Hausdorff dimension of invariant measure of circle diffeomorphisms with breaks

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Abstract

We prove that, for almost all irrational $\rho \in (0,1)$, Hausdorff dimension of invariant measure of circle diffeomorphisms with breaks, with rotation number ρ, is zero. This result cannot be extended to all irrational rotation numbers.

1 Introduction

The investigation of dimensional properties of invariant sets and measures originated in mathematical physics and is a part of a new important area — dimension theory of dynamical systems [10]. In this paper, we focus on the dimension of invariant measures for circle diffeomorphisms with breaks, i.e, diffeomorphisms of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, with a singular point where the derivative has a jump discontinuity. Circle maps with breaks are an active area of research for the following two reasons. Firstly, they provide the simplest nontrivial extension of circle diffeomorphisms, whose renormalizations converge to nonlinear maps. Secondly, they form a class of cyclic generalized (nonlinear) interval exchange transformations, i.e., piecewise smooth homeomorphisms of the circle [9, 1].

Dimensional properties of invariant measures of circle homeomorphisms are closely related to their rigidity properties. Both are determined by the lengths and distribution of intervals of dynamical partitions of the circle (see next section), and depend strongly on the rotation numbers of these maps.

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For rational rotation numbers, there may be many invariant measures, but any ergodic invariant measure μ is a uniform δ-measure on a periodic orbit. This immediately gives that the pointwise dimension of the measure $d_\mu(x) = 0$, for μ-almost all $x \in \mathbb{S}^1$. The latter implies that $\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = 0$. Here, $\dim_H \mu$, $\underline{\dim}_B \mu$ and $\overline{\dim}_B \mu$ are the Hausdorff and lower and upper box dimensions of μ, respectively. On the contrary, for irrational rotation numbers, circle homeomorphisms are uniquely ergodic [2].

The dimension of the invariant measure of (sufficiently-smooth) circle diffeomorphisms with an irrational rotation number ρ is closely related with the regularity of their conjugacies to a rotation $R_\rho : x \mapsto x + \rho \,(\mathrm{mod}\ 1)$. It follows from the work of Herman [3] and Yoccoz [12] that smooth circle diffeomorphisms with Diophantine rotation numbers are smoothly conjugate to the rotation R_ρ. In a recent formulation [6], $C^{2+\alpha}$-smooth circle diffeomorphisms, $\alpha \in (0, 1)$, with a Diophantine rotation number $\rho \in (0, 1)$ of class $D(\delta)$, with $\delta \in [0, \alpha)$, are $C^{1+\alpha-\delta}$-smoothly conjugate to the rotation R_ρ. A number $\rho \in (0, 1)$ is called Diophantine of class $D(\delta)$ if there exists $\mathcal{C} > 0$ such that, for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, $|\rho - p/q| \geq \mathcal{C}/q^{2+\delta}$. In particular, this immediately implies that, for their unique invariant measure μ, we have: the pointwise dimension $d_\mu(x) = 1$, for every $x \in \mathbb{S}^1$, and $\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = 1$. Since, for $\delta > 0$, Diophantine numbers of class $D(\delta)$ form a set of full Lebesgue measure, these properties are generic for circle diffeomorphisms.

In [11], for any Liouville (non-Diophantine) number $\rho \in (0, 1)$ and any $\beta \in [0, 1]$, Sadovskaya constructed examples of C^∞-smooth circle diffeomorphisms with a rotation number ρ whose unique invariant measure μ has Hausdorff dimension $\dim_H \mu = \beta$. In particular, for $\beta < 1$, this implies that their invariant measure is singular.

The main result of this paper is the following.

Theorem 1.1 *For almost all irrational $\rho \in (0, 1)$ and all $\alpha \in (0, 1)$ and $c \in \mathbb{R}_+ \backslash \{1\}$, and for any $C^{2+\alpha}$-smooth circle diffeomorphism with a break T, with a break of size c and rotation number ρ, we have $\dim_H \mu = 0$, where μ is the unique invariant measure for T.*

Remark 1 The set of irrational numbers for which theorem Theorem 1.1 holds contains a set $\mathcal{S}$ of all irrational numbers of full Lebesgue measure for which denominators of rational convergents q_n (see next section) do not grow faster than exponentially in n, and for which there is a subsequence of partial quotients k_{σ_n+1}, with σ_n odd, if $c > 1$, or even, if $c < 1$, which grows faster than linearly in σ_n.

Remark 2 The result of Theorem 1.1 stands in sharp contrast to the case of circle diffeomorphisms, for which the Hausdorff dimension of the invariant measure is, generically, nonzero. This is the consequence of the *strongly unbounded geometry* of these maps — the ratio of lengths of neighboring intervals of the dynamical partitions $\mathcal{P}_n$ (see next section) can be exponentially small in the corresponding partial quotients k_{n+1}. The same phenomenon is responsible for generic C^1 but not $C^{1+\varepsilon}$-rigidity of these maps. It follows from the extension of Herman's theory to circle diffeomorphisms with a break [4, 5, 8] that, any two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break, with the same irrational rotation number $\rho \in \mathcal{S}$ and the same size of the break $c \neq 1$, are C^1 but not $C^{1+\varepsilon}$-smoothly conjugate to each other, for any $\varepsilon > 0$.

Remark 3 This result can be extended to piecewise smooth homeomorphisms of a circle with finitely many break points.

The above result is optimal in the following sense. Let $\mathcal{B}$ be the set of irrational numbers in $(0, 1)$ with bounded partial quotients (see next section), i.e., such that, for each $\rho \in \mathcal{B}$, there exists a constant $K > 0$ such that $k_n \leq K$, for all $n \in \mathbb{N}$.

Proposition 1.2 *For every $\alpha \in (0, 1)$, $c \in \mathbb{R}_+ \setminus \{1\}$ and $\rho \in \mathcal{B}$, and for any $C^{2+\alpha}$-smooth circle diffeomorphism with a break T, with a break of size c and rotation number ρ, we have $\dim_H \mu > 0$, where μ is the unique invariant measure for T.*

In Section 2, we give some definitions of dimensions of an invariant measure, and introduce the dynamical partitions of the circle and renormalizations of circle homeomorphisms, which are used to prove our results. In Section 3, we prove Theorem 1.1. In Section 4, we prove Proposition 1.2.

2 Preliminaries

Let μ be a probability measure on a space X. For any set $E \subset X$, we define the d-dimensional Hausdorff content of E as $H^d_r(E) := \inf\{\sum_{i=1}^{\infty} r_i^d : E \subset \cup_{i=1}^{\infty} B(x_i, r_i), r_i < r\}$. The d-dimensional Hausdorff measure of E is given by $H^d(E) := \sup_{r>0} H^d_r(E)$. The Hausdorff dimension of the set E is defined as $\dim_H E := \inf\{d : H^d(E) = 0\}$.

The Hausdorff dimension of the measure μ is defined as

$$\dim_H \mu := \inf\{\dim_H E : \mu(E) = 1\}. \tag{2.1}$$

The lower and upper pointwise dimensions of the measure μ are defined by

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\ln \mu(B(x, r))}{\ln r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\ln \mu(B(x, r))}{\ln r}, \tag{2.2}$$

respectively, for any $x \in \mathbb{S}^1$. The pointwise dimension is defined by $d_\mu(x) := \underline{d}_\mu(x) = \overline{d}_\mu(x)$, if the latter equality holds.

Poincaré showed that for every orientation-preserving homeomorphism $T : \mathbb{S}^1 \to \mathbb{S}^1$ there is a unique rotation number $\rho \in (0, 1)$, given by the limit

$$\rho := \lim_{n\to\infty} \frac{\mathcal{T}^n(x) - x}{n} \mod 1, \tag{2.3}$$

where $\mathcal{T}$ is any lift of T to $\mathbb{R}$. The continued fraction expansion of $\rho \in (0,1)$ is given by

$$\rho = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \dots}}}, \tag{2.4}$$

that we also write as $\rho = [k_1, k_2, k_3, \dots]$. The sequence of integers $(k_n)_{n\in\mathbb{N}}$, called *partial quotients*, is infinite if and only if ρ is irrational. Every irrational ρ defines uniquely the sequence of partial quotients. Conversely, every infinite sequence of partial quotients defines uniquely an irrational number ρ as the limit of the sequence of rational convergents $p_n/q_n = [k_1, k_2, \dots, k_n]$. It is well-known that p_n/q_n form a sequence of best rational approximations of an irrational ρ, i.e., there are no rational numbers, with denominators smaller or equal to q_n, that are closer to ρ than p_n/q_n. The rational convergents can also be defined recursively by $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$, starting with $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$.

To define the renormalizations of an orientation-preserving homeomorphism $T : \mathbb{S}^1 \to \mathbb{S}^1$, with an irrational rotation number ρ, we start with a *marked point* $x_0 \in \mathbb{S}^1$, and consider the marked orbit $x_i = T^i x_0$, with $i \in \mathbb{N}$. The subsequence $(x_{q_n})_{n\in\mathbb{N}}$, indexed by the denominators q_n of the sequence of rational convergents of the rotation number ρ, are called the sequence of *dynamical convergents*. It follows from the simple arithmetic properties of the rational convergents that the sequence of dynamical convergents $(x_{q_n})_{n\in\mathbb{N}}$ for the rigid rotation R_ρ has the property that its subsequence with n odd approaches x_0 from the left and the subsequence with n even approaches x_0 from the right. Since all circle homeomorphisms with the same irrational rotation number are combinatorially equivalent, the order of the dynamical convergents of T is the same.

The intervals $[x_{q_n}, x_0]$, for n odd, and $[x_0, x_{q_n}]$, for n even, will be denoted by $\Delta_0^{(n)}$ and called the n-th renormalization segment associated to x_0. The n-th renormalization segment associated to x_i will be denoted by $\Delta_i^{(n)}$. We also have the following important property: the only points of the orbit $\{x_i : 0 < i \le q_{n+1}\}$ that belong to $\Delta_0^{(n-1)}$ are $\{x_{q_{n-1}+iq_n} : 0 \le i \le k_{n+1}\}$.

Certain number of images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$, under the iterates of a map T, cover the whole circle without overlapping beyond the end points and form the n-th *dynamical partition* of the circle

$$\mathcal{P}_n := \{T^i \Delta_0^{(n-1)} : 0 \le i < q_n\} \cup \{T^i \Delta_0^{(n)} : 0 \le i < q_{n-1}\}. \tag{2.5}$$

The intervals $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ will be called the *fundamental intervals* of $\mathcal{P}_n$.

The n-th *renormalization* of an orientation-preserving homeomorphism $T : \mathbb{S}^1 \to \mathbb{S}^1$, with rotation number ρ, with respect to the marked point $x_0 \in \mathbb{S}^1$, is a function $f_n : [-1, 0] \to \mathbb{R}$, obtained from the restriction of T^{q_n} to $\Delta_0^{(n-1)}$, by rescaling the coordinates. More precisely, if τ_n is the affine change of coordinates that maps $x_{q_{n-1}}$ to -1 and x_0 to

0, then

$$f_n := \tau_n \circ T^{q_n} \circ \tau_n^{-1}. \tag{2.6}$$

If we identify x_0 with zero, then τ_n is exactly the multiplication by $(-1)^n/|\Delta_0^{(n-1)}|$. Here, and in what follows, $|I|$ denotes the length of an interval I on $\mathbb{S}^1$.

This paper concerns circle diffeomorphisms (maps) with a break, i.e., homeomorphisms of a circle for which there exists a point $x_{br} \in \mathbb{S}^1$, such that $T \in C^r(\mathbb{S}^1 \backslash \{x_{br}\})$, $T'(x)$ is bounded from below by a positive constant and the one-sided derivatives at x_{br} are such that the *size of the break*

$$c := \sqrt{\frac{T'_-(x_{br})}{T'_+(x_{br})}} \neq 1.$$

The renormalizations that we use in this paper are defined with the marked point $x_0 = x_{br}$.

Since, for circle maps with breaks T, $\ln T'$ is differentiable and has a bounded derivative on $\mathbb{S}^1 \backslash \{x_{br}\}$, $\ln T'$ is of bounded variation. Let $V := \mathrm{Var}_{\mathbb{S}^1} \ln T'$. Consequently, the Denjoy lemma holds, and

$$|\ln (T^{q_n})'| \leq V. \tag{2.7}$$

It was proved in [7] that the length of the elements Δ, of dynamical partitions $\mathcal{P}_n$, decreases exponentially with n, i.e., $|\Delta| \leq \bar{\lambda}^{n-1}$, where $\bar{\lambda} := (1+e^{-V})^{-1/2}$. Furthermore, the sequence of renormalizations $(f_n)_{n \in \mathbb{N}_0}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, of any $C^{2+\alpha}$-smooth ($\alpha \in (0,1)$) circle map with a break T, with a break of size $c \in \mathbb{R}_+ \backslash \{1\}$, converges to a sequence of fractional linear transformations $F_n := F_{a_n, b_n, M_n, c_n} : [-1, 0] \to \mathbb{R}$,

$$F_n(z) := \frac{a_n + (a_n + b_n M_n) z}{1 - (M_n - 1) z}, \tag{2.8}$$

where

$$a_n := \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|}, \quad b_n := \frac{|\Delta_0^{(n-1)}| - |\Delta_{q_{n-1}}^{(n)}|}{|\Delta_0^{(n-1)}|}, \quad M_n := \exp\left(\sum_{i=0}^{q_n - 1} \int_{x_{q_{n-1}+i}}^{x_i} \frac{T''(x)}{2T'(x)}\, dx \right). \tag{2.9}$$

In fact, it was shown in [4], that there exists a universal constant $\lambda \in (0,1)$ (depending on c and α only) and, for any $C^{2+\alpha}$-smooth circle map with a break T, with an irrational rotation number ρ and a break of size c, a constant $C > 0$, such that

$$\|f_n - F_n\|_{C^2[-1,0]} \leq C \lambda^n. \tag{2.10}$$

It was also shown in [4] (see Proposition 3.5 therein) that there exists $\delta > 0$ such that, for every $\varepsilon_0 > 0$ and sufficiently large $n \in \mathbb{N}$, the parameters M_n, characterizing the nonlinearity of the maps, satisfy

$$\frac{M_n - 1}{c_n - 1} \in (\delta - \varepsilon_0, 1 + \varepsilon_0). \tag{2.11}$$

Here $c_n := c^{(-1)^n}$. In particular, this implies that M_n is bounded, bounded away from both 0 and 1, and larger than 1, if $c_n > 1$, and smaller than 1, if $c_n < 1$.

The following properties of renormalizations are crucial for the proof of Theorem 1.1. For sufficiently large n and $c_n < 1$, the renormalizations f_n satisfy:

(i) f_n'' is negative and bounded away from zero, uniformly in n;

(ii) $f_n'(-1) \to c_n^{-1}$ and $f_n'(0) \to c_n$, as $k_{n+1} \to \infty$.

The claim (i) follows from (2.10), and the explicit expression for the second derivative

$$F_n''(z) = 2(M_n - 1)\frac{(a_n + b_n)M_n}{(1-(M_n-1)z)^3}, \tag{2.12}$$

taking into account the estimate (2.11) and that, by the Denjoy lemma (2.7), a_n is bounded and of the order of $1 - b_n$. The claim (ii) follows from the fact that as $k_{n+1} \to \infty$, we have $a_n \to 0$, $b_n \to 1$ and $M_n \to c_n$.

At the end of this section, let us give a few comments about notation. For positive sequences A_n and B_n, $n \in \mathbb{N}$, we say that $A_n = \Theta(B_n)$, if there exists a constant $\mathfrak{C} > 0$ such that $\mathfrak{C}^{-1}B_n \le A_n \le \mathfrak{C}B_n$.

3 Proof of Theorem 1.1

Let $\mathcal{S}_{\exp}$ be the set of irrational $\rho \in (0,1)$ for which there exists $C_1 > 0$ and $\nu \in (0,1)$ such that $q_n \le C_1 \nu^{-n}$.

Let $\mathcal{S}^{(o)}$ and $\mathcal{S}^{(e)}$ be the sets of irrational $\rho \in (0,1)$ for which, there exist a sequence $A(m) > 0$, sequences $\sigma_n \in 2\mathbb{N} - 1$ and $\sigma_n \in 2\mathbb{N}$, respectively, and $n_0 \in \mathbb{N}$ such that $k_{\sigma_n+1} \ge A(\sigma_n)\sigma_n$, for all $n \ge n_0$, the sequence $A(m)$ diverges to infinity as $m \to \infty$, and the series $\sum 1/(A(m)m)$ diverges.

We define $\mathcal{S}_{\mathrm{sl}} := \mathcal{S}^{(o)}$ if $c > 1$ and $\mathcal{S}_{\mathrm{sl}} := \mathcal{S}^{(e)}$ if $c < 1$. We define $\mathcal{S} := \mathcal{S}_{\exp} \cap \mathcal{S}_{\mathrm{sl}}$ (the labels exp and sl stand for *exponential* and *super-linear*, respectively).

Proposition 3.1 *The set $\mathcal{S}$ has Lebesgue measure* 1.

Proof. It follows from a theorem of Khintchine that for almost all irrational $\rho \in (0,1)$, $\frac{\ln q_n}{n} \to \frac{\pi^2}{12\ln 2}$, as $n \to \infty$. This implies that, for any $\delta > 0$, almost surely, there exists $C_1 > 0$, such that $q_n \le C_1\nu^{-n}$, with $\nu = \exp\left(-\frac{\pi^2}{12\ln 2} - \delta\right)$, for all $n \in \mathbb{N}$. Consequently, $\mathcal{S}_{\exp}$ has Lebesgue measure equal to 1. The condition that the series $\sum 1/(A(m)m)$ diverges assures that the sets $\mathcal{S}^{(o)}$ and $\mathcal{S}^{(e)}$ also have Lebesgue measure equal to 1. The proof that $\mathcal{S}^{(o)}$ and $\mathcal{S}^{(e)}$ have full Lebesgue measure is similar to that of Proposition 4.1 of [8]. The claim follows. **QED**

Let $c \in \mathbb{R}_+\backslash\{1\}$, $\alpha \in (0,1)$ and $\rho \in \mathcal{S}$ be given, and let T be a $C^{2+\alpha}$-smooth circle map with a break, with rotation number ρ and the size of the break c. Let σ_n be the sequence associated with ρ described above. Let $\epsilon_n := e^{-k_{\sigma_n+1}\eta(\sigma_n)}$, where $\eta(m) > 0$ is any sequence converging to zero such that $A(m)\eta(m) \to \infty$, as $m \to \infty$. For each $n \in \mathbb{N}$, let

$$E_{n,0} := \tau_{\sigma_n}^{-1}([-1, -1+\epsilon_n] \cup [-\epsilon_n, 0]), \tag{3.1}$$

and let

$$E_{n,i} := T^i(E_{n,0}), \quad \text{for} \quad i = 1, \dots, q_{\sigma_n} - 1. \tag{3.2}$$

We define

$$E_n := \bigcup_{i=0}^{q_{\sigma_n}-1} E_{n,i}, \tag{3.3}$$

and

$$E := \limsup_{n\to\infty} E_n = \bigcap_{n\geq 1} \bigcup_{j\geq n} E_j. \tag{3.4}$$

Proposition 3.2 $\mu(E) = 1$.

Proof. For sufficiently large n, the renormalization graphs f_{σ_n} are uniformly concave downwards, due to (i). It follows that the iterates $f_{\sigma_n}^i(-1)$, for $i = 0, \dots, k_{n+1}$ can only accumulate near -1 or 0. Moreover, this accumulation is geometrical, since the derivatives $f'_{\sigma_n}(-1)$ and $f'_{\sigma_n}(0)$ are bounded away from 1, due to (ii). Consequently, the length of the longest of the intervals $[f_{\sigma_n}^{i-1}(-1), f_{\sigma_n}^i(-1)]$, which is a subset of $\tau_{\sigma_n}(E_{n,0})$, is of the order of ϵ_n, and the number of the elements of the set $\{f_{\sigma_n}^i(-1) : i = 0, \dots, k_{\sigma_n+1}\}$ outside $\tau_{\sigma_n}(E_{n,0})$ is of the order of $|\ln\epsilon_n|$. Since the invariant measure of the intervals $\tau_{\sigma_n}^{-1}([f_{\sigma_n}^{i-1}(-1), f_{\sigma_n}^i(-1)])$ is independent of i, $\mu(E_{n,0})/\mu(\tau_{\sigma_n}^{-1}([-1,0])) = 1 - \Theta(\frac{|\ln\epsilon_n|}{k_{\sigma_n+1}})$. By the invariance of the measure μ, $\mu(E_{n,i})/\mu(\Delta_i^{(\sigma_n-1)}) = 1 - \Theta(\frac{|\ln\epsilon_n|}{k_{\sigma_n+1}})$. Since

$$\sum_{i=0}^{q_{\sigma_n}-1} \mu(\Delta_i^{(\sigma_n-1)}) + \sum_{i=0}^{q_{\sigma_n-1}-1} \mu(\Delta_i^{(\sigma_n)}) = q_{\sigma_n}\mu(\Delta_0^{(\sigma_n-1)}) + q_{\sigma_n-1}\mu(\Delta_0^{(\sigma_n)}) = 1, \tag{3.5}$$

$q_{\sigma_n-1} \leq q_{\sigma_n}$ and $\mu(\Delta_0^{(\sigma_n)}) = \mu(\tau_{\sigma_n}^{-1}([-1, f_{\sigma_n}(-1)]))$, we have $\mu(E_n) = 1 - \Theta(\frac{|\ln\epsilon_n|}{k_{\sigma_n+1}})$. Since $\mu(\cup_{j\geq n}E_j) \geq \mu(E_i)$, for any $i \geq n$, and $\mu(E_i) \to 1$ as $i \to \infty$, it follows that $\mu(\cup_{j\geq n}E_j) = 1$, for any $n \in \mathbb{N}$. The claim follows. **QED**

Proposition 3.3 *For any $d \in [0,1)$, $H^d(E) = 0$.*

Proof. The set $\{E_{n,i} : i = 0, \ldots, q_{\sigma_n} - 1\}$ is a closed cover of E_n. Since the length of the intervals $\Delta_i^{(\sigma_n-1)}$ of dynamical partitions $\mathcal{P}_{\sigma_n}$ satisfies $|\Delta_i^{(\sigma_n-1)}| \leq \bar{\lambda}^{\sigma_n-1}$, for every $r > 0$, there exists $n_1 \in \mathbb{N}$ such that, for $n \geq n_1$,

$$\begin{aligned} H_r^d(E_n) &\leq \sum_{i=0}^{q_{\sigma_n}-1} \left(\epsilon_n \bar{\lambda}^{\sigma_n-1}\right)^d \leq q_{\sigma_n} e^{-dk_{\sigma_n+1}\eta(\sigma_n)} \bar{\lambda}^{d(\sigma_n-1)} \\ &\leq C_1 \nu^{-\sigma_n} e^{-d\sigma_n A(\sigma_n)\eta(\sigma_n)} \bar{\lambda}^{d(\sigma_n-1)} \leq C_2 e^{-\varkappa_1 \sigma_n} \leq C_2 e^{-\varkappa_2 n}, \end{aligned} \tag{3.6}$$

where $\varkappa_1, \varkappa_2, C_2 > 0$. It follows that

$$H_r^d(E) \leq \sum_{j=n}^{\infty} H_r^d(E_j) \leq \sum_{j=n}^{\infty} C_2 e^{-\varkappa_2 j} \leq \frac{C_2}{1 - e^{-\varkappa_2}} e^{-\varkappa_2 n}. \tag{3.7}$$

Since the right hand side goes to 0 as $n \to \infty$, for any $r > 0$, we have $H_r^d(E) = 0$. By taking the supremum over all $r > 0$, the claim follows. **QED**

An immediate corollary of Proposition 3.3 is the following.

Corollary 3.4 $\dim_H E = 0$.

This implies

Corollary 3.5 $\dim_H \mu = 0$.

This proves Theorem 1.1.

4 Proof of Proposition 1.2

The following holds for any $C^{2+\alpha}$-smooth circle map with a break T, with a rotation number $\rho \in \mathcal{B}$, and with a break of size $c \in \mathbb{R}_+ \backslash \{1\}$.

Proposition 4.1 *For every $\rho \in \mathcal{B}$, there exists $\kappa > 0$ such that, for every $n \in \mathbb{N}$, every interval $\Delta \in \mathcal{P}_n$ and every $\bar{\Delta} \in \mathcal{P}_{n+1}$ satisfying $\bar{\Delta} \subset \Delta$, we have $|\bar{\Delta}| > \kappa |\Delta|$.*

Proof. Each element $\Delta_i^{(n)}$, $i = 0, \ldots, q_{n-1} - 1$, of partition $\mathcal{P}_n$ is also an element of partition $\mathcal{P}_{n+1}$. Each element $\Delta_i^{(n)}$, $i = 0, \ldots, q_n - 1$, of partition $\mathcal{P}_n$ is a union of intervals $\Delta_{i+q_{n-1}+jq_n}^{(n)}$, for $j = 0, \ldots, k_{n+1} - 1$, and the interval $\Delta_i^{(n+1)}$ of partition $\mathcal{P}_{n+1}$. Therefore, since $\rho \in \mathcal{B}$, there exists $K > 0$ such that each element of partition $\mathcal{P}_n$ contains at most $k_{n+1} + 1 \leq K + 1$ elements of partition $\mathcal{P}_{n+1}$. Since $\Delta_{i+q_{n-1}+(j+1)q_n}^{(n)} = T^{q_n}(\Delta_{i+q_{n-1}+jq_n}^{(n)})$ and $\Delta_i^{(n+1)} \subset \Delta_{i+q_{n+1}}^{(n)}$, it follows from the Denjoy estimate (2.7) that the lengths of

$\Delta_{i+q_{n-1}+jq_n}^{(n)}$, for $j = 0, \ldots, k_{n+1} - 1$, are of the same order as the length of $\Delta_i^{(n-1)}$. Since $\Delta_{i+q_{n-1}}^{(n)} = T^{q_{n-1}}(\Delta_i^{(n)})$, the length of $\Delta_i^{(n)}$ is of the same order as the length of $\Delta_i^{(n-1)}$ as well. Applying the same arguments for $n+1$ instead of n gives that the length of the interval $\Delta_i^{(n+1)}$ is also of the same order. The claim follows. **QED**

Proposition 4.1 immediately implies

Corollary 4.2 *For every $\rho \in \mathcal{B}$, every $n \in \mathbb{N}$ and every interval $\Delta \in \mathcal{P}_n$, we have $|\Delta| > \kappa^n$.*

Proposition 4.3 *For every $\rho \in \mathcal{B}$, there exists $C_3 > 1$ such that, for every $n \in \mathbb{N}$ and every two neighboring intervals $\Delta, \Delta' \in \mathcal{P}_n$, we have $C_3^{-1} < |\Delta|/|\Delta'| < C_3$.*

Proof. Notice that the neighboring intervals Δ and Δ' of $\mathcal{P}_n$ must satisfy one of the following statements: either $\Delta' = T^{\pm q_{n-1}}(\Delta)$ or there is an interval $\bar{\Delta} \in \mathcal{P}_{n+1}$, such that either $\bar{\Delta} \subset \Delta$ and $\bar{\Delta} = T^{q_{n-1}}(\Delta')$ or $\bar{\Delta} \subset \Delta'$ and $\bar{\Delta} = T^{q_{n-1}}(\Delta)$. The claim now follows by applying the Denjoy estimate (2.7) or Proposition 4.1. **QED**

Proposition 4.4 *For every $\rho \in \mathcal{B}$,*

$$\mu(\Delta_0^{(n)}) \leq \frac{1}{q_{n+1}}. \tag{4.1}$$

Proof. Since T is topologically conjugate to the rotation R_ρ, $\mu(\Delta_0^{(n)}) = |q_n\rho - p_n|$. The claim follows from the properties of the continued fractions. **QED**

Proof of Theorem 1.2. Let $x \in \mathbb{S}^1$ and let $r_n := \frac{|\Delta_{n,\min}|}{2}$, where $\Delta_{n,\min}$ is the smallest element of partition $\mathcal{P}_n$, for every $n \in \mathbb{N}_0$. Let $r > 0$. If $r_n \leq r \leq r_{n-1}$, for some $n \in \mathbb{N}_0$, then the open ball of radius r around x, i.e. $B(x,r)$, cannot have a nonempty intersection with more than two intervals of partition $\mathcal{P}_{n-1}$. Consequently, by Proposition 4.4, $\mu(B(x,r)) \leq \frac{2}{q_n}$. Using Corollary 4.2, it follows that

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\ln \mu(B(x,r))}{\ln r} \geq \liminf_{n\to\infty} \frac{\ln \frac{2}{q_n}}{\ln r_n} \geq \liminf_{n\to\infty} \frac{\ln \frac{2}{q_n}}{\ln \frac{\kappa^n}{2}} \geq \frac{\ln \gamma}{\ln \kappa^{-1}}, \tag{4.2}$$

where $\gamma = \frac{1+\sqrt{5}}{2}$ is the golden mean. In the last equality, we have used that $q_n \geq \gamma^{n-1}$. Since $\underline{d}_\mu(x) \geq \frac{\ln\gamma}{\ln\kappa^{-1}}$ for every $x \in \mathbb{S}^1$, $\dim_H \mu \geq \frac{\ln\gamma}{\ln\kappa^{-1}}$, as follows from a result of Young [13]. **QED**

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