delta} \to 0$$

as $m \to \infty$ by means of the one of our assumptions. Similarly to the above, we have

$$\widehat{G}(p) = \int_{-\sqrt{a}}^{p} \frac{d\widehat{G}(q)}{dq}dq, \quad \widehat{G}_m(p) = \int_{-\sqrt{a}}^{p} \frac{d\widehat{G}_m(q)}{dq}dq, \quad m \in \mathbb{N},$$

such that

$$\left\|\frac{\widehat{G}_m(p) - \widehat{G}(p)}{p^2 - a}\chi_{I_\delta^-}\right\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}}\frac{\|xG_m(x) - xG(x)\|_{L^1(\mathbb{R})}}{2\sqrt{a} - \delta} \to 0$$

as $m \to \infty$ according to the one of our assumptions. This proves that

$$\frac{\widehat{G}_m(p)}{p^2 - a} \to \frac{\widehat{G}(p)}{p^2 - a}, \quad m \to \infty$$

in $L^\infty(\mathbb{R})$. Thus by virtue of the triangle inequality, we have

$$\left\|\frac{\widehat{G}_m(p)}{p^2 - a}\right\|_{L^\infty(\mathbb{R})} \to \left\|\frac{\widehat{G}(p)}{p^2 - a}\right\|_{L^\infty(\mathbb{R})}, \quad m \to \infty.$$

Clearly,

$$\frac{p^2\widehat{G}_m(p)}{p^2 - a} - \frac{p^2\widehat{G}(p)}{p^2 - a} = \widehat{G}_m(p) - \widehat{G}(p) + a\left[\frac{\widehat{G}_m(p)}{p^2 - a} - \frac{\widehat{G}(p)}{p^2 - a}\right],$$

such that

$$\left\|\frac{p^2\widehat{G}_m(p)}{p^2 - a} - \frac{p^2\widehat{G}(p)}{p^2 - a}\right\|_{L^\infty(\mathbb{R})} \le \|\widehat{G}_m(p) - \widehat{G}(p)\|_{L^\infty(\mathbb{R})} + a\left\|\frac{\widehat{G}_m(p)}{p^2 - a} - \frac{\widehat{G}(p)}{p^2 - a}\right\|_{L^\infty(\mathbb{R})} \to 0, \quad m \to \infty$$

via the results obtained above. Thus,

$$\frac{p^2\widehat{G}_m(p)}{p^2 - a} \to \frac{p^2\widehat{G}(p)}{p^2 - a}, \quad m \to \infty$$

in $L^\infty(\mathbb{R})$. By virtue of the triangle inequality,

$$\left\|\frac{p^2\widehat{G}_m(p)}{p^2 - a}\right\|_{L^\infty(\mathbb{R})} \to \left\|\frac{p^2\widehat{G}(p)}{p^2 - a}\right\|_{L^\infty(\mathbb{R})}, \quad m \to \infty.$$

Inequality (6.10) comes from (2.6) as a result of the trivial limiting argument.

Then we turn our attention to the proof of part b) of the lemma. Via the simple limiting argument below, we have

$$(G(x),1)_{L^2(\mathbb{R})}=0 \quad \textit{and} \quad (G(x),x)_{L^2(\mathbb{R})}=0. \tag{6.18}$$

The proof of the first identity in (6.18) is analogous to establishing (6.16). Obviously,

$$|(G(x),x)_{L^2(\mathbb{R})}|=|(G(x)-G_m(x),x)_{L^2(\mathbb{R})}|\leq \int_{|x|\leq 1}|G(x)-G_m(x)|dx+$$

$$+\int_{|x|>1}|x^2G(x)-x^2G_m(x)|dx\leq \|G_m(x)-G(x)\|_{L^1(\mathbb{R})}+\|x^2G_m(x)-x^2G(x)\|_{L^1(\mathbb{R})}\to 0$$

as $m\to\infty$ by virtue of our assumptions. Then by means of the result of part b) of Lemma A1 of [20], we have

$$N_{0,\ 1}<\infty.$$

We express

$$\frac{\widehat{G}_m(p)}{p^2}-\frac{\widehat{G}(p)}{p^2}=\frac{\widehat{G}_m(p)-\widehat{G}(p)}{p^2}\chi_{\{|p|\leq 1\}}+\frac{\widehat{G}_m(p)-\widehat{G}(p)}{p^2}\chi_{\{|p|>1\}}. \tag{6.19}$$

Using (6.15), we easily obtain

$$\left\|\frac{\widehat{G}_m(p)-\widehat{G}(p)}{p^2}\chi_{\{|p|>1\}}\right\|_{L^\infty(\mathbb{R})}\leq \|\widehat{G}_m(p)-\widehat{G}(p)\|_{L^\infty(\mathbb{R})}\leq \frac{1}{\sqrt{2\pi}}\|G_m(x)-G(x)\|_{L^1(\mathbb{R})}, \tag{6.20}$$

which tends to zero as $m\to\infty$ as assumed. Orthogonality relations (6.11) and (6.18) imply that

$$\widehat{G}(0)=0,\quad \frac{d\widehat{G}}{dp}(0)=0,\quad \widehat{G}_m(0)=0,\quad \frac{d\widehat{G}_m}{dp}(0)=0,\quad m\in\mathbb{N},$$

which yields

$$\widehat{G}_m(p)=\int_0^p\left(\int_0^s\frac{d^2\widehat{G}_m(q)}{dq^2}dq\right)ds,\quad \widehat{G}(p)=\int_0^p\left(\int_0^s\frac{d^2\widehat{G}(q)}{dq^2}dq\right)ds.$$

By means of the definition of the Fourier transform (6.1), we have

$$\left|\frac{d^2\widehat{G}_m(p)}{dp^2}-\frac{d^2\widehat{G}(p)}{dp^2}\right|\leq \frac{1}{\sqrt{2\pi}}\|x^2G_m(x)-x^2G(x)\|_{L^1(\mathbb{R})},$$

such that

$$|\widehat{G}_m(p)-\widehat{G}(p)|\leq \frac{1}{\sqrt{2\pi}}\|x^2G_m(x)-x^2G(x)\|_{L^1(\mathbb{R})}\frac{p^2}{2}.$$

Thus

$$\left\| \frac{\widehat{G}_m(p) - \widehat{G}(p)}{p^2} \chi_{\{|p|\le 1\}} \right\|_{L^\infty(\mathbb{R})} \le \frac{1}{2\sqrt{2\pi}} \|x^2 G_m(x) - x^2 G(x)\|_{L^1(\mathbb{R})} \to 0, \quad m \to \infty \tag{6.21}$$

due to one of the assumptions of the lemma. Therefore,

$$\frac{\widehat{G}_m(p)}{p^2} \to \frac{\widehat{G}(p)}{p^2}, \quad m \to \infty$$

in $L^\infty(\mathbb{R})$. By means of the triangle inequality, we have

$$\left\| \frac{\widehat{G}_m(p)}{p^2} \right\|_{L^\infty(\mathbb{R})} \to \left\| \frac{\widehat{G}(p)}{p^2} \right\|_{L^\infty(\mathbb{R})}, \quad m \to \infty.$$

Also,

$$\|\widehat{G}_m(p)\|_{L^\infty(\mathbb{R})} \to \|\widehat{G}(p)\|_{L^\infty(\mathbb{R})}, \quad m \to \infty,$$

which comes from (6.15) via the triangle inequality. Estimate (6.14) stems from (2.7) as a result of the elementary limiting argument. ■

The proposition above can be generalized to higher dimensions in the following statement.

Lemma A2. *Let the assumptions of Theorem 1 hold in dimensions* $d = 2, 3$.
a) If $a > 0$, *let*

$$\Big(G_m(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\Big)_{L^2(\mathbb{R}^d)} = 0 \quad \textit{for} \quad p \in S^d_{\sqrt{a}} \quad \textit{a.e.}, \quad m \in \mathbb{N}. \tag{6.22}$$

Then

$$\frac{\widehat{G}_m(p)}{p^2 - a} \to \frac{\widehat{G}(p)}{p^2 - a}, \quad \frac{p^2\widehat{G}_m(p)}{p^2 - a} \to \frac{p^2\widehat{G}(p)}{p^2 - a}, \quad m \to \infty \tag{6.23}$$

in $L^\infty(\mathbb{R}^d)$, *such that*

$$\left\| \frac{\widehat{G}_m(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)} \to \left\| \frac{\widehat{G}(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)}, \quad \left\| \frac{p^2\widehat{G}_m(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)} \to \left\| \frac{p^2\widehat{G}(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)} \tag{6.24}$$

as $m \to \infty$. *Furthermore,*

$$\sqrt{2}(2\pi)^{\frac{d}{2}} N_{a,\ d} l \le 1 - \varepsilon \tag{6.25}$$

holds.
b) When $a = 0$, *let*

$$(G_m(x), 1)_{L^2(\mathbb{R}^d)} = 0 \quad \textit{and} \quad (G_m(x), x_k)_{L^2(\mathbb{R}^d)} = 0, \quad 1 \le k \le d, \quad m \in \mathbb{N}. \tag{6.26}$$

Then

$$\frac{\widehat{G}_m(p)}{p^2} \to \frac{\widehat{G}(p)}{p^2}, \quad \widehat{G}_m(p) \to \widehat{G}(p), \quad m \to \infty \tag{6.27}$$

in $L^\infty(\mathbb{R}^d)$, *such that*

$$\left\| \frac{\widehat{G}_m(p)}{p^2} \right\|_{L^\infty(\mathbb{R}^d)} \to \left\| \frac{\widehat{G}(p)}{p^2} \right\|_{L^\infty(\mathbb{R}^d)}, \quad \|\widehat{G}_m(p)\|_{L^\infty(\mathbb{R}^d)} \to \|\widehat{G}(p)\|_{L^\infty(\mathbb{R}^d)}, \quad m \to \infty. \tag{6.28}$$

Moreover,

$$\sqrt{2}(2\pi)^{\frac{d}{2}} N_{0,\ d} l \le 1 - \varepsilon \tag{6.29}$$

holds.

Proof. Let us first establish part a) of the lemma. Via the trivial limiting argument similar to the proof of (6.16), we obtain

$$\left(G(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0 \quad for \quad p \in S^d_{\sqrt{a}} \quad a.e. \tag{6.30}$$

Then by virtue of part a) of Lemma A2 of [20], we have

$$N_{a,\ d} < \infty.$$

We will use the auxiliary spherical layer in the space of $d = 2, 3$ dimensions

$$A_\delta := \{p \in \mathbb{R}^d \mid \sqrt{a} - \delta < |p| < \sqrt{a} + \delta\}, \quad 0 < \delta < \sqrt{a},$$

such that

$$\frac{\widehat{G}_m(p)}{p^2 - a} - \frac{\widehat{G}(p)}{p^2 - a} = \frac{\widehat{G}_m(p) - \widehat{G}(p)}{p^2 - a}\chi_{A_\delta} + \frac{\widehat{G}_m(p) - \widehat{G}(p)}{p^2 - a}\chi_{A_\delta^c}. \tag{6.31}$$

The second term in the right side of (6.31) can be estimated above in the absolute value as

$$\left| \frac{\widehat{G}_m(p) - \widehat{G}(p)}{p^2 - a}\chi_{A_\delta^c} \right| \le \frac{|\widehat{G}_m(p) - \widehat{G}(p)|}{\sqrt{a}\delta}.$$

By virtue of the analog of inequality (6.15) in dimensions $d = 2, 3$, we obtain

$$\left\| \frac{\widehat{G}_m(p) - \widehat{G}(p)}{p^2 - a}\chi_{A_\delta^c} \right\|_{L^\infty(\mathbb{R}^d)} \le \frac{\|G_m(x) - G(x)\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}\sqrt{a}\delta} \to 0, \quad m \to \infty$$

due to one of our assumptions. By means of (6.30) and (6.22), we have

$$\widehat{G}(\sqrt{a}, \sigma) = 0, \quad \widehat{G}_m(\sqrt{a}, \sigma) = 0, \quad m \in \mathbb{N}.$$

Here and below σ denotes the angle variables on the sphere. This enables us to express

$$\widehat{G}(p) = \int_{\sqrt{a}}^{|p|} \frac{\partial \widehat{G}(s,\sigma)}{\partial s} ds, \quad \widehat{G}_m(p) = \int_{\sqrt{a}}^{|p|} \frac{\partial \widehat{G}_m(s,\sigma)}{\partial s} ds, \quad m \in \mathbb{N}.$$

Evidently, using the definition of the Fourier transform (6.1), we arrive at

$$\left| \frac{\partial \widehat{G}_m(|p|,\sigma)}{\partial |p|} - \frac{\partial \widehat{G}(|p|,\sigma)}{\partial |p|} \right| \le \frac{1}{(2\pi)^{\frac{d}{2}}} \|xG_m(x) - xG(x)\|_{L^1(\mathbb{R}^d)}.$$

Therefore,

$$\left\| \frac{\widehat{G}_m(p) - \widehat{G}(p)}{p^2 - a} \chi_{A_\delta} \right\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{(2\pi)^{\frac{d}{2}} \sqrt{a}} \|xG_m(x) - xG(x)\|_{L^1(\mathbb{R}^d)} \to 0, \quad m \to \infty$$

by virtue of the one of our assumptions. This implies that

$$\frac{\widehat{G}_m(p)}{p^2 - a} \to \frac{\widehat{G}(p)}{p^2 - a}, \quad m \to \infty$$

in $L^\infty(\mathbb{R}^d)$. By means of the triangle inequality

$$\left\| \frac{\widehat{G}_m(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)} \to \left\| \frac{\widehat{G}(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)}, \quad m \to \infty,$$

which is analogous to the first statement of (6.9) of Lemma A1 in one dimension. Note that

$$\frac{p^2 \widehat{G}_m(p)}{p^2 - a} \to \frac{p^2 \widehat{G}(p)}{p^2 - a}, \quad m \to \infty$$

in $L^\infty(\mathbb{R}^d)$ and

$$\left\| \frac{p^2 \widehat{G}_m(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)} \to \left\| \frac{p^2 \widehat{G}(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R}^d)}, \quad m \to \infty$$

holds here as well, which can be proven analogously to corresponding statements of Lemma A1 in one dimension. By means of the trivial limiting argument, we arrive at

$$\sqrt{2}(2\pi)^{\frac{d}{2}} N_{a,\ d} l \le 1 - \varepsilon.$$

Then we turn our attention to the proof of part b) of the lemma. By virtue of the straightforward limiting argument similarly to the proof of (6.18) in one dimension, we arrive at

$$(G(x), 1)_{L^2(\mathbb{R}^d)} = 0, \quad (G(x), x_k)_{L^2(\mathbb{R}^d)} = 0, \quad 1 \le k \le d. \tag{6.32}$$

By means of part b) of Lemma A2 of [20], we obtain

$$N_{0,\ d} < \infty.$$

Identities (6.26) and (6.32) imply that

$$\widehat{G}(0)=0, \quad \frac{\partial \widehat{G}}{\partial |p|}(0,\sigma)=0, \quad \widehat{G}_m(0)=0, \quad \frac{\partial \widehat{G}_m}{\partial |p|}(0,\sigma)=0, \quad m\in \mathbb{N}.$$

This enables us to express

$$\widehat{G}(p)=\int_0^{|p|}\Bigg(\int_0^s \frac{\partial^2 \widehat{G}(q,\sigma)}{\partial q^2}dq\Bigg)ds, \quad \widehat{G}_m(p)=\int_0^{|p|}\Bigg(\int_0^s \frac{\partial^2 \widehat{G}_m(q,\sigma)}{\partial q^2}dq\Bigg)ds, \quad m\in \mathbb{N}.$$

Using the definition of the standard Fourier transform (6.1), we derive

$$\left|\frac{\partial^2 \widehat{G}_m(|p|,\sigma)}{\partial |p|^2}-\frac{\partial^2 \widehat{G}(|p|,\sigma)}{\partial |p|^2}\right| \leq \frac{1}{(2\pi)^{\frac{d}{2}}}\|x^2G_m(x)-x^2G(x)\|_{L^1(\mathbb{R}^d)},$$

such that

$$|\widehat{G}_m(p)-\widehat{G}(p)|\leq \frac{1}{(2\pi)^{\frac{d}{2}}}\|x^2G_m(x)-x^2G(x)\|_{L^1(\mathbb{R}^d)}\frac{p^2}{2}.$$

We will use the analog of formula (6.19) in dimensions $d=2,3$. Clearly, the estimate similar to (6.20) holds here as well. The analog of (6.21) is valid here due to the argument presented above. This proves (6.27) along with (6.28). Inequality (6.29) can be easily established via the limiting argument. ■

Let the function $G(x):I\to \mathbb{R}$, $G(0)=G(2\pi)$ and its Fourier image on the finite interval is given by

$$G_n:=\int_0^{2\pi} G(x)\frac{e^{-inx}}{\sqrt{2\pi}}dx, \quad n\in \mathbb{Z}, \tag{6.33}$$

such that $G(x)=\sum_{n=-\infty}^{\infty} G_n\frac{e^{inx}}{\sqrt{2\pi}}$. Clearly we have the bound

$$\|G_n\|_{l^\infty}\leq \frac{1}{\sqrt{2\pi}}\|G\|_{L^1(I)}. \tag{6.34}$$

Analogously to the whole space case we introduce for $m\in \mathbb{N}$

$$\mathcal{N}_{a,\ m}:=max\Bigg\{\Bigg\|\frac{G_{m,n}}{n^2-a}\Bigg\|_{l^\infty}, \quad \Bigg\|\frac{n^2G_{m,n}}{n^2-a}\Bigg\|_{l^\infty}\Bigg\} \tag{6.35}$$

for $a>0$. In the case of $a=0$

$$\mathcal{N}_{0,\ m}:=max\Bigg\{\Bigg\|\frac{G_{m,n}}{n^2}\Bigg\|_{l^\infty}, \quad \Bigg\|G_{m,n}\Bigg\|_{l^\infty}\Bigg\}. \tag{6.36}$$

In the limiting case

$$\mathcal{N}_a := max\left\{\left\|\frac{G_n}{n^2-a}\right\|_{l^\infty}, \quad \left\|\frac{n^2 G_n}{n^2-a}\right\|_{l^\infty}\right\}, \quad a>0 \tag{6.37}$$

and

$$\mathcal{N}_0 := max\left\{\left\|\frac{G_n}{n^2}\right\|_{l^\infty}, \quad \left\|G_n\right\|_{l^\infty}\right\}, \quad a=0. \tag{6.38}$$

We have the following technical statement.

Lemma A3. *Let the assumptions of Theorem 2 hold.*
a) If $a>0$ and $a\neq n^2,\ n\in\mathbb{Z}$ then

$$\frac{G_{m,n}}{n^2-a}\to\frac{G_n}{n^2-a}, \quad \frac{n^2G_{m,n}}{n^2-a}\to\frac{n^2G_n}{n^2-a}, \quad m\to\infty \tag{6.39}$$

in l^∞, such that

$$\left\|\frac{G_{m,n}}{n^2-a}\right\|_{l^\infty}\to\left\|\frac{G_n}{n^2-a}\right\|_{l^\infty}, \quad \left\|\frac{n^2G_{m,n}}{n^2-a}\right\|_{l^\infty}\to\left\|\frac{n^2G_n}{n^2-a}\right\|_{l^\infty}, \quad m\to\infty. \tag{6.40}$$

Moreover,

$$2\sqrt{\pi}\mathcal{N}_a l\leq 1-\varepsilon \tag{6.41}$$

holds.
b) When $a=n_0^2,\ n_0\in\mathbb{N}$, let

$$\left(G_m(x), \frac{e^{\pm in_0x}}{\sqrt{2\pi}}\right)_{L^2(I)}=0, \quad m\in\mathbb{N}. \tag{6.42}$$

Then

$$\frac{G_{m,n}}{n^2-n_0^2}\to\frac{G_n}{n^2-n_0^2}, \quad \frac{n^2G_{m,n}}{n^2-n_0^2}\to\frac{n^2G_n}{n^2-n_0^2}, \quad m\to\infty \tag{6.43}$$

in l^∞, such that

$$\left\|\frac{G_{m,n}}{n^2-n_0^2}\right\|_{l^\infty}\to\left\|\frac{G_n}{n^2-n_0^2}\right\|_{l^\infty}, \quad \left\|\frac{n^2G_{m,n}}{n^2-n_0^2}\right\|_{l^\infty}\to\left\|\frac{n^2G_n}{n^2-n_0^2}\right\|_{l^\infty}, \quad m\to\infty. \tag{6.44}$$

Furthermore,

$$2\sqrt{\pi}\mathcal{N}_{n_0^2} l\leq 1-\varepsilon \tag{6.45}$$

holds.
c) If $a=0$, let

$$(G_m(x),1)_{L^2(I)}=0, \quad m\in\mathbb{N}. \tag{6.46}$$

Then

$$\frac{G_{m,n}}{n^2} \to \frac{G_n}{n^2}, \quad G_{m,n} \to G_n, \quad m \to \infty \tag{6.47}$$

in l^∞, *such that*

$$\left\| \frac{G_{m,n}}{n^2} \right\|_{l^\infty} \to \left\| \frac{G_n}{n^2} \right\|_{l^\infty}, \quad \|G_{m,n}\|_{l^\infty} \to \|G_n\|_{l^\infty}, \quad m \to \infty. \tag{6.48}$$

Moreover,

$$2\sqrt{\pi}\mathcal{N}_0 l \leq 1 - \varepsilon \tag{6.49}$$

holds.

Proof. Obviously,

$$|G(0)-G(2\pi)| \leq |G(0)-G_m(0)|+|G_m(2\pi)-G(2\pi)| \leq 2\|G_m(x)-G(x)\|_{L^\infty(I)} \to 0, \quad m \to \infty$$

as assumed, such that $G(0) = G(2\pi)$. As noted in the proof of Theorem 2 above, under the given conditions $G_m(x) \in L^1(I),\ m \in \mathbb{N}$ and $G_m(x) \to G(x)$ in $L^1(I)$ as $m \to \infty$. By means of (6.34), we have

$$\|G_{m,n} - G_n\|_{l^\infty} \leq \frac{1}{\sqrt{2\pi}} \|G_m(x) - G(x)\|_{L^1(I)} \to 0, \quad m \to \infty, \tag{6.50}$$

such that

$$G_{m,n} \to G_n, \quad m \to \infty$$

in l^∞. Let us first address case a) when $a > 0,\ a \neq n^2,\ n \in \mathbb{Z}$. Part a) of Lemma A3 of [20] implies that $\mathcal{N}_a < \infty$. We define

$$\gamma := \min_{n\in\mathbb{Z}} |n^2 - a| > 0.$$

Apparently,

$$\left\| \frac{G_{m,n}}{n^2 - a} - \frac{G_n}{n^2 - a} \right\|_{l^\infty} \leq \frac{1}{\sqrt{2\pi}\gamma} \|G_m(x) - G(x)\|_{L^1(I)} \to 0, \quad m \to \infty,$$

such that

$$\frac{G_{m,n}}{n^2 - a} \to \frac{G_n}{n^2 - a}, \quad m \to \infty$$

in l^∞. A trivial calculation yields

$$\frac{n^2 G_{m,n}}{n^2 - a} - \frac{n^2 G_n}{n^2 - a} = G_{m,n} - G_n + a\frac{G_{m,n} - G_n}{n^2 - a},$$

such that

$$\left\| \frac{n^2 G_{m,n}}{n^2 - a} - \frac{n^2 G_n}{n^2 - a} \right\|_{l^\infty} \leq \|G_{m,n} - G_n\|_{l^\infty} + a \left\| \frac{G_{m,n}}{n^2 - a} - \frac{G_n}{n^2 - a} \right\|_{l^\infty} \to 0, \quad m \to \infty.$$

Hence

$$\frac{n^2 G_{m,n}}{n^2-a} \to \frac{n^2 G_n}{n^2-a}, \quad m \to \infty \tag{6.51}$$

in l^∞. Therefore, (6.40) holds by means of the triangle inequality. We obtain (6.41) via an easy limiting argument.

Then we turn our attention to establishing part b) of the lemma. By virtue of the limiting argument analogous to the proof of (6.16), we derive

$$\left(G(x), \frac{e^{\pm in_0 x}}{\sqrt{2\pi}}\right)_{L^2(I)} = 0. \tag{6.52}$$

Then $\mathcal{N}_{n_0^2} < \infty$ due to part b) of Lemma A3 of [20]. We obtain

$$\left\|\frac{G_{m,n}}{n^2-n_0^2} - \frac{G_n}{n^2-n_0^2}\right\|_{l^\infty} \leq \frac{\|G_m(x)-G(x)\|_{L^1(I)}}{\sqrt{2\pi}(2n_0-1)} \to 0, \quad m \to \infty,$$

such that

$$\frac{G_{m,n}}{n^2-n_0^2} \to \frac{G_n}{n^2-n_0^2}, \quad m \to \infty$$

in l^∞. Note that $G_{m,\pm n_0}$, $m \in \mathbb{N}$ and $G_{\pm n_0}$ vanish due to orthogonality conditions (6.42) and (6.52). Similarly to the proof of (6.51) above, we obtain

$$\frac{n^2 G_{m,n}}{n^2-n_0^2} \to \frac{n^2 G_n}{n^2-n_0^2}, \quad m \to \infty$$

in l^∞. By virtue of the triangle inequality, we easily arrive at (6.44). Inequality (6.45) stems from the trivial limiting argument.

We conclude the proof of the lemma with considering case c). The limiting argument analogous to the proof of (6.16) yields

$$(G(x), 1)_{L^2(I)} = 0. \tag{6.53}$$

Part c) of Lemma A3 of [20] gives us $\mathcal{N}_0 < \infty$. Evidently,

$$\left\|\frac{G_{m,n}}{n^2} - \frac{G_n}{n^2}\right\|_{l^\infty} \leq \frac{1}{\sqrt{2\pi}}\|G_m(x)-G(x)\|_{L^1(I)} \to 0, \quad m \to \infty.$$

Note that $G_{m,0}$, $m \in \mathbb{N}$ and G_0 vanish due to orthogonality conditions (6.46) and (6.53). Hence,

$$\frac{G_{m,n}}{n^2} \to \frac{G_n}{n^2}, \quad m \to \infty$$

in l^∞. The triangle inequality yields (6.48). Inequality (6.49) is a result of a simple limiting argument. ■

Let $G(x)$ be a function on the product of spaces studied in Theorem 3, $G(x) : \Omega = I \times \mathbb{R}^d \to \mathbb{R},\ d = 1, 2,\ G(0, x_\perp) = G(2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e. and its Fourier transform on the product of spaces is given by

$$\widehat{G}_n(p) := \frac{1}{(2\pi)^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} dx_\perp e^{-ipx_\perp} \int_0^{2\pi} G(x_1, x_\perp) e^{-inx_1} dx_1, \quad p \in \mathbb{R}^d,\ n \in \mathbb{Z} \tag{6.54}$$

such that for the norm

$$\|\widehat{G}_n(p)\|_{L^\infty_{n,p}} := \sup_{\{p \in \mathbb{R}^d,\ n \in \mathbb{Z}\}} |\widehat{G}_n(p)| \le \frac{1}{(2\pi)^{\frac{d+1}{2}}} \|G(x)\|_{L^1(\Omega)} \tag{6.55}$$

and $G(x) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} \widehat{G}_n(p) e^{ipx_\perp} e^{inx_1} dp$. It is also useful to consider the Fourier transform only in the first variable, such that

$$G_n(x_\perp) := \int_0^{2\pi} G(x_1, x_\perp) \frac{e^{-inx_1}}{\sqrt{2\pi}} dx_1, \quad n \in \mathbb{Z}.$$

Evidently, under the assumptions of Theorem 3 by means of (6.55), we have

$$\|\widehat{G}_{m,n}(p) - \widehat{G}_n(p)\|_{L^\infty_{n,p}} \le \frac{1}{(2\pi)^{\frac{d+1}{2}}} \|G_m(x) - G(x)\|_{L^1(\Omega)} \to 0, \quad m \to \infty. \tag{6.56}$$

We define the auxiliary quantities for $a \ge 0$

$$\xi^a_n(p) := \frac{\widehat{G}_n(p)}{p^2 + n^2 - a}, \quad \xi^a_{m,n}(p) := \frac{\widehat{G}_{m,n}(p)}{p^2 + n^2 - a}, \quad m \in \mathbb{N} \tag{6.57}$$

and introduce for $m \in \mathbb{N}$

$$M_{a,\ m} := max\{\|\xi^a_{m,n}(p)\|_{L^\infty_{n,p}}, \quad \|(p^2 + n^2)\xi^a_{m,n}(p)\|_{L^\infty_{n,p}}\} \tag{6.58}$$

if $a > 0$ and

$$M_{0,\ m} := max\left\{\left\|\frac{\widehat{G}_{m,n}(p)}{p^2 + n^2}\right\|_{L^\infty_{n,p}}, \quad \|\widehat{G}_{m,n}(p)\|_{L^\infty_{n,p}}\right\} \tag{6.59}$$

when $a = 0$. Similarly, in the limiting case we define

$$M_a := max\{\|\xi^a_n(p)\|_{L^\infty_{n,p}}, \quad \|(p^2 + n^2)\xi^a_n(p)\|_{L^\infty_{n,p}}\} \tag{6.60}$$

when $a > 0$ and

$$M_0 := max\left\{\left\|\frac{\widehat{G}_n(p)}{p^2 + n^2}\right\|_{L^\infty_{n,p}}, \quad \|\widehat{G}_n(p)\|_{L^\infty_{n,p}}\right\} \tag{6.61}$$

if $a = 0$. Here the momentum vector $p \in \mathbb{R}^d$.

Lemma A4. *Let the assumptions of Theorem 3 hold, $a=0$ and for all $m \in \mathbb{N}$*

$$(G_m(x), 1)_{L^2(\Omega)} = 0, \ (G_m(x), x_{\perp, \ k})_{L^2(\Omega)} = 0, \ 1 \le k \le d, \ d = 1, 2. \tag{6.62}$$

Then

$$\frac{\widehat{G}_{m,n}(p)}{p^2+n^2} \to \frac{\widehat{G}_n(p)}{p^2+n^2}, \quad \widehat{G}_{m,n}(p) \to \widehat{G}_n(p), \quad m \to \infty \tag{6.63}$$

in $L^\infty_{n,p}$, such that

$$\left\| \frac{\widehat{G}_{m,n}(p)}{p^2+n^2} \right\|_{L^\infty_{n,p}} \to \left\| \frac{\widehat{G}_n(p)}{p^2+n^2} \right\|_{L^\infty_{n,p}}, \quad \|\widehat{G}_{m,n}(p)\|_{L^\infty_{n,p}} \to \|\widehat{G}_n(p)\|_{L^\infty_{n,p}}, \quad m \to \infty. \tag{6.64}$$

Moreover,

$$\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_0 l \le 1 - \varepsilon \tag{6.65}$$

holds.

Proof. Evidently, $\|G(0, x_\perp) - G(2\pi, x_\perp)\|_{L^\infty(\mathbb{R}^d)}$ can be bounded above by

$$\|G(0, x_\perp) - G_m(0, x_\perp)\|_{L^\infty(\mathbb{R}^d)} + \|G_m(2\pi, x_\perp) - G(2\pi, x_\perp)\|_{L^\infty(\mathbb{R}^d)} \to 0, \quad m \to \infty$$

as assumed, such that $G(0, x_\perp) = G(2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e.. The anologous reasoning is valid for Lemmas A5 and A6 below. A trivial limiting argument similar to the proof of (6.32) gives us

$$(G(x), 1)_{L^2(\Omega)} = 0, \ (G(x), x_{\perp, \ k})_{L^2(\Omega)} = 0, \ 1 \le k \le d, \ d = 1, 2. \tag{6.66}$$

By means of the result of Lemma A4 of [20], we obtain $M_0 < \infty$. We express

$$\frac{\widehat{G}_{m,n}(p)}{p^2+n^2} - \frac{\widehat{G}_n(p)}{p^2+n^2}$$

as

$$\frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2+n^2} \chi_{\{p \in \mathbb{R}^d, \ n=0\}} + \frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2+n^2} \chi_{\{p \in \mathbb{R}^d, \ n \in \mathbb{Z}, \ n \ne 0\}}. \tag{6.67}$$

Clearly, the second term in (6.67) can be bounded above in the absolute value by

$$\|\widehat{G}_{m,n}(p) - \widehat{G}_n(p)\|_{L^\infty_{n,p}} \to 0, \quad m \to \infty$$

due to (6.56). Let us write the first term in (6.67) as

$$\frac{\widehat{G}_{m,0}(p) - \widehat{G}_0(p)}{p^2} \chi_{\{|p| \le 1\}} + \frac{\widehat{G}_{m,0}(p) - \widehat{G}_0(p)}{p^2} \chi_{\{|p| > 1\}}. \tag{6.68}$$

Using (6.56), we estimate the second term in (6.68) from above in the norm as

$$\left\| \frac{\widehat{G}_{m,0}(p) - \widehat{G}_0(p)}{p^2} \chi_{\{|p|>1\}} \right\|_{L^\infty_{n,p}} \le \frac{1}{(2\pi)^{\frac{d+1}{2}}} \|G_m(x) - G(x)\|_{L^1(\Omega)} \to 0, \quad m \to \infty$$

as assumed. Let us first study the first term in (6.68) in dimension $d = 1$. By virtue of relations (6.62) and (6.66), we have

$$\widehat{G}_0(0) = 0, \quad \frac{d\widehat{G}_0}{dp}(0) = 0, \quad \widehat{G}_{m,0}(0) = 0, \quad \frac{d\widehat{G}_{m,0}}{dp}(0) = 0, \quad m \in \mathbb{N}.$$

This yields the representations

$$\widehat{G}_0(p) = \int_0^p \left(\int_0^s \frac{d^2\widehat{G}_0(q)}{dq^2} dq \right) ds, \quad \widehat{G}_{m,0}(p) = \int_0^p \left(\int_0^s \frac{d^2\widehat{G}_{m,0}(q)}{dq^2} dq \right) ds, \quad m \in \mathbb{N}.$$

Our definition (6.54) of the Fourier transform easily implies the upper bound

$$\left| \frac{d^2\widehat{G}_{m,0}(p)}{dp^2} - \frac{d^2\widehat{G}_0(p)}{dp^2} \right| \le \frac{1}{2\pi} \|x_\perp^2 G_m(x) - x_\perp^2 G(x)\|_{L^1(\Omega)}.$$

Therefore,

$$|\widehat{G}_{m,0}(p) - \widehat{G}_0(p)| \le \frac{1}{2\pi} \|x_\perp^2 G_m(x) - x_\perp^2 G(x)\|_{L^1(\Omega)} \frac{p^2}{2},$$

such that

$$\left\| \frac{\widehat{G}_{m,0}(p) - \widehat{G}_0(p)}{p^2} \chi_{\{|p| \le 1\}} \right\|_{L^\infty_{n,p}} \le \frac{1}{4\pi} \|x_\perp^2 G_m(x) - x_\perp^2 G(x)\|_{L^1(\Omega)} \to 0, \quad m \to \infty$$

due to one of our assumptions. Finally, we consider the case of the dimension $d = 2$. By virtue of conditions (6.62) and (6.66), we derive

$$\widehat{G}_0(0) = 0, \quad \frac{\partial \widehat{G}_0}{\partial |p|}(0, \sigma) = 0, \quad \widehat{G}_{m,0}(0) = 0, \quad \frac{\partial \widehat{G}_{m,0}}{\partial |p|}(0, \sigma) = 0, \quad m \in \mathbb{N},$$

such that

$$\widehat{G}_0(p) = \int_0^{|p|} \left(\int_0^s \frac{\partial^2 \widehat{G}_0}{\partial q^2}(q, \sigma) dq \right) ds, \quad \widehat{G}_{m,0}(p) = \int_0^{|p|} \left(\int_0^s \frac{\partial^2 \widehat{G}_{m,0}}{\partial q^2}(q, \sigma) dq \right) ds, \quad m \in \mathbb{N}.$$

Using definition (6.54) of the Fourier transform, we arrive at

$$\left| \frac{\partial^2 \widehat{G}_{m,0}}{\partial |p|^2}(|p|, \sigma) - \frac{\partial^2 \widehat{G}_0}{\partial |p|^2}(|p|, \sigma) \right| \le \frac{1}{(2\pi)^{\frac{3}{2}}} \|x_\perp^2 G_m(x) - x_\perp^2 G(x)\|_{L^1(\Omega)}.$$

Hence

$$|\widehat{G}_{m,0}(p) - \widehat{G}_0(p)| \le \frac{1}{(2\pi)^{\frac{3}{2}}} \|x_\perp^2 G_m(x) - x_\perp^2 G(x)\|_{L^1(\Omega)} \frac{p^2}{2}.$$

Therefore, we arrive at

$$\left\| \frac{\widehat{G}_{m,0}(p) - \widehat{G}_0(p)}{p^2} \chi_{\{|p| \le 1\}} \right\|_{L^\infty_{n,p}} \le \frac{1}{2(2\pi)^{\frac{3}{2}}} \|x_\perp^2 G_m(x) - x_\perp^2 G(x)\|_{L^1(\Omega)} \to 0, \quad m \to \infty$$

according to one of our assumptions. Hence

$$\frac{\widehat{G}_{m,n}(p)}{p^2+n^2} \to \frac{\widehat{G}_n(p)}{p^2+n^2}, \quad m\to\infty$$

in $L^\infty_{n,p}$ in dimensions $d=1,2$. Using the triangle inequality, we easily obtain (6.64). A trivial limiting argument yields (6.65). ■

Next we turn our attention to the cases when the parameter a does not vanish.

Lemma A5. *Let the conditions of Theorem 3 hold, $a=n_0^2$, $n_0\in\mathbb{N}$ and for all $m\in\mathbb{N}$*

$$\left(G_m(x_1,x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{\pm i\sqrt{n_0^2-n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad |n|\le n_0-1, \quad d=1, \tag{6.69}$$

$$\left(G_m(x_1,x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0, \quad p\in S^2_{\sqrt{n_0^2-n^2}} \quad a.e., \quad |n|\le n_0-1, \quad d=2, \tag{6.70}$$

$$\left(G_m(x_1,x_\perp), \frac{e^{\pm in_0x_1}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad \left(G_m(x_1,x_\perp), \frac{e^{\pm in_0x_1}}{\sqrt{2\pi}}x_{\perp,\,k}\right)_{L^2(\Omega)} = 0, \quad 1\le k\le d. \tag{6.71}$$

Then

$$\xi^{n_0^2}_{m,n}(p)\to\xi^{n_0^2}_n(p), \quad (p^2+n^2)\xi^{n_0^2}_{m,n}(p)\to(p^2+n^2)\xi^{n_0^2}_n(p), \quad m\to\infty \tag{6.72}$$

in $L^\infty_{n,p}$, such that when $m\to\infty$

$$\|\xi^{n_0^2}_{m,n}(p)\|_{L^\infty_{n,p}}\to\|\xi^{n_0^2}_n(p)\|_{L^\infty_{n,p}}, \quad \|(p^2+n^2)\xi^{n_0^2}_{m,n}(p)\|_{L^\infty_{n,p}}\to\|(p^2+n^2)\xi^{n_0^2}_n(p)\|_{L^\infty_{n,p}}. \tag{6.73}$$

Moreover,

$$\sqrt{2}(2\pi)^{\frac{d+1}{2}}M_{n_0^2}l\le 1-\varepsilon \tag{6.74}$$

holds.

Proof. Apparently, (6.73) will follow from (6.72) by virtue of the standard triangle inequality. Let us prove that the first statement in (6.72) will yield the second one. Indeed, it can be trivially shown that

$$(p^2+n^2)\xi^{n_0^2}_{m,n}(p)-(p^2+n^2)\xi^{n_0^2}_n(p)=[\widehat{G}_{m,n}(p)-\widehat{G}_n(p)]+n_0^2[\xi^{n_0^2}_{m,n}(p)-\xi^{n_0^2}_n(p)],$$

such that due to (6.56)

$$\|(p^2+n^2)\xi^{n_0^2}_{m,n}(p)-(p^2+n^2)\xi^{n_0^2}_n(p)\|_{L^\infty_{n,p}}\le\|\widehat{G}_{m,n}(p)-\widehat{G}_n(p)\|_{L^\infty_{n,p}}+n_0^2\|\xi^{n_0^2}_{m,n}(p)-\xi^{n_0^2}_n(p)\|_{L^\infty_{n,p}}\to 0,$$

$m\to\infty$. By means of the elementary limiting argument, similarly to the proof of (6.66), we derive

$$\left(G(x_1,x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{\pm i\sqrt{n_0^2-n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad |n|\le n_0-1, \quad d=1, \tag{6.75}$$

$$\left(G(x_1,x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0, \quad p\in S^2_{\sqrt{n_0^2-n^2}} \quad a.e., \quad |n|\le n_0-1, \quad d=2, \tag{6.76}$$

$$\left(G(x_1,x_\perp), \frac{e^{\pm in_0x_1}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad \left(G(x_1,x_\perp), \frac{e^{\pm in_0x_1}}{\sqrt{2\pi}}x_{\perp,\ k}\right)_{L^2(\Omega)} = 0, \quad 1\le k\le d. \tag{6.77}$$

Note that it can be easily checked that

$$x_\perp G_m(x) \to x_\perp G(x), \quad m\to\infty \tag{6.78}$$

in $L^1(\Omega)$ under the given conditions. Indeed, $\|x_\perp G_m(x) - x_\perp G(x)\|_{L^1(\Omega)}$ can be expressed as

$$\int_0^{2\pi} dx_1 \int_{|x_\perp|\le 1} |x_\perp||G_m(x)-G(x)|dx_\perp + \int_0^{2\pi} dx_1 \int_{|x_\perp|>1} |x_\perp||G_m(x)-G(x)|dx_\perp \le$$

$$\le \|G_m(x)-G(x)\|_{L^1(\Omega)} + \|x_\perp^2 G_m(x) - x_\perp^2 G(x)\|_{L^1(\Omega)} \to 0, \quad m\to\infty$$

as assumed. By the similar reasoning, it can be trivially shown that

$$\|x_\perp G_m(x)\|_{L^1(\Omega)} \le \|G_m(x)\|_{L^1(\Omega)} + \|x_\perp^2 G_m(x)\|_{L^1(\Omega)} < \infty, \quad m\in\mathbb{N}$$

due to our assumptions, such that $x_\perp G_m(x), x_\perp G(x) \in L^1(\Omega)$. By virtue of the result of Lemma A5 of [20], we obtain $M_{n_0^2} < \infty$. Inequality (6.74) can be easily established via a limiting argument.

Let us use the representation of $\xi_{m,n}^{n_0^2}(p) - \xi_n^{n_0^2}(p),\ m\in\mathbb{N},\ n\in\mathbb{Z},\ p\in\mathbb{R}^d$ as the sum

$$[\xi_{m,n}^{n_0^2}(p) - \xi_n^{n_0^2}(p)]\chi_{\{p\in\mathbb{R}^d,\ |n|>n_0\}} + [\xi_{m,n}^{n_0^2}(p) - \xi_n^{n_0^2}(p)]\chi_{\{p\in\mathbb{R}^d,\ |n|<n_0\}} +$$

$$+[\xi_{m,n}^{n_0^2}(p) - \xi_n^{n_0^2}(p)]\chi_{\{p\in\mathbb{R}^d,\ n=n_0\}} + [\xi_{m,n}^{n_0^2}(p) - \xi_n^{n_0^2}(p)]\chi_{\{p\in\mathbb{R}^d,\ n=-n_0\}}. \tag{6.79}$$

Evidently, the first term in (6.79) can be estimated from above in the absolute value by

$$\|\widehat{G}_{m,n}(p) - \widehat{G}_n(p)\|_{L^\infty_{n,p}} \to 0, \quad m\to\infty$$

by virtue of (6.56). Let us first study the third term in (6.79) in dimension $d=1$. Orthogonality conditons (6.71) and (6.77) imply that

$$\widehat{G}_{n_0}(0)=0, \quad \frac{d\widehat{G}_{n_0}}{dp}(0)=0, \quad \widehat{G}_{m,n_0}(0)=0, \quad \frac{d\widehat{G}_{m,n_0}}{dp}(0)=0, \quad m\in\mathbb{N},$$

such that

$$\widehat{G}_{n_0}(p) = \int_0^p \left(\int_0^s \frac{d^2\widehat{G}_{n_0}(q)}{dq^2}dq\right)ds, \quad \widehat{G}_{m,n_0}(p) = \int_0^p \left(\int_0^s \frac{d^2\widehat{G}_{m,n_0}(q)}{dq^2}dq\right)ds, \quad m\in\mathbb{N}.$$

Definition (6.54) of our Fourier transform easily yields the upper bound

$$\left|\frac{d^2\widehat{G}_{m,n_0}(p)}{dp^2} - \frac{d^2\widehat{G}_{n_0}(p)}{dp^2}\right| \le \frac{1}{2\pi}\|x_\perp^2 G_m(x) - x_\perp^2 G(x)\|_{L^1(\Omega)}.$$

Hence, we obtain

$$|\widehat{G}_{m,n_0}(p)-\widehat{G}_{n_0}(p)|\le \frac{p^2}{4\pi}\|x_{\perp}^2G_m(x)-x_{\perp}^2G(x)\|_{L^1(\Omega)}.$$

Therefore, when the dimension $d=1$, the third term in (6.79) can be bounded from above in the absolute value by

$$\frac{1}{4\pi}\|x_{\perp}^2G_m(x)-x_{\perp}^2G(x)\|_{L^1(\Omega)}\to 0,\quad m\to\infty$$

due to one of our assumptions. Then we turn our attention to the analogous estimates on the third term in (6.79) in dimension $d=2$. By means of orthogonality conditions (6.71) and (6.77), we arrive at

$$\widehat{G}_{n_0}(0)=0,\quad \frac{\partial \widehat{G}_{n_0}}{\partial |p|}(0,\sigma)=0,\quad \widehat{G}_{m,n_0}(0)=0,\quad \frac{\partial \widehat{G}_{m,n_0}}{\partial |p|}(0,\sigma)=0,\quad m\in\mathbb{N}.$$

This yields the representations

$$\widehat{G}_{n_0}(p)=\int_0^{|p|}\Big(\int_0^s\frac{\partial^2\widehat{G}_{n_0}(q,\sigma)}{\partial q^2}dq\Big)ds,\quad \widehat{G}_{m,n_0}(p)=\int_0^{|p|}\Big(\int_0^s\frac{\partial^2\widehat{G}_{m,n_0}(q,\sigma)}{\partial q^2}dq\Big)ds,$$

with $m\in\mathbb{N}$. By virtue of our definition of the Fourier transform (6.54), we easily estimate

$$\left|\frac{\partial^2\widehat{G}_{m,n_0}(|p|,\sigma)}{\partial |p|^2}-\frac{\partial^2\widehat{G}_{n_0}(|p|,\sigma)}{\partial |p|^2}\right|\le\frac{1}{(2\pi)^{\frac{3}{2}}}\|x_{\perp}^2G_m(x)-x_{\perp}^2G(x)\|_{L^1(\Omega)}.$$

This enables us to obtain the upper bound

$$|\widehat{G}_{m,n_0}(p)-\widehat{G}_{n_0}(p)|\le\frac{1}{(2\pi)^{\frac{3}{2}}}\|x_{\perp}^2G_m(x)-x_{\perp}^2G(x)\|_{L^1(\Omega)}\frac{p^2}{2}.$$

Therefore, the third term in (6.79) in dimension $d=2$ can be estimated from above in the absolute value by

$$\frac{1}{2(2\pi)^{\frac{3}{2}}}\|x_{\perp}^2G_m(x)-x_{\perp}^2G(x)\|_{L^1(\Omega)}\to 0,\quad m\to\infty$$

as assumed. Then we turn our attention to the studies of the fourth term of (6.79), first in dimension $d=1$. Orthogonality conditions (6.71) and (6.77) enable us to obtain

$$\widehat{G}_{-n_0}(0)=0,\quad \frac{d\widehat{G}_{-n_0}}{dp}(0)=0,\quad \widehat{G}_{m,-n_0}(0)=0,\quad \frac{d\widehat{G}_{m,-n_0}}{dp}(0)=0,\quad m\in\mathbb{N}.$$

Thus

$$\widehat{G}_{-n_0}(p)=\int_0^p\Big(\int_0^s\frac{d^2\widehat{G}_{-n_0}(q)}{dq^2}dq\Big)ds,\quad \widehat{G}_{m,-n_0}(p)=\int_0^p\Big(\int_0^s\frac{d^2\widehat{G}_{m,-n_0}(q)}{dq^2}dq\Big)ds,$$

with $m \in \mathbb{N}$. Definition (6.54) of our Fourier transform easily implies the estimate from above

$$\left| \frac{d^2 \widehat{G}_{m,-n_0}(p)}{dp^2} - \frac{d^2 \widehat{G}_{-n_0}(p)}{dp^2} \right| \le \frac{1}{2\pi} \| x_\perp^2 G_m(x) - x_\perp^2 G(x) \|_{L^1(\Omega)}.$$

Thus, we derive

$$|\widehat{G}_{m,-n_0}(p) - \widehat{G}_{-n_0}(p)| \le \frac{p^2}{4\pi} \| x_\perp^2 G_m(x) - x_\perp^2 G(x) \|_{L^1(\Omega)}.$$

Hence, when the dimension $d = 1$, the fourth term in (6.79) can be estimated from above in the absolute value by

$$\frac{1}{4\pi} \| x_\perp^2 G_m(x) - x_\perp^2 G(x) \|_{L^1(\Omega)} \to 0, \quad m \to \infty$$

via one of our assumptions. Then we perform the similar estimates on the fourth term in (6.79) when the dimension $d = 2$. By virtue of orthogonality relations (6.71) and (6.77), we derive

$$\widehat{G}_{-n_0}(0) = 0, \quad \frac{\partial \widehat{G}_{-n_0}}{\partial |p|}(0, \sigma) = 0, \quad \widehat{G}_{m,-n_0}(0) = 0, \quad \frac{\partial \widehat{G}_{m,-n_0}}{\partial |p|}(0, \sigma) = 0, \quad m \in \mathbb{N}.$$

This gives us the representations

$$\widehat{G}_{-n_0}(p) = \int_0^{|p|} \left(\int_0^s \frac{\partial^2 \widehat{G}_{-n_0}(q, \sigma)}{\partial q^2} dq \right) ds, \quad \widehat{G}_{m,-n_0}(p) = \int_0^{|p|} \left(\int_0^s \frac{\partial^2 \widehat{G}_{m,-n_0}(q, \sigma)}{\partial q^2} dq \right) ds,$$

with $m \in \mathbb{N}$. By means of our definition of the Fourier transform (6.54), we easily estimate

$$\left| \frac{\partial^2 \widehat{G}_{m,-n_0}(|p|, \sigma)}{\partial |p|^2} - \frac{\partial^2 \widehat{G}_{-n_0}(|p|, \sigma)}{\partial |p|^2} \right| \le \frac{1}{(2\pi)^{\frac{3}{2}}} \| x_\perp^2 G_m(x) - x_\perp^2 G(x) \|_{L^1(\Omega)}.$$

This allows us to derive the upper bound

$$|\widehat{G}_{m,-n_0}(p) - \widehat{G}_{-n_0}(p)| \le \frac{1}{(2\pi)^{\frac{3}{2}}} \| x_\perp^2 G_m(x) - x_\perp^2 G(x) \|_{L^1(\Omega)} \frac{p^2}{2}.$$

Hence, the fourth term in (6.79) in dimension $d = 2$ can be bounded from above in the absolute value by

$$\frac{1}{2(2\pi)^{\frac{3}{2}}} \| x_\perp^2 G_m(x) - x_\perp^2 G(x) \|_{L^1(\Omega)} \to 0, \quad m \to \infty$$

due to one of the assumptions. Finally, it remains to treat the second term in (6.79). Let us consider the situation when the dimension $d = 1$ first. With a slight abuse of notations, for all $|n| \le n_0 - 1$, we introduce $0 < \gamma < \sqrt{n_0^2 - n^2}$ and define the intervals on the real line

$$I_{n,\gamma}^+ := \left[\sqrt{n_0^2 - n^2} - \gamma, \sqrt{n_0^2 - n^2} + \gamma \right], \quad I_{n,\gamma}^- := \left[-\sqrt{n_0^2 - n^2} - \gamma, -\sqrt{n_0^2 - n^2} + \gamma \right],$$

such that $I^c_{n,\gamma}$ will stand for the complement of $I^+_{n,\gamma} \cup I^-_{n,\gamma}$. Let us first consider the term

$$\frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2 - (n_0^2 - n^2)} \chi_{I^+_{n,\gamma}}, \quad |n| \leq n_0 - 1. \tag{6.80}$$

Orthogonality relations (6.69) and (6.75) imply for $|n| \leq n_0 - 1$ that

$$\widehat{G}_n\left(\sqrt{n_0^2 - n^2}\right) = 0, \quad \widehat{G}_{m,n}\left(\sqrt{n_0^2 - n^2}\right) = 0, \quad m \in \mathbb{N},$$

such that we can write

$$\widehat{G}_n(p) = \int_{\sqrt{n_0^2 - n^2}}^p \frac{d\widehat{G}_n(s)}{ds} ds, \quad \widehat{G}_{m,n}(p) = \int_{\sqrt{n_0^2 - n^2}}^p \frac{d\widehat{G}_{m,n}(s)}{ds} ds, \quad m \in \mathbb{N}, \quad |n| \leq n_0 - 1.$$

By virtue of definition (6.54) of our Fourier transform, we obtain

$$\left|\frac{d\widehat{G}_{m,n}(p)}{dp} - \frac{d\widehat{G}_n(p)}{dp}\right| \leq \frac{1}{2\pi} \|x_\perp G_m(x) - x_\perp G(x)\|_{L^1(\Omega)}, \tag{6.81}$$

such that

$$|\widehat{G}_{m,n}(p) - \widehat{G}_n(p)| \leq \frac{1}{2\pi} \|x_\perp G_m(x) - x_\perp G(x)\|_{L^1(\Omega)} \left|p - \sqrt{n_0^2 - n^2}\right|, \quad m \in \mathbb{N}, \quad |n| \leq n_0 - 1.$$

This enables us to estimate (6.80) in the absolute value from above by

$$\frac{\|x_\perp G_m(x) - x_\perp G(x)\|_{L^1(\Omega)}}{2\pi(2\sqrt{2n_0 - 1} - \gamma)} \to 0, \quad m \to \infty$$

due to (6.78) with $0 < \gamma < \sqrt{2n_0 - 1}$. Similarly, we treat the term

$$\frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2 - (n_0^2 - n^2)} \chi_{I^-_{n,\gamma}}, \quad |n| \leq n_0 - 1. \tag{6.82}$$

Orthogonality conditions (6.69) and (6.75) give us for $|n| \leq n_0 - 1$ that

$$\widehat{G}_n\left(-\sqrt{n_0^2 - n^2}\right) = 0, \quad \widehat{G}_{m,n}\left(-\sqrt{n_0^2 - n^2}\right) = 0, \quad m \in \mathbb{N}.$$

This enables us to express

$$\widehat{G}_n(p) = \int_{-\sqrt{n_0^2 - n^2}}^p \frac{d\widehat{G}_n(s)}{ds} ds, \quad \widehat{G}_{m,n}(p) = \int_{-\sqrt{n_0^2 - n^2}}^p \frac{d\widehat{G}_{m,n}(s)}{ds} ds, \quad m \in \mathbb{N}, \quad |n| \leq n_0 - 1$$

and to bound (6.82) in the absolute value from above by

$$\frac{\|x_\perp G_m(x) - x_\perp G(x)\|_{L^1(\Omega)}}{2\pi(2\sqrt{2n_0-1}-\gamma)} \to 0, \quad m \to \infty$$

via (6.78). Finally for the studies in dimension $d = 1$, we consider the term

$$\frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2 - (n_0^2 - n^2)} \chi_{I^c_{n,\gamma}}, \quad |n| \le n_0 - 1. \tag{6.83}$$

Evidently, (6.83) can be estimated from above in the absolute value by

$$\frac{\|\widehat{G}_{m,n}(p) - \widehat{G}_n(p)\|_{L^\infty_{n,p}}}{\gamma^2} \to 0, \quad m \to \infty$$

by means of (6.56). Let us conclude the proof of the lemma with the studies of the second term in (6.79) when the dimension $d = 2$. For $|n| \le n_0 - 1$ we introduce the sets

$$A_{n,\gamma} := \left\{ p \in \mathbb{R}^2 \mid \sqrt{n_0^2 - n^2} - \gamma \le |p| \le \sqrt{n_0^2 - n^2} + \gamma \right\}$$

with $0 < \gamma < \sqrt{2n_0 - 1}$. Let us first analyze the term

$$\frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2 - (n_0^2 - n^2)} \chi_{A^c_{n,\gamma}}, \quad |n| \le n_0 - 1.$$

Evidently, it can be trivially estimated from above in the absolute value by

$$\frac{\|\widehat{G}_{m,n}(p) - \widehat{G}_n(p)\|_{L^\infty_{n,p}}}{\gamma\sqrt{2n_0 - 1}} \to 0, \quad m \to \infty$$

due to (6.56). At last, let us treat the term

$$\frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2 - (n_0^2 - n^2)} \chi_{A_{n,\gamma}}, \quad |n| \le n_0 - 1. \tag{6.84}$$

Orthogonality conditions (6.70) and (6.76) imply that for $|n| \le n_0 - 1$

$$\widehat{G}_n\left(\sqrt{n_0^2 - n^2}, \sigma\right) = 0, \quad \widehat{G}_{m,n}\left(\sqrt{n_0^2 - n^2}, \sigma\right) = 0, \quad m \in \mathbb{N},$$

which enables us to express

$$\widehat{G}_n(p) = \int_{\sqrt{n_0^2 - n^2}}^{|p|} \frac{\partial \widehat{G}_n(s, \sigma)}{\partial s} ds, \quad \widehat{G}_{m,n}(p) = \int_{\sqrt{n_0^2 - n^2}}^{|p|} \frac{\partial \widehat{G}_{m,n}(s, \sigma)}{\partial s} ds, \quad m \in \mathbb{N},$$

with $|n| \le n_0 - 1$. By virtue of our definition (6.54), we easily arrive at

$$\left| \frac{\partial \widehat{G}_{m,n}(p)}{\partial |p|} - \frac{\partial \widehat{G}_n(p)}{\partial |p|} \right| \le \frac{1}{(2\pi)^{\frac{3}{2}}} \|x_\perp G_m(x) - x_\perp G(x)\|_{L^1(\Omega)}. \tag{6.85}$$

This allows us to estimate expression (6.84) from above in the absolute value by

$$\frac{\|x_\perp G_m(x) - x_\perp G(x)\|_{L^1(\Omega)}}{(2\pi)^{\frac{3}{2}}\sqrt{2n_0 - 1}} \to 0, \quad m \to \infty$$

in dimension $d = 2$ due to (6.78). Therefore, (6.72) holds. ∎

We conclude the article with the studies of the case when the parameter a is located on an open interval between the squares of two consecutive nonnegative integers.

Lemma A6. *Let the assumptions of Theorem 3 hold,* $n_0^2 < a < (n_0+1)^2$, $n_0 \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ *and for all* $m \in \mathbb{N}$

$$\left(G_m(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{\pm i\sqrt{a-n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad |n| \le n_0, \quad d = 1, \tag{6.86}$$

$$\left(G_m(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0, \quad p \in S^2_{\sqrt{a-n^2}} \ a.e., \quad |n| \le n_0, \quad d = 2. \tag{6.87}$$

Then

$$\xi^a_{m,n}(p) \to \xi^a_n(p), \quad (p^2+n^2)\xi^a_{m,n}(p) \to (p^2+n^2)\xi^a_n(p), \quad m \to \infty \tag{6.88}$$

in $L^\infty_{n,p}$, *such that when* $m \to \infty$

$$\|\xi^a_{m,n}(p)\|_{L^\infty_{n,p}} \to \|\xi^a_n(p)\|_{L^\infty_{n,p}}, \quad \|(p^2+n^2)\xi^a_{m,n}(p)\|_{L^\infty_{n,p}} \to \|(p^2+n^2)\xi^a_n(p)\|_{L^\infty_{n,p}}. \tag{6.89}$$

Moreover,

$$\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_a l \le 1 - \varepsilon \tag{6.90}$$

holds.

Proof. Obviously, (6.89) will follow from (6.88) by means of the standard triangle inequality. Let us show that the first statement in (6.88) will imply the second one. Indeed, it can be easily verified that

$$(p^2+n^2)\xi^a_{m,n}(p) - (p^2+n^2)\xi^a_n(p) = [\widehat{G}_{m,n}(p) - \widehat{G}_n(p)] + a[\xi^a_{m,n}(p) - \xi^a_n(p)],$$

such that via (6.56)

$$\|(p^2+n^2)\xi^a_{m,n}(p) - (p^2+n^2)\xi^a_n(p)\|_{L^\infty_{n,p}} \le \|\widehat{G}_{m,n}(p) - \widehat{G}_n(p)\|_{L^\infty_{n,p}} + a\|\xi^a_{m,n}(p) - \xi^a_n(p)\|_{L^\infty_{n,p}} \to 0,$$

$m \to \infty$. By virtue of the trivial limiting argument, similarly to the proof of (6.16), we arrive at

$$\left(G(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{\pm i\sqrt{a-n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad |n| \le n_0, \quad d = 1, \tag{6.91}$$

$$\left(G(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0, \quad p \in S^2_{\sqrt{a-n^2}} \ a.e., \quad |n| \le n_0, \quad d = 2. \tag{6.92}$$

Then by means of the result of Lemma A6 of [20], we have $M_a < \infty$. Let us express $\xi^a_{m,n}(p) - \xi^a_n(p)$ as the sum of two terms

$$\frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2 + n^2 - a} \chi_{\{p\in\mathbb{R}^d, \ n\in\mathbb{Z}, \ |n|\ge n_0+1\}} + \frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2 + n^2 - a} \chi_{\{p\in\mathbb{R}^d, \ n\in\mathbb{Z}, \ |n|\le n_0\}}, \tag{6.93}$$

such that the absolute value of the first one can be estimated from above by

$$\frac{\|\widehat{G}_{m,n}(p) - \widehat{G}_n(p)\|_{L^\infty_{n,p}}}{(n_0+1)^2 - a} \to 0, \quad m \to \infty$$

due to (6.56). Let us first study the second term in (6.93) when the dimension $d = 1$. Orthogonality relations (6.86) and (6.91) with $|n| \le n_0$ yield

$$\widehat{G}_n(\pm\sqrt{a-n^2}) = 0, \quad \widehat{G}_{m,n}(\pm\sqrt{a-n^2}) = 0, \quad m \in \mathbb{N},$$

such that we have the representations for $|n| \le n_0$

$$\widehat{G}_n(p) = \int_{\pm\sqrt{a-n^2}}^{p} \frac{d\widehat{G}_n(s)}{ds} ds, \quad \widehat{G}_{m,n}(p) = \int_{\pm\sqrt{a-n^2}}^{p} \frac{d\widehat{G}_{m,n}(s)}{ds} ds, \quad m \in \mathbb{N}. \tag{6.94}$$

With a slight abuse of notations, let us introduce $0 < \gamma < \sqrt{a-n^2}$ for all $|n| \le n_0$ and related to it the intervals on the real line

$$I^+_{n,\gamma} := [\sqrt{a-n^2} - \gamma, \sqrt{a-n^2} + \gamma], \quad I^-_{n,\gamma} := [-\sqrt{a-n^2} - \gamma, -\sqrt{a-n^2} + \gamma].$$

Their union on the real line will be denoted as $I_{n,\gamma} := I^+_{n,\gamma} \cup I^-_{n,\gamma}$ and the complement is $I^c_{n,\gamma}$, such that $\mathbb{R} = I^+_{n,\gamma} \cup I^-_{n,\gamma} \cup I^c_{n,\gamma}$. Therefore, for $|n| \le n_0$, it remains to study the sum of the three terms

$$\frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2 - (a-n^2)} \chi_{I^+_{n,\gamma}} + \frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2 - (a-n^2)} \chi_{I^-_{n,\gamma}} + \frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2 - (a-n^2)} \chi_{I^c_{n,\gamma}}. \tag{6.95}$$

Evidently, the last term in (6.95) can be easily estimated in the absolute value from above by

$$\frac{\|\widehat{G}_{m,n}(p) - \widehat{G}_n(p)\|_{L^\infty_{n,p}}}{\gamma^2} \to 0, \quad m \to \infty$$

via (6.56). The first term in (6.95) can be bounded from above in the absolute value using (6.94) along with (6.81) by

$$\frac{\|x_\perp G_m(x) - x_\perp G(x)\|_{L^1(\Omega)}}{2\pi(2\sqrt{a-n_0^2}-\gamma)} \to 0, \quad m \to \infty$$

due to one of our assumptions with $0 < \gamma < \sqrt{a-n_0^2}$. Apparently, the second term in (6.95) can be estimated similarly to the first one. Therefore, (6.88) holds when the dimension $d = 1$.

Let us conclude the proof of the lemma by treating the case of $d = 2$. By virtue of orthogonality conditions (6.87) and (6.92), for $|n| \leq n_0$ we have

$$\widehat{G}_n(\sqrt{a-n^2}, \sigma) = 0, \quad \widehat{G}_{m,n}(\sqrt{a-n^2}, \sigma) = 0, \quad m \in \mathbb{N}.$$

This yields the representations for $|n| \leq n_0$

$$\widehat{G}_n(p) = \int_{\sqrt{a-n^2}}^{|p|} \frac{\partial \widehat{G}_n(s,\sigma)}{\partial s} ds, \quad \widehat{G}_{m,n}(p) = \int_{\sqrt{a-n^2}}^{|p|} \frac{\partial \widehat{G}_{m,n}(s,\sigma)}{\partial s} ds, \quad m \in \mathbb{N}. \tag{6.96}$$

With a slight abuse of notations, we introduce $0 < \delta < \sqrt{a-n^2}$ for all $|n| \leq n_0$ and define the sets

$$A_{n,\delta} := \{p \in \mathbb{R}^2 \mid \sqrt{a-n^2} - \delta \leq |p| \leq \sqrt{a-n^2} + \delta\}, \quad |n| \leq n_0.$$

The complement of $A_{n,\delta}$ on the plane is denoted as $A_{n,\delta}^c$. Thus, for $|n| \leq n_0$ it remains to study the sum of the two terms

$$\frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2 - (a-n^2)} \chi_{A_{n,\delta}} + \frac{\widehat{G}_{m,n}(p) - \widehat{G}_n(p)}{p^2 - (a-n^2)} \chi_{A_{n,\delta}^c}. \tag{6.97}$$

Clearly, the second term in (6.97) can be estimated from above in the absolute value by

$$\frac{\|\widehat{G}_{m,n}(p) - \widehat{G}_n(p)\|_{L^\infty_{n,p}}}{\sqrt{a-n_0^2}\,\delta} \to 0, \quad m \to \infty$$

due to (6.56). By virtue of (6.96) along with (6.85), the first term in (6.97) can be bounded from above in the absolute value by

$$\frac{\|x_\perp G_m(x) - x_\perp G(x)\|_{L^1(\Omega)}}{(2\pi)^{\frac{3}{2}}\sqrt{a-n_0^2}} \to 0, \quad m \to \infty$$

as assumed. This proves that (6.88) holds when the dimension $d = 2$ as well. A straightforward limiting argument gives us (6.90). ■

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