SOLVABILITY OF SOME INTEGRO-DIFFERENTIAL EQUATIONS WITH ANOMALOUS DIFFUSION

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Abstract: The paper is devoted to the existence of solutions of an integro-differential equation in the case of anomalous diffusion with the Laplace operator in a fractional power. The proof of existence of solutions relies on a fixed point technique. Solvability conditions for elliptic operators without Fredholm property in unbounded domains are used.

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1. Introduction

The present article deals with the existence of stationary solutions of the following nonlocal reaction-diffusion equation

$$\frac{\partial u}{\partial t} = -D\left(-\frac{\partial^2}{\partial x^2}\right)^s u + \int_{-\infty}^{\infty} K(x-y)g(u(y,t))dy + f(x), \quad 0 < s < \frac{1}{4}, \tag{1.1}$$

which appears in cell population dynamics. The space variable x here corresponds to the cell genotype, $u(x,t)$ stands for the cell density as a function of their genotype and time. The right side of this equation describes the evolution of cell density via cell proliferation, mutations and cell influx. The anomalous diffusion term here corresponds to the change of genotype due to small random mutations, and the nonlocal term describes large mutations. Function $g(u)$ designates the rate of cell birth which depends on u (density dependent proliferation), and the kernel $K(x-y)$ gives the proportion of newly born cells changing their genotype from y to x. We assume that it depends on the distance between the genotypes. Finally, the last term in the right side of this equation stands for the influx of cells for different genotypes.

The operator $\left(-\frac{\partial^2}{\partial x^2}\right)^s$ in problem (1.1) describes a particular case of anomalous diffusion actively treated in the context of different applications in plasma physics and turbulence [11], [12], surface diffusion [13], [15], semiconductors [16] and so on. Anomalous diffusion can be described as a random process of particle motion characterized by the probability density distribution of jump length. The moments of this density distribution are finite in the case of normal diffusion, but this is not the case for superdiffusion. Asymptotic behavior at infinity of the probability density function determines the value s of the power of the Laplacian [14]. The operator $\left(-\frac{\partial^2}{\partial x^2}\right)^s$ is defined by virtue of the spectral calculus. In our work we will treat the case of $0 < s < 1/4$. A similar equation in the case of the standard Laplacian in the diffusion term was investigated recently in [26].

We set $D = 1$ and prove the existence of solutions of the equation

$$-\left(-\frac{d^2}{dx^2}\right)^s u + \int_{-\infty}^{\infty} K(x-y)g(u(y))dy + f(x) = 0, \quad 0 < s < \frac{1}{4}. \tag{1.2}$$

We will consider the case where the linear part of this operator does not satisfy the Fredholm property. Consequently, conventional methods of nonlinear analysis may not be applicable. We use solvability conditions for non Fredholm operators along with the method of contraction mappings.

Consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either zero identically or converges to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of equation (1.3) contains the origin. Consequently, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present article deals with the studies of certain properties of the operators of this kind. Note that elliptic equations with non Fredholm operators were treated actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The non Fredholm Schrödinger type operators were studied with the methods of the spectral and the scattering theory in [17], [19], [21]. The Laplace operator with drift from the point of view of non Fredholm operators was treated in [20] and linearized Cahn-Hilliard equations in [22] and [24]. Nonlinear non Fredholm elliptic problems were studied in [23] and [25]. Important applications to the theory of reaction-diffusion equations were developed in [8], [9]. Non Fredholm operators arise also when studying wave systems with an infinite number of localized traveling waves (see [1]). In particular, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is significantly different and the method developed in these works cannot be applied. Front propagation equations with anomalous diffusion were treated actively in recent years (see e.g. [27], [28]).

Let us set $K(x) = \varepsilon\mathcal{K}(x)$ with $\varepsilon \geq 0$ and suppose that the assumption below is satisfied.

Assumption 1. *Consider* $0 < s < \dfrac{1}{4}$. *Let* $f(x) : \mathbb{R} \to \mathbb{R}$ *be nontrivial, such that* $f(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *and* $\left(-\dfrac{d^2}{dx^2}\right)^{\frac{1}{2}-s} f(x) \in L^2(\mathbb{R})$. *Assume also that* $\mathcal{K}(x) : \mathbb{R} \to \mathbb{R}$ *and* $\mathcal{K}(x) \in L^1(\mathbb{R})$. *In addition,* $\left(-\dfrac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \mathcal{K}(x) \in L^2(\mathbb{R})$, *such that*

$$Q := \left\| \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \mathcal{K}(x) \right\|_{L^2(\mathbb{R})} > 0.$$

We choose the space dimension $d = 1$, which is related to the solvability conditions for the linear Poisson type problem (4.31) proved in Lemma 6. From the perspective of applications, the space dimension is not restricted to $d = 1$ because the space variable corresponds to cell genotype but not to the usual physical space. Let us use the Sobolev spaces

$$H^{2s}(\mathbb{R}) := \left\{ u(x) : \mathbb{R} \to \mathbb{R} \,|\, u(x) \in L^2(\mathbb{R}), \ \left(-\frac{d^2}{dx^2}\right)^s u \in L^2(\mathbb{R}) \right\}, \quad 0 < s \leq 1$$

equipped with the norm

$$\|u\|^2_{H^{2s}(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \left\| \left(-\frac{d^2}{dx^2}\right)^s u \right\|^2_{L^2(\mathbb{R})}. \tag{1.4}$$

By virtue of the standard Sobolev inequality in one dimension (see e.g. Section 8.5 of [10]) we have

$$\|u\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \|u\|_{H^1(\mathbb{R})}. \tag{1.5}$$

When the nonnegative parameter ε vanishes, we arrive at the linear Poisson type equation (4.31). By means of Lemma 6 below along with Assumption 1 problem

(4.31) admits a unique solution

$$u_0(x) \in H^{2s}(\mathbb{R}), \quad 0 < s < \frac{1}{4},$$

such that no orthogonality relations are required. By virtue of Lemma 6, when $\frac{1}{4} \leq s < 1$, certain orthogonality conditions (4.33) and (4.34) are required to be able to solve equation (4.31) in $H^{2s}(\mathbb{R})$. By virtue of Assumption 1, since

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u(x) = \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} f(x) \in L^2(\mathbb{R}),$$

we have for the unique solution of linear problem (4.31) that $u_0(x) \in H^1(\mathbb{R})$. Let us seek the resulting solution of nonlinear equation (1.2) as

$$u(x) = u_0(x) + u_p(x). \tag{1.6}$$

Evidently, we obtain the perturbative equation

$$\left(-\frac{d^2}{dx^2}\right)^{s} u_p = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g(u_0(y) + u_p(y)) dy, \quad 0 < s < \frac{1}{4}. \tag{1.7}$$

Let us introduce a closed ball in the Sobolev space

$$B_\rho := \{u(x) \in H^1(\mathbb{R}) \mid \|u\|_{H^1(\mathbb{R})} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{1.8}$$

We look for the solution of problem (1.7) as the fixed point of the auxiliary nonlinear equation

$$\left(-\frac{d^2}{dx^2}\right)^{s} u = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g(u_0(y) + v(y)) dy, \quad 0 < s < \frac{1}{4} \tag{1.9}$$

in ball (1.8). For a given function $v(y)$ this is an equation with respect to $u(x)$. The left side of (1.9) involves the non Fredholm operator

$$\left(-\frac{d^2}{dx^2}\right)^{s} : H^{2s}(\mathbb{R}) \to L^2(\mathbb{R}).$$

Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Therefore, such operator has no bounded inverse. The similar situation appeared in works [23] and [25] but as distinct from the present situation, the equations studied there required orthogonality conditions. The fixed point technique was used in [18] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear equation there had the Fredholm property (see Assumption 1 of [18], also [7]). Let us define the interval on the real line

$$I := \Big[-\frac{1}{\sqrt{2}}\|u_0\|_{H^1(\mathbb{R})} - \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\|u_0\|_{H^1(\mathbb{R})} + \frac{1}{\sqrt{2}}\Big] \tag{1.10}$$

along with the closed ball in the space of $C_2(I)$ functions, namely

$$D_M := \{g(s) \in C_2(I) \mid \|g\|_{C_2(I)} \le M\}, \quad M > 0. \tag{1.11}$$

Here the norm

$$\|g\|_{C_2(I)} := \|g\|_{C(I)} + \|g'\|_{C(I)} + \|g''\|_{C(I)}, \tag{1.12}$$

where $\|g\|_{C(I)} := max_{s\in I}|g(s)|$. We make the following assumption on the nonlinear part of equation (1.2).

Assumption 2. *Let $g(z) : \mathbb{R} \to \mathbb{R}$, such that $g(0) = 0$ and $g'(0) = 0$. It is also assumed that $g(z) \in D_M$ and it does not vanish identically on the interval I.*

Let us introduce the operator T_g, such that $u = T_g v$, where u is a solution of equation (1.9). Our first main statement is as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Then equation (1.9) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon*$ for some $\varepsilon* > 0$. The unique fixed point $u_p(x)$ of this map T_g is the only solution of problem (1.7) in B_ρ.*

Clearly, the resulting solution of equation (1.2) given by (1.6) will be nontrivial because the source term $f(x)$ is nontrivial and $g(0) = 0$ due to our assumptions. We make use of the following elementary lemma.

Lemma 4. *For $R \in (0, +\infty)$ consider the function*

$$\varphi(R) := \alpha R^{1-4s} + \frac{\beta}{R^{4s}}, \quad 0 < s < \frac{1}{4}, \quad \alpha, \beta > 0.$$

It attains the minimal value at $R^* := \dfrac{4\beta s}{\alpha(1-4s)}$, which is given by

$$\varphi(R^*) = \frac{(1-4s)^{4s-1}}{(4s)^{4s}}\alpha^{4s}\beta^{1-4s}.$$

Our second main result is about the continuity of the fixed point of the map T_g which existence was established in Theorem 3 above with respect to the nonlinear function g.

Theorem 5. *Let $j = 1, 2$, the assumptions of Theorem 3 hold, such that $u_{p,j}(x)$ is the unique fixed point of the map $T_{g_j} : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon_j^*$ and $\delta := min(\varepsilon_1^*, \varepsilon_2^*)$. Then for all $0 < \varepsilon < \delta$ the inequality*

$$\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R})} \le C\|g_1 - g_2\|_{C_2(I)} \tag{1.13}$$

holds, where $C > 0$ is a constant.

We proceed to the proof of our first main result.

2. The existence of the perturbed solution

Proof of Theorem 3. Let us choose arbitrarily $v(x) \in B_\rho$ and denote the term involved in the integral expression in the right side of problem (1.9) as

$$G(x) := g(u_0(x) + v(x)).$$

Let us use the standard Fourier transform

$$\widehat{\phi}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{-ipx} dx. \tag{2.14}$$

Evidently, we have the inequality

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}} \|\phi(x)\|_{L^1(\mathbb{R})}. \tag{2.15}$$

We apply (2.14) to both sides of problem (1.9) and arrive at

$$\widehat{u}(p) = \varepsilon\sqrt{2\pi} \frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{|p|^{2s}}.$$

Hence for the norm we obtain

$$\|u\|_{L^2(\mathbb{R})}^2 = 2\pi\varepsilon^2 \int_{-\infty}^{\infty} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{|p|^{4s}} dp. \tag{2.16}$$

As distinct from works [23] and [25] including the standard Laplace operator in the diffusion term, here we do not try to control the norm

$$\left\| \frac{\widehat{\mathcal{K}}(p)}{|p|^{2s}} \right\|_{L^\infty(\mathbb{R})}.$$

Instead, let us estimate the right side of (2.16) using the analog of inequality (2.15) applied to functions $\mathcal{K}$ and G with $R > 0$ as

$$2\pi\varepsilon^2 \int_{|p|\le R} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{|p|^{4s}} dp + 2\pi\varepsilon^2 \int_{|p|>R} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{|p|^{4s}} dp \le$$

$$\leq \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R})} \left\{ \frac{1}{\pi} \|G(x)\|^2_{L^1(\mathbb{R})} \frac{R^{1-4s}}{1-4s} + \frac{1}{R^{4s}} \|G(x)\|^2_{L^2(\mathbb{R})} \right\}. \tag{2.17}$$

Since $v(x) \in B_\rho$, we have

$$\|u_0 + v\|_{L^2(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})} + 1.$$

Sobolev inequality (1.5) yields

$$|u_0 + v| \leq \frac{1}{\sqrt{2}} (\|u_0\|_{H^1(\mathbb{R})} + 1).$$

Formula $G(x) = \int_0^{u_0+v} g'(z)dz$ with the interval I defined in (1.10) implies

$$|G(x)| \leq sup_{z \in I} |g'(z)||u_0 + v| \leq M|u_0 + v|.$$

Thus

$$\|G(x)\|_{L^2(\mathbb{R})} \leq M\|u_0 + v\|_{L^2(\mathbb{R})} \leq M(\|u_0\|_{H^1(\mathbb{R})} + 1).$$

Obviously, $G(x) = \int_0^{u_0+v} dy \Big[\int_0^y g''(z)dz \Big]$. This gives us

$$|G(x)| \leq \frac{1}{2} sup_{z \in I} |g''(z)||u_0 + v|^2 \leq \frac{M}{2} |u_0 + v|^2,$$

$$\|G(x)\|_{L^1(\mathbb{R})} \leq \frac{M}{2} \|u_0 + v\|^2_{L^2(\mathbb{R})} \leq \frac{M}{2} (\|u_0\|_{H^1(\mathbb{R})} + 1)^2. \tag{2.18}$$

Thus we obtain the estimate from above for the right side of (2.17) as

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R})} M^2 (\|u_0\|_{H^1(\mathbb{R})} + 1)^2 \left\{ \frac{(\|u_0\|_{H^1(\mathbb{R})} + 1)^2 R^{1-4s}}{4\pi(1-4s)} + \frac{1}{R^{4s}} \right\},$$

where $R \in (0, +\infty)$. By means of Lemma 4 we obtain the minimal value of the expression above. Thus,

$$\|u\|^2_{L^2(\mathbb{R})} \leq \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R})} (\|u_0\|_{H^1(\mathbb{R})} + 1)^{2+8s} \frac{M^2}{(1-4s)(16\pi s)^{4s}}. \tag{2.19}$$

Obviously, via (1.9) we have

$$\left(-\frac{d^2}{dx^2} \right)^{\frac{1}{2}} u(x) = \varepsilon \left(-\frac{d^2}{dx^2} \right)^{\frac{1}{2}-s} \int_{-\infty}^{\infty} \mathcal{K}(x-y) G(y) dy.$$

By virtue of the analog of inequality (2.15) applied to function G along with (2.18) we arrive at

$$\left\| \frac{du}{dx} \right\|^2_{L^2(\mathbb{R})} \leq \varepsilon^2 \|G\|^2_{L^1(\mathbb{R})} Q^2 \leq \varepsilon^2 \frac{M^2}{4} (\|u_0\|_{H^1(\mathbb{R})} + 1)^4 Q^2. \tag{2.20}$$

Hence, by means of the definition of the norm (1.4) with $s = \dfrac{1}{2}$ along with inequalities (2.19) and (2.20) we obtain the estimate from above for $\|u\|_{H^1(\mathbb{R})}$ as

$$\varepsilon(\|u_0\|_{H^1(\mathbb{R})} + 1)^2 M \left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R})}(\|u_0\|_{H^1(\mathbb{R})} + 1)^{8s-2}}{(1-4s)(16\pi s)^{4s}} + \frac{Q^2}{4} \right]^{\frac{1}{2}} \leq \rho$$

for all $\varepsilon > 0$ small enough. Thus, $u(x) \in B_\rho$ as well. If for some $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of problem (1.9), their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R})$ satisfies

$$\left(-\frac{d^2}{dx^2}\right)^s w = 0.$$

Since the operator $\left(-\dfrac{d^2}{dx^2}\right)^s$ considered on the whole real line does not have non-trivial square integrable zero modes, $w(x)$ vanishes a.e. on $\mathbb{R}$. Therefore, problem (1.9) defines a map $T_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ sufficiently small.

Our goal is to prove that this map is a strict contraction. We choose arbitrarily $v_{1,2}(x) \in B_\rho$. The argument above yields $u_{1,2} := T_g v_{1,2} \in B_\rho$ as well. By virtue of (1.9) we have

$$\left(-\frac{d^2}{dx^2}\right)^s u_1 = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g(u_0(y) + v_1(y)) dy, \tag{2.21}$$

$$\left(-\frac{d^2}{dx^2}\right)^s u_2 = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g(u_0(y) + v_2(y)) dy, \tag{2.22}$$

$0 < s < \dfrac{1}{4}$. Let us introduce

$$G_1(x) := g(u_0(x) + v_1(x)), \quad G_2(x) := g(u_0(x) + v_2(x))$$

and apply the standard Fourier transform (2.14) to both sides of problems (2.21) and (2.22). We arrive at

$$\widehat{u_1}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G_1}(p)}{|p|^{2s}}, \quad \widehat{u_2}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G_2}(p)}{|p|^{2s}}.$$

Evidently,

$$\|u_1 - u_2\|^2_{L^2(\mathbb{R})} = \varepsilon^2 2\pi \int_{-\infty}^{\infty} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G_1}(p) - \widehat{G_2}(p)|^2}{|p|^{4s}} dp.$$

Apparently, it can be bounded from above by means of inequality (2.15) by

$$\varepsilon^2\|\mathcal{K}\|_{L^1(\mathbb{R})}^2\Bigg\{\frac{1}{\pi}\|G_1(x)-G_2(x)\|_{L^1(\mathbb{R})}^2\frac{R^{1-4s}}{1-4s}+\|G_1(x)-G_2(x)\|_{L^2(\mathbb{R})}^2\frac{1}{R^{4s}}\Bigg\},$$

with $R\in(0,+\infty)$. Let us use the formula

$$G_1(x)-G_2(x)=\int_{u_0+v_2}^{u_0+v_1}g'(z)dz.$$

Thus

$$|G_1(x)-G_2(x)|\le sup_{z\in I}|g'(z)||v_1-v_2|\le M|v_1-v_2|.$$

Therefore,

$$\|G_1(x)-G_2(x)\|_{L^2(\mathbb{R})}\le M\|v_1-v_2\|_{L^2(\mathbb{R})}\le M\|v_1-v_2\|_{H^1(\mathbb{R})}.$$

Obviously,

$$G_1(x)-G_2(x)=\int_{u_0+v_2}^{u_0+v_1}dy\Big[\int_0^y g''(z)dz\Big].$$

We derive the upper bound for $G_1(x)-G_2(x)$ in the absolute value as

$$\frac{1}{2}sup_{z\in I}|g''(z)||(v_1-v_2)(2u_0+v_1+v_2)|\le\frac{M}{2}|(v_1-v_2)(2u_0+v_1+v_2)|.$$

The Schwarz inequality implies the estimate from above for the norm $\|G_1(x)-G_2(x)\|_{L^1(\mathbb{R})}$ as

$$\frac{M}{2}\|v_1-v_2\|_{L^2(\mathbb{R})}\|2u_0+v_1+v_2\|_{L^2(\mathbb{R})}\le M\|v_1-v_2\|_{H^1(\mathbb{R})}(\|u_0\|_{H^1(\mathbb{R})}+1). \quad (2.23)$$

Hence we obtain the upper bound for the norm $\|u_1(x)-u_2(x)\|_{L^2(\mathbb{R})}^2$ given by

$$\varepsilon^2\|\mathcal{K}\|_{L^1(\mathbb{R})}^2M^2\|v_1-v_2\|_{H^1(\mathbb{R})}^2\Big\{\frac{1}{\pi}(\|u_0\|_{H^1(\mathbb{R})}+1)^2\frac{R^{1-4s}}{1-4s}+\frac{1}{R^{4s}}\Big\}.$$

Lemma 4 enables us to minimize the expression above over $R\in(0,+\infty)$ to derive the estimate from above for $\|u_1(x)-u_2(x)\|_{L^2(\mathbb{R})}^2$ as

$$\varepsilon^2\|\mathcal{K}\|_{L^1(\mathbb{R})}^2M^2\|v_1-v_2\|_{H^1(\mathbb{R})}^2\frac{(\|u_0\|_{H^1(\mathbb{R})}+1)^{8s}}{(1-4s)(4\pi s)^{4s}}. \quad (2.24)$$

Formulas (2.21) and (2.22) yield

$$\Bigg(-\frac{d^2}{dx^2}\Bigg)^{\frac{1}{2}}(u_1-u_2)=\varepsilon\Bigg(-\frac{d^2}{dx^2}\Bigg)^{\frac{1}{2}-s}\int_{-\infty}^{\infty}\mathcal{K}(x-y)[G_1(y)-G_2(y)]dy.$$

Inequalities (2.15) and (2.23) imply

$$\left\|\frac{d}{dx}(u_1-u_2)\right\|^2_{L^2(\mathbb{R})}\le \varepsilon^2 Q^2\|G_1-G_2\|^2_{L^1(\mathbb{R})}\le$$

$$\le \varepsilon^2 Q^2 M^2\|v_1-v_2\|^2_{H^1(\mathbb{R})}(\|u_0\|_{H^1(\mathbb{R})}+1)^2. \tag{2.25}$$

By means of (2.24) and (2.25) the norm $\|u_1-u_2\|_{H^1(\mathbb{R})}$ can be bounded from above by the expression

$$\varepsilon M(\|u_0\|_{H^1(\mathbb{R})}+1)\left\{\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R})}(\|u_0\|_{H^1(\mathbb{R})}+1)^{8s-2}}{(1-4s)(4\pi s)^{4s}}+Q^2\right\}^{\frac{1}{2}}\|v_1-v_2\|_{H^1(\mathbb{R})}. \tag{2.26}$$

This implies that our map $T_g : B_\rho \to B_\rho$ defined by problem (1.9) is a strict contraction for all values of $\varepsilon > 0$ sufficiently small. Its unique fixed point $u_p(x)$ is the only solution of equation (1.7) in the ball B_ρ. The resulting $u(x) \in H^1(\mathbb{R})$ given by (1.6) is a solution of problem (1.2). ■

Then we turn our attention to the proof of the second main result of our work.

3. The continuity of the fixed point of the map T_g

Proof of Theorem 5. Clearly, for all $0 < \varepsilon < \delta$ we have

$$u_{p,1} = T_{g_1}u_{p,1}, \quad u_{p,2} = T_{g_2}u_{p,2},$$

such that

$$u_{p,1}-u_{p,2} = T_{g_1}u_{p,1}-T_{g_1}u_{p,2}+T_{g_1}u_{p,2}-T_{g_2}u_{p,2}.$$

Hence,

$$\|u_{p,1}-u_{p,2}\|_{H^1(\mathbb{R})}\le\|T_{g_1}u_{p,1}-T_{g_1}u_{p,2}\|_{H^1(\mathbb{R})}+\|T_{g_1}u_{p,2}-T_{g_2}u_{p,2}\|_{H^1(\mathbb{R})}.$$

By virtue of estimate (2.26), we have

$$\|T_{g_1}u_{p,1}-T_{g_1}u_{p,2}\|_{H^1(\mathbb{R})}\le\varepsilon\sigma\|u_{p,1}-u_{p,2}\|_{H^1(\mathbb{R})},$$

where $\varepsilon\sigma < 1$ due to the fact that the map $T_{g_1} : B_\rho \to B_\rho$ under our assumptions is a strict contraction. Here the positive constant

$$\sigma := M(\|u_0\|_{H^1(\mathbb{R})}+1)\left\{\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R})}(\|u_0\|_{H^1(\mathbb{R})}+1)^{8s-2}}{(1-4s)(4\pi s)^{4s}}+Q^2\right\}^{\frac{1}{2}}.$$

Thus, we arrive at

$$(1-\varepsilon\sigma)\|u_{p,1}-u_{p,2}\|_{H^1(\mathbb{R})}\le\|T_{g_1}u_{p,2}-T_{g_2}u_{p,2}\|_{H^1(\mathbb{R})}. \tag{3.27}$$

Note that for our fixed point $T_{g_2}u_{p,2} = u_{p,2}$ and denote $\xi(x) := T_{g_1}u_{p,2}$. We obtain

$$\left(-\frac{d^2}{dx^2}\right)^s \xi(x) = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y)g_1(u_0(y)+u_{p,2}(y))dy, \tag{3.28}$$

$$\left(-\frac{d^2}{dx^2}\right)^s u_{p,2}(x) = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y)g_2(u_0(y)+u_{p,2}(y))dy, \tag{3.29}$$

with $0 < s < \frac{1}{4}$. Let $G_{1,2}(x) := g_1(u_0(x)+u_{p,2}(x))$ and $G_{2,2}(x) := g_2(u_0(x)+u_{p,2}(x))$. By applying the standard Fourier transform (2.14) to both sides of equations (3.28) and (3.29) above, we easily arrive at

$$\widehat{\xi}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G}_{1,2}(p)}{|p|^{2s}}, \quad \widehat{u}_{p,2}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G}_{2,2}(p)}{|p|^{2s}}.$$

Obviously,

$$\|\xi(x)-u_{p,2}(x)\|^2_{L^2(\mathbb{R})} = \varepsilon^2 2\pi \int_{-\infty}^{\infty} \frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}_{1,2}(p)-\widehat{G}_{2,2}(p)|^2}{|p|^{4s}}dp.$$

Evidently, it can be estimated from above by virtue of (2.15) by

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R})}\left\{\frac{1}{\pi}\|G_{1,2}-G_{2,2}\|^2_{L^1(\mathbb{R})}\frac{R^{1-4s}}{1-4s} + \|G_{1,2}-G_{2,2}\|^2_{L^2(\mathbb{R})}\frac{1}{R^{4s}}\right\},$$

where $R \in (0,+\infty)$. Let us use the formula

$$G_{1,2}(x) - G_{2,2}(x) = \int_0^{u_0(x)+u_{p,2}(x)} [g_1'(s) - g_2'(s)]ds.$$

Hence

$$|G_{1,2}(x) - G_{2,2}(x)| \le sup_{s\in I}|g_1'(s) - g_2'(s)||u_0(x)+u_{p,2}(x)| \le$$
$$\le \|g_1 - g_2\|_{C_2(I)}|u_0(x)+u_{p,2}(x)|,$$

such that

$$\|G_{1,2} - G_{2,2}\|_{L^2(\mathbb{R})} \le \|g_1 - g_2\|_{C_2(I)}\|u_0 + u_{p,2}\|_{L^2(\mathbb{R})} \le$$
$$\le \|g_1 - g_2\|_{C_2(I)}(\|u_0\|_{H^1(\mathbb{R})} + 1).$$

Another useful representation formula is

$$G_{1,2}(x) - G_{2,2}(x) = \int_0^{u_0(x)+u_{p,2}(x)} dy\Big[\int_0^y (g_1''(z) - g_2''(z))dz\Big].$$

Thus

$$|G_{1,2}(x) - G_{2,2}(x)| \leq \frac{1}{2} sup_{z \in I} |g_1''(z) - g_2''(z)| |u_0(x) + u_{p,2}(x)|^2 \leq$$

$$\leq \frac{1}{2} \|g_1 - g_2\|_{C_2(I)} |u_0(x) + u_{p,2}(x)|^2.$$

Therefore,

$$\|G_{1,2} - G_{2,2}\|_{L^1(\mathbb{R})} \leq \frac{1}{2} \|g_1 - g_2\|_{C_2(I)} \|u_0 + u_{p,2}\|^2_{L^2(\mathbb{R})} \leq$$

$$\leq \frac{1}{2} \|g_1 - g_2\|_{C_2(I)} (\|u_0\|_{H^1(\mathbb{R})} + 1)^2. \tag{3.30}$$

This enables us to estimate the norm $\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R})}$ from above by

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R})} (\|u_0\|_{H^1(\mathbb{R})} + 1)^2 \|g_1 - g_2\|^2_{C_2(I)} \Big[\frac{1}{4\pi} (\|u_0\|_{H^1(\mathbb{R})} + 1)^2 \frac{R^{1-4s}}{1-4s} + \frac{1}{R^{4s}} \Big].$$

This expression can be easily minimized over $R \in (0, +\infty)$ by means of Lemma 4. We arrive at the inequality

$$\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R})} \leq \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R})} (\|u_0\|_{H^1(\mathbb{R})} + 1)^{2+8s} \frac{\|g_1 - g_2\|^2_{C_2(I)}}{(1-4s)(16\pi s)^{4s}}.$$

By virtue of (3.28) and (3.29) we have

$$\left(-\frac{d^2}{dx^2} \right)^{\frac{1}{2}} \xi(x) = \varepsilon \left(-\frac{d^2}{dx^2} \right)^{\frac{1}{2}-s} \int_{-\infty}^{\infty} \mathcal{K}(x-y) G_{1,2}(y) dy,$$

$$\left(-\frac{d^2}{dx^2} \right)^{\frac{1}{2}} u_{p,2}(x) = \varepsilon \left(-\frac{d^2}{dx^2} \right)^{\frac{1}{2}-s} \int_{-\infty}^{\infty} \mathcal{K}(x-y) G_{2,2}(y) dy,$$

such that via (2.15) and (3.30) the norm $\|\xi'(x) - u'_{p,2}(x)\|^2_{L^2(\mathbb{R})}$ can be bounded above by

$$\varepsilon^2 \|G_{1,2} - G_{2,2}\|^2_{L^1(\mathbb{R})} Q^2 \leq \frac{\varepsilon^2 Q^2}{4} (\|u_0\|_{H^1(\mathbb{R})} + 1)^4 \|g_1 - g_2\|^2_{C_2(I)}.$$

Therefore, $\|\xi(x) - u_{p,2}(x)\|_{H^1(\mathbb{R})} \leq$

$$\leq \varepsilon \|g_1 - g_2\|_{C_2(I)} (\|u_0\|_{H^1(\mathbb{R})} + 1)^2 \left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R})} (\|u_0\|_{H^1(\mathbb{R})} + 1)^{8s-2}}{(1-4s)(16\pi s)^{4s}} + \frac{Q^2}{4} \right]^{\frac{1}{2}}.$$

By means of inequality (3.27), the norm $\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R})}$ can be estimated from above by

$$\frac{\varepsilon}{1-\varepsilon\sigma}(\|u_0\|_{H^1(\mathbb{R})}+1)^2 \left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R})}(\|u_0\|_{H^1(\mathbb{R})}+1)^{8s-2}}{(1-4s)(16\pi s)^{4s}} + \frac{Q^2}{4}\right]^{\frac{1}{2}} \|g_1 - g_2\|_{C_2(I)},$$

which completes the proof of the theorem. ∎

4. Auxiliary results

Below we derive the solvability conditions for the linear Poisson type equation with a square integrable right side

$$\left(-\frac{d^2}{dx^2}\right)^s u = f(x), \quad x \in \mathbb{R}, \quad 0 < s < 1. \tag{4.31}$$

Let us denote the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx, \tag{4.32}$$

with a slight abuse of notations when the functions involved in (4.32) are not square integrable, like for instance the one present in orthogonality condition (4.33) of Lemma 6 below. Indeed, if $f(x) \in L^1(\mathbb{R})$ and $g(x) \in L^\infty(\mathbb{R})$, then the integral in the right side of (4.32) is well defined. The left side of relation (4.34) makes sense as well under the given conditions. We have the following technical statement.

Lemma 6. *Let $f(x) : \mathbb{R} \to \mathbb{R}$ and $f(x) \in L^2(\mathbb{R})$.*
1) When $0 < s < \frac{1}{4}$ and in addition $f(x) \in L^1(\mathbb{R})$, equation (4.31) admits a unique solution $u(x) \in H^{2s}(\mathbb{R})$.

2) When $\frac{1}{4} \le s < \frac{3}{4}$ and additionally $|x|f(x) \in L^1(\mathbb{R})$, problem (4.31) possesses a unique solution $u(x) \in H^{2s}(\mathbb{R})$ if and only if the orthogonality relation

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{4.33}$$

holds.

3) When $\frac{3}{4} \le s < 1$ and in addition $x^2 f(x) \in L^1(\mathbb{R})$, equation (4.31) has a unique solution $u(x) \in H^{2s}(\mathbb{R})$ if and only if orthogonality conditions (4.33) and

$$(f(x), x)_{L^2(\mathbb{R})} = 0 \tag{4.34}$$

hold.

Proof. First we observe that by means of norm definition (1.4) along with the square integrability of the right side of (4.31), it would be sufficient to prove the solvability of problem (4.31) in $L^2(\mathbb{R})$. The solution $u(x) \in L^2(\mathbb{R})$ will obviously belong to $H^{2s}(\mathbb{R}),\ 0 < s < 1$ as well.

Let us establish the uniqueness of solutions for equation (4.31). If $u_{1,2}(x) \in H^{2s}(\mathbb{R})$ both satisfy (4.31), then the difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R})$ solves the homogeneous problem

$$\left(-\frac{d^2}{dx^2}\right)^s w = 0.$$

Since the operator $\left(-\frac{d^2}{dx^2}\right)^s$ on $\mathbb{R}$ does not have nontrivial square integrable zero modes, $w(x)$ vanishes a.e. on the real line.

Let us apply (2.14) to both sides of equation (4.31). This gives us

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s}}\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}} + \frac{\widehat{f}(p)}{|p|^{2s}}\chi_{\{p\in\mathbb{R}\ |\ |p|>1\}}, \tag{4.35}$$

where χ_A is the characteristic function of a set $A \subseteq \mathbb{R}$. Obviously, for all $0 < s < 1$ the second term in the right side of (4.35) is square integrable by virtue of the bound

$$\int_{-\infty}^{\infty} \frac{|\widehat{f}(p)|^2}{|p|^{4s}}\chi_{\{p\in\mathbb{R}\ |\ |p|>1\}}dp \le \|f\|^2_{L^2(\mathbb{R})} < \infty.$$

To prove the square integrability of the first term in the right side of (4.35) when $0 < s < \frac{1}{4}$, we apply estimate (2.15), which gives

$$\int_{-\infty}^{\infty} \frac{|\widehat{f}(p)|^2}{|p|^{4s}}\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}}dp \le \frac{\|f(x)\|^2_{L^1(\mathbb{R})}}{\pi(1-4s)} < \infty,$$

which completes the proof of part 1) of the lemma.

To establish the solvability of equation (4.31) when $\frac{1}{4} \le s < \frac{3}{4}$, we use the formula

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^p \frac{d\widehat{f}(s)}{ds}ds.$$

This allows us to express the first term in the right side of (4.35) as

$$\frac{\widehat{f}(0)}{|p|^{2s}}\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}} + \frac{\int_0^p \frac{d\widehat{f}(s)}{ds}ds}{|p|^{2s}}\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}}. \tag{4.36}$$

By virtue of definition (2.14)

$$\left|\frac{d\widehat{f}(p)}{dp}\right| \le \frac{1}{\sqrt{2\pi}}\||x|f(x)\|_{L^1(\mathbb{R})} < \infty$$

due to one of our assumptions. Hence,

$$\left|\frac{\int_0^p \frac{d\widehat{f}(s)}{ds}ds}{|p|^{2s}}\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}}\right| \le \frac{1}{\sqrt{2\pi}}\||x|f(x)\|_{L^1(\mathbb{R})}|p|^{1-2s}\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}} \in L^2(\mathbb{R}).$$

The remaining term in (4.36) $\frac{\widehat{f}(0)}{|p|^{2s}}\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}} \in L^2(\mathbb{R})$ if and only if $\widehat{f}(0) = 0$, which implies orthogonality relation (4.33) in case 2) of the lemma.

Finally, it remains to investigate the situation when $\frac{3}{4} \le s < 1$. For that purpose, we use the representation

$$\widehat{f}(p) = \widehat{f}(0) + p\frac{d\widehat{f}}{dp}(0) + \int_0^p \Big(\int_0^r \frac{d^2\widehat{f}(q)}{dq^2}dq\Big)dr,$$

which enables us to write the first term in the right side of (4.35) as

$$\left[\frac{\widehat{f}(0)}{|p|^{2s}} + \frac{p\frac{d\widehat{f}}{dp}(0)}{|p|^{2s}} + \frac{\int_0^p \Big(\int_0^r \frac{d^2\widehat{f}(q)}{dq^2}dq\Big)dr}{|p|^{2s}}\right]\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}}. \tag{4.37}$$

Definition (2.14) yields

$$\left|\frac{d^2\widehat{f}(p)}{dp^2}\right| \le \frac{1}{\sqrt{2\pi}}\|x^2 f(x)\|_{L^1(\mathbb{R})} < \infty$$

as assumed. This enables us to estimate

$$\left|\frac{\int_0^p \Big(\int_0^r \frac{d^2\widehat{f}(q)}{dq^2}dq\Big)dr}{|p|^{2s}}\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}}\right| \le \frac{1}{2\sqrt{2\pi}}\|x^2 f(x)\|_{L^1(\mathbb{R})}|p|^{2-2s}\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}},$$

which clearly belongs to $L^2(\mathbb{R})$. The sum of the first and the second terms in (4.37) is not square integrable unless both $\widehat{f}(0)$ and $\frac{d\widehat{f}}{dp}(0)$ vanish, which gives us orthogonality relations (4.33) and (4.34) respectively. ■

Notably for the lower values of the power of the negative second derivative operator $0 < s < \frac{1}{4}$ under the assumptions given above no orthogonality conditions are required to solve the linear Poisson type equation (4.31) in $H^{2s}(\mathbb{R})$.

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