NECKPINCH SINGULARITIES
IN FRACTIONAL MEAN CURVATURE FLOWS

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ABSTRACT. In this paper we consider the evolution of sets by a fractional mean curvature flow. Our main result states that for any dimension $n \geqslant 2$, there exists an embedded surface in $\mathbb{R}^n$ evolving by fractional mean curvature flow, which developes a singularity before it can shrink to a point. When $n \geqslant 3$ this result generalizes the analogue result of Grayson [18] for the classical mean curvature flow. Interestingly, when $n = 2$, our result provides instead a counterexample in the nonlocal framework to the well known Grayson Theorem [19], which states that any smooth embedded curve in the plane evolving by (classical) MCF shrinks to a point.

1. INTRODUCTION

This paper is concerned with the study of a *nonlocal mean curvature flow*. More precisely, we want to study the evolution E_t, for time $t > 0$ of an inital set E_0, such that the velocity of a point $x \in \partial E_t$ in the outer normal direction ν is given by the quantity $-H_E^s$, where H_E^s denotes the fractional mean curvature of E, that is, for any $x \in \partial E_t$ we have

$$\partial_t x \cdot \nu = -H_{E_t}^s. \tag{1}$$

For a real parameter $s \in (0, 1)$, we recall that the fractional mean curvature of a set E at a point $x \in \partial E$ is defined as follows

$$H_E^s(x) := \int_{\mathbb{R}^n} \frac{\chi_{\mathcal{C}E}(y) - \chi_E(y)}{|x - y|^{n+s}} \, dy, \tag{2}$$

where χ_A denotes the characteristic function of the set A, $\mathcal{C}A$ denotes the complement of A, and the integral above has to be understood in the principal value sense.

This evolution is the natural analogue in the nonlocal setting of the classical mean curvature flow, which has been widely studied in the last decades, see e.g. [15, 27].

While the classical mean curvature flow is the L^2-gradient flow of the usual perimeter functional, it can be proved that (1) is the L^2-gradient flow of the *fractional perimeter*, which was first introduced in [5], where suitable density estimates, a monotonicity formula, and some regularity results for minimizers were established. The question of the regularity for minimizers of the fractional perimeter was also addressed in several recent works (see [3, 14, 8, 29]). Further motivations for the study of (1) come from dislocation dynamics and phasefield theory for fractional reaction-diffusion equations, see [26].

The classical mean curvature flow is a quasilinear parabolic problem and a local existence result holds for smooth solutions starting from any compact regular initial surface, see [17, 27]. As time evolves, solutions typically develop singularities in finite time due to curvature blowup. Several notions of generalized solutions have been introduced to study the flow after the onset of singularities. Particularly relevant for our purposes is the definition by Chen, Giga, Goto [13] and by Evans and Spruck [16], based on the level set approach and the notion of viscosity solutions.

For the fractional mean curvature flow, local smooth solutions are also expected to exist, but no proof of this is available yet. There are existence results for viscosity solutions, first obtained by Imbert [26], and later extended to more general nonlocal flows by Chambolle, Morini, Ponsiglione in [10, 9] and by Chambolle, Novaga, Ruffini [12].

The analysis of the formation of singularities is an important topic in classical mean curvature flow. A pioneering result in this framework was obtained in [17] and [20], where it was shown that a closed convex surface remains convex along the evolution and shrinks to a point in finite time. If convexity is dropped, then other kinds of singular behaviour may occur. A standard example is the so called *neckpinch*. The idea is to consider a surface which looks like two large balls connected by a cylindrical very thin neck, so that, in dimension $n > 2$, the mean curvature in the neck is much larger than the one in the balls, and hence the radius of the neck must go to zero faster than the radius of the balls. The existence of this type of surfaces was first proved by Grayson [18] and later considered with a simplified proof by Ecker [15], see also Angenent [2]. In the 2-dimensional case a result by Grayson [19] ensures instead that *any smooth embedded curve in the plane evolving by mean curvature flow shrinks to a point.*

In this paper we construct examples of neckpinch singularities for the fractional mean curvature flow. More precisely, we obtain the following result

Theorem 1. *Let $n \geqslant 2$. There exists an embedded surface $\mathcal{M}_0$ in $\mathbb{R}^n$ such that the viscosity solution of the fractional mean curvature flow* (1) *starting from $\mathcal{M}_0$, developes a singularity before it shrinks to a point.*

We stress that, in contrast to the classical case, our construction can be made in any dimension, in the particular for $n = 2$, showing that Grayson's theorem fails in the nonlocal case. It also shows that the distance comparison property for curves for the classical flow proved by Huisken [22] no longer holds in the nonlocal case. Our result has some analogies with the one of [4], where Delaunay-type periodic curves in the plane with constant fractional mean curvature are constructed, while in the classical case such objects exist only in dimension $n \geqslant 3$.

A crucial step in the proof of Theorem 1 is provided by the following result, of independent interest: if a set E is contained in a strip and its boundary ∂E has sufficiently small slope and small classical curvatures, then the fractional mean curvature of E is bounded below by a positive universal constant, which depends only on s and on the dimension n. This is the content of Proposition 5. In particular, a thin set can have all negative classical principal curvature at some point, but positive fractional mean curvature.

Other recent contributions in the study of the fractional (and more general nonlocal) mean curvature flows are the articles [6, 12, 28]. In [6], the convergence of a class of threshold dynamics approximations to moving fronts was established. In particular, threshold

dynamics associated to fractional powers of the Laplacian of order $s \in (0,2)$ were considered: interestingly, when $s \in [1,2)$ the resulting interface moves by a (weighted) mean curvature flows, while when $s \in (0,1)$ it moves by a fractional mean curvature flow. In [12], the results contained in [6] have been extended to the anisotropic case. Finally, in [28] smooth solutions to the fractional mean curvature flow were studied and the evolution equations for several geometric local and nonlocal quantities were computed. The particular cases of entire graphs and star-shaped surfaces were considered, obtaining striking analogies with the properties of the classical case.

The paper is organized as follows:

- In Section 2 we describe the level set approach in the study of nonlocal mean curvature flows, and we recall the notion of viscosity solutions and a classical comparison principle;
- In Section 3 we establish some estimates on the nonlocal mean curvature of perturbed strips, which will be a crucial step in the proof of our main result;
- In Section 4, we prove our main result Theorem 1.

2. Viscosity solutions and comparison principles via the level set approach

In this Section we recall the notion of viscosity solutions for the fractional mean curvature flow, which is based on the *level sets approach*. The idea is the following: given an inital surface $\mathcal{M}_0 = \partial E_0$, we choose any continuous function $u_0 : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mathcal{M}_0 = \{x \in \mathbb{R}^n \ : \ u_0(x) = 0\}. \tag{3}$$

The geometric equation satisfied by the evolution $\mathcal{M}_t$ of $\mathcal{M}_0$ can then be translated into an equation satisfied by a function $u(x,t)$, where $u(x,0) = u_0(x)$ and at each time

$$\mathcal{M}_t = \{x \in \mathbb{R}^n \ : \ u(\cdot,t) = 0\}. \tag{4}$$

More precisely, the *level set equation* satisfied by u is

$$\partial_t u + H^s[x, u(\cdot,t)]|Du(x,t)| = 0 \quad \text{in } \mathbb{R}^n \times (0,+\infty), \tag{5}$$

where u satisfies the inital condition

$$u(x,t) = u_0(x) \quad \text{in } \mathbb{R}^n.$$

Here and in the following we denote by $H_s[x, u(\cdot,t)]$ the fractional mean curvature of the superlevel set of $u(\cdot,t)$ at the point x, i.e.

$$H^s[x, u(\cdot,t)] = H^s_{\{y \in \mathbb{R}^n \, : \, u(y,t) > u(x,t)\}}(x).$$

Of course the defintion of $\mathcal{M}_t$ is well posed if one shows that equation (5) has a unique solution, and definition (3) does not depend on the inital choice of the function u_0. These basic properties were established in [16] for the classical mean curvature flow and in [26] for the fractional one.

Since the nonlocal case is not so standard and for the sake of completeness, we recall here below all the rigourous definitions and the basic results in [26] (see also [10]). Let

$\mathcal{M} = \{x \in \mathbb{R}^n : u(x) = 0\} = \partial\{x \in \mathbb{R}^n : u(x) > 0\}$. If $u \in C^{1,1}$ and $Du \neq 0$, we can define the following quantities

$$
\begin{aligned}
k^*[x, \mathcal{M}] = k^*[x, u] &= \int_{\mathbb{R}^n} \frac{\chi_{\{u(x+z) \geqslant u(x)\}}(z) - \chi_{\{u(x+z) < u(x)\}}(z)}{|z|^{n+s}} \, dz, \\
k_*[x, \mathcal{M}] = k_*[x, u] &= \int_{\mathbb{R}^n} \frac{\chi_{\{u(x+z) > u(x)\}}(z) - \chi_{\{u(x+z) \leqslant u(x)\}}(z)}{|z|^{n+s}} \, dz.
\end{aligned}
\tag{6}
$$

It is easy to see that if $u \in C^{1,1}$ and its gradient Du does not vanish on $\{z \in \mathbb{R}^n : u(z) = u(x)\}$, then k^* are finite and

$$
k^*[x, u] = k_*[x, u] = -H_s[x, u].
$$

We can now give the definition of *viscosity solution* for (5) (see [26], Sec. 3).

Definition 2.   i) *An upper semicontinuous function $u : [0, T] \times \mathbb{R}^n$ is a viscosity subsolution of (5) if for every smooth test function ϕ such that $u - \phi$ admits a global zero maximum at (t, x), we have*

$$
\partial_t \phi \leqslant k^*[x, \phi(\cdot, t)] |D\phi|(x, t)
\tag{7}
$$

if $D\phi(x, t) \neq 0$, and $\partial_t \phi(x, t) \leqslant 0$ if not.
ii) *A lower semicontinuous function $u : [0, T] \times \mathbb{R}^n$ is a viscosity supersolution of (5) if for every smooth test function ϕ such that $u - \phi$ admits a global zero minimum at (t, x), we have*

$$
\partial_t \phi \geqslant k_*[x, \phi(\cdot, t)] |D\phi|(x, t)
\tag{8}
$$

if $D\phi(x, t) \neq 0$, and $\partial_t \phi(x, t) \geqslant 0$ if not.
iii) *A locally bounded function u is a viscosity solution of (5) if its upper semicontinuous envelope is a subsolution and its lower semicontinuous envelope is a supersolution of (5).*

Remark 3. It is easy to verify that any classical subsolution (respectively supersolution) is in particular a viscosity subsolution (respectively supersolution).

We can now state the comparison principles, that we will use later on in Section 4.

Proposition 4 (Theorem 2 in [26]). *Suppose that the initial datum u_0 is a bounded and Lipschitz continuous function. Let u (respectively v) be a bounded viscosity subsolution (respectively supersolution) of (5).*
If $u(x, 0) \leqslant u_0(x) \leqslant v(x, 0)$, then $u \leqslant v$ on $\mathbb{R}^n \times (0, +\infty)$.

In Theorem 3 of [26] existence and uniqueness of viscosity solutions of (5) were proven. The prove of existence uses the Perron's method, while uniqueness relies on the comparison principle stated in Proposition 4. Finally in [26], Theorem 6, the consistency of Definition (4) is established, showing that if u and v are two viscosity solutions of (5) with two different initial data u_0 and v_0 which have the same zero-level set, then for every time $t > 0$ also $u(\cdot, t)$ and $v(\cdot, t)$ have the same zero level set.

The uniqueness and the consistency result allow to define the fractional mean curvature flow for $\mathcal{M}_t$ by using the solution $u(\cdot, t)$ of (5).

3. The nonlocal curvature of perturbed strips

The goal of this section is to give estimates on the nonlocal curvature of sets with small slopes and small classical curvatures which are contained in a strip. The result is rather general, but we focus on a particular case for the sake of concreteness:

Proposition 5. *Let κ, $\eta > 0$. Let E_-, $E_+ \subset \mathbb{R}^n$. Assume that $E_+ \cap E_- = \varnothing$, that*

$$E_- \supseteq \{x_n \leqslant -1\} \quad and \quad E_+ \supseteq \{x_n \geqslant 1\}. \tag{9}$$

Suppose also that the boundaries of E_- and E_+ are of class C^2, with classical directional curvatures bounded by κ.

In addition, let ν_- and ν_+ be the exterior normals of E_- and E_+ respectively, and assume that

$$|\nu_- \cdot e_i| \leqslant \eta \quad and \quad |\nu_+ \cdot e_i| \leqslant \eta \quad for\ every\ i \in \{1, \dots, n-1\}. \tag{10}$$

Let $E := \mathbb{R}^n \setminus (E_- \cup E_+)$. Then, there exist c_0, κ_0, $\eta_0 > 0$, depending on n and s, such that for any $x \in \partial E$

$$H_E^s(x) \geqslant c_0,$$

provided that $\kappa \in [0, \kappa_0]$ and $\eta \in [0, \eta_0]$.

Before giving the proof of Proposition 5, we state the following Lemma that will be useful later. Basically, it states that a set, whose boundary has small enough slopes and classical curvatures bounded by κ, can be trapped between paraboloids of opening a multilple of κ.

Lemma 6. *Let κ, $\eta > 0$ and $G \subset \mathbb{R}^n$, with boundary of class C^2, with classical directional curvatures bounded by κ and exterior normal ν such that $|\nu \cdot e_i| \leqslant \eta$ for any $i \in \{1, \dots, n-1\}$.*

Suppose also that $0 \in \partial G$. Then, fixed $\rho > 0$, there exist $\eta_0 > 0$, depending on n, and $\kappa_0(\rho)$, depending on n and ρ, such that, if $\kappa \in [0, \kappa_0(\rho)]$ and $\eta \in [0, \eta_0]$, then

$$\mathcal{R}(G \cap B_\rho) \subseteq \{x_n \leqslant C\kappa |x'|^2\},$$

for a suitable rotation $\mathcal{R}$, which dists from the identity less than $C\eta$, for a suitable $C > 0$, depending on n.

Proof. We notice that the result is true for some ρ_0 universal, due to the Implicit Function Theorem, and so for all $\rho \in (0, \rho_0]$.

Let now $\rho > \rho_0$. Let

$$G_\rho := \frac{\rho_0}{\rho} G = \{y \in \mathbb{R}^n \text{ s.t. } \rho \rho_0^{-1} y \in G\}.$$

Notice that the slope of the normal of G_ρ is the same as the one of G, and so it is bounded by η. On the other hand, the curvatures of G_ρ are $\rho \rho_0^{-1}$ times the curvatures of G, and therefore are bounded by $\rho \rho_0^{-1} \kappa$.

Accordingly, if $\eta \in [0, \eta_0]$ and $\rho \rho_0^{-1} \kappa \in [0, \kappa_0(\rho)]$ (i.e. if $\kappa \in [0, \rho_0 \rho^{-1} \kappa_0(\rho_0)]$), we have that the claim holds true for G_ρ in B_{ρ_0}, namely

$$\mathcal{R}(G_\rho \cap B_{\rho_0}) \subseteq \{y_n \leqslant C \rho \rho_0^{-1} \kappa |y'|^2\},$$

where $\mathcal{R}$ is a rotation as in the statement of Lemma 6.

Using the change of scale $x := \rho\,\rho_0^{-1}\,y$, this gives that

$$\mathcal{R}(G \cap B_\rho) = \frac{\rho}{\rho_0}\,\mathcal{R}(G_\rho \cap B_{\rho_0}) \subseteq \frac{\rho}{\rho_0}\,\{y_n \leqslant C\,\rho\,\rho_0^{-1}\,\kappa\,|y'|^2\} = \{x_n \leqslant C\kappa\,|x'|^2\},$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition 5. We fix $x \in \partial E$. Without loss of generality, we may suppose that $x \in \partial E_+$. We use the notation $\nu := \nu_+(x)$. Also, we set $\chi := \chi_E - \chi_{\mathcal{C}E}$. For short, we also denote by C a positive constant, depending on n and s, which we take the liberty of modifying from line to line.

We consider the two slabs

$$S := \{p \in \mathbb{R}^n \text{ s.t. } |(p - x) \cdot \nu| \leqslant 4\}$$
$$\text{and}\qquad S_\star := \{p \in \mathbb{R}^n \text{ s.t. } |p \cdot e_n| \leqslant 1\}.$$

Notice that, by (9),

$$E \subseteq S_\star. \tag{11}$$

We also observe that

$$S_\star \setminus S \subseteq \mathbb{R}^n \setminus B_{C/\eta}(x). \tag{12}$$

Indeed, if $p \in S_\star \setminus S$, we have that $|(p - x) \cdot \nu| > 4$ and $|p \cdot e_n| \leqslant 1$. Thus, we recall (10) and (11), we write $\nu = (\nu \cdot e_1)e_1 + \cdots + (\nu \cdot e_n)e_n$ and we find that

$$
\begin{aligned}
4 \;&<\; \left|(p - x) \cdot \sum_{i=1}^n (\nu \cdot e_i)e_i\right| \\
&\leqslant\; \eta \sum_{i=1}^{n-1} |(p - x) \cdot e_i| + |(p - x) \cdot e_n| \\
&\leqslant\; \eta\,(n-1)\,|p - x| + |p \cdot e_n| + |x \cdot e_n| \\
&\leqslant\; \eta\,(n-1)\,|p - x| + 1 + 1,
\end{aligned}
$$

which implies (12) as long as η is sufficiently small.

Then, in view of (11) and (12), we have that

$$E \setminus S \subseteq S_\star \setminus S \subseteq \mathbb{R}^n \setminus B_{C/\eta}(x)$$

and therefore

$$\int_{E \setminus S} \frac{dy}{|x - y|^{n+s}} \leqslant \int_{\mathbb{R}^n \setminus B_{C/\eta}(x)} \frac{dy}{|x - y|^{n+s}} \leqslant C\eta^s. \tag{13}$$

Now we set

$$G := \{p \in B_{10}(x) \text{ s.t. } (p - x) \cdot \nu > 4\}.$$

Notice that G is a circular sector, so, in particular, we have that

$$|G| \geqslant C. \tag{14}$$

Also, by construction,

$$G \cap S = \varnothing. \tag{15}$$

We claim that

$$G \subseteq \mathbb{R}^n \setminus E. \tag{16}$$

To check this, let $p \in G$. Then, by (10) and (11),

$$
\begin{aligned}
4 \; &< \; \sum_{i=1}^{n} (\nu \cdot e_i)(p - x) \cdot e_i \\
&\leqslant \; \eta\,(n-1)\,|p - x| + (p - x) \cdot e_n \\
&\leqslant \; 10\eta\,(n-1) + p \cdot e_n + 1,
\end{aligned}
$$

and so, if η is sufficiently small, we deduce that $p \cdot e_n \geqslant 2$. Hence, we have that $p \in E_+$, which in turn implies (16).

Now, from (15) and (16), we get that

$$
(\mathbb{R}^n \setminus E) \setminus S \;\supseteq\; G \setminus S \;=\; G.
$$

Notice also that if $y \in G$ then $|x - y| \leqslant 10$. As a consequence of these observations and (14), we infer that

$$
\int_{(\mathbb{R}^n \setminus E) \setminus S} \frac{dy}{|x - y|^{n+s}} \;\geqslant\; \int_G \frac{dy}{|x - y|^{n+s}} \;\geqslant\; \frac{|G|}{10^{n+s}} \;\geqslant\; C.
$$

Combining this with (13), we conclude that

$$
\int_{\mathbb{R}^n \setminus S} \frac{\chi(y)\,dy}{|x - y|^{n+s}} \;=\; \int_{\mathbb{R}^n \setminus S} \frac{\chi_E(y) - \chi_{\mathcal{C}E}(y)}{|x - y|^{n+s}}\,dy \;\leqslant\; C\eta^s - C_\star. \tag{17}
$$

Here we denote by $C_\star > 0$ a "special" constant (to distinguish it from the "other" constants just denoted by C): indeed, $C_\star$ will produce the desired result after small perturbations, as we will see in the sequel.

Now we analyze the contributions inside S. For this purpose, we take an additional parameter

$$
\rho \in \left(0, \; \min\left\{ |\log \kappa|, \frac{1}{\eta} \right\} \right). \tag{18}
$$

Such ρ will be taken appropriately large (in dependence of $C_\star$): then κ and η have to be chosen sufficiently small, in such a way that ρ belongs to the interval stated in (18). That is, the parameter ρ is taken large with respect to the constants, then κ and η will be taken to be small in dependence on the constants and on ρ.

Thanks to the cuvature assumption on ∂E_+, we know that the boundary of E_+ inside $B_\rho(x)$ is trapped between paraboloids of opening $C\kappa$: more precisely, by Lemma 6 there exists a rigid motion $\mathcal{R}$ with $\mathcal{R}(x) = 0$ and such that

$$
\begin{aligned}
\mathcal{R}\big(E_+ \cap B_\rho(x)\big) &\supseteq \big\{ p = (p', p_n) \in B_\rho \text{ s.t. } p_n \geqslant C\kappa\,|p'|^2 \big\} =: \mathcal{P}_1, \\
\mathcal{R}\big((\partial E_+) \cap B_\rho(x)\big) &\subseteq \big\{ p = (p', p_n) \in B_\rho \text{ s.t. } |p_n| \leqslant C\kappa\,|p'|^2 \big\} =: \mathcal{P}_2 \\
\text{and} \quad \mathcal{R}\big(E_+ \cap B_\rho(x)\big) &\subseteq \mathcal{P}_1 \cup \mathcal{P}_2.
\end{aligned} \tag{19}
$$

Notice that $\mathcal{R}$ is the composition of a translation that sends x to 0 and then a rotation which sets ν in the vertical direction. Therefore, we can write $\mathcal{R}(y) := \mathcal{R}_\star(y - x)$, where $\mathcal{R}_\star$ is a rotation which dists from the identity less than $C\eta$, due to (10).
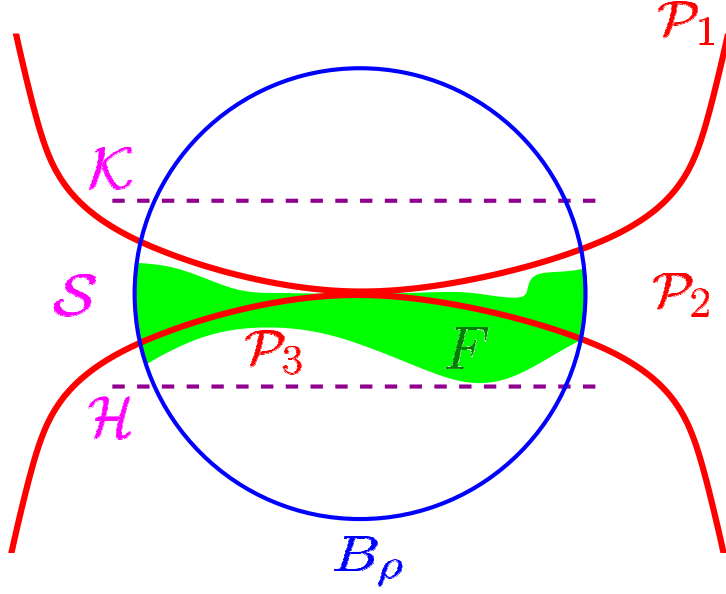
FIGURE 1

We set

$$F := \mathcal{R}\big(E \cap B_\rho(x)\big),$$
$$\mathcal{H} := \{p = (p', p_n) \in B_\rho \text{ s.t. } p_n < -4\},$$
$$\mathcal{K} := \{p = (p', p_n) \in B_\rho \text{ s.t. } p_n > 4\}$$
$$\text{and} \qquad \mathcal{P}_3 := B_\rho \setminus \big(\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{H}\big).$$

By construction, we see that

$$\mathcal{R}(S \cap B_\rho(x)) = B_\rho \setminus (\mathcal{H} \cup \mathcal{K}) = \{p = (p', p_n) \in B_\rho \text{ s.t. } |p_n| \leqslant 4\} =: \mathcal{S}. \qquad (20)$$

We also observe that

$$\mathcal{P}_3 \cap \mathcal{S} = -(\mathcal{P}_1 \cap \mathcal{S}), \qquad (21)$$

up to sets of null measure. To check this, let $p \in \mathcal{P}_3 \cap \mathcal{S}$ and $q := -p$. Then we have that $|q_n| = |p_n| \leqslant 4$, thus $q \in \mathcal{S}$. Furthermore, since $p \notin \mathcal{P}_1 \cup \mathcal{P}_2$, we have that $p_n < -C\kappa |p'|^2$. Accordingly,

$$q_n = -p_n > C\kappa |p'|^2 = C\kappa |q'|^2,$$

and so $q \in \mathcal{P}_1$. This shows that $q \in \mathcal{P}_1 \cap \mathcal{S}$, and so

$$\mathcal{P}_3 \cap \mathcal{S} \subseteq -(\mathcal{P}_1 \cap \mathcal{S}). \qquad (22)$$

Now, let us take $q \in (\mathcal{P}_1 \cap \mathcal{S})$ and set $p := -q$. We have that $|p_n| = |q_n| \leqslant 4$, hence $p \in \mathcal{S}$. Moreover,

$$p_n = -q_n \leqslant -C\kappa |q'|^2 = -C\kappa |p'|^2.$$

Since we can remove the case in which $p_n = -C\kappa |p'|^2$ (being this a set of null measure), we can conclude that $p_n < -C\kappa |p'|^2$ and so $p \notin \mathcal{P}_1 \cup \mathcal{P}_2$. This says that

$$p \in \mathcal{S} \setminus \big(\mathcal{P}_1 \cup \mathcal{P}_2\big) \subseteq \big(B_\rho \setminus \mathcal{H}\big) \setminus \big(\mathcal{P}_1 \cup \mathcal{P}_2\big) \subseteq B_\rho \setminus \big(\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{H}\big) = \mathcal{P}_3.$$

Hence $p \in \mathcal{P}_3 \cap \mathcal{S}$, up to negligible sets. Thus, we have shown that

$$-(\mathcal{P}_1 \cap \mathcal{S}) \subseteq \mathcal{P}_3 \cap \mathcal{S},$$

up to negligible sets. Combining this with (22), we conclude the proof of (21).

Also, we notice that

$$\mathcal{H} \cap \left(\mathcal{P}_1 \cup \mathcal{P}_2 \right) = \varnothing \quad \text{and} \quad \mathcal{K} \subseteq \mathcal{P}_1. \tag{23}$$

Indeed, if $p \in \left(\mathcal{P}_1 \cup \mathcal{P}_2 \right)$, then

$$p_n \geqslant -C\kappa \, |p'|^2 \geqslant -C\kappa \rho^2 \geqslant -1$$

if κ sufficiently small (possibly in dependence of ρ), hence $p \notin \mathcal{H}$. Similarly, if $p \in \mathcal{K}$ then

$$p_n - C\kappa \, |p'|^2 \geqslant 4 - C\kappa \rho^2 > 0$$

and so $p \in \mathcal{P}_1$. These observations prove (23).

We also point out that

$$F = \mathcal{R}\big(E \cap B_\rho(x) \big) \subseteq \mathcal{R}\big((\mathbb{R}^n \setminus E_+) \cap B_\rho(x) \big) = \mathbb{R}^n \setminus \Big(\mathcal{R}\big(E_+ \cap B_\rho(x) \big) \Big) \subseteq \mathbb{R}^n \setminus \mathcal{P}_1, \tag{24}$$

thanks to (19), and therefore

$$B_\rho \setminus F \supseteq \mathcal{P}_1. \tag{25}$$

In addition, we claim that

$$F \subseteq \mathcal{P}_2 \cup \mathcal{P}_3. \tag{26}$$

Indeed, if $p \in F$, then $p = \mathcal{R}(y) = \mathcal{R}_\star (y-x)$ for some $y \in E$. Therefore, if $I_n \in \mathrm{Mat}(n \times n)$ denotes the identity matrix, recalling (11) we have that

$$
\begin{aligned}
p_n &= \left(\mathcal{R}_\star (y-x) \right) \cdot e_n \\
&\geqslant (y-x) \cdot e_n - \left| \left((\mathcal{R}_\star - I_n)(y-x) \right) \cdot e_n \right| \\
&\geqslant -2 - \eta \rho \\
&> -4,
\end{aligned}
$$

provided that η is sufficiently small (possibly in dependence of ρ). This shows that $p \notin \mathcal{H}$. As a consequence, we see that

$$p \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3.$$

This and (24) establish the validity of (26).

Now, we use the change of variable $z = \mathcal{R}(y)$ and formulas (20), (25) and (26) to see that

$$
\begin{aligned}
\int_{\mathcal{S} \cap B_\rho(x)} \frac{\chi(y) \, dy}{|x-y|^{n+s}} &= \int_{\mathcal{S}} \frac{\chi_F(z) - \chi_{B_\rho \setminus F}(z) \, dz}{|z|^{n+s}} \\
&\leqslant \int_{\mathcal{P}_2} \frac{dz}{|z|^{n+s}} + \int_{\mathcal{P}_3 \cap \mathcal{S}} \frac{dz}{|z|^{n+s}} - \int_{\mathcal{P}_1 \cap \mathcal{S}} \frac{dz}{|z|^{n+s}}.
\end{aligned} \tag{27}
$$

Moreover, from (21) and a reflection in the vertical variable, we have that

$$\int_{\mathcal{P}_3 \cap \mathcal{S}} \frac{dz}{|z|^{n+s}} = \int_{\mathcal{P}_1 \cap \mathcal{S}} \frac{dz}{|z|^{n+s}}.$$

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Accordingly, (27) becomes

$$\int_{S\cap B_\rho(x)} \frac{\chi(y)\,dy}{|x-y|^{n+s}} \leqslant \int_{\mathcal{P}_2} \frac{dz}{|z|^{n+s}}$$

$$\leqslant \int_{|z'|\leqslant\rho} dz' \int_{|z_n|\leqslant C\kappa\,|z'|^2} dz_n\,|z'|^{-n-s} = C\kappa \int_{|z'|\leqslant\rho} dz'\,|z'|^{2-n-s} \qquad (28)$$

$$= C\kappa\rho^{2-s}.$$

On the other hand, we have that

$$\int_{S\setminus B_\rho(x)} \frac{\chi(y)\,dy}{|x-y|^{n+s}} \leqslant \int_{\mathbb{R}^n\setminus B_\rho(x)} \frac{dy}{|x-y|^{n+s}} = \frac{C}{\rho^s}.$$

This and (28) yield

$$\int_S \frac{\chi(y)\,dy}{|x-y|^{n+s}} \leqslant C\left(\kappa\rho^{2-s} + \frac{1}{\rho^s}\right).$$

Combining this with (17), we see that

$$-H_E^s(x) = \int_{\mathbb{R}^n\setminus S} \frac{\chi(y)\,dy}{|x-y|^{n+s}} \leqslant C\left(\eta^s + \kappa\rho^{2-s} + \frac{1}{\rho^s}\right) - C_\star \leqslant C\left(\eta^s + \kappa\rho^{2-s}\right) - \frac{C_\star}{2},$$

provided that ρ is sufficiently large.

Hence, if η and κ are small enough, we obtain that $-H_E^s(x) \leqslant -C_\star/4$, as desired. $\qquad\square$

Corollary 7. *Let $\epsilon,\ \delta > 0$ and*

$$E := \left\{x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } |x_n| < \epsilon + \frac{2}{\pi}\arctan\left(\delta\,|x'|^2\right)\right\}.$$

Then, if ϵ and δ are sufficiently small, we have that

$$\inf_{x\in\partial E} H_E^s(x) \geqslant c_0 > 0,$$

for some c_0 depending only on s and n.

Proof. We define

$$E_- := \left\{x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } x_n \leqslant -\epsilon - \frac{2}{\pi}\arctan\left(\delta\,|x'|^2\right)\right\}$$

$$\text{and}\qquad E_+ := \left\{x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } x_n \geqslant \epsilon + \frac{2}{\pi}\arctan\left(\delta\,|x'|^2\right)\right\}.$$

Notice that the horizontal component of the normal to ∂E is of size $O(\delta)$ and the curvatures are of size $O(\delta^2)$. Thus, we are in the position of exploiting Proposition 5, from which we obtain the desired result. $\qquad\square$

4. Neckpinch

In this section we prove our main result, that is Theorem 1. More precisely, we provide an example of surface evolving by fractional mean curvature flow, which developes a singularity before it can shrink to a point. For the classical mean curvature flow, this phenomenon appears in dimensions $n \geqslant 3$ (that is, for at least 2-dimensional surfaces) and a standard example is a set made by the union of two large balls, connected with a very thin (cylindrical) neck (see [18]). The idea is that the mean curvature in the neck is much larger than the one in the two balls and hence the radius of the neck goes to zero faster than the radius of the balls. Of course the fact that the dimension of the surface is larger than 2 is crucial to have positive mean curvature in the neck (indeed for a cylindrical neck we would have one direction in which the principal curvature is zero, and the others in which all principal curvatures are strictly positive).

Interestingly, when we consider the fractional mean curvature flow, this neckpinch singularity can be observed also in dimension $n = 2$, providing hence a counterexample to the Grayson Theorem [19], which states that any smooth embedded curve in the plane evolving by (classical) MCF shrinks to a point. The heuristich reason for this is that, thanks to its nonlocality, the fractional mean curvature of a very thin neck, is strictly positive also in dimension 2: Indeed if we "sit" on the boundary of the neck we see much more complement of E than E itself.

To build our example, we start by recalling the following fact that was proved in [28].

Lemma 8 (Lemma 2 and Corollary 3 in [28]). *The fractional mean curvature of the ball of radius R is equal, up to dimensional constants, to R^{-s}.*

More precisely, for any $x \in \partial B_1$

$$H^s_{B_1}(x) = \bar{\omega},$$

for some $\bar{\omega} > 0$, and for any $x \in \partial B_R(0)$,

$$H^s_{B_R}(x) = \bar{\omega} R^{-s}.$$

Moreover, if we set $R(t) := (R_0^{s+1} - (\bar{\omega}(1+s))t)^{\frac{1}{s+1}}$, then $B_{R(t)}$ is a solution to the fractional mean curvature flow starting from B_{R_0} and it collapses to a point in the finite time

$$T_{B_{R_0}} = \frac{R_0^{s+1}}{\bar{\omega}(s+1)}. \tag{29}$$

Observe that, while for the classical mean curvature flow, the extinction time of a sphere of radius R_0 is proportional to R_0^2, in the fractional case it is proportional to R_0^{s+1}.

We can now give the proof of our main result.

Proof of Theorem 4. We consider now the set E_ϵ defined in Corollary 7:

$$E_\epsilon := \left\{ x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } |x_n| < \epsilon + \frac{2}{\pi} \arctan\left(\delta |x'|^2\right) \right\}.$$

We know that there exists $\bar{\epsilon}$ and $\bar{\delta}$ positive such that, for any $0 < \epsilon \leqslant \bar{\epsilon}$ and $0 < \delta \leqslant \bar{\delta}$

$$\inf_{x \in \partial E_\epsilon} H^s_{E_\epsilon}(x) \geqslant c_0 > 0, \tag{30}$$

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for some c_0 depending only on n and s.

Let now κ and ϵ_0 be two positive parameters satisfying

$$\kappa < c_0 \quad \text{and} \quad \epsilon_0 < \min\left\{\bar{\epsilon}, \frac{1}{4}\kappa T_{B_1}\right\}, \tag{31}$$

where T_{B_1} is the extinction time of the ball of radius 1 given in (29).

The idea is to consider the set E_{ϵ_0} and to make it evolve with constant velocity κ in the inner vertical direction. More precisely, we set

$$\epsilon(t) := \epsilon_0 - \kappa t,$$

and, for any t, we consider the set

$$E_{\epsilon(t)} := \left\{ x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } |x_n| < \epsilon(t) + \frac{2}{\pi} \arctan\left(\delta |x'|^2\right) \right\}. \tag{32}$$

Hence, we have that any point $x \in \partial E_{\epsilon(t)}$ satisfies

$$\partial_t x \cdot \nu = V \cdot \nu,$$

where

$$V = \begin{cases} -\kappa e_n & \text{if } x_n > 0 \\ \kappa e_n & \text{if } x_n < 0. \end{cases}$$

Thus,

$$\partial_t x \cdot \nu \geqslant -\kappa > -c_0 \geqslant -H^s_{E_{\epsilon(t)}},$$

where in the last inequality we have used (30) and the fact that $E_{\epsilon(t)} \subset E_{\epsilon_0}$ for any $t > 0$. Therefore, the set $E_{\epsilon(t)}$ is a smooth supersolution (and hence in particular a viscosity supersolution) to (1).

By the definition of the set E_{ϵ_0} we have that the infimum distance between the two disconnected components of its boundary $\{(x', x_n) \in \mathbb{R}^n \text{ s.t. } x_n = \epsilon_0 + \arctan\left(\delta|x'|^2\right)\}$ and $\{(x', x_n) \in \mathbb{R}^n \text{ s.t. } x_n = -\epsilon_0 - \arctan\left(\delta|x'|^2\right)\}$ is attained at the points $(0, \ldots, 0, \epsilon_0)$ and $(0, \ldots, 0, -\epsilon_0)$. Since $E_{\epsilon(t)}$ evolves with constant negative velocity κ along the inner vertical direction, we deduce that the singular time for $E_{\epsilon(t)}$ is given by

$$T_{E_{\epsilon(t)}} = \frac{2\epsilon_0}{\kappa}. \tag{33}$$

Let now define a closed set A_0 with the following properties:

(1) A_0 is contained in E_{ϵ_0};
(2) A_0 contains two balls B_1^- and B_1^+ of radius 1 centered at $(-L, 0, \ldots, 0)$ and $(L, 0, \ldots, 0)$ respectively, where L is chosen large enough so that (1) and (2) are both satisfied.

We consider now the fractional mean curvature flow A_t starting from A_0. By the comparison principle (Proposition 4), A_t must be contained in $E_{\epsilon(t)}$. Moreover it must contains the evolutions $B_{1,t}^-$ and $B_{1,t}^+$ of the two balls B_1^- and B_1^+.

On the one hand, since A_t must be contained in $E_{\epsilon(t)}$, using (33) and the choice of ϵ_0 (31), we deduce that the singular time of A_t satisfies

$$T_A \leqslant \frac{2\epsilon_0}{\kappa} \leqslant \frac{1}{2}T_{B_1}.$$

On the other hand, by assumption (2), A_t does not shrink to a point as $t \to T_A$, since it contains two balls with positive radius. This concludes the proof of Theorem 1. $\square$

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