Transitions and Anti-integrable Limits for Multi-hole Sturmian Systems and Denjoy Counterexamples

Yi-Chiuan Chen
Institute of Mathematics, Academia Sinica
6th Floor, Astronomy-Mathematics Building
No. 1, Section 4, Roosevelt Road, Taipei 10617, Taiwan
Email: YCChen@math.sinica.edu.tw
TEL: +886 2 23685999 ext. 630; FAX: +886 2 23689771

September 22, 2016

Dedicated to Robert MacKay on the occasion of his 60th birthday

Abstract

For a Denjoy homeomorphism f of the circle S, we call a pair of distinct points of the ω-limit set $\omega(f)$ whose forward and backward orbits converge together a *gap*, and call an orbit of gaps a *hole*. In this paper, we generalise the Sturmian system of Morse and Hedlund and show that the dynamics of any Denjoy minimal set of finite number of holes is conjugate to a generalised Sturmian system. Moreover, for any Denjoy homeomorphism f having a finite number of holes and for any transitive orientation-preserving homeomorphism f_1 of the circle with the same rotation number $\rho(f_1)$ as $\rho(f)$, we construct a family f_ϵ of Denjoy homeomorphisms of rotation number $\rho(f)$ containing f such that $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to $(\omega(f), f)$ for $0 < \epsilon < \tilde{\epsilon} < 1$ but the number of holes changes at $\epsilon = \tilde{\epsilon}$, that $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to $(\omega(f_{\tilde{\epsilon}}), f_{\tilde{\epsilon}})$ for $\tilde{\epsilon} \leq \epsilon < 1$ but $\lim_{\epsilon \nearrow 1} f_\epsilon(t) = f_1(t)$ for any $t \in S$, and that f_ϵ has a singular limit when $\epsilon \searrow 0$. We show this singular limit is an anti-integrable limit in the sense of Aubry. That is, the Denjoy minimal system reduces to a symbolic dynamical system. The anti-integrable limit can be degenerate or non-degenerate. All transitions can be precisely described in terms of the generalised Sturmian systems.

Key words: Denjoy counterexample, Denjoy minimal set, generalised Sturmian system, anti-integrable limit

2010 Mathematics Subject Classification: 37E10, 37B10

1 Introduction

Let $S = \{z \in \mathbb{C} |\, |z| = 1\}$ be the unit circle. We identify S with $\mathbb{R}/\mathbb{Z}$ and have the identification $[0, 1) \ni t$ with $\{z \in \mathbb{C} |\, |z| = 1\} \ni e^{2\pi it}$. We shall freely use the representation of S which is most convenient. Let $\beta \in (0, 1)$ and $R_\beta : S \to S$, $t \mapsto t + \beta \pmod 1$, be the rotation with angle β. Denjoy proved by constructing examples that there exist circle diffeomorphisms which have irrational rotation number β but are not conjugate to R_β. The ω-limit sets of Denjoy's examples are Cantor sets. We refer to any orientation-preserving homeomorphism (OPH) of S with irrational rotation number that is not conjugate to a rotation as a *Denjoy homeomorphism* or *Denjoy counterexample*. When we say two dynamical systems are conjugate or semi-conjugate we means topologically conjugate or topologically semi-conjugate, respectively.

For a monotone twist map of an annulus, the celebrated Aubry-Mather theory tells that one can always find invariant minimal closed subsets of the annulus on which the twist map has irrational rotation numbers. These closed subsets are either Lipschitz circles or Cantor sets on Lipschitz circles. In the latter case, they are also called *cantori* or *Denjoy minimal sets*.

Baesens and MacKay [8, 19] showed that cantori of a given rotation number may form an interval in the vague topology for multiharmonic maps, for example, of the following form

$$x_{i+1} = x_i + y_{i+1} \pmod 1, \tag{1}$$

$$y_{i+1} = y_i - \frac{a}{2\pi} \sin 2\pi x_i - \frac{b}{4\pi} \sin 4\pi x_i, \tag{2}$$

near enough an anti-integrable limit (AI-limit). (For the concept of AI-limit, see e.g. [4, 5, 7, 12, 18].) This is because the cantori may have multiple number of holes. Following their terminology, we call a pair of distinct points of a Denjoy minimal set whose forward and backward orbits converge together a *gap*. The gaps come in orbits. We call an orbit of gaps a *hole*. Aubry calls it a *discontinuity class* [6]. For the two-harmonic family (1) and (2), the cantori depend on parameters a and b. Baesens and MacKay proved that there are parameter regimes such that on passing different regimes there exists a bifurcation in which a one-hole cantorus gains a second hole or there exists an invariant circle to one-hole cantorus transition. See also [9] for numerical demonstration.

The Aubry-Mather theory indicates that an invariant circle breaks by the conjugacy from an irrational rotation becoming discontinuous. For a large class of annulus maps and rotation numbers, the breakup boundary in parameter space is believed to be smooth. But for the two-harmonic family of maps (and for multiharmonic maps in general), the breakup boundary exhibits a fractal structure. Baesens and MacKay [8] believe that this is because the space of cantori of this family of fixed rotation number contains an interval, and thus the breakup boundary is composed of many pieces, each one corresponding to a point in the interval.

Note that for an area-preserving monotone twist map, Mather [22] showed that if there is no invariant circle of a given irrational rotation number β, then there exist uncountably many Denjoy minimal sets of that rotation number. Moreover, as pointed out by Boyland [10], these are n-fold Denjoy minimal sets, i.e. they wraps n-times around the annulus, with average speed β for all n loops with $n \geq 2$. The n-fold Denjoy minimal sets showed by Mather have dimension $n-1$ in the vague topology.

There are circle diffeomorphisms whose Denjoy minimal sets have multiple holes. These diffeomorphisms can be constructed, for example, by "blowing up" points in a multiple number of orbits of R_β, instead of only in one orbit. It is natural and interesting to investigate whether similar bifurcation and transition phenomena studied in [8, 9] also happen in the minimal sets for Denjoy homeomorphisms of the circle. If it does, can one describe the bifurcations or transitions for multi-hole Denjoy minimal sets in terms of symbolic dynamics? More importantly, what is the AI-limit for Denjoy homeomorphisms? Theses questions motivated this paper and are the central issues to be addressed.

In this paper, we generalise the Sturmian system of Morse and Hedlund [23] by coding irrational rotations with respect to an arbitrary finite partition on the circle and show that the dynamics of any Denjoy minimal set of finite number of holes is conjugate to a generalised Sturmian system. Notice that it is known (see e.g. [17]) that the restriction of a one-hole Denjoy homeomorphism to its ω-limit set is conjugate to the restriction of the full two-shift homeomorphism to a closed invariant subset. We call a generalised Sturmian system a *multi-hole Sturmian system*. Moreover, for any Denjoy homeomorphism f having a finite number of holes and for any transitive OPH f_1 of the circle with the same rotation number $\rho(f_1)$ as $\rho(f)$, we construct a one-parameter family f_ϵ of Denjoy homeomorphisms of rotation number $\rho(f)$ having the following properties. The first property is that $f_{\epsilon_0} = f$ and $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to $(\omega(f), f)$ when $0 < \epsilon < \tilde{\epsilon}$ with some $0 < \epsilon_0 < \tilde{\epsilon} < 1$. The second is that the number of holes changes at $\epsilon = \tilde{\epsilon}$, corresponding to a transition of cantorus in which a cantorus gains or loses a certain number of holes, and that $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to $(\omega(f_{\tilde{\epsilon}}), f_{\tilde{\epsilon}})$ when $\tilde{\epsilon} \leq \epsilon < 1$. The third is that $\lim_{\epsilon \nearrow 1} f_\epsilon(t) = f_1(t)$ pointwisely for any $t \in S$, corresponding to the circle to cantorus transition, and that f_ϵ has a singular limit when $\epsilon \searrow 0$. We show that this singular limit is an AI-limit in the sense of Aubry [4, 5]. That is, the Denjoy minimal set collapses to a set of finite point and the Denjoy minimal system reduces to a symbolic dynamical system. The AI-limit can be degenerate or non-degenerate. All transitions can be precisely described in terms of multi-hole Sturmian sequences.

Roughly speaking, near an AI-limit, $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to a multi-hole Sturmian system but reduces to a factor of that multi-hole Sturmian system at the AI-limit. (When we say a dynamical system is a factor of another, we mean a topological factor.) For instance, suppose that f_ϵ is constructed by blowing up orbit points $R_\beta^n(0)$ of the origin into wandering intervals $I_n^{(1)}$ and the length $|I_n^{(1)}|$ of $I_n^{(1)}$ depends on ϵ for every $n \in \mathbb{Z}$. Then, an AI-limit will correspond to the limit $|I_n^{(1)}| \to 0$ for all n except $|I_1^{(1)}| \to 1$ as $\epsilon \to 0$.

Analogously, if f_ϵ is obtained by blowing up points of the two orbits $\{R_\beta^n(0) |\, n \in \mathbb{Z}\}$ and $\{R_\beta^n(1/2) |\, n \in \mathbb{Z}\}$ into wandering intervals $I_n^{(1)}$ and $I_n^{(2)}$, respectively. Again, if the lengths of these intervals depend on ϵ. Then, a two to one-hole transition of cantorus will occur at $\epsilon = \tilde{\epsilon}$ provided that the length of $I_n^{(2)}$ shrinks to zero (the gap corresponding to the boundary of $I_n^{(2)}$ is annihilated) for every n when $\epsilon = \tilde{\epsilon}$ but the union $\bigcup_{n\in\mathbb{Z}} I_n^{(1)}$ remains constituting the wandering intervals.

The rest of this paper is organized as follows. In the next section, we briefly review fundamental properties of Denjoy homeomorphisms. Before describing a way to code symbolically a Denjoy minimal set in Section 4, we establish in Section 3 the multi-hole Sturmian systems that code irrational rotations with respect to arbitrary partitions on the circle. Section 5 is devoted to the transitions and AI-limits of Denjoy minimal systems. The transitions and AI-limits will be described in terms of quotients of multi-hole Sturmian systems. We postpone all proofs of our theorems until the final section.

2 Denjoy counterexample

The purpose of this section is twofold. On the one hand, it provides a brief review of well-known facts about Denjoy counterexamples. (For a detailed account, the reader may refer to [13, 17, 25], for instance.) On the other hand, it introduces our assumption on the Denjoy minimal sets to be studied.

If f is a homeomorphism of S having $\omega(f)$ a Cantor set, then f is semi-conjugate to R_β for some irrational number β. In other words, f has R_β as a factor. More precisely, there is a unique (up to a rotation) continuous non-decreasing surjection h of degree one such that the diagram below commutes:

$$\begin{array}{ccc} S & \xrightarrow{f} & S \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ S & \xrightarrow{R_\beta} & S. \end{array} \tag{*}$$

The image of $\omega(f)$ under h is S. The complement $S \setminus \omega(f) = \bigcup_{n\in\mathbb{Z}} I_n$ consists of countable pairwise disjoint open sets I_n, which are invariant under f and for which $h(I_n)$ is a single point for every n. Thus, $h(S \setminus \omega(f))$ is a countable invariant set of R_β. The semi-conjugacy h is one-to-one on $S \setminus \bigcup_{n\in\mathbb{Z}} \operatorname{cl} I_n$, where $\operatorname{cl} I_n$ denotes the closure of I_n.

The topological classification of Denjoy homeomorphisms with a given irrational rotation number β is due to Markley given by a finite or countable collection of orbits of the rotation R_β up to a simultaneous translation of all these orbits. For a Denjoy homeomorphism, define

$$\mathcal{D}(f) := h(\bigcup_{n\in\mathbb{Z}} \operatorname{cl} I_n).$$

We call the number of disjoint orbits of $\mathcal{D}(f)$ under R_β the *number of holes* of $\omega(f)$. The number of holes of a Denjoy minimal set is at least one, and may be infinite. Markley [20] proved the following.

Theorem 2.1 (Markley 1970)**.** *A Denjoy homeomorphism f is semi-conjugate to another $\tilde{f}$ via an orientation-preserving surjection if and only if they have the same rotation number and*

$$\mathcal{D}(\tilde{f}) \subseteq R_\alpha\left(\mathcal{D}(f)\right) \tag{3}$$

for some $0 \le \alpha < 1$. The surjection is a homeomorphism if and only if equality holds in (3)

On the other way round, for any countable R_β-invariant subset $D \subset S$, one can choose pairwise disjoint open intervals I_d, $d \in D$, which have the same cyclic ordering as points in D and whose union is dense in S. Then there is a continuous surjection h of S such that $h^{-1}(d) = \operatorname{cl} I_d$ for all $d \in D$ and which is one-to-one on $h^{-1}(S \setminus D)$. Moreover, one can construct a homeomorphism f of S with rotation number β so that h satisfies (*) and that $S \setminus \bigcup_{d\in D} I_d$ is a Cantor set and is the unique minimal invariant set (equal to $\omega(f)$) under f.

Given a Cantor set $\mathcal{C}$ on S, the complement of $\mathcal{C}$ can be described as $S\setminus\mathcal{C} = \bigcup_{n\in\mathbb{Z}} I_n$ where I_n's are pairwise disjoint open sets. A continuous function $P : S \to S$ is called a *Cantor function* associated to $\mathcal{C}$ provided that

$$P(x) = P(y) \quad \Longleftrightarrow \quad x = y \text{ or } x, y \in \operatorname{cl} I_n$$

for some $n \in \mathbb{Z}$ and that the function collapses every I_n into a point in such a way that the cyclic ordering is preserved. Therefore, the semi-conjugacy h satisfying (*) is a Cantor function. It is worth noticing that for every given Cantor set $\mathcal{C} \subset S$, irrational $\beta \in (0, 1)$, and a countable R_β-invariant subset $D \subset S$, there is a Denjoy homeomorphism f with rotation number β, $\mathcal{D}(f) = D$, and $\omega(f) = \mathcal{C}$. This is because the components of $S \setminus \mathcal{C}$ and the set D can be put in a one-to-one correspondence which preserves the cyclic ordering.

Let X be a topological space and f an invertible map of X. Denote the orbit of a point $x \in X$ under the iteration of f by $\mathcal{O}(x; f) := \{f^n(x) |\, n \in \mathbb{Z}\}$. If Y is a subset of X, let $\mathcal{O}(Y; f) := \bigcup_{x\in Y} \mathcal{O}(x; f)$.

Denote by $\rho(f)$ the rotation number of a homeomorphism f of the unit circle S. If two OPHs $\tilde{f}$ and f of the circle are conjugate by an orientation preserving (resp. reversing) homeomorphism, then $\rho(\tilde{f}) = \rho(f)$ (resp. $\rho(\tilde{f}) = -\rho(f) \mod 1$). Note that R_β and $R_{-\beta}$ are conjugate via reflection $t \mapsto -t \mod 1$. (The orbit $\mathcal{O}(t; R_\beta)$ of a point t under R_β is identical to the one $\mathcal{O}(t; R_{1-\beta}^{-1})$ under inverse iteration of $R_{1-\beta}$.) In fact, two Denjoy homeomorphisms $\tilde{f}$ and f are conjugate via an orientation-reversing conjugacy if and only if $\rho(\tilde{f}) = 1-\rho(f)$ and $\mathcal{D}(\tilde{f}) = 1-R_\alpha(\mathcal{D}(f))$ for some $0 \le \alpha < 1$ [20]. For these reasons, in this paper we concentrate on those Denjoy homeomorphisms of rotation number less than $1/2$.

If the number of holes is finite for a Denjoy minimal set, without loss of generality, we make the following assumption throughout this paper.

Assumption A. Let f be a Denjoy homeomorphism. Assume that the number of holes of $\omega(f)$ is finite and equal to some integer $K \ge 1$. Assume that

- $0 < \rho(f) = \beta < 1/2$,
- there is a set $\Theta = \{\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(K)}\}$ of K points, $0 = \theta^{(1)} < \theta^{(2)} < \dots < \theta^{(K)} < 1$, with $\mathcal{O}(\theta^{(i)}; R_\beta) \cap \mathcal{O}(\theta^{(j)}; R_\beta) = \emptyset$ for all $1 \le i < j \le K$,
- there are open sets $I_n^{(k)}$, $n \in \mathbb{Z}$, $1 \le k \le K$, such that

$$\begin{aligned} h^{-1}(\theta^{(k)}) &= \operatorname{cl} I_0^{(k)} \\ &= [a_0^{(k)}, b_0^{(k)}], \\ f^n(I_0^{(k)}) &= I_n^{(k)} \\ &= (a_n^{(k)}, b_n^{(k)}), \\ \overline{\bigcup_{1\le k\le K} \bigcup_{n\in\mathbb{Z}} I_n^{(k)}} &= S, \end{aligned}$$

where h is the semi-conjugacy satisfying (*), $a_n^{(k)}, b_n^{(k)}$ are points in $\omega(f)$, and $(a_n^{(k)}, b_n^{(k)})$ and $[a_n^{(k)}, b_n^{(k)}]$ denote the open and closed (anti-clockwise) intervals from $a_n^{(k)}$ to $b_n^{(k)}$ in S. We assume $0 \in I_0^{(1)}$.

Remark 2.2.

(i) We call the minimal system $(\omega(f), f)$ in Assumption A a *K-hole Denjoy minimal system*.

(ii) If a Denjoy homeomorphism f satisfies Assumption A, then $\mathcal{D}(f) = \mathcal{O}(\Theta; R_\beta)$.

Clearly, $f^n(a_0^{(k)}) = a_n^{(k)}$, $f^n(b_0^{(k)}) = b_n^{(k)}$, and $\lim_{|n|\to\infty} |f^n(a_0^{(k)}) - f^n(b_0^{(k)})| = 0$. Therefore, the pair of points $a_n^{(k)}$ and $b_n^{(k)}$ is a gap. Define the following equivalence relation on $\omega(f)$: For points $x, y \in \omega(f)$, and a subset $\widehat{\Theta} \subseteq \Theta$, we say

$$x \sim_{\widehat{\Theta}} y \tag{4}$$

if $\lim_{|n|\to\infty} |f^n(x) - f^n(y)| = 0$, $x, y \in \left\{\mathcal{O}(a_0^{(k)}; f), \mathcal{O}(b_0^{(k)}; f)\right\}$, and if $\theta^{(k)} \in \widehat{\Theta}$. Note that $x \sim_\Theta y$ if $\lim_{|n|\to\infty} |f^n(x) - f^n(y)| = 0$ or equivalent if $h(x) = h(y)$. It is necessary that $x \sim_{\{\theta\}} y$ for some $\theta \in \widehat{\Theta}$ if $x \sim_{\widehat{\Theta}} y$. Hence, two distinct points x and y in $\omega(f)$ form a gap if and only if $x \sim_\Theta y$ or if and only if $h(x) = h(y)$.

For the sake of convenience of notation, in the sequel, we use (Y, f) to denote the restriction $f|_Y$ of a continuous map f of a topological space X to an invariant subset $Y \subseteq X$. Also, we use $(Y, f)/\sim$ instead of $(Y/\sim, f_\sim)$ to denote dynamical system of the induced map $f_\sim$ of f of the quotient of Y by an equivalence relation $\sim$ on Y.

The following is well-known.

Theorem 2.3. *Let f be a Denjoy homeomorphism satisfying Assumption A. The quotient space $\omega(f)/\sim_\Theta$ of $\omega(f)$ by the equivalence relation $\sim_\Theta$ is homeomorphic to S. The quotient dynamics $(\omega(f), f)/\sim_\Theta$ is conjugate to (S, R_β).*

3 Coding of irrational rotation

First, we describe a way to characterize symbolic codes of a irrational rotation of the unit circle S. It is a generalisation of Morse and Hedlund's construction of Sturmian sequences in 1940 [23]. Given irrational $\beta \in (0, 1/2)$ and $t \in S$, we investigate the coding of the orbit $\mathcal{O}(t; R_\beta)$ in this section.

Let $Q \subset S$ be a finite set of real numbers having cardinality $N \geq 2$. Suppose $Q = \{q_1, q_2, \ldots, q_N\}$ with the ordering

$$0 = q_1 < q_2 < \ldots < q_N < 1$$

is a set of N consecutive points on S. We call such a finite set Q a *partition set* or a *set of partition points* on the circle S. Partition S into N number of intervals:

$$\begin{aligned} J_1^+ &= [0, q_2) \\ J_2^+ &= [q_2, q_3) \\ &\vdots \\ J_{N-1}^+ &= [q_{N-1}, q_N) \\ J_N^+ &= [q_N, 1), \end{aligned} \qquad \text{or} \qquad \begin{aligned} J_1^- &= (0, q_2] \\ J_2^- &= (q_2, q_3] \\ &\vdots \\ J_{N-1}^- &= (q_{N-1}, q_N] \\ J_N^- &= (q_N, 1]. \end{aligned}$$

Denote by $\sharp(Q)$ the cardinality of Q. Given a partition set Q, we define $\Phi = \{\phi_1, \phi_2, \ldots, \phi_N\}$ to be a finite real number set of the same cardinality as Q, $\sharp(\Phi) = \sharp(Q) = N$, with the ordering

$$0 \leq \phi_1 < \phi_2 < \ldots < \phi_N \leq 1.$$

Associated with the rotation R_β, define two maps $\boldsymbol{\nu}^+(\cdot; \beta, Q, \Phi)$ and $\boldsymbol{\nu}^-(\cdot; \beta, Q, \Phi)$ from the circle S to the product space $\Phi^{\mathbb{Z}}$,

$$\begin{aligned} \boldsymbol{\nu}^+(t; \beta, Q, \Phi) &= (\cdots, \boldsymbol{\nu}^+(t; \beta, Q, \Phi)_{-1}, \boldsymbol{\nu}^+(t; \beta, Q, \Phi)_0, \boldsymbol{\nu}^+(t; \beta, Q, \Phi)_1, \cdots), \\ \boldsymbol{\nu}^-(t; \beta, Q, \Phi) &= (\cdots, \boldsymbol{\nu}^-(t; \beta, Q, \Phi)_{-1}, \boldsymbol{\nu}^-(t; \beta, Q, \Phi)_0, \boldsymbol{\nu}^-(t; \beta, Q, \Phi)_1, \cdots), \end{aligned}$$

by

$$\boldsymbol{\nu}^{\pm}(t;\beta,Q,\Phi)_n = \phi_i \qquad \text{if } R_\beta^n(t) \in J_i^{\pm} \text{ for } 1 \le i \le N,\ n \in \mathbb{Z}.$$

In other words, $\boldsymbol{\nu}^{\pm}(t;\beta,Q,\Phi)$ give the *itinerary sequences* of the orbit of t under R_β with respect to the partition Q. We call such a finite set Φ a *symbol set* or a *set of symbols*, and call (Q,Φ) a *partition-symbol pair* or a *pair of partition and symbol sets*.

Endow the finite set Φ with the discrete topology, and the set of sequences $\mathbf{u} = (\ldots,u_{-1},u_0,u_1,\ldots) \in \Phi^{\mathbb{Z}}$ with the product topology. Define a set $X_{\beta,Q,\Phi}$ by

$$X_{\beta,Q,\Phi} := \bigcup_{t\in S} \left(\boldsymbol{\nu}^{-}(t;\beta,Q,\Phi) \cup \boldsymbol{\nu}^{+}(t;\beta,Q,\Phi)\right). \tag{5}$$

Let $\sigma = \sigma_N : \Phi^{\mathbb{Z}} \to \Phi^{\mathbb{Z}}$, $(u_i)_{i\in\mathbb{Z}} \mapsto (v_i)_{i\in\mathbb{Z}}$ with $v_i = u_{i+1}$, be the usual shift automorphism. We call the subshift $(X_{\beta,Q,\Phi},\sigma)$ of $(\Phi^{\mathbb{Z}},\sigma)$ an *N-symbol Sturmian system of partition points Q with symbols Φ and rotation number β* (where $N = \sharp(Q) = \sharp(\Phi)$). For the sake of simplicity, $(\Phi^{\mathbb{Z}},\sigma)$ instead of $(\Phi^{\mathbb{Z}},\sigma_N)$ is used in the rest of this paper provided no ambiguity is caused.

A sequence $\mathbf{u} \in \Phi^{\mathbb{Z}}$ is called a *rotation sequence of partition Q* with irrational rotation number $\beta \in (0,1/2)$ if there exists $t \in S$ such that either $\boldsymbol{\nu}^{+}(t;\beta,Q,\Phi) = \mathbf{u}$ or $\boldsymbol{\nu}^{-}(t;\beta,Q,\Phi) = \mathbf{u}$. A sequence $\mathbf{u} \in \Phi^{\mathbb{Z}}$ is called a *rotation sequence* if it is a rotation sequence of some partition with some rotation number.

By the definition above, a rotation sequence of partition $\{0,\beta\}$ or $\{0,1-\beta\}$ with irrational rotation number β gives rise to a *Sturmian sequence*. See subsection 3.1 for a brief account of the Sturmian sequence. We remark that for $0 < c < 1-\beta$ the partition $\{0,c,c+\beta/2,1-\beta/2\}$ that divides the circle into four arcs was ever studied by Hockett and Holmes [16], but they used two symbols rather than four to characterize a rotation. See also [11] for coding rotations with two symbols by more general partitions.

Remark 3.1. Suppose $(u_n)_{n\in\mathbb{Z}} = \boldsymbol{\nu}^{+}(t;\beta,Q,\Phi)$ (or $\boldsymbol{\nu}^{-}(t;\beta,Q,\Phi)$), then

$$u_n = \phi_{i_n} \quad \Longleftrightarrow \quad R_{1-\beta}^{-n}(t) \in J_{i_n}^{+} \text{ (resp. } J_{i_n}^{-}\text{)}$$

for all $1 \le i_n \le N = \sharp(Q)$ and $n \in \mathbb{Z}$. Hence, with respect to the partition Q, the sequence $(u_n)_{n\in\mathbb{Z}}$ is also the itinerary sequence of the orbit of t under the reverse rotation with angle $1-\beta$.

Theorem 3.2. *Given an irrational number $\beta \in (0,1/2)$, the set $X_{\beta,Q,\Phi}$ is a Cantor set in $\Phi^{\mathbb{Z}}$, the shift σ is a homeomorphism of $X_{\beta,Q,\Phi}$, and the system $(X_{\beta,Q,\Phi},\sigma)$ is invariant and minimal.*

The minimality of the set $X_{\beta,Q,\Phi}$ means that the set can be defined alternatively to be the orbit closure

$$X_{\beta,Q,\Phi} := \overline{\{\sigma^n(\mathbf{u}) \mid n \in \mathbb{Z}\}} \tag{6}$$

of *any* rotation sequence $\mathbf{u}$ of partition points Q with symbols Φ and rotation number β. (Actually, we prove the minimality in Theorem 3.2 by showing that (6) holds.) The reader can refer to [15] for the equivalence of the two definitions for the Sturmian system cases $X_{\beta,\{0,\beta\},\{0,1\}}$ and $X_{\beta,\{0,1-\beta\},\{0,1\}}$. For the Sturmian cases, Theorem 3.2 has been known in [15].

Theorem 3.3. *Given any irrational numbers β, $\widetilde{\beta} \in (0, 1/2)$, partition-symbol pairs (Q, Φ) and $(\widetilde{Q}, \widetilde{\Phi})$, the system $(X_{\widetilde{\beta},\widetilde{Q},\widetilde{\Phi}}, \sigma)$ is a factor of $(X_{\beta,Q,\Phi}, \sigma)$ if and only if $\widetilde{\beta} = \beta$ and*

$$\mathcal{O}(\widetilde{Q}; R_{\widetilde{\beta}}) \subseteq \mathcal{O}(Q; R_\beta). \tag{7}$$

The two systems are conjugate if and only if equality holds in (7).

In virtue of the above theorem, it is necessary that $\beta = \widetilde{\beta}$ for the two systems to be conjugate. Hence, we shall concentrate on a fixed irrational β and, when no ambiguity is caused, write $\boldsymbol{\nu}^\pm(t; Q, \Phi) = \boldsymbol{\nu}^\pm(t; \beta, Q, \Phi)$ and $X_{Q,\Phi} = X_{\beta,Q,\Phi}$ to simplify notation.

Remark 3.4. *Theorem 3.3 is no longer true if one allows rotation numbers to belong to* $(0, 1)$.

Given a partition set Q and a symbol set Φ, let $\widetilde{Q}$ be a subset of Q. Assume that $\mathbf{u}$, $\mathbf{v}$ belong to $X_{Q,\Phi}$. Define the following equivalence relation: $\mathbf{u} \sim_{\widetilde{Q}} \mathbf{v}$ if $\mathbf{u}$, $\mathbf{v} \in \{\boldsymbol{\nu}^-(t; Q, \Phi), \boldsymbol{\nu}^+(t; Q, \Phi)\}$ for some $t \in \mathcal{O}(\widetilde{Q}; R_\beta)$. It is easy to check that the equivalence relation defined is indeed an equivalence relation. For any two subsets $\widetilde{\Theta}$ and Θ' of Θ, the union $\sim_{\widetilde{\Theta}} \cup \sim_{\Theta'}$ is again an equivalence relation, and is equal to $\sim_{\widetilde{\Theta}\cup\Theta'}$.

Theorem 3.5. *Given a partition-symbol pair (Q, Φ), the system $(X_{Q,\Phi}, \sigma)$ is semi-conjugate to (S, R_β) in such a way that $(X_{Q,\Phi}, \sigma)$ is a 2-to-1 extension of (S, R_β) and the semi-conjugacy is 1-to-1 except on the countable subset $\bigcup_{q\in Q} \{\sigma^n \circ \boldsymbol{\nu}^\pm(q; Q, \Phi) |\, n \in \mathbb{Z}\}$. The quotient space $X_{Q,\Phi}/{\sim_Q}$ of $X_{Q,\Phi}$ by the equivalence relation $\sim_Q$ is topologically a circle, and $(X_{Q,\Phi}, \sigma)/{\sim_Q}$ is conjugate to (S, R_β).*

The above theorem has been known (e.g. [3]) for the Sturmian case $Q = \{0, \beta\}$ or $\{0, 1 - \beta\}$.

3.1 The Sturmian system

The material in this subsection can be found, for example, in [3, 14, 23].

Given a sequence $\mathbf{u}$ over a finite alphabet $\mathcal{A}$, the *complexity* function $p = p_{\mathbf{u}} : \mathbb{N} \to \mathbb{N}$, $n \mapsto p(n)$, is defined as the number of distinct words of length n occurring in $\mathbf{u}$. If U is a finite word over $\mathcal{A}$, denote by $|U|_a$ the number of occurrence of the letter $a \in \mathcal{A}$ in U. A sequence $\mathbf{u}$ over a two-letter alphabet $\{0, 1\}$ is called *balanced* if for any pair of words U, V of the same length in $\mathbf{u}$, we have $||U|_1 - |V|_1| \leq 1$ or equivalently $||U|_0 - |V|_0| \leq 1$. A theorem of Morse and Hedlund states that a binary sequence $\mathbf{u}$ is periodic if and only if $p(n) \leq n$ for some n. A binary sequence $\mathbf{u}$ is called *Sturmian* if it is balanced and not eventually periodic. It can be shown that a binary sequence $\mathbf{u}$ is Sturmian if and only if it has complexity $p(n) = n + 1$ and is not eventually periodic. Thus, among all non-eventually periodic binary sequences, Sturmian sequences are those having the smallest possible complexity.

The *frequency* of letter 0 (or 1) in a Sturmian sequence $\mathbf{u} = (u_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ defined as the limit

$$\lim_{n\to\infty} \frac{|u_{-n} \dots u_0 \dots u_n|_0}{2n+1} \qquad (\lim_{n\to\infty} \frac{|u_{-n} \dots u_0 \dots u_n|_1}{2n+1}, \text{ resp.})$$

is an irrational number. If the frequency of letter 0 in a Sturmian sequence is β, the frequency of letter 1 in that sequence is $1-\beta$. The following has been known [23].

Theorem 3.6 (Morse & Hedlund 1940). *Let* $\beta \in (0, 1/2)$ *and* $\mathbf{u} \in \{0,1\}^{\mathbb{Z}}$.

- $\mathbf{u}$ *is a Sturmian sequence and the frequency of* 0 *in* $\mathbf{u}$ *is* β *if and only if* $\mathbf{u}$ *coincides with* $\boldsymbol{\nu}^+(t; \{0,\beta\}, \{0,1\})$ *or* $\boldsymbol{\nu}^-(t; \{0,\beta\}, \{0,1\})$ *for some* $t \in S$.
- $\mathbf{u}$ *is a Sturmian sequence and the frequency of* 1 *in* $\mathbf{u}$ *is* β *if and only if* $\mathbf{u}$ *coincides with* $\boldsymbol{\nu}^+(t; \{0,1-\beta\}, \{0,1\})$ *or* $\boldsymbol{\nu}^-(t; \{0,1-\beta\}, \{0,1\})$ *for some* $t \in S$.

If a Sturmian sequence $\mathbf{u}$ differs from another $\mathbf{v}$ in exactly two positions, then precisely $\mathbf{u}$ differs from $\mathbf{v}$ in exactly two consecutive positions. Therefore, if $\mathbf{u} = \boldsymbol{\nu}^+(t; \{0,\beta\}, \{0,1\})$ for some $t \in S$, then $\mathbf{v}$ must be $\boldsymbol{\nu}^-(t; \{0,\beta\}, \{0,1\})$, and vice versa. Also, $\mathbf{u} \sim_{\{0\}} \mathbf{v}$ if $\mathbf{u} = \mathbf{v}$ or $\mathbf{u}$ differs from $\mathbf{v}$ in exactly two positions.

We remark that Sturmian sequences over a two-letter alphabet are also codings of trajectories of irrational initial slope in a unit square billiard obtained by labeling horizontal sides by one letter and vertical sides by the other, namely the so-called *billiard sequences*. Equivalently, they are also the so-called *cutting sequences*: Write the letter 0 each time when the line $y = \frac{\beta}{1-\beta}x + \frac{t}{1-\beta}$ on the x-y plane cuts a vertical line $x = integer$, and the letter 1 each time it cuts a horizontal line $y = integer$. Then the cutting sequence is a rotation sequence $\nu^+(t; \{0, 1-\beta\}, \{0,1\})$ or $\nu^-(t; \{0,1-\beta\}, \{0,1\})$

3.2 Multi-hole Sturmian system

Given a partition set Q, we can find a subset $\widetilde{\Theta} \subseteq Q$,

$$\widetilde{\Theta} = \left\{\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(L)}\right\}$$

consisting of L points in S satisfying

$$0 = \theta^{(1)} < \dots < \theta^{(L-1)} < \theta^{(L)} < 1$$

as well as

$$\mathcal{O}(\theta^{(i)}; R_\beta) \cap \mathcal{O}(\theta^{(j)}; R_\beta) = \emptyset \qquad \forall 1 \le i < j \le L,$$

(i.e. orbits of elements of $\widetilde{\Theta}$ under R_β are mutually disjoint) and can find integers

$$M_1 \ge 1,\ M_2 \ge 1, \ldots, M_L \ge 1,$$

$$\begin{gathered} T_1^{(1)} < T_2^{(1)} < \ldots < T_{M_1}^{(1)}, \\ T_1^{(2)} < T_2^{(2)} < \ldots < T_{M_2}^{(2)}, \\ \vdots \\ T_1^{(L)} < T_2^{(L)} < \ldots < T_{M_L}^{(L)} \end{gathered}$$

such that

$$Q = \bigcup_{k=1}^{L} \bigcup_{i=1}^{M_k} R_\beta^{T_i^{(k)}}(\theta^{(k)}).$$

Note that for each k one element of the set $\{T_1^{(k)}, T_2^{(k)}, \ldots, T_{M_k}^{(k)}\}$ must be zero, i.e. $0 \in \{T_1^{(k)}, T_2^{(k)}, \ldots, T_{M_k}^{(k)}\}$ $\forall 1 \le k \le L$. Note also that $M_L \ge 2$ if $L = 1$. The choice of the subset Θ for a given Q is finite but *not* unique, whereas the choice of the integers $M_1, \ldots, M_L$ is unique. In particular, $\sharp(Q) = \sum_{k=1}^{L} M_k$. Moreover, the cardinality of $\widetilde{\Theta}$ is fixed for any possible choice. We call the subset $\widetilde{\Theta}$ just described a *least equivalent sub-partition* of Q, and call the cardinality $\sharp(\widetilde{\Theta})$ of a least equivalent sub-partition $\widetilde{\Theta}$ the *number of holes* of the subshift $(X_{Q,\Phi}, \sigma)$ of $(\Phi^{\mathbb{Z}}, \sigma)$.

The subset $\widetilde{\Theta}$ is called a "sub-partition" because it is a subset of the partition set Q and itself can be used as a partition set provided that $L \ge 2$; it is called "equivalent" because the resulting subshift $X_{\widetilde{\Theta},\widetilde{\Theta}}$ is conjugate to $X_{Q,Q}$ (by Theorem 3.3); it is called "least" because if any point is removed from $\widetilde{\Theta}$, then the resulting subshift cannot be conjugate to the original one, i.e. $X_{\widehat{\Theta},\widehat{\Theta}}$ is not conjugate to $X_{Q,Q}$ if $\widehat{\Theta}$ which contains zero is a proper subset of $\widetilde{\Theta}$.

In fact, we have the following result, which is an immediate consequence of Theorem 3.3.

Corollary 3.7.

(i) *Any two systems $(X_{Q,\Phi}, \sigma)$ and $(X_{\widetilde{Q},\widetilde{\Phi}}, \sigma)$ are not conjugate if their numbers of holes are different.*

(ii) *The system $(X_{Q,\Phi}, \sigma)$ is conjugate to $(X_{\{0,\beta\},\{0,1\}}, \sigma)$ if it has only one hole.*

(iii) *The system $(X_{Q,\Phi}, \sigma)$ is conjugate to $(X_{\widetilde{\Theta},\widetilde{\Theta}}, \sigma)$ if it has more than one hole and $\widetilde{\Theta}$ is a least equivalent sub-partition of Q.*

An example of Corollary 3.7 is given below.

Example 3.8. $(X_{\{0,\beta\},\{0,1\}}, \sigma)$ is not conjugate to $(X_{\{0,\alpha\},\{0,1\}}, \sigma)$ if $\alpha \notin \mathcal{O}(0, R_\beta)$ because the former has one hole whereas the latter has two holes. Conversely, $(X_{\{0,\alpha\},\{0,1\}}, \sigma)$ is conjugate to $(X_{\{0,\beta\},\{0,1\}}, \sigma)$ if $\alpha \in \mathcal{O}(0, R_\beta)$.

We remark that, by our construction, an L-hole Sturmian system must have a least $\max\{2, L\}$ symbols.

We learnt that Masui [21] ever constructed a partition of the unit circle similar to our $\widetilde{\Theta}$ here, but it requires $\beta \in \widetilde{\Theta}$. And, a version of Theorem 4.3(i) to come in the next section in this paper was also proved in [21]. The version proved there is a special case of ours when the semi-conjugacy is a conjugacy. Note that partitions similar to our Q here were also appeared in [1, 2], but they did not associate their partitions with the Denjoy minimal system. The complexity of an irrational rotation sequence of partition $Q = \{0, q_2, q_3, \dots, q_N\}$ has the form $p(n) = an + b$ with $a \leq N$ for n large enough. If β, q_2, $q_3, \dots, q_N$ are rationally independent, then $a = N$, $b = 0$ (see [2]). In particular, if $Q = \{0, 1/2\}$, then $p(n) = 2n$ for all integer n (see [26]).

The following result is an analogous of Theorem 3.5.

Theorem 3.9. *Given a partition-symbol pair (Q, Φ) and a least equivalent sub-partition $\widetilde{\Theta}$ of Q, suppose $\sharp(\widetilde{\Theta}) \geq 2$.*

(i) $(X_{Q,\Phi}, \sigma)$ is semi-conjugate to $(X_{\{0,\beta\},\{0,1\}}, \sigma)$. The semi-conjugacy is 1-to-1 except on the countable set $\bigcup_{\theta \in \widetilde{\Theta} \setminus \{0\}} \{\sigma^n \circ \boldsymbol{\nu}^\pm(\theta; Q, \Phi) |\, n \in \mathbb{Z}\}$, where it is 2-to-1. The quotient system $(X_{Q,\Phi}, \sigma)/{\sim_{\widetilde{\Theta} \setminus \{0\}}}$ is conjugate to $(X_{\{0,\beta\},\{0,1\}}, \sigma)$.

(ii) Suppose $\widehat{\Theta}$ is a proper subset of $\widetilde{\Theta}$ not containing zero. If $\sharp(\widetilde{\Theta} \setminus \widehat{\Theta}) \geq 2$, then $(X_{Q,\Phi}, \sigma)$ is semi-conjugate to $(X_{\widetilde{\Theta} \setminus \widehat{\Theta}, \widetilde{\Theta} \setminus \widehat{\Theta}}, \sigma)$. The semi-conjugacy is 1-to-1 except on the countable set $\bigcup_{\theta \in \widehat{\Theta}} \{\sigma^n \circ \boldsymbol{\nu}^\pm(\theta; Q, \Phi) |\, n \in \mathbb{Z}\}$, where it is 2-to-1. The quotient system $(X_{Q,\Phi}, \sigma)/{\sim_{\widehat{\Theta}}}$ is conjugate to $(X_{\widetilde{\Theta} \setminus \widehat{\Theta}, \widetilde{\Theta} \setminus \widehat{\Theta}}, \sigma)$.

It is worth noticing a corollary of the statement (i) of the theorem above: The quotient dynamical system $(X_{\{0,\alpha\},\{0,1\}}, \sigma)/{\sim_{\{\alpha\}}}$ is conjugate to $(X_{\{0,\beta\},\{0,1\}}, \sigma)$ for any $\alpha \notin \mathcal{O}(0; R)$. Providing that $\sharp(\widetilde{\Theta}) \geq 2$, the statement (ii) says that if any non-zero partition point θ is eliminated from $\widetilde{\Theta}$, then the resulting subshift $(X_{\widetilde{\Theta} \setminus \{\theta\}, \widetilde{\Theta} \setminus \{\theta\}}, \sigma)$ is a factor of the original one $(X_{\widetilde{\Theta}, \widetilde{\Theta}}, \sigma)$.

4 Coding of Denjoy minimal set

Assume that the ω-limit set $\omega(f)$ of a Denjoy homeomorphism f satisfying Assumption A is a K-hole Cantor set. Let (Q, Φ) be a partition-symbol pair with a least equivalent sub-partition $\widetilde{\Theta}$ of Q. Assume

$$\widetilde{\Theta} \subseteq \Theta, \tag{8}$$

$\sharp(Q) = N$, and $Q = \{q_1, q_2, \ldots, q_N\}$, with $0 = q_1 < q_2 < \ldots < q_N < 1$. For each $0 \le i \le N$, let

$$z_i \in h^{-1}(q_i) \tag{9}$$

be *any* point in $h^{-1}(q_i)$. Define a set A,

$$A = \{A_1, A_2, \ldots, A_N\}, \tag{10}$$

of open intervals A_i delimited by theses z_i's on S by

$$\begin{aligned} A_1 &= (z_1, z_2)\,, \\ A_2 &= (z_2, z_3)\,, \\ &\vdots \\ A_{N-1} &= (z_{N-1}, z_N)\,, \\ A_N &= (z_N, z_1)\,. \end{aligned}$$

With the given symbol set $\Phi = \{\phi_1, \phi_2, \ldots, \phi_N\}$ and the intervals just constructed, define the *coding sequence* $E(x; Q, \Phi) = (E(x; Q, \Phi)_n)_{n\in\mathbb{Z}}$ of a point $x \in \omega(f)$ by

$$E(x; Q, \Phi)_n = \phi_i \quad \text{if } f^n(x) \in A_i \tag{11}$$

for all $n \in \mathbb{Z}$ and some $1 \le i \le N$. Remark that since the set $\{z_1, z_2, \ldots, z_N\}$ does not intersect the ω-limit set $\omega(f)$, the above definition is well defined.

Proposition 4.1. *Suppose $x \in \omega(f)$.*

(i) $E(f(x); Q, \Phi) = \sigma(E(x; Q, \Phi))$.

(ii) $E(x; Q, \Phi) = \boldsymbol{\nu}^-(h(x); Q, \Phi)$ *if and only if* $x = \inf(h^{-1}(h(x)))$*;*
$E(x; Q, \Phi) = \boldsymbol{\nu}^+(h(x); Q, \Phi)$ *if and only if* $x = \sup(h^{-1}(h(x)))$.
In particular, $E(x; Q, \Phi) = \boldsymbol{\nu}^-(h(x); Q, \Phi) = \boldsymbol{\nu}^+(h(x); Q, \Phi)$ *if and only if* $h(x) \notin \mathcal{O}(Q; R_\beta)$.

Proof. (i) The assertion clearly holds.

(ii) Let $1 \le i \le N = \sharp(Q)$, $q_{N+1} = 1$, and suppose $y \in A_i$. Then, by our construction, we have $h(y) \in J_i^+$ (or J_i^-) if and only if $y \notin \inf\left(h^{-1}(q_{i+1})\right)$ (or $\sup\left(h^{-1}(q_i)\right)$, respectively.)

If $h(x) \notin \mathcal{O}(Q; R_\beta)$, then $R_\beta^n(h(x))$ does not locate on the boundary of $J_i^\pm$ for every $n \in \mathbb{Z}$ and $1 \le i \le N$. Thus, $E(x; Q, \Phi) = \boldsymbol{\nu}^-(h(x); Q, \Phi) = \boldsymbol{\nu}^+(h(x); Q, \Phi)$. On the other hand, if $h(x) = R_\beta^n(q_i)$ for some integer n and $1 \le i \le N$, then

$$\begin{aligned} f^{-n}(x) &= \inf\left(h^{-1} \circ R_\beta^{-n} \circ h(x)\right) && \text{or} \quad \sup\left(h^{-1} \circ R_\beta^{-n} \circ h(x)\right) \\ &= \inf\left(h^{-1}(q_i)\right) && \text{or} \quad \sup\left(h^{-1}(q_i)\right), \text{ respectively} \\ &\in A_{i-1} \quad (\text{with } A_0 = A_N) && \text{or} \quad \in A_i, \text{ respectively.} \end{aligned}$$

Therefore, by the paragraph above, it is necessary and sufficient that $h(f^{-n}(x)) \in J_{i-1}^-$ so that $E(x; Q, \Phi)_{-n} = \phi_{i-1}$, or it is necessary and sufficient that $h(f^{-n}(x)) \in J_i^+$ so that $E(x; Q, \Phi)_{-n} = \phi_i$. $\square$

Proposition 4.2.

(i) The mapping $x \mapsto E(x; Q, \Phi)$ is continuous in $\omega(f)$ and is 1-to-1 except on the countable set $\omega(f) \cap \bigcup_{t \in \mathcal{O}(\Theta \setminus \widetilde{\Theta}; R_\beta)} h^{-1}(t)$ where it is 2-to-1.

(ii) $\overline{\{\sigma^n \circ E(x; Q, \Phi) |\, n \in \mathbb{Z}\}} = E(\omega(f); Q, \Phi) = X_{Q,\Phi}$ for any $x \in \omega(f)$.

Proof. (i) Because S is compact, f is uniformly continuous. Given any positive integer M, there exists $\delta > 0$ such that if $|y - x| < \delta$ then $|f^n(y) - f^n(x)| < \min_{i=1}^N |h^{-1}(q_i)|/2$ for all $|n| \leq M$ and $q_i \in Q$, where $|h^{-1}(q_i)|$ is the length of $h^{-1}(q_i)$ and $N = \sharp(Q)$. It follows that if $x \in \omega(f)$ and $f^n(x) \in A_i$ for some $1 \leq i \leq N$ then $f^n(y) \in A_i$ for all $|n| \leq M$ for any point y whose distance from x is within δ, for otherwise $f^n(y) \in S \setminus \omega(f)$. This proves the continuity.

In view of Proposition 4.1(ii), $E(x; Q, \Phi)$ is 1-to-1 in x if $h(x) \in \mathcal{O}(Q; R_\beta)$. Otherwise, it is 2-to1 since $E(x; Q, \Phi) = E(y; Q, \Phi)$ for distinct x and y if $h(x) = h(y)$. But, $h(x) = h(y) \notin \mathcal{O}(Q; R_\beta)$ if and only if $h(x) = h(y) \in \mathcal{O}(\Theta \setminus \widetilde{\Theta}; R_\beta)$.

(ii) Because of (i), $E(\omega(f); Q, \Phi)$ is a continuous image of the compact set $\omega(f)$ thus is compact. And, the set $\mathcal{O}(x; f)$ is dense in $\omega(f)$, so is $E(\mathcal{O}(x; f); Q, \Phi)$ in $E(\omega(f); Q, \Phi)$. The first equality follows. Because h is surjective, the second equality follows from Proposition 4.1(ii) and the definition (5). (Alternatively, the second equality can also be obtained by using (6).) $\square$

It is known (see e.g. [17]) that the restriction of a one-hole Denjoy homeomorphism to its ω-limit set is conjugate to the restriction of the full two-shift homeomorphism to a closed invariant subset. In view of Propositions 4.1 and 4.2 we arrive at the following conclusion.

Theorem 4.3. *Assume that $\omega(f)$ of a Denjoy homeomorphism f satisfies Assumption A. Let (Q, Φ) be a partition-symbol pair with a least equivalent sub-partition $\widetilde{\Theta}$. Assume $\widetilde{\Theta} \subseteq \Theta$. Then,*

(i) $(\omega(f), f)$ is semi-conjugate to $(X_{Q,\Phi}, \sigma)$ via the coding $E(\cdot; Q, \Phi)$. In particular, the coding is injective if and only if $\widetilde{\Theta} = \Theta$.

(ii) $(\omega(f), f)/\!\sim_{\Theta \setminus \widetilde{\Theta}}$ is conjugate to $(X_{Q,\Phi}, \sigma)$.

Because for any set Θ containing zero on S, there exists a Denjoy homeomorphism f of irrational rotation number β such that $\mathcal{D}(f)$ coincides with $\mathcal{O}(\Theta; R_\beta)$, we have an immediate corollary.

Corollary 4.4. *For any partition-symbol pair (Q, Φ) with a least equivalent sub-partition Θ, there exists a Denjoy homeomorphism f satisfying Assumption A such that $(\omega(f), f)$ is conjugate to $(X_{Q,\Phi})$.*

Remark 4.5. Theorem 4.3(i) says that $(X_{Q,\Phi}, \sigma)$ is always a factor of $(\omega(f), f)$ if the condition (8) holds. Of course, one could construct a partition set Q' with a least equivalent sub-partition Θ' such that Θ is a proper subset of Θ'. Then, $(\omega(f), f)$ would be a factor of $(X_{Q',Q'}, \sigma)$ via a multi-valued coding $E(\cdot; Q', Q')$. The coding is multi-valued because there must be some interval in the set A whose boundary points contains a point of $(\omega(f), f)$. Using a set like this A as a partition to code a Cantor set is not natural.

5 Transitions and anti-integrable limits

Theorem 5.1 (Cantorus to circle transition)**.** *Assume that $\omega(f)$ of a Denjoy homeomorphism f satisfies Assumption A. Let (Q, Φ) be a partition-symbol pair with Θ a least equivalent sub-partition of Q. Let $0 < \epsilon_0 < 1$ be a real number, and f_1 be any transitive OPH of S with $\rho(f_1) = \rho(f) = \beta$. We can construct a family of OPHs f_ϵ parametrized by ϵ with $f_{\epsilon_0} = f$ and $\lim_{\epsilon \nearrow 1} f_\epsilon(t) = f_1(t)$ for all $t \in S$ so that $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to $(X_{Q,\Phi}, \sigma)$ for $\epsilon_0 \leq \epsilon < 1$, but $(\omega(f_1), f_1)$ is conjugate to (S, R_β).*

In Theorem 5.1, a Denjoy minimal system $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to the $\sharp(\Theta)$-hole Sturmian system $(X_{Q,\Phi}, \sigma)$ when ϵ is slightly less than 1, but to the irrational rotation (S, R_β) when ϵ is equal to 1. In this situation, up to a conjugacy, the system $(\omega(f_\epsilon), f_\epsilon)$ bifurcates or degenerates to the irrational rotation (S, R_β) at $\epsilon = 1$.

Theorem 5.2 ($\sharp(\Theta)$ to $\sharp(\widetilde{\Theta})$-hole cantorus transition)**.** *Assume that $\omega(f)$ of a Denjoy homeomorphism f satisfies Assumption A. Suppose $\sharp(\Theta) \geq 2$. Let $\widetilde{\Theta}$ containing zero be a proper subset of Θ, $0 < \epsilon_0 < \tilde{\epsilon}$ be real numbers, and $f_{\tilde{\epsilon}}$ be any Denjoy homeomorphism satisfying $\rho(f_{\tilde{\epsilon}}) = \beta$ and $\mathcal{D}(f_{\tilde{\epsilon}}) = \mathcal{O}(\widetilde{\Theta}, R_\beta)$. We can construct a family of Denjoy homeomorphisms f_ϵ with $f_{\epsilon_0} = f$ and $\lim_{\epsilon \nearrow \tilde{\epsilon}} f_\epsilon(t) = f_{\tilde{\epsilon}}(t)$ for all $t \in S$ so that $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to $(X_{\Theta,\Theta}, \sigma)$ for $\epsilon_0 \leq \epsilon < \tilde{\epsilon}$, but $(\omega(f_{\tilde{\epsilon}}), f_{\tilde{\epsilon}})$ is conjugate to $(X_{\widetilde{\Theta},\widetilde{\Theta}}, \sigma)$ if $\sharp(\widetilde{\Theta}) \geq 2$ or to $(X_{\{0,\beta\},\{0,1\}}, \sigma)$ if $\sharp(\widetilde{\Theta}) = 1$.*

In Theorem 5.2, $\sharp(\widetilde{\Theta}) < \sharp(\Theta)$, and $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to the $\sharp(\Theta)$-hole Sturmian system $(X_{\Theta,\Theta}, \sigma)$ when ϵ is slightly less than $\tilde{\epsilon}$, but to the $\sharp(\widetilde{\Theta})$-hole Sturmian system $(X_{\widetilde{\Theta},\widetilde{\Theta}}, \sigma)$ when ϵ is equal to $\tilde{\epsilon}$. In this situation, the Denjoy minimal system $(\omega(f_\epsilon), f_\epsilon)$ undergoes a $\sharp(\Theta)$ to $\sharp(\widetilde{\Theta})$-hole transition at $\epsilon = \tilde{\epsilon}$.

Theorem 5.3 (AI-limit)**.** *Assume that $\omega(f)$ of a Denjoy homeomorphism f satisfies Assumption A. Let $0 < \epsilon_0 < 1$ be a real number, and $\widetilde{\Theta}$ containing zero a subset of Θ. For any partition-symbol pair (Q, Φ) with $\widetilde{\Theta}$ a least equivalent sub-partition of Q, we can construct a continuous family of Denjoy homeomorphisms f_ϵ so that $f_{\epsilon_0} = f$*

and that $(\omega(f_\epsilon), f_\epsilon)$ *is semi-conjugate to* $(X_{Q,\Phi}, \sigma)$ *via a family of codings* $E_\epsilon(\cdot; Q, \Phi) : \omega(f_\epsilon) \to X_{Q,\Phi}$*, which is injective if and only if* $\widetilde{\Theta} = \Theta$*, for* $0 < \epsilon \le \epsilon_0$ *with the property: for all* $\mathbf{u} \in X_{Q,\Phi}$ *we have*

$$\lim_{\epsilon \searrow 0} \mathcal{O}\left(E_\epsilon^{-1}(\mathbf{u}; Q, \Phi), f_\epsilon\right) = \mathbf{u} \tag{12}$$

in the uniform topology.

In Theorem 5.3, $(\omega(f_\epsilon), f_\epsilon)$ is semi-conjugate to the $\sharp(\widetilde{\Theta})$-hole Sturmian system $(X_{Q,\Phi}, \sigma)$ when ϵ is slightly larger than zero. As ϵ tends to zero from above, in the light of (12), $(\omega(f_\epsilon), f_\epsilon)$ reduces to the $\sharp(\Phi)$-symbol $\sharp(\widetilde{\Theta})$-hole Sturmian system $(X_{Q,\Phi}, \sigma)$ of partition Q. We say that the limit $\epsilon \searrow 0$ is the *anti-integrable limit* (AI-limit) for the family of Denjoy homeomorphisms f_ϵ.

If $\widetilde{\Theta} = \Theta$, we call the AI-limit in Theorem 5.3 a *non-degenerate* AI-limit. Because in this situation the semi-conjugacy is in fact a conjugacy, and when $\epsilon \searrow 0$ the Denjoy minimal system $(\omega(f_\epsilon), f_\epsilon)$ reduces to a symbolic dynamical system that is conjugate to $(\omega(f_\epsilon), f_\epsilon)$ of small ϵ. If, when $\epsilon \searrow 0$, a Denjoy minimal system $(\omega(f_\epsilon), f_\epsilon)$ reduces to a symbolic dynamical system that is *not* conjugate to but a factor of $(\omega(f_\epsilon), f_\epsilon)$ of small ϵ, we call such a limit a *degenerate* AI-limit. The limit $\epsilon \searrow 0$ in Theorem 5.3 is a degenerate AI-limit if and only if $\widetilde{\Theta} \neq \Theta$.

5.1 Examples

We close this section by providing examples in this subsection.

Let f be a Denjoy homeomorphism satisfying Assumption A. Let the length $|I_n^{(k)}|$ of $I_n^{(k)}$ be $l_n^{(k)} > 0$. For instance, $l_n^{(k)}$ can be chosen such that $\sum_{k=1}^K l^{(k)} = 1$ with $l^{(k)} = \sum_{n\in\mathbb{Z}} l_n^{(k)}$, and the pairwise disjoint open intervals $I_n^{(k)}$ can be chosen as

$$\begin{aligned} a_n^{(k)} &:= \eta + \sum_{1\le j\le K} \sum_{i: R_\beta^i(\theta^{(j)}) \in \left[0, R_\beta^n(\theta^{(k)})\right)} l_i^{(j)} \pmod 1, \\ b_n^{(k)} &:= a_n^{(k)} + l_n^{(k)}, \end{aligned}$$

where η satisfying $0 < \eta + l_0^{(1)} < 1$ is a real number to control the position of $a_0^{(1)}$. Actually, $(a_0^{(1)}, b_0^{(1)}) = (\eta, \eta + l_0^{(1)})$. It is easy to see that $I_n^{(k)}$ defined by the above $a_n^{(k)}$ and $b_n^{(k)}$ has the same cyclic ordering as $R_\beta^n(\theta^{(k)})$. Because $\sum_{k=1}^K l^{(k)} = 1$, the union $\bigcup_{1\le k\le K,\, n\in\mathbb{Z}} I_n^{(k)}$ is open and dense in S.

Suppose η and $l_n^{(k)}$ depend continuously on a parameter ϵ, then f, which has $S \setminus \bigcup_{1\le k\le K, n\in\mathbb{Z}} I_n^{(k)}$ as its ω-limit set, depends on ϵ as well. Write it as f_ϵ.

Partitions on S and schematic illustrations of the construction of families of Denjoy homeomorphisms by changing the size of wandering intervals are shown in Figure 1. In Figure 1(a), $\Theta = \{0, 1/2\}$, $Q = \{0, 1/2\}$, $\Phi = \{0, 1\}$. There is a circle to one-hole cantorus transition occurring at $\epsilon = 1$, a one to two-hole cantorus transition at $\epsilon = \tilde{\epsilon}$. The limit $\epsilon \searrow 0$ is a non-degenerate AI-limit, at which the dynamics is $(X_{\{0,1/2\},\{0,1\}}, \sigma)$. In Figure 1(b), $\Theta = \{0, 1/2\}$, $Q = \{0, 1/2, R_\beta^n(0)\}$ for some $n \in \mathbb{Z}$, $\Phi = \{\phi_1, \phi_2, 1\}$ for some $0 < \phi_1 < \phi_2 < 1$. There is a circle to two-hole cantorus transition occurring at $\epsilon = 1$. The limit $\epsilon \searrow 0$ is a non-degenerate AI-limit, at which the dynamics is $(X_{\{0,1/2,R_\beta^n(0)\},\{\phi_1,\phi_2,1\}}, \sigma)$. In Figure 1(c), $\Theta = \{0, 1/2\}$, $Q = \{0, R_\beta^n(0)\}$, $\Phi = \{0, 1\}$. There is a circle to two-hole cantorus transition occurring at $\epsilon = 1$. The limit $\epsilon \searrow 0$ is a degenerate AI-limit, at which the dynamics is $(X_{\{0,R_\beta^n(0)\},\{0,1\}}, \sigma)$.

Example 5.4. Set $\Theta = \{0, 1/2\}$, $Q = \{0, 1/2\}$ and $\Phi = \{0, 1\}$. Then, $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to $(X_{\{0,1/2\},\{0,1\}}, \sigma)$ via the coding $E_\epsilon(\cdot; \{0, 1/2\}, \{0, 1\})$ when $0 < \epsilon < \tilde{\epsilon}$. See Figure 1(a).

Example 5.5. Set $\Theta = \{0, 1/2\}$, $Q = \{0, 1/2, 1 - \beta\}$ and $\Phi = \{\phi_1, \phi_2, 1\}$ for $0 < \phi_1 < \phi_2 < 1$. Then, $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to $(X_{\{0,1/2,1-\beta\},\{\phi_1,\phi_2,1\}}, \sigma)$ via the coding $E_\epsilon(\cdot; \{0, 1/2, 1 - \beta\}, \{\phi_1, \phi_2, 1\})$ when $0 < \epsilon < 1$. See Figure 1(b).

Certainly, $(X_{\{0,1/2\},\{0,1\}}, \sigma)$ is conjugate to $(X_{\{0,1/2,1-\beta\},\{\phi_1,\phi_2,1\}}, \sigma)$ by Theorem 3.3.

Example 5.6. Set $\Theta = \{0, 1/2\}$, $Q = \{0, 1/2\}$ and $\Phi = \{0, 1\}$. Can choose the positive numbers $l_n^{(k)}$'s in such a way that $l_n^{(k)} \to 0$ as $\epsilon \nearrow 1$ for all $n \in \mathbb{Z}$ and $1 \le k \le 2$. Then, $f_\epsilon(x) \to R_\beta(x) =: f_1(x)$ for all $x \in S$ as $\epsilon \nearrow 1$, and $(\omega(f_1), f_1)$ is conjugate to $(X_{\{0,1/2\},\{0,1\}}, \sigma)/\sim_{\{0,1/2\}}$. See Figure 1(a).

Example 5.7. Set $\Theta = \{0, 1/2\}$, $Q = \{0, 1/2, 1-\beta\}$, and $\Phi = \{\phi_1, \phi_2, 1\}$. Can choose the positive numbers $l_n^{(k)}$'s in such a way that $l_n^{(k)} \to 0$ as $\epsilon \nearrow 1$ for all $n \in \mathbb{Z}$ and $1 \le k \le 2$. Then, $f_\epsilon(x) \to R_\beta(x) =: f_1(x)$ for all $x \in S$ as $\epsilon \nearrow 1$, and $(\omega(f_1), f_1)$ is conjugate to $(X_{\{0,1/2,1-\beta\},\{\phi_1,\phi_2,1\}}, \sigma)/\sim_{\{0,1/2\}}$. See Figure 1(b).

Certainly, $(X_{\{0,1/2\},\{0,1\}}, \sigma)/\sim_{\{0,1/2\}}$ is conjugate to $(X_{\{0,1/2,1-\beta\},\{\phi_1,\phi_2,1\}}, \sigma)/\sim_{\{0,1/2\}}$ by Theorem 3.3.

Example 5.8. Set $\Theta = \{0, 1/2\}$, $Q = \{0, 1/2\}$ and $\Phi = \{0, 1\}$. Can choose $l_n^{(k)}$'s in such a way that $l_n^{(2)} \to 0$ but $I_n^{(1)}$ remains a component of wandering intervals for every integer n as $\epsilon \nearrow \tilde{\epsilon}$. Then, $(\omega(f_{\tilde{\epsilon}}), f_{\tilde{\epsilon}})$ is conjugate to $(X_{\{0,1/2\},\{0,1\}}, \sigma)/\sim_{\{1/2\}}$. The latter itself is conjugate to $(X_{\{0,\beta\},\{0,1\}}, \sigma)$. See Figure 1(a).

Example 5.9. Set $\Theta = \{0, 1/2\}$, $Q = \{0, 1/2\}$ and $\Phi = \{0, 1\}$. Can choose $l_n^{(k)}$'s in such a way that $l_n^{(k)} \to 0$ but $l_0^{(2)} \to 1$ as $\epsilon \searrow 0$ for every $n \in \mathbb{Z}$ and $1 \le k \le 2$. Consequently, as $\epsilon \searrow 0$, the two-hole Denjoy minimal system $(\omega(f_\epsilon), f_\epsilon)$ reduces to the two-hole Sturmian system $(X_{\{0,1/2\},\{0,1\}}, \sigma)$ in the sense that

$$\lim_{\epsilon \searrow 0} \mathcal{O}\left(E_\epsilon^{-1}(\mathbf{u}; \{0, 1/2\}, \{0, 1\}), f_\epsilon\right) = \mathbf{u}$$

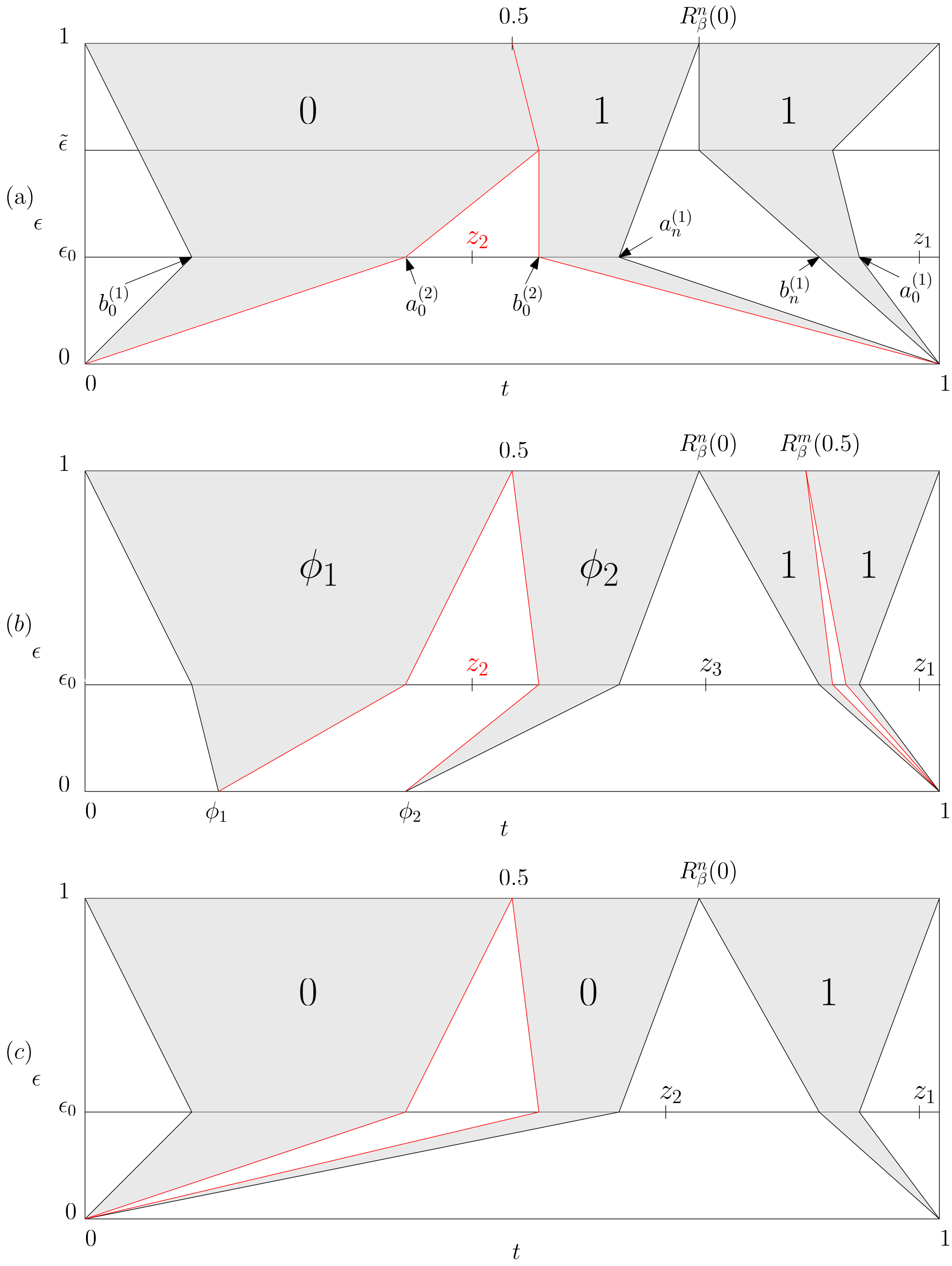

Figure 1: Partitions on S with which orbits of irrational rotations and Denjoy homeomorphisms are coded, and schematic illustrations of the idea of the construction of families of Denjoy homeomorphisms by changing the size of wandering intervals. (a) $\Theta = \{0, 1/2\}$, $Q = \{0, 1/2\}$, $\Phi = \{0, 1\}$. (b) $\Theta = \{0, 1/2\}$, $Q = \{0, 1/2, R_\beta^n(0)\}$, $\Phi = \{\phi_1, \phi_2, 1\}$. (c) $\Theta = \{0, 1/2\}$, $Q = \{0, R_\beta^n(0)\}$, $\Phi = \{0, 1\}$.

for any $\mathbf{u} \in X_{\{0,1/2\},\{0,1\}}$. The limit $\epsilon \searrow 0$ is a non-degenerate AI-limit. See Figure 1(a).

Example 5.10. Set $\Theta = \{0, 1/2\}$, $Q = \{0, 1/2, 1-\beta\}$, and $\Phi = \{\phi_1, \phi_2, 1\}$. Can design $l_n^{(k)}$'s in such a way that $l_n^{(k)} \to 0$ for every $n \in \mathbb{Z}$ and $1 \le k \le 2$ but $l_0^{(1)} \to \phi_1$, $l_0^{(2)} \to \phi_2 - \phi_1$, and $l_{-1}^{(1)} \to 1 - \phi_2$ as $\epsilon \searrow 0$. Then, as $\epsilon \searrow 0$, the two-hole Denjoy minimal system $(\omega(f_\epsilon), f_\epsilon)$ reduces to the two-hole Sturmian system $(X_{\{0,1/2,1-\beta\},\{\phi_1,\phi_2,1\}}, \sigma)$ in the sense that

$$\lim_{\epsilon \searrow 0} \mathcal{O}\left(E_\epsilon^{-1}(\mathbf{u}; \{0, 1/2, 1-\beta\}, \{\phi_1, \phi_2, 1\}), f_\epsilon\right) = \mathbf{u}$$

for any $\mathbf{u}$ belonging to the subshift $X_{\{0,1/2,1-\beta\},\{\phi_1,\phi_2,1\}}$. The limit $\epsilon \searrow 0$ is a non-degenerate AI-limit. See Figure 1(b).

Example 5.11. Set $\Theta = \{0, 1/2\}$, $Q = \{0, 1-\beta\}$ and $\Phi = \{0, 1\}$. Can design $l_n^{(k)}$'s in such a way that $l_n^{(k)} \to 0$ except $l_{-1}^{(1)} \to 1$ as $\epsilon \searrow 0$ for every $n \in \mathbb{Z}$ and $1 \le k \le 2$. Then, as $\epsilon \searrow 0$, the two-hole Denjoy minimal system $(\omega(f_\epsilon), f_\epsilon)$ reduces to the one-hole Sturmian system $(X_{\{0,1-\beta\},\{0,1\}}, \sigma)$ in the sense that

$$\lim_{\epsilon \searrow 0} \mathcal{O}\left(E_\epsilon^{-1}(\mathbf{u}; \{0, 1-\beta\}, \{0, 1\}), f_\epsilon\right) = \mathbf{u}$$

for any $\mathbf{u} \in X_{\{0,1-\beta\},\{0,1\}}$. The limit $\epsilon \searrow 0$ is a degenerate AI-limit. See Figure 1(c).

6 Proofs of Theorems

For the Sturmian cases, the partition set is $\{0, \beta\}$ or $\{0, 1-\beta\}$. From [15] we know that each of the four mappings $t \mapsto \boldsymbol{\nu}^\pm(t; \{0,\beta\}, \{0,1\})$ or $t \mapsto \boldsymbol{\nu}^\pm(t; \{0, 1-\beta\}, \{0,1\})$ is 1-to-1 everywhere in S, and is continuous in S except at a countable set consisting of the orbit $\mathcal{O}(0; R_\beta)$ of 0 under R_β. Each of the inverses $\boldsymbol{\nu}^\pm(t; \{0,\beta\}, \{0,1\}) \mapsto t$ or $\boldsymbol{\nu}^\pm(t; \{0, 1-\beta\}, \{0,1\}) \mapsto t$, however, is continuous. All these properties can be extended to the multi-hole Sturmian cases for a general, arbitrary partition set Q.

Proposition 6.1. *For any partition-symbol pair (Q, Φ), both maps $\boldsymbol{\nu}^-(\cdot; Q, \Phi)$ and $\boldsymbol{\nu}^+(\cdot; Q, \Phi)$ are 1-to-1 everywhere in S:*
(i) If $\boldsymbol{\nu}^+(s; Q, \Phi) = \boldsymbol{\nu}^+(t; Q, \Phi)$ or $\boldsymbol{\nu}^-(s; Q, \Phi) = \boldsymbol{\nu}^-(t; Q, \Phi)$, then $s = t$.
(ii) $\boldsymbol{\nu}^+(s; Q, \Phi) = \boldsymbol{\nu}^-(t; Q, \Phi)$ if and only if $R_\beta^n(s) = R_\beta^n(t) \notin Q$ for all integer n.

Proof. (i) If the statement is not true, then $s \neq t$. Subsequently, there exists an integer l such that $R_\beta^l(t)$ lies in the interior of $J_1^\pm$ while $R_\beta^l(s)$ lies in the interior of $J_j^\pm$ for some $2 \le j \le N = \sharp(Q)$. Consequently, $\boldsymbol{\nu}^\pm(t; Q, \Phi)_l = 1 \neq j = \boldsymbol{\nu}^\pm(s; Q, \Phi)$, contradicting to the hypothesis of the proposition.

(ii) If $s = t$ and $\mathcal{O}(s; R_\beta) \cap Q = \emptyset$, then for every integer n the orbit point $R_\beta^n(s)$ does not located on the boundary of $J_i^\pm$ for all $1 \le i \le N$. Hence, $\boldsymbol{\nu}^+(s; Q, \Phi) =$

$\boldsymbol{\nu}^-(s;Q,\Phi) = \boldsymbol{\nu}^-(t;Q,\Phi) = \boldsymbol{\nu}^+(t;Q,\Phi)$. On the other hand, if $\boldsymbol{\nu}^+(s;Q,\Phi) = \boldsymbol{\nu}^-(t;Q,\Phi)$, then $s = t$. (For if $s \neq t$, then it follows by the same argument used to prove (i), there exists $l \in \mathbb{Z}$ such that $\boldsymbol{\nu}^+(s;Q,\Phi)_l = 1$ but $\boldsymbol{\nu}^-(t;Q,\Phi)_l = j$ for some $j \neq 1$.) Suppose $R_\beta^m(t) = q_k$ for some $m \in \mathbb{Z}$ and $1 \leq k \leq N$. Then, $\boldsymbol{\nu}^+(t;Q,\Phi)_m = k$ while $\boldsymbol{\nu}^-(t;Q,\Phi)_m = k-1$ (or $= N$ if $k = 1$), contradicting to the hypothesis. $\square$

Proposition 6.2. *For any partition-symbol pair (Q,Φ), both maps $\boldsymbol{\nu}^-(\cdot;Q,\Phi)$ and $\boldsymbol{\nu}^+(\cdot;Q,\Phi)$ are continuous except at the countable set $\mathcal{O}(Q;R_\beta)$. More precisely, if $t \notin \mathcal{O}(Q;R_\beta)$, then*

$$\lim_{s\to t} \boldsymbol{\nu}^-(s;Q,\Phi) = \boldsymbol{\nu}^-(t;Q,\Phi) = \boldsymbol{\nu}^+(t;Q,\Phi) = \lim_{s\to t} \boldsymbol{\nu}^+(s;Q,\Phi);$$

if $t \in \mathcal{O}(Q;R_\beta)$, then

$$\lim_{s\to t^-} \boldsymbol{\nu}^-(s;Q,\Phi) = \lim_{s\to t^-} \boldsymbol{\nu}^+(s;Q,\Phi) = \boldsymbol{\nu}^-(t;Q,\Phi) \tag{13}$$

and

$$\lim_{s\to t^+} \boldsymbol{\nu}^-(s;Q,\Phi) = \lim_{s\to t^+} \boldsymbol{\nu}^+(s;Q,\Phi) = \boldsymbol{\nu}^+(t;Q,\Phi). \tag{14}$$

Proof. If $t \notin \mathcal{O}(Q;R_\beta)$, then for every integer n the orbit point $R_\beta^n(t)$ does not locate at any boundary point of $J_i^\pm$ for all $1 \leq i \leq N = \sharp(Q)$. Thus, $\boldsymbol{\nu}^-(t;Q,\Phi) = \boldsymbol{\nu}^+(s;Q,\Phi)$. Given any integer $M > 0$, there exists $\delta > 0$ such that for every $|n| \leq M$ both orbit points $R_\beta^n(s)$ and $R_\beta^n(t)$ lie in the same interior of intervals J_i^+ and J_i^- for some i provided $|s-t| < \delta$. This means that $\boldsymbol{\nu}^-(s;Q,\Phi)_n = \boldsymbol{\nu}^-(t;Q,\Phi)_n = \boldsymbol{\nu}^+(s;Q,\Phi)_n = \boldsymbol{\nu}^+(t;Q,\Phi)_n$ for all $|n| \leq M$, and implies the continuity at t.

If $t = R_\beta^m(q_j)$ for some $m \in \mathbb{Z}$ and $1 \leq j \leq N$, then there are N_j number of integers $m_1, m_2, \ldots, m_{N_j}$ (all depending on m and j) with $1 \leq N_j \leq N$ and $m_1 = -m$ for which $R_\beta^{m_1}(t)$, $R_\beta^{m_2}(t), \ldots, R_\beta^{m_{N_j}}(t) \in Q$, and $R_\beta^n(t) \notin Q$ for any other integer n. Therefore, none of the points in $\{R_\beta^n(t) |\, n \in \mathbb{Z} \setminus \{m_1, \ldots, m_{N_j}\}\}$ is a boundary point of $J_i^\pm$ for all $1 \leq i \leq N$. Thus, for any integer $M > 0$ there exists $\delta > 0$ such that for every $|n| \leq M$ and $n \notin \{m_1, \ldots, m_{N_j}\}$ both orbit points $R_\beta^n(s)$ and $R_\beta^n(t)$ lie in the same interior of both intervals J_i^+ and J_i^- and that for every $n \in \{m_1, \ldots, m_{N_j}\}$ points $R_\beta^n(s)$ and $R_\beta^n(t)$ lie in the same interval J_i^+ for some $1 \leq i \leq N$ provided $0 < s-t < \delta$. This implies the property (14). Similarly, the property (13) can be proved. $\square$

Proposition 6.3. *For any partition-symbol pair (Q,Φ), both the inverses $\boldsymbol{\nu}^-(t;Q,\Phi) \mapsto t$ and $\boldsymbol{\nu}^+(t;Q,\Phi) \mapsto t$ are continuous in S: Suppose $\mathbf{u}_\infty$, $\mathbf{u}_1$, $\mathbf{u}_2, \ldots$ all belong to $X_{Q,\Phi}$ with $\lim_{n\to\infty} \mathbf{u}_n = \mathbf{u}_\infty$, and suppose t_∞, t_1, $t_2, \ldots$ are corresponding points in S given* *by the injectivity of each of the mappings* $t \mapsto \boldsymbol{\nu}^{\pm}(t;Q,\Phi)$. *Then*

$$\begin{aligned}\lim_{n\to\infty} t_n = t_\infty \quad &\textit{if} \quad \mathbf{u}_\infty = \boldsymbol{\nu}^{+}(t_\infty;Q,\Phi) = \boldsymbol{\nu}^{-}(t_\infty;Q,\Phi);\\ \lim_{n\to\infty} t_n = t_\infty^{+} \quad &\textit{if} \quad \mathbf{u}_\infty = \boldsymbol{\nu}^{+}(t_\infty;Q,\Phi) \neq \boldsymbol{\nu}^{-}(t_\infty;Q,\Phi);\\ \lim_{n\to\infty} x_n = t_\infty^{-} \quad &\textit{if} \quad \mathbf{u}_\infty = \boldsymbol{\nu}^{-}(t_\infty;Q,\Phi) \neq \boldsymbol{\nu}^{+}(t_\infty;Q,\Phi).\end{aligned}$$

Proof. $R_\beta^n(t_\infty) \notin Q$ for all integer n if $\mathbf{u}_\infty = \boldsymbol{\nu}^{+}(t_\infty;Q,\Phi) = \boldsymbol{\nu}^{-}(t_\infty;Q,\Phi)$. In this case, t_∞ is contained in the interior of an interval J_j^+ or J_j^- for some $1 \leq j \leq N = \sharp(Q)$. Suppose $(t_n)_{n\geq 1}$ does not converge to t_∞. Then, it contains a subsequence that converges to another point, say, $\bar{t}$. It follows from Proposition 6.2 that the sequence $(\mathbf{u})_{n\geq 1}$ must converge either to $\boldsymbol{\nu}^{+}(\bar{t};Q,\Phi)$ or to $\boldsymbol{\nu}^{-}(\bar{t};Q,\Phi)$. In other word, $\mathbf{u}_\infty = \boldsymbol{\nu}^{+}(\bar{t};Q,\Phi)$ or $\boldsymbol{\nu}^{-}(\bar{t};Q,\Phi)$. But, it follows from Proposition 6.1(i) that $\boldsymbol{\nu}^{+}(\bar{t};Q,\Phi) \neq \boldsymbol{\nu}^{+}(t_\infty;Q,\Phi) = \boldsymbol{\nu}^{-}(t_\infty;Q,\Phi) \neq \boldsymbol{\nu}^{-}(\bar{t};Q,\Phi)$, a contradiction.

If $\mathbf{u}_\infty = \boldsymbol{\nu}^{+}(t_\infty;Q,\Phi) \neq \boldsymbol{\nu}^{-}(t_\infty;Q,\Phi)$, then $t_\infty \in \mathcal{O}(Q;R_\beta)$. If $(t_n)_{n\geq 1}$ converges to t_∞, then $\lim_{n\to\infty} t_n = t_\infty^+$ by Proposition 6.2. If $(t_n)_{n\geq 1}$ does not converge, there is a subsequence converging to another point $\bar{t} \neq t_\infty$. And, there is a corresponding subsequence of $(\mathbf{u}_n)_{n\geq 1}$ that converges to $\boldsymbol{\nu}^{+}(\bar{t};Q,\Phi)$ or $\boldsymbol{\nu}^{-}(\bar{t};Q,\Phi)$. Therefore, $\boldsymbol{\nu}^{+}(t_\infty;Q,\Phi) = \boldsymbol{\nu}^{+}(\bar{t};Q,\Phi)$ or $\boldsymbol{\nu}^{+}(t_\infty;Q,\Phi) = \boldsymbol{\nu}^{-}(\bar{t};Q,\Phi)$, but according to Proposition 6.1(ii), the latter is impossible, and the former implies $t_\infty = \bar{t}$ by Proposition 6.1(i).

The final case $\mathbf{u}_\infty = \boldsymbol{\nu}^{-}(t_\infty;Q,\Phi) \neq \boldsymbol{\nu}^{+}(t_\infty;Q,\Phi)$ can be treated similarly. □

Proof of Theorem 3.2.

It is well-known that the shift σ is a homeomorphism of $\Phi^{\mathbb{Z}}$. Thus, σ is also a homeomorphism of $X_{Q,\Phi}$ if $X_{Q,\Phi}$ is a compact invariant subset of $\Phi^{\mathbb{Z}}$. Since the latter is compact, it is enough to show that $X_{Q,\Phi}$ is invariant and closed. Because

$$\sigma^{\pm 1}(\boldsymbol{\nu}^{+}(t;Q,\Phi)) = \boldsymbol{\nu}^{+}(R_\beta^{\pm 1}(t);Q,\Phi) \tag{15}$$

and

$$\sigma^{\pm 1}(\boldsymbol{\nu}^{-}(t;Q,\Phi)) = \boldsymbol{\nu}^{-}(R_\beta^{\pm 1}(t);Q,\Phi) \tag{16}$$

for any $t \in S$, the shift σ is a bijection of $X_{Q,\Phi}$ and $X_{Q,\Phi}$ is invariant under σ.

Now, we show that $X_{Q,\Phi}$ is a closed subset. Any infinite sequence of points in $X_{Q,\Phi}$ must contain an infinite subsequence of points of the form $(\boldsymbol{\nu}^{+}(t_n;Q,\Phi))_{n\geq 1}$ or of the form $(\boldsymbol{\nu}^{-}(t_n;Q,\Phi))_{n\geq 1}$. Without loss of generality, we can assume that the first case happens. Taking a subsequence if necessary, we assume that the sequence $(t_n)_{n\geq 1}$ converges to a point t_∞ by the compactness of S. Then, $(t_n)_{n\geq 1}$ contains either a subsequence that converges to t_∞ from the left (anti-clockwise) or a subsequence that converges to t_∞ from the right (clockwise). If the first case holds for $(t_n)_{n\geq 1}$, that is, $\lim_{n\to\infty} t_n = t_\infty^-$ (by taking a subsequence again if necessary), then by Proposition 6.2, we infer that $\lim_{n\to\infty} \boldsymbol{\nu}^+(t_n; Q, \Phi) = \boldsymbol{\nu}^-(t_\infty; Q, \Phi)$. If the second case holds, that is, $\lim_{n\to\infty} t_n = t_n^+$, then $\lim_{n\to\infty} \boldsymbol{\nu}^+(t_n; Q, \Phi) = \boldsymbol{\nu}^+(t_\infty; Q, \Phi)$. This proves that $X_{Q,\Phi}$ is closed.

Now, $X_{Q,\Phi}$ is a subset of the totally disconnected set $\Phi^{\mathbb{Z}}$, so is itself totally disconnected. Proposition 6.2 implies that every point in $X_{Q,\Phi}$ is a limit point of points in $X_{Q,\Phi}$. Since $X_{Q,\Phi}$ is compact, it is a Cantor set.

Because $\mathcal{O}(s; R_\beta)$ is dense in S for any $s \in S$, it follows from (15) and (16) and Proposition 6.2 again that $\mathcal{O}(\mathbf{u}; \sigma)$ is dense in $X_{Q,\Phi}$ for any $\mathbf{u} \in X_{Q,\Phi}$. This completes the proof of the minimality. $\square$

Proof of Theorem 3.3.

From Corollary 4.4, there are Denjoy homeomorphisms f and $\tilde{f}$ satisfying Assumption A such that $\rho(f) = \beta$, $\rho(\tilde{f}) = \widetilde{\beta}$, $(\omega(f), f)$ is conjugate to $(X_{\beta,Q,\Phi})$, and that $(\omega(\tilde{f}), \tilde{f})$ is conjugate to $(X_{\widetilde{\beta},\widetilde{Q},\widetilde{\Phi}})$. By Remark 2.2(ii), we have $\mathcal{D}(f) = \mathcal{O}(Q; R_\beta)$ and $\mathcal{D}(\tilde{f}) = \mathcal{O}(\widetilde{Q}; R_{\widetilde{\beta}})$. Therefore, by Theorem 2.1, $(X_{\beta,Q,\Phi}, \sigma)$ is semi-conjugate to $(X_{\widetilde{\beta},\widetilde{Q},\widetilde{\Phi}}, \sigma)$ if and only if $\widetilde{\beta} = \beta$ and

$$\mathcal{O}(\widetilde{Q}; R_\beta) \subseteq R_\alpha(\mathcal{O}(Q; R_{\widetilde{\beta}}))$$

for some $0 \leq \alpha < 1$. But, because both $\mathcal{O}(Q; R_\beta)$ and $\mathcal{O}(\widetilde{Q}; R_\beta)$ contain $\mathcal{O}(0; R_\beta)$, the values of α satisfying the above equality are those $R_\alpha(\mathcal{O}(Q; R_\beta)) = \mathcal{O}(Q; R_\beta)$. $\square$

Let X and Y be topological spaces. Recall that a surjective map $p : X \to Y$ is called a *quotient map* if a subset U of Y is open (or closed) in Y if and only if $p^{-1}(U)$ is open (resp. closed) in X. We shall use the following result, the statement of which is slightly modified from Theorem 22.2 and Corollary 22.3 of [24].

Theorem 6.4. *Let X, Z be topological spaces, $g : X \to Z$ a continuous surjection, and $X^* = \{g^{-1}(z) |\ z \in Z\}$ a collection of subsets of X. Let $p : X \to X^*$ be the quotient map and give X^* the quotient topology induced by p.*
(i) The map g induces a continuous bijection $\xi : X^* \to Z$ satisfying $\xi \circ p = g$, which is a homeomorphism if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* \arrow[r, "\xi"'] & Z \end{tikzcd}$$

(ii) If Z is Hausdorff, so is the quotient space X^*.

Proof of Theorem 3.5.

It follows from Proposition 6.3 that a map $\boldsymbol{\nu}^{-1} : X_{Q,\Phi} \to S$ defined by

$$\boldsymbol{\nu}^{-1}(\mathbf{u}) = t \quad \text{if } \mathbf{u} = \boldsymbol{\nu}^{+}(t; Q, \Phi) \text{ or } \boldsymbol{\nu}^{-}(t; Q, \Phi)$$

is a continuous surjection. And, because

$$\begin{aligned} & \boldsymbol{\nu}^{-1} \circ \sigma \left(\boldsymbol{\nu}^{\pm}(t; Q, \Phi)\right) \\ = \; & \boldsymbol{\nu}^{-1} \left(\boldsymbol{\nu}^{\pm}(R_\beta(t); Q, \Phi)\right) \\ = \; & R_\beta(t) \\ = \; & R_\beta \circ \boldsymbol{\nu}^{-1} \left(\boldsymbol{\nu}^{\pm}(t; Q, \Phi)\right), \end{aligned}$$

the map $\boldsymbol{\nu}^{-1}$ is a semi-conjugacy. From Proposition 6.1, the map $\boldsymbol{\nu}^{-1}$ is 1-to-1 except when $\boldsymbol{\nu}^{+}(t; Q, \Phi) \neq \boldsymbol{\nu}^{-}(t; Q, \Phi)$ and this occurs only if $t \in \mathcal{O}(Q, R_\beta)$. This proves the first part of the theorem.

$$\begin{array}{ccc} X_{Q,\Phi} & & \\ \downarrow & \searrow^{\boldsymbol{\nu}^{-1}} & \\ X^*_{Q,\Phi} & \longrightarrow & S \end{array}$$

For the second part, notice that $\boldsymbol{\nu}^{-1}$ is a quotient map: It sends closed sets, which are compact in $X_{Q,\Phi}$, to closed sets in S, since compact sets in a Hausdorff space are closed. Now, let $X^*_{Q,\Phi} = \left\{ \left(\boldsymbol{\nu}^{-1}\right)^{-1}(t) \middle|\, t \in S \right\}$. It is clear that $X^*_{Q,\Phi} = X_{Q,\Phi}/{\sim_Q}$. Then, by virtue of Theorem 6.4 and the first part of the theorem, $\boldsymbol{\nu}^{-1}$ induces a homeomorphism for which $(X_{Q,\Phi}, \sigma)/{\sim_Q}$ is conjugate to (S, R_β). □

Proof of Theorem 3.9.

In view of Theorem 3.3, it is enough to prove the theorem for the case $Q = \widetilde{\Theta}$. Moreover, we are going to prove the case (ii) only. Case (i) can be proved almost exactly the same as case (ii).

We would like to show that a surjection $g : X_{\widetilde{\Theta},\widetilde{\Theta}} \to X_{\widetilde{\Theta}\setminus\widehat{\Theta},\widetilde{\Theta}\setminus\widehat{\Theta}}$ defined by

$$g : \mathbf{u} \mapsto \begin{cases} \boldsymbol{\nu}^{+}(t; \widetilde{\Theta} \setminus \widehat{\Theta}, \widetilde{\Theta} \setminus \widehat{\Theta}) & \text{if } \mathbf{u} = \boldsymbol{\nu}^{+}(t; \widetilde{\Theta}, \widetilde{\Theta}) \\ \boldsymbol{\nu}^{-}(t; \widetilde{\Theta} \setminus \widehat{\Theta}, \widetilde{\Theta} \setminus \widehat{\Theta}) & \text{if } \mathbf{u} = \boldsymbol{\nu}^{-}(t; \widetilde{\Theta}, \widetilde{\Theta}) \end{cases}$$

for all $t \in S$ acts as a semi-conjugacy. First, it is easy to verify that

$$
\begin{aligned}
g \circ \sigma \circ \boldsymbol{\nu}^{\pm}(t; \widetilde{\Theta}, \widetilde{\Theta}) &= g \circ \boldsymbol{\nu}^{\pm}(R_\beta(t); \widetilde{\Theta}, \widetilde{\Theta}) \\
&= \boldsymbol{\nu}^{\pm}(R_\beta(t); \widetilde{\Theta} \setminus \widehat{\Theta}, \widetilde{\Theta} \setminus \widehat{\Theta}) \\
&= \sigma \circ \boldsymbol{\nu}^{\pm}(t; \widetilde{\Theta} \setminus \widehat{\Theta}, \widetilde{\Theta} \setminus \widehat{\Theta}) \\
&= \sigma \circ g \circ \boldsymbol{\nu}^{\pm}(t; \widetilde{\Theta}, \widetilde{\Theta}).
\end{aligned}
$$

By Proposition 6.1, the map g is 1-to-1 at $\boldsymbol{\nu}^{\pm}(t; \widetilde{\Theta}, \widetilde{\Theta})$ if t is such a point in S that $R_\beta(t) \notin \widetilde{\Theta}$ for all integer n or if $R^n_\beta(t) \in \widetilde{\Theta} \setminus \widehat{\Theta}$ for some n, otherwise it is 2-to-1. To show it is continuous, suppose $\mathbf{u}_\infty$, $\mathbf{u}_1$, $\mathbf{u}_2, \ldots$ all belong to $X_{\widetilde{\Theta},\widetilde{\Theta}}$ with $\lim_{n\to\infty} \mathbf{u}_n = \mathbf{u}_\infty$, and suppose t_∞, t_1, $t_2, \ldots$ are corresponding points in S and $\mathbf{v}_\infty$, $\mathbf{v}_1$, $\mathbf{v}_2, \ldots$ are corresponding points in $X_{\widetilde{\Theta}\setminus\widehat{\Theta},\widetilde{\Theta}\setminus\widehat{\Theta}}$ given by the injectivity of each of the mappings $\boldsymbol{\nu}^+(t; \widetilde{\Theta}, \widetilde{\Theta}) \mapsto t \mapsto \boldsymbol{\nu}^+(t; \widetilde{\Theta} \setminus \widehat{\Theta}, \widetilde{\Theta} \setminus \widehat{\Theta})$ or $\boldsymbol{\nu}^-(t; \widetilde{\Theta}, \widetilde{\Theta}) \mapsto t \mapsto \boldsymbol{\nu}^-(t; \widetilde{\Theta} \setminus \widehat{\Theta}, \widetilde{\Theta} \setminus \widehat{\Theta})$. From Proposition 6.3, it follows that $\lim_{n\to\infty} t_n = t_\infty$. If $t_\infty \notin \mathcal{O}(\widetilde{\Theta} \setminus \widehat{\Theta}; R_\beta)$, then $\lim_{n\to\infty} \mathbf{v}_n = \mathbf{v}_\infty$ by Proposition 6.2. If $t_\infty \in \mathcal{O}(\widetilde{\Theta} \setminus \widehat{\Theta}; R_\beta)$, then $t_\infty \in \mathcal{O}(\widetilde{\Theta}; R_\beta)$, and $\lim_{n\to\infty} t_n \to t_\infty^+$ provided $\mathbf{u}_\infty = \boldsymbol{\nu}^+(t_\infty; \widetilde{\Theta}, \widetilde{\Theta})$ by Proposition 6.3. In this situation, $\mathbf{v}_\infty = \boldsymbol{\nu}^+(t_\infty; \widetilde{\Theta} \setminus \widehat{\Theta}, \widetilde{\Theta} \setminus \widehat{\Theta})$. Consequently, $\lim_{n\to\infty} \mathbf{v}_n = \mathbf{v}_\infty$ by using Proposition 6.2 again. The other situation that $t_\infty \in \mathcal{O}(\widetilde{\Theta} \setminus \widehat{\Theta}; R_\beta)$ and $\lim_{n\to\infty} \to t_\infty^-$ can be treated similarly. This proves the continuity of g.

$$
\begin{tikzcd}
X_{\widetilde{\Theta},\widetilde{\Theta}} \arrow[d] \arrow[dr, "g"] & \\
X^*_{\widetilde{\Theta},\widetilde{\Theta}} \arrow[r] & X_{\widetilde{\Theta}\setminus\widehat{\Theta},\widetilde{\Theta}\setminus\widehat{\Theta}}
\end{tikzcd}
$$

Now, let $X^*_{\widetilde{\Theta},\widetilde{\Theta}} = \left\{ g^{-1}(\mathbf{v}) | \ \mathbf{v} \in X_{\widetilde{\Theta}\setminus\widehat{\Theta},\widetilde{\Theta}\setminus\widehat{\Theta}} \right\}$. It is clear that $X^*_{\widetilde{\Theta},\widetilde{\Theta}} = X_{\widetilde{\Theta},\widetilde{\Theta}}/\sim_{\widehat{\Theta}}$, and that g is a quotient map. Then, by Theorem 6.4, g induces a homeomorphism via which $(X_{\widetilde{\Theta},\widetilde{\Theta}}, \sigma)/\sim_{\widehat{\Theta}}$ is conjugate to $(X_{\widetilde{\Theta}\setminus\widehat{\Theta},\widetilde{\Theta}\setminus\widehat{\Theta}}, \sigma)$. $\square$

Recall that a *lift* of an OPH $f : S \to S$ is a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ which satisfies $f(x) = F(x) \mod 1$ for $x \in [0, 1)$ and $F(x + 1) = F(x) + 1$ for every $x \in \mathbb{R}$. Such a lift is unique up to an additive constant: If $\tilde{F}$ is another lift, then $\tilde{F}(x) = F(x) + m$ for some integer m.

Proof of Theorem 5.1

f_1 is transitive, thus is conjugate to R_β. That is, there exists an OPH g of S such that

$$R_\beta \circ g = g \circ f_1.$$

Subsequently, f is semi-conjugate to f_1:

$$g^{-1} \circ h \circ f = f_1 \circ g^{-1} \circ h.$$

Let $h_1 = g^{-1} \circ h$, the orientation-preserving semi-conjugacy. Without loss of generality, we can assume $g(0) = 0$, thence $h_1(0) = g^{-1} \circ h(0) = g^{-1}(0) = 0$.

Let $H_1 : \mathbb{R} \to \mathbb{R}$ be the unique lift of h_1 satisfying $H_1(0) = 0$. For $\epsilon_0 \le \epsilon \le 1$, define a continuous map of $\mathbb{R}$:

$$H_\epsilon : \bar{x} \mapsto H_1(\bar{x}) + \frac{1-\epsilon}{1-\epsilon_0}\left(\bar{x} - H_1(\bar{x})\right).$$

We claim that H_ϵ is an OPH when $\epsilon_0 \le \epsilon < 1$. To see this, it is sufficient to show that it is strictly increasing. Suppose $\bar{x} < \bar{y}$, then $H_1(\bar{x}) \le H_1(\bar{y})$, and

$$\begin{aligned} H_\epsilon(\bar{x}) - H_\epsilon(\bar{y}) &= \frac{\epsilon-\epsilon_0}{1-\epsilon_0}\left(H_1(\bar{x}) - H_1(\bar{y})\right) + \frac{1-\epsilon}{1-\epsilon_0}\left(\bar{x} - \bar{y}\right) \\ &< 0. \end{aligned}$$

Now, because $H_\epsilon(\bar{x}+1) = H_1(\bar{x}+1) + (\bar{x}+1-H_1(\bar{x}+1))\,(1-\epsilon)/(1-\epsilon_0) = H_\epsilon(\bar{x})+1$, the map H_ϵ is a lift of an OPH $h_\epsilon : S \to S$. Clearly, maps h_ϵ form a continuous family of homeomorphisms when $\epsilon_0 \le \epsilon < 1$, and $h_\epsilon \to h_1$ uniformly on S as $\epsilon \nearrow 1$. Notice that the map $H(\epsilon, \cdot) := H_\epsilon$ acts as a straight-line homotopy for which $H(\epsilon_0, \cdot) = \mathrm{id}_{\mathbb{R}}$, and $H(1, \cdot) = H_1$, and that the map $h(\epsilon, \cdot) := h_\epsilon$ is a straight-line homotopy for which $h(\epsilon_0, \cdot) = \mathrm{id}_S$, and $h(1, \cdot) = h_1$.

Let $F : \mathbb{R} \to \mathbb{R}$ be the unique lift of f satisfying $F(0) = f(0)$. Define a family of OPHs F_ϵ of $\mathbb{R}$ by

$$F_\epsilon := H_\epsilon \circ F \circ H_\epsilon^{-1} \qquad \text{for } \epsilon_0 \le \epsilon < 1,$$

and define a family of OPHs f_ϵ of S by

$$f_\epsilon := h_\epsilon \circ f \circ h_\epsilon^{-1} \qquad \text{for } \epsilon_0 \le \epsilon < 1.$$

Clearly, F_ϵ is a lift of f_ϵ, satisfying $F_\epsilon(0) = f_\epsilon(0)$.

Now, from the Proposition 6.5 below and Theorem 4.3, it follows that $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to $(X_{Q,\Phi}, \sigma)$ when $\epsilon_0 \le \epsilon < 1$. And, the proof of Theorem 5.1 will be complete if we prove the Proposition 6.6 below. $\square$

Let $I_{n,\epsilon}^{(k)} = h_\epsilon(I_n^{(k)})$.

Proposition 6.5. *f_ϵ is a Denjoy homeomorphism having rotation number β for every $\epsilon_0 \le \epsilon < 1$. The set $S \setminus \bigcup_{n\in\mathbb{Z}} \bigcup_{1\le k\le K} I_{n,\epsilon}^{(k)}$ is equal to $\omega(f_\epsilon)$, and $\mathcal{D}(f_\epsilon) = \mathcal{D}(f)$.*

Proof. f_ϵ is conjugate to f via h_ϵ^{-1}, and $I_{n,\epsilon}^{(k)}$'s are the wandering intervals. In particular, $\bigcup_{n\in\mathbb{Z}} \bigcup_{1\le k\le K} I_{n,\epsilon}^{(k)}$ is a continuous image of $\bigcup_{n\in\mathbb{Z}} \bigcup_{1\le k\le K} I_n^{(k)}$ under h_ϵ, thus is dense in S. $\square$

Proposition 6.6. $\lim_{\epsilon\nearrow 1} f_\epsilon(t) \to f_1(t)$ *for all t in S.*

Proof. It is convenient to prove the proposition by showing that $\lim_{\epsilon\nearrow 1} F_\epsilon(\bar{t}) \to F_1(\bar{t})$ for all $\bar{t} \in \mathbb{R}$, where F_1 is the unique lift of f_1 satisfying $F_1(0) = f_1(0)$.

Let $\pi : \mathbb{R} \to S$, $\bar{t} \mapsto \bar{t} \mod 1$, be the projection, $\bar{a}_n^{(k)}$, $\bar{b}_n^{(k)}$ be real numbers, and $\bar{I}_{n,\epsilon}^{(k)}$, $\bar{I}_n^{(k)}$ be open intervals on $\mathbb{R}$ such that $\bar{I}_{n,\epsilon_0}^{(k)} = \bar{I}_n^{(k)} = (\bar{a}_n^{(k)}, \bar{b}_n^{(k)})$, $\pi(\bar{I}_{n,\epsilon}^{(k)}) = I_{n,\epsilon}^{(k)}$, $F_\epsilon^n(\bar{I}_{0,\epsilon}^{(k)}) = \bar{I}_{n,\epsilon}^{(k)}$, $H_\epsilon(\bar{I}_n^{(k)}) = \bar{I}_{n,\epsilon}^{(k)}$ for all $\epsilon_0 \le \epsilon < 1$, $n \in \mathbb{Z}$, and $1 \le k \le K$.

There are two cases: $H_1^{-1}(\bar{t}) = \bar{I}_n^{(k)} + p$ for some n, $p \in \mathbb{Z}$, $1 \le k \le K$, or $H_1^{-1}(\bar{t}) = \bar{x}$ for some $x \in \mathbb{R}$.

There are two sub-cases for the first case: $\bar{t} \in \operatorname{cl} \bar{I}_n^{(k)} + p$ or not. If $\bar{t} \in \operatorname{cl} \bar{I}_n^{(k)} + p$, then $\bar{t} \in \operatorname{cl} \bar{I}_{n,\epsilon}^{(k)} + p$ for all $\epsilon_0 \le \epsilon < 1$. Consequently, $H_\epsilon \circ F \circ H_\epsilon^{-1}(\bar{t}) \in \bar{I}_{n+1,\epsilon}^{(k)} + p$ for all $\epsilon_0 \le \epsilon < 1$. And, $\lim_{\epsilon\nearrow 1} \bar{I}_{n+1,\epsilon}^{(k)} + p = F_1(\bar{t})$. If $\bar{t} \notin \operatorname{cl} \bar{I}_n^{(k)} + p$, then $\bar{t} \notin \operatorname{cl} \bar{I}_{n,\epsilon}^{(k)} + p$ for all $\epsilon_0 \le \epsilon < 1$. In this situation, suppose $\bar{b}_n^{(k)} + p < \bar{t}$. (The other situation $a_n^{(k)} + p > \bar{t}$ can be treated similarly.) Let $\bar{y}_\epsilon = H_\epsilon^{-1}(\bar{t})$. Then $\bar{y}_\epsilon > \bar{b}_n^{(k)} + p$ and $\bar{y}_\epsilon \to \bar{b}_n^{(k)+} + p$ as $\epsilon \nearrow 1$. Now,

$$F_\epsilon(\bar{t}) = H_1(F(\bar{y}_\epsilon)) + \frac{1-\epsilon}{1-\epsilon_0}\left(F(\bar{y}_\epsilon) - H_1(F(\bar{y}_\epsilon))\right), \tag{17}$$

$$F_1(\bar{t}) = H_1(F(\bar{b}_n^{(k)} + p). \tag{18}$$

Because the distance between $\bar{y}$ and $H_1(\bar{y})$ is bounded above by 1 for any $\bar{y} \in \mathbb{R}$ and because F and H_1 are continuous, $F_\epsilon(\bar{t}) \to F_1(\bar{t})$ as $\epsilon \nearrow 1$.

For the second case $H_1^{-1}(\bar{t}) = \bar{x}$, the proof is essentially the same as the first case. There are two sub-cases: $\bar{t} = \bar{x}$ or not. If $\bar{t} = \bar{x}$, then $\bar{t} = H_\epsilon^{-1}(\bar{t}) = \bar{x}$ for all $\epsilon_0 \le \epsilon < 1$, hence $\lim_{\epsilon\nearrow 1} H_\epsilon \circ F \circ H_\epsilon^{-1}(t) = H_1 \circ F(\bar{t}) = F_1 \circ H_1(\bar{t}) = F_1(\bar{t})$. If $\bar{t} \ne \bar{x}$, then $\bar{t} \ne H_\epsilon^{-1}(\bar{t}) \ne \bar{x}$ for all $\epsilon_0 \le \epsilon < 1$. In this situation, suppose $\bar{x} < \bar{t}$. (The alternative situation $\bar{x} > \bar{t}$ can be treated similarly.) Let $\bar{x}_\epsilon = H_\epsilon^{-1}(\bar{t})$. Then $\bar{x}_\epsilon \to \bar{x}^+$ as $\epsilon \nearrow 1$. Then repeating calculations (17) and (18) but replacing $\bar{y}_\epsilon$ by $\bar{x}_\epsilon$, $\bar{b}_n^{(k)} + p$ by $\bar{x}$, and using continuity of F and H_1 again, we conclude $F_\epsilon(\bar{t}) \to F_1(\bar{t})$ as $\epsilon \nearrow 1$. □

Proof of Theorem 5.2

The proof of this theorem is similar to that of Theorem 5.1. $f_{\tilde{\epsilon}}$ is semi-conjugate to R_β, thus there exists an orientation-preserving surjection $h_{\tilde{\epsilon}}$ of S such that

$$R_\beta \circ h_{\tilde{\epsilon}} = h_{\tilde{\epsilon}} \circ f_{\tilde{\epsilon}}.$$

The wandering intervals of $f_{\tilde{\epsilon}}$ consists of the union $\bigcup_{\theta \in \mathcal{O}(\tilde{\Theta}; R_\beta)} h_{\tilde{\epsilon}}^{-1}(\theta)$. There exists an orientation-preserving semi-conjugacy g such that

$$g \circ f = f_{\tilde{\epsilon}} \circ g$$

and that

$$h = h_{\tilde{\epsilon}} \circ g$$

by choosing an appropriate $h_{\tilde{\epsilon}}$. Except on the set $\bigcup_{\theta \in \mathcal{O}(\Theta \setminus \widetilde{\Theta}; R_\beta)} h^{-1}(\theta)$, the semi-conjugacy g is 1-to-1. If $\theta \in \mathcal{O}(\Theta \setminus \widetilde{\Theta}; R_\beta)$, $h^{-1}(\theta)$ consists of two points, but the image of $h^{-1}(\theta)$ under g is a single point.

Let $G : \mathbb{R} \to \mathbb{R}$ be the lift of g satisfying $G(0) = 0$. For $\epsilon_0 \leq \epsilon \leq \tilde{\epsilon}$, define a continuous map of $\mathbb{R}$:

$$G_\epsilon : \bar{x} \mapsto G(\bar{x}) + \frac{\tilde{\epsilon} - \epsilon}{\tilde{\epsilon} - \epsilon_0} (\bar{x} - G(\bar{x})) .$$

We claim that G_ϵ is an OPH when $\epsilon_0 \leq \epsilon < \tilde{\epsilon}$. To see this, it is sufficient to show that it is strictly increasing. Suppose $\bar{x} < \bar{y}$, then $G(\bar{x}) \leq G(\bar{y})$, and

$$\begin{aligned} G_\epsilon(\bar{x}) - G_\epsilon(\bar{y}) &= \frac{\epsilon - \epsilon_0}{\tilde{\epsilon} - \epsilon_0} (G(\bar{x}) - G(\bar{y})) + \frac{\tilde{\epsilon} - \epsilon}{\tilde{\epsilon} - \epsilon_0} (\bar{x} - \bar{y}) \\ &< 0. \end{aligned}$$

Now, because $G_\epsilon(\bar{x}+1) = G(\bar{x}+1) + (\bar{x}+1 - G(\bar{x}+1))(\tilde{\epsilon}-\epsilon)/(\tilde{\epsilon}-\epsilon_0) = G_\epsilon(\bar{x})+1$, the map G_ϵ is a lift of an OPH $g_\epsilon : S \to S$. Clearly, maps g_ϵ form a continuous family of homeomorphisms when $\epsilon_0 \leq \epsilon < \tilde{\epsilon}$, and $g_\epsilon \to g$ uniformly on S as $\epsilon \nearrow \tilde{\epsilon}$. Notice that the map $G(\epsilon, \cdot) := G_\epsilon$ acts as a straight-line homotopy for which $G(\epsilon_0, \cdot) = \mathrm{id}_{\mathbb{R}}$, and $H(\tilde{\epsilon}, \cdot) = G$, and that the map $g(\epsilon, \cdot) := g_\epsilon$ is a straight-line homotopy for which $g(\epsilon_0, \cdot) = \mathrm{id}_S$, and $h(\tilde{\epsilon}, \cdot) = g$.

Let $F : \mathbb{R} \to \mathbb{R}$ be the unique lift of f satisfying $F(0) = f(0)$. Define a family of OPHs F_ϵ of $\mathbb{R}$ by

$$F_\epsilon := G_\epsilon \circ F \circ G_\epsilon^{-1} \qquad \text{for } \epsilon_0 \leq \epsilon < \tilde{\epsilon},$$

and define a family of OPHs f_ϵ of S by

$$f_\epsilon := g_\epsilon \circ f \circ g_\epsilon^{-1} \qquad \text{for } \epsilon_0 \leq \epsilon < \tilde{\epsilon}.$$

Clearly, F_ϵ is a lift of f_ϵ, satisfying $F_\epsilon(0) = f_\epsilon(0)$.

Now, from the Proposition 6.7 below and Theorem 4.3, it follows that $(\omega(f_\epsilon), f_\epsilon)$ is conjugate to $(X_{\Theta,\Theta}, \sigma)$ when $\epsilon_0 \leq \epsilon < \tilde{\epsilon}$. And, the proof of Theorem 5.2 will be complete if we prove the Proposition 6.8 below. □

Let $I_{n,\epsilon}^{(k)} = g_\epsilon(I_n^{(k)})$.

Proposition 6.7. *f_ϵ is a Denjoy homeomorphism having rotation number β when $\epsilon_0 \leq \epsilon < \tilde{\epsilon}$. The set $S \setminus \bigcup_{n \in \mathbb{Z}} \bigcup_{1 \leq k \leq K} I_{n,\epsilon}^{(k)}$ is equal to $\omega(f_\epsilon)$, and $\mathcal{D}(f_\epsilon) = \mathcal{D}(f)$.*

Proof. f_ϵ is conjugate to f via g_ϵ^{-1}, and $I_{n,\epsilon}^{(k)}$'s are the wandering intervals. In particular, $\bigcup_{n \in \mathbb{Z}} \bigcup_{1 \leq k \leq K} I_{n,\epsilon}^{(k)}$ is a continuous image of $\bigcup_{n \in \mathbb{Z}} \bigcup_{1 \leq k \leq K} I_n^{(k)}$ under g_ϵ, thus is dense in S. □

Proposition 6.8. $\lim_{\epsilon \nearrow \tilde{\epsilon}} f_\epsilon(t) \to f_{\tilde{\epsilon}}(t)$ *for all* t *in* S.

Proof. It is convenient to prove the proposition by showing that $\lim_{\epsilon \nearrow \tilde{\epsilon}} F_\epsilon(\bar{t}) \to F_{\tilde{\epsilon}}(\bar{t})$ for all $\bar{t} \in \mathbb{R}$, where $F_{\tilde{\epsilon}}$ is the unique lift of $f_{\tilde{\epsilon}}$ satisfying $F_{\tilde{\epsilon}}(0) = f_{\tilde{\epsilon}}(0)$.

Let $\pi : \mathbb{R} \to S$, $\bar{t} \mapsto \bar{t} \mod 1$, be the projection, $\bar{a}_n^{(k)}$, $\bar{b}_n^{(k)}$ be real numbers, and $\bar{I}_{n,\epsilon}^{(k)}$, $\bar{I}_n^{(k)}$ be open intervals on $\mathbb{R}$ such that $\bar{I}_{n,\epsilon_0}^{(k)} = \bar{I}_n^{(k)} = (\bar{a}_n^{(k)}, \bar{b}_n^{(k)})$, $\pi(\bar{I}_{n,\epsilon}^{(k)}) = I_{n,\epsilon}^{(k)}$, $F_\epsilon^n(\bar{I}_{0,\epsilon}^{(k)}) = \bar{I}_{n,\epsilon}^{(k)}$, $G_\epsilon(\bar{I}_n^{(k)}) = \bar{I}_{n,\epsilon}^{(k)}$ for all $\epsilon_0 \le \epsilon < \tilde{\epsilon}$, $n \in \mathbb{Z}$, and $1 \le k \le K$.

There are two cases: $G^{-1}(\bar{t}) = \bar{I}_n^{(k)} + p$ for some n, $p \in \mathbb{Z}$, $1 \le k \le K$, or $G^{-1}(\bar{t}) = \bar{x}$ for some $x \in \mathbb{R}$.

There are two sub-cases for the first case: $\bar{t} \in \operatorname{cl} \bar{I}_n^{(k)} + p$ or not. If $\bar{t} \in \operatorname{cl} \bar{I}_n^{(k)} + p$, then $\bar{t} \in \operatorname{cl} \bar{I}_{n,\epsilon}^{(k)} + p$ for all $\epsilon_0 \le \epsilon < \tilde{\epsilon}$. Consequently, $G_\epsilon \circ F \circ G_\epsilon^{-1}(\bar{t}) \in \bar{I}_{n+1,\epsilon}^{(k)} + p$ for all $\epsilon_0 \le \epsilon < \tilde{\epsilon}$, and $\lim_{\epsilon \nearrow \tilde{\epsilon}} \bar{I}_{n+1,\epsilon}^{(k)} + p = F_{\tilde{\epsilon}}(\bar{t})$. If $\bar{t} \notin \operatorname{cl} \bar{I}_n^{(k)} + p$, then $\bar{t} \notin \operatorname{cl} \bar{I}_{n,\epsilon}^{(k)} + p$ for all $\epsilon_0 \le \epsilon < \tilde{\epsilon}$. In this situation, suppose $\bar{b}_n^{(k)} + p < \bar{t}$. (The other situation $a_n^{(k)} + p > \bar{t}$ can be treated similarly.) Let $\bar{y}_\epsilon = G_\epsilon^{-1}(\bar{t})$. Then $\bar{y}_\epsilon > \bar{b}_n^{(k)} + p$ and $\bar{y}_\epsilon \to \bar{b}_n^{(k)+} + p$ as $\epsilon \nearrow \tilde{\epsilon}$. Now,

$$F_\epsilon(\bar{t}) = G(F(\bar{y}_\epsilon)) + \frac{\tilde{\epsilon} - \epsilon}{\tilde{\epsilon} - \epsilon_0} \left(F(\bar{y}_\epsilon) - G(F(\bar{y}_\epsilon)) \right), \tag{19}$$

$$F_{\tilde{\epsilon}}(\bar{t}) = G(F(\bar{b}_n^{(k)} + p). \tag{20}$$

Because the distance between $\bar{y}$ and $G(\bar{y})$ is bounded above by 1 for any $\bar{y} \in \mathbb{R}$ and because F and G are continuous, $F_\epsilon(\bar{t}) \to F_{\tilde{\epsilon}}(\bar{t})$ as $\epsilon \nearrow \tilde{\epsilon}$.

For the second case $G^{-1}(\bar{t}) = \bar{x}$, the proof is essentially the same as the first case. There are two sub-cases: $\bar{t} = \bar{x}$ or not. If $\bar{t} = \bar{x}$, then $\bar{t} = G_\epsilon^{-1}(\bar{t}) = \bar{x}$ for all $\epsilon_0 \le \epsilon < \tilde{\epsilon}$, hence $\lim_{\epsilon \nearrow \tilde{\epsilon}} G_\epsilon \circ F \circ G_\epsilon^{-1}(t) = G \circ F(\bar{t}) = F_{\tilde{\epsilon}} \circ G(\bar{t}) = F_{\tilde{\epsilon}}(\bar{t})$. If $\bar{t} \neq \bar{x}$, then $\bar{t} \neq G_\epsilon^{-1}(\bar{t}) \neq \bar{x}$ for all $\epsilon_0 \le \epsilon < \tilde{\epsilon}$. In this situation, suppose $\bar{x} < \bar{t}$. (The alternative situation $\bar{x} > \bar{t}$ can be treated similarly.) Let $\bar{x}_\epsilon = G_\epsilon^{-1}(\bar{t})$. Then $\bar{x}_\epsilon \to \bar{x}^+$ as $\epsilon \nearrow \tilde{\epsilon}$. Subsequently, repeating calculations (19) and (20) but replacing $\bar{y}_\epsilon$ by $\bar{x}_\epsilon$, $\bar{b}_n^{(k)} + p$ by $\bar{x}$, and using continuity of F and G again, we conclude $F_\epsilon(\bar{t}) \to F_{\tilde{\epsilon}}(\bar{t})$ as $\epsilon \nearrow \tilde{\epsilon}$. ☐

Proof of Theorem 5.3

By Theorem 4.3, $(\omega(f), f)$ is semi-conjugate to $(X_{Q,\Phi}, \sigma)$ via the semi-conjugacy $E(\cdot; Q, \Phi)$. Let $H : \mathbb{R} \to \mathbb{R}$ be the unique lift of h satisfying $H(0) = h(0)$. For $0 \le \epsilon \le \epsilon_0$, define a continuous map G_ϵ of $\mathbb{R}$:

$$G_\epsilon : \bar{x} \mapsto \begin{cases} \left(1 - \dfrac{\epsilon}{\epsilon_0}\right) \left(\phi_{i-1} + \dfrac{\bar{x} - \inf H^{-1}(q_i)}{\sup H^{-1}(q_i) - \inf H^{-1}(q_i)} (\phi_i - \phi_{i-1}) \right) + \dfrac{\epsilon}{\epsilon_0} \bar{x} \\ \qquad\qquad\qquad\qquad \text{if } \inf H^{-1}(q_i) \le \bar{x} \le \sup H^{-1}(q_i) \text{ and } 1 \le i \le N \\ \left(1 - \dfrac{\epsilon}{\epsilon_0}\right) \phi_i + \dfrac{\epsilon}{\epsilon_0} \bar{x} \\ \qquad\qquad\qquad\qquad \text{if } \sup H^{-1}(q_i) \le \bar{x} \le \inf H^{-1}(q_{i+1}) \text{ and } 1 \le i \le N, \end{cases}$$

where $N = \sharp(Q)$, $q_i \in Q$, $q_{N+1} = q_1 + 1$, $\phi_i \in \Phi$ and $\phi_0 = \phi_N - 1$. By using the property $H(\bar{x} + 1) = H(\bar{x}) + 1$, the map G_ϵ can be defined on the entire real numbers, and has the property $G_\epsilon(\bar{x} + 1) = G_\epsilon(\bar{x}) + 1$. It is clear that G_ϵ is strictly increasing on both intervals $[\inf H^{-1}(q_i), \sup H^{-1}(q_i)]$ and $[\sup H^{-1}(q_i), \inf H^{-1}(q_{i+1})]$ for non-zero ϵ. Consequently, G_ϵ is an OPH when $0 < \epsilon \leq \epsilon_0$.

Note that the image of the interval $[\inf H^{-1}(q_i), \sup H^{-1}(q_i)]$ under G_0 is the interval $[\phi_{i-1}, \phi_i]$, while the image of $[\sup H^{-1}(q_i), \inf H^{-1}(q_{i+1})]$ is the single point ϕ_i.

Let $F : \mathbb{R} \to \mathbb{R}$ be the unique lift of f satisfying $F(0) = f(0)$. Define a continuous family of OPHs F_ϵ of $\mathbb{R}$ by

$$F_\epsilon := G_\epsilon \circ F \circ G_\epsilon^{-1} \qquad \text{for } 0 < \epsilon \leq \epsilon_0.$$

The map G_ϵ is a lift of an OPH g_ϵ of S. Notice that $G_{\epsilon_0} = \mathrm{id}_{\mathbb{R}}$ and $g_{\epsilon_0} = \mathrm{id}_S$. Define also a continuous family of OPHs f_ϵ of S by

$$f_\epsilon := g_\epsilon \circ f \circ g_\epsilon^{-1} \qquad \text{for } 0 < \epsilon \leq \epsilon_0.$$

Clearly, F_ϵ is a lift of f_ϵ, satisfying $F_\epsilon(0) = f_\epsilon(0)$.

Let $I_{n,\epsilon}^{(k)} = g_\epsilon(I_n^{(k)})$. Because f_ϵ is conjugate to f via g_ϵ^{-1}, the map f_ϵ is a Denjoy homeomorphism having rotation number β when $0 < \epsilon \leq \epsilon_0$. The set $S \backslash \bigcup_{n\in\mathbb{Z}} \bigcup_{1\leq k\leq K} I_{n,\epsilon}^{(k)}$ is equal to $\omega(f_\epsilon)$. In particular, $\bigcup_{n\in\mathbb{Z}} \bigcup_{1\leq k\leq K} I_{n,\epsilon}^{(k)}$ is a continuous image of $\bigcup_{n\in\mathbb{Z}} \bigcup_{1\leq k\leq K} I_n^{(k)}$, thus is dense in S. Hence, $(\omega(f_\epsilon), f_\epsilon)$ is semi-conjugate to $(X_{Q,\Phi}, \sigma)$ when $0 < \epsilon \leq \epsilon_0$. The proof of the theorem will be complete if we prove the Proposition 6.9 below. □

Proposition 6.9. $\lim_{\epsilon\searrow 0} \mathcal{O}\left(E_\epsilon^{-1}(\mathbf{u}; Q, \Phi), f_\epsilon\right) = \mathbf{u}$ *in the uniform topology for all* $\mathbf{u} \in X_{Q,\Phi}$.

Proof. Let $z_{i,\epsilon}$, $1 \leq i \leq N = \sharp(Q)$, be any point in $g_\epsilon(h^{-1}(q_i))$ and $A_{i,\epsilon}$ be open intervals delimited by $z_{i,\epsilon}$'s on S: $A_{1,\epsilon} = (z_{1,\epsilon}, z_{2,\epsilon})$, $A_{2,\epsilon} = (z_{2,\epsilon}, z_{3,\epsilon}), \ldots, A_{N,\epsilon} = (z_{N,\epsilon}, z_{1,\epsilon})$. Then, with the partition-symbol pair (Q, Φ), a family of coding sequences $E_\epsilon(\cdot; Q, \Phi)$ can be constructed as in (9)–(11) (replacing the sets A_i's in (10) by $A_{i,\epsilon}$'s here), via which f_ϵ is semi-conjugate to $(X_{Q,\Phi}, \sigma)$. Given $\mathbf{u} = (u_n)_{n\in\mathbb{Z}} \in X_{Q,\Phi}$, let $x_\epsilon = E_\epsilon^{-1}(\mathbf{u}; Q, \Phi)$, and $f_\epsilon^n(x_\epsilon) = x_{n,\epsilon}$ for all integer n. Then $x_{n,\epsilon}$ belongs to the closed interval $[g_\epsilon\left(\sup h^{-1}(q_{i_n})\right), g_\epsilon\left(\inf h^{-1}(q_{i_n+1})\right)]$, where every i_n satisfies $\phi_{i_n} = u_n$. Subsequently, by our construction of g_ϵ, both points $g_\epsilon\left(\sup h^{-1}(q_{i_n})\right)$ and $g_\epsilon\left(\inf h^{-1}(q_{i_n+1})\right)$ converge to point ϕ_{i_n} as $\epsilon \searrow 0$ uniformly in n. □

Acknowledgments

The author was supported by NSC 97-2119-M-001-008, 98-2115-M-001-012, 102-2115-M-001-009, and by MOST 104-2115-M-001-007. He thanks Fumiaki Sugisaki for helpful discussions, and Kumamoto University for kind hospitality during his visit in 2013. An earlier version of the results of this paper were first announced at The Dynamics of Complex Systems: A meeting in honour of the 60th birthday of Robert MacKay FRS.

References

[1] Akiyama, S., Shirasaka, M.: Recursively renewable words and coding of irrational rotations. *J. Math. Soc. Japan* **59** (2007), 1199–1234.

[2] Alessandri, P., Berthé, V.: Three distance theorems and combinatorics on words. *Enseign. Math. (2)* **44** (1998), 103–132.

[3] Arnoux, P.: Sturmian sequences. In: Berthé, V., Ferenczi, S., Mauduit, C., Siegel, A. (eds.) *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Math., 1794, pp. 143–198, Springer, Berlin (2002)

[4] Aubry, S.: Anti-integrability in dynamical and variational problems. *Physica D* **86** (1995), 284–296.

[5] Aubry, S., Abramovici, G.: Chaotic trajectories in the standard map: the concept of anti-integrability. *Physica D* **43** (1990), 199–219.

[6] Aubry, S., Gosso, J.-P., Abramovici, G., Raimbault, J.-L., Quemerais, P.: Effective discommensurations in the incommensurate ground-states of the extended Frenkel-Kontorova models. *Physica D* **47** (1991), 461–497.

[7] Baesens, C., Chen, Y.-C., MacKay, R.S.: Abrupt bifurcations in chaotic scattering: view from the anti-integrable limit. *Nonlinearity* **26** (2013), 2703–2730.

[8] Baesens, C., MacKay, R.S.: Cantori for multiharmonic maps. *Physica D* **69** (1993), 59–76.

[9] Baesens, C., MacKay, R.S.: The one to two-hole transition for cantori. *Physica D* **71** (1994), 372–389.

[10] Boyland, P.L.: Invariant circles and rotation bands in monotone twist maps. *Comm. Math. Phys.* **113** (1987), 67–77.

[11] Boyland, P.: Weak disks of Denjoy minimal sets. *Ergodic Theory Dynam. Systems* **13** (1993), 597–614.

[12] Chen, Y.-C.: On topological entropy of billiard tables with small inner scatterers. *Advances in Mathematics* **224** (2010), 432–460.

[13] Cornfeld, I.P., Fomin, S.V., Sinai, Ya. G.: *Ergodic theory*, Grundlehren der Mathematischen Wissenschaften, **245**, Springer-Verlag, New York (1982)

[14] Coven, E.M., Hedlund, G.A.: Sequences with minimal block growth. *Math. Systems Theory* **7** (1973), 138–153.

[15] Hedlund, G.A.: Sturmian minimal sets. *Amer. J. Math.* **66** (1944), 605–620.

[16] Hockett, K., Holmes, P.: Josephson's junction, annulus maps, Birkhoff attractors, horseshoes and rotation sets. *Ergodic Theory Dynam. Systems* **6** (1986), 205–239.

[17] Katok, A., Hasselblatt, B.: *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press (1995)

[18] MacKay, R.S., Meiss, J.D.: Cantori for symplectic maps near the anti-integrable limit. *Nonlinearity* **5** (1992), 149–160.

[19] MacKay, R.S.: Greene's residue criterion. *Nonlinearity* **5** (1992), 161–187.

[20] Markley, N.G.: Homeomorphisms of the circle without periodic points. *Proc. London Math. Soc.* (3) **20** (1970), 688–698.

[21] Masui, K.: Denjoy Systems and Substitutions. *Tokyo Journal of Mathematics* **32** (2009), 33–53.

[22] Mather, J.N.: More Denjoy minimal sets for area preserving diffeomorphisms. *Comment. Math. Helv.* **60** (1985), 508–557.

[23] Morse, M., Hedlund, G.A.: Symbolic dynamics II. Sturmian trajectories. *Amer. J. Math.* **62** (1940), 1–42.

[24] Munkres, J.R.: *Topology* 2nd. Edition, Prentice Hall, NJ (2000)

[25] Nitecki, Z.: *Differentiable dynamics. An introduction to the orbit structure of diffeomorphisms.* The MIT Press, Cambridge, Mass.-London (1971)

[26] Rote, G.: Sequences with subword complexity $2n$. *J. Number Theory* **46** (1994), 196–213.