Existence and nonlinear stability of stationary states for the magnetic Schrödinger-Poisson system

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9th March 2016

Abstract

The article is devoted to the studies of the stationary states of the magnetic Schrödinger-Poisson system in the repulsive (plasma physics) Coulomb case. Particularly, we prove the existence and the nonlinear stability of a wide class of stationary states by virtue of the energy-Casimir method. We generalize the global well-posedness result for the Schrödinger-Poisson system obtained in [9] to the case when a magnetic field is turned on.

2010 *Mathematics Subject Classification*: 82D10, 82C10
Keywords: Magnetic field, Schrödinger-Poisson system, Hartree-von Neumann equation, stationary solutions, nonlinear stability, global existence and uniqueness.

1 Introduction

In the present article, we establish the existence and the nonlinear stability for a certain class of stationary solutions of the magnetic Schrödinger-Poisson system in a bounded domain with Dirichlet boundary conditions. This system describes the mean-field dynamics of non relativistic quantum particles in the case of plasma in a magnetic field. We consider quantum particles confined in a domain $\Omega \subset \mathbb{R}^3$ which is an open, bounded set with a C^2 boundary, such that $|\Omega| < \infty$. The particles are interacting by virtue of the electrostatic field they collectively generate. In the mean-field limit, the density matrix describes the *mixed* state of the system and satisfies the Hartree-von Neumann equation

$$\begin{cases} i\partial_t \rho(t) = [H_{A,\ V}, \rho(t)], \quad x \in \Omega, \quad t \geq 0 \\ -\Delta V = n(t,x), \quad n(t,x) = \rho(t,x,x), \ \rho(0) = \rho_0 \end{cases}, \tag{1.1}$$

with Dirichlet boundary conditions, $\rho(t,x,y) = 0$ if x or $y \in \partial\Omega$, for $t \geq 0$. Our single particle Hamiltonian is given by

$$H_{A,\ V} := (-i\nabla + A)^2 + V(t,x), \tag{1.2}$$

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where $A(x) \in C^1(\bar{\Omega}, \mathbb{R}^3)$ is the magnetic vector potential and $divA = 0$. In system (1.1) and further down, $(-i\nabla + A)^2$ denotes the magnetic Dirichlet Laplacian on $L^2(\Omega)$. Let us refer to [5] and [6] for a derivation of the analogous system of equations in the *non magnetic* case. Since $\rho(t)$ is a nonnegative, self-adjoint and trace-class operator acting on $L^2(\Omega)$, we are able to expand its kernel, for every $t \in \mathbb{R}_+$, with respect to an orthonormal basis of $L^2(\Omega)$. Let us designate this kernel at the initial time $t = 0$ by ρ_0,

$$\rho_0(x, y) = \sum_{k\in\mathbb{N}} \lambda_k \psi_k(x)\overline{\psi_k(y)}. \tag{1.3}$$

Here $\{\psi_k\}_{k\in\mathbb{N}}$ denotes an orthonormal basis of $L^2(\Omega)$, such that $\psi_k|_{\partial\Omega} = 0$ for all $k \in \mathbb{N}$, and the coefficients are given by

$$\underline{\lambda} := \{\lambda_k\}_{k\in\mathbb{N}} \in l^1, \quad \lambda_k \geq 0, \quad \sum_{k\in\mathbb{N}} \lambda_k = 1. \tag{1.4}$$

In Lemma 13 of the Appendix below, we prove that there exists a one-parameter family of complete orthonormal bases of $L^2(\Omega)$, $\{\psi_k(t)\}_{k\in\mathbb{N}}$, with $\psi_k(t)|_{\partial\Omega} = 0$ for all $k \in \mathbb{N}$, and for $t \in \mathbb{R}_+$, such that the kernel of the density matrix $\rho(t)$, which satisfies system (1.1), can be expressed as

$$\rho(t, x, y) = \sum_{k\in\mathbb{N}} \lambda_k \psi_k(t, x)\overline{\psi_k(t, y)}. \tag{1.5}$$

Consequently of the particular commutator structure of (1.1) (where $\rho(t)$ and $-iH_{A,\ V}$ satisfy the conditions of a Lax pair), the corresponding flow of $\rho(t)$ leaves its spectrum invariant. Therefore, the coefficients $\underline{\lambda}$ are *independent* of t. This isospectrality is crucial for the stability analysis of stationary states based on the Casimir energy method used in this article; see also [7, 11, 12, 18, 20]. Similar ideas were exploited recently in the analysis of the semi-relativistic Schrödinger-Poisson system without a magnetic field in [1], [2], [3] describing the heated plasma.

When substituting expression (1.5) in the system (1.1), one can verify that the one-parameter family of orthonormal vectors $\{\psi_k(t)\}_{k\in\mathbb{N}}$ solves the magnetic Schrödinger-Poisson system equivalent to (1.1) and given by

$$i\frac{\partial \psi_k}{\partial t} = (-i\nabla + A)^2\psi_k + V[\Psi]\psi_k, \quad k \in \mathbb{N}, \tag{1.6}$$

$$-\Delta V[\Psi] = n[\Psi], \tag{1.7}$$

$$\psi_k(t = 0, .) = \psi_k(0), \quad \forall k \tag{1.8}$$

and

$$\psi_k(t, x) = 0\ , \quad V(t, x) = 0,\ t \geq 0,\ \forall x \in \partial\Omega,\ \forall k \in \mathbb{N}, \tag{1.9}$$

where we used the notations

$$\Psi := \{\psi_k\}_{k=1}^{\infty} \quad \text{and} \quad n[\Psi(t, x)] := \sum_{k=1}^{\infty} \lambda_k |\psi_k(t, x)|^2. \tag{1.10}$$

Here $\{\psi_k(0)\}_{k=1}^{\infty}$ is the initial data, the potential function $V[\Psi]$ solves the Poisson equation (1.7) and both $V[\Psi]$ and $\psi_k(t)$, for all $k \in \mathbb{N}$, satisfy the Dirichlet boundary conditions (1.9).

The global well posedness for system (1.6)-(1.9) is proved in the Appendix below. Similar results without a magnetic field were obtained before in a finite volume domain with Dirichlet boundary conditions in [9], and in the whole space of $\mathbb{R}^3$ in [9] and [12].

In the article, we are interested in the properties of *stationary states* occurring when $\rho(t) = f(H_{A,\ V})$ for a certain function f. When substituting the latter in (1.1), the commutator on the right side of the first equation of system (1.1) vanishes, such that the density matrix is time independent. The exact properties of the distribution function f will be discussed further down. The solution of the Schrödinger-Poisson system which corresponds to the stationary states is

$$\psi_k(t,x) = e^{-i\mu_k t}\psi_k(x), \quad k \in \mathbb{N},$$

such that the potential function $V[\Psi]$ is time independent, $\mu_k \in \mathbb{R}$ are the eigenvalues of the Hamiltonian (1.2) and $\psi_k(x)$ are the corresponding eigenfunctions.

Note that our results are relevant to recent work on stellar dynamics, see [8, 17].

Our article is organized as follows. In Section 2, we describe the class of stationary states we will study, and state our hypotheses and main results about nonlinear stability and existence of stationary states. Section 3 is devoted to the derivation of some preliminary results. In Section 4, we establish the nonlinear stability of the stationary states of the magnetic Schrödinger-Poisson system via the energy-Casimir functional as a Lyapunov function (see the statement of Theorem 1. In Section 5 we define the dual functional and in Section 6 study its properties using the methods of convex analysis, and show that it admits a unique maximizer (see Theorem 2), which implies the existence of a stationary state for our magnetic Schrödinger-Poisson system. In the Appendix, we prove the global well-posedness for our magnetic Schrödinger-Poisson system.

2 The Model and Statement of the Main Results

We define the state space for the magnetic Schrödinger-Poisson system as

$$\mathcal{L} := \{(\Psi, \underline{\lambda}) \mid \Psi = \{\psi_k\}_{k=1}^{\infty} \subset H^1_{0,\ A}(\Omega) \cap H^2_A(\Omega) \ \ \textit{is a complete orthonormal system}$$

$$\textit{in } L^2(\Omega),\ \underline{\lambda} = \{\lambda_k\}_{k=1}^{\infty} \in l^1, \quad \lambda_k \geq 0,\ k \in \mathbb{N}, \quad \sum_{k=1}^{\infty} \lambda_k \int_{\Omega} |(-i\nabla + A)^2 \psi_k|^2 dx < \infty\},$$

see [18] in the non magnetic case. The magnetic Sobolev spaces $H^1_{0,\ A}(\Omega)$ and $H^2_A(\Omega)$ here are the standard ones.

For the precise definition of the class of stationary states we will study, we introduce the *Casimir class* of functions. Let us say that a function $f : \mathbb{R} \to \mathbb{R}$ is of Casimir class $\mathcal{C}$ if and only if it has the following properties:

(i) f is continuous, such that $f(s) > 0$ for $s < s_0$ and $f(s) = 0$ when $s \geq s_0$, with some $s_0 \in]0, \infty]$,

(ii) f is strictly decreasing on $]-\infty, s_0]$, such that $\lim_{s \to -\infty} f(s) = \infty$,

(iii) there exist constants $\varepsilon > 0$ and $C > 0$, such that for $s \geq 0$ the estimate

$$f(s) \leq C(1+s)^{-\frac{7}{2}-\varepsilon} \tag{2.1}$$

holds.

Throughout the article C will stand for a finite, positive constant. Note that the rate of decay assumed in (2.1) is the same one as in the non magnetic case treated in [18]. Assumption (iii) along with Weyl asymptotics for the Laplacian and the comparison of magnetic and non magnetic Dirichlet eigenvalues following from the result of Lemma 14 of the Appendix, yield that $f((-i\nabla + A)^2 + V)$ and $F((-i\nabla + A)^2 + V)$ are trace-class, for smooth enough and positive V, a smooth vector potential A and F defined in (2.5) (see Lemma 14 in Section 3).

Let us consider the quadruple $(\Psi_0, \underline{\lambda}_0, \mu_0, V_0)$ with $(\Psi_0, \underline{\lambda}_0) \in \mathcal{L}$, $\mu_0 = \{\mu_{0,k}\}_{k=1}^{\infty}$ real valued, and the potential function $V_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, such that the stationary magnetic Schrödinger-Poisson system is given by

$$((-i\nabla + A)^2 + V_0)\psi_{0,k} = \mu_{0,k}\psi_{0,k}, \quad k \in \mathbb{N}, \tag{2.2}$$

$$-\Delta V_0 = n_0 = \sum_{k=1}^{\infty} \lambda_{0,k} |\psi_{0,k}|^2 , \tag{2.3}$$

with

$$\lambda_{0,k} = f(\mu_{0,k}), \quad k \in \mathbb{N}, \tag{2.4}$$

and $f \in \mathcal{C}$. Then, the corresponding density matrix $\rho_0 = f((-i\nabla + A)^2 + V_0)$ satisfies the stationary state Hartree-von Neumann equation

$$[H_{A,\ V_0}, \rho_0] = 0.$$

Remark. *In the semi-relativistic case studied in [3], the Casimir class was defined analogously but the rate of decay of the distribution function f was assumed to be higher. A good example of $f \in \mathcal{C}$ is the function decaying exponentially as $s \to \infty$ with the cut-off level $s_0 = \infty$. This is precisely the Boltzmann distribution $f(s) := e^{-\beta s}$, $\beta > 0$.*

To establish the nonlinear stability of the stationary states, we will rely on the energy-Casimir method. This method was used in [7] for fluid problems, and in [11, 20] for treating stationary states of kinetic equations, in particular, Vlasov-Poisson systems. In the present work, we extend the energy-Casimir functional used in [18] to the magnetic case. For $f \in \mathcal{C}$, we define

$$F(s) := \int_s^{\infty} f(\sigma) d\sigma, \quad s \in \mathbb{R}. \tag{2.5}$$

Clearly, the function defined via (2.5) is decreasing, continuously differentiable, nonnegative and is strictly convex on its support. Furthermore, for $s \geq 0$

$$F(s) \leq C(1+s)^{-\frac{5}{2}-\varepsilon}. \tag{2.6}$$

Its Legendre (Fenchel) transform is given by

$$F^*(s) := \sup_{\lambda \in \mathbb{R}} (\lambda s - F(\lambda)), \quad s \leq 0. \tag{2.7}$$

Let us define the energy-Casimir functional for a fixed f as

$$\mathcal{H}_C(\Psi, \underline{\lambda}) := \sum_{k=1}^{\infty} F^*(-\lambda_k) + \mathcal{H}(\Psi, \underline{\lambda}), \quad (\Psi, \underline{\lambda}) \in \mathcal{L}, \tag{2.8}$$

where $\mathcal{H}(\Psi, \underline{\lambda})$ is defined as

$$\begin{aligned} \mathcal{H}(\Psi, \underline{\lambda}) &:= \sum_{k=1}^{\infty} \lambda_k \int_\Omega |(-i\nabla + A)\psi_k|^2 dx + \frac{1}{2} \int_\Omega n_{\psi,\lambda} V_{\psi,\lambda} dx \\ &= \sum_{k=1}^{\infty} \lambda_k \int_\Omega |(-i\nabla + A)\psi_k|^2 dx + \frac{1}{2} \int_\Omega |\nabla V_{\psi,\lambda}|^2 dx. \end{aligned} \tag{2.9}$$

Particularly, $\mathcal{H}_C$ is conserved along solutions of our magnetic Schrödinger-Poisson system, due to the isospectrality of the flow of $\rho(t)$, which is equivalent to the t- independence of λ_k and the conservation of energy $\mathcal{H}(\Psi, \underline{\lambda})$ established in Lemma 19 of the Appendix. The main results of this work address the existence and stability of stationary states given by (2.2)-(2.4), for $f \in \mathcal{C}$. The stability is established in the first main theorem.

Theorem 1. *Let* $(\Psi_0, \underline{\lambda}_0, \mu_0, V_0)$ *be a stationary state of the magnetic Schrödinger-Poisson system, where*

$$\lambda_{0,k} = f(\mu_{0,k}), \quad k \in \mathbb{N}$$

with some $f \in \mathcal{C}$ *and* $(\Psi_0, \underline{\lambda}_0) \in \mathcal{L}$. *Let* $(\Psi(t), \underline{\lambda})$ *be a solution of the magnetic Schrödinger-Poisson system, with the initial datum* $(\Psi(0), \underline{\lambda}) \in \mathcal{L}$. *Then, for all* $t \geq 0$, *the estimate*

$$\frac{1}{2}\|\nabla V_{\psi(t),\lambda} - \nabla V_0\|^2_{L^2(\Omega)} \leq \mathcal{H}_C(\Psi(0), \underline{\lambda}) - \mathcal{H}_C(\Psi_0, \underline{\lambda}_0) \tag{2.10}$$

holds, such that the stationary state is nonlinearly stable.

Suppose for some $f \in \mathcal{C}$ there are two stationary states $(\Psi_0, \underline{\lambda}_0, \mu_0, V_0)$ and $(\Psi_1, \underline{\lambda}_1, \mu_1, V_1)$ with $(\Psi_0, \underline{\lambda}_0)$, $(\Psi_1, \underline{\lambda}_1) \in \mathcal{L}$. Then both sides of (2.10) will vanish and these stationary states will coincide.

To establish the existence of stationary states, we exploit the dual of the energy-Casimir functional. For $\Lambda > 0$ fixed, we define $\mathcal{G}(\Psi, \underline{\lambda}, V, \sigma)$ as

$$\sum_{k=1}^{\infty}[F^*(-\lambda_k) + \lambda_k \int_{\Omega}[|(-i\nabla + A)\psi_k|^2 + V|\psi_k|^2]dx] - \frac{1}{2}\int_{\Omega}|\nabla V|^2 dx + \sigma\Big[\sum_{k=1}^{\infty}\lambda_k - \Lambda\Big],$$

where $\sigma \in \mathbb{R}$ is a Lagrange multiplier. The dual functional to $\mathcal{H}_C$ is given by

$$\Phi(V, \sigma) := \inf_{\Psi,\underline{\lambda}} \mathcal{G}(\Psi, \underline{\lambda}, V, \sigma). \tag{2.11}$$

We take the infimum in the formula above over all $\underline{\lambda} \in l^1_+$ and all $\Psi = \{\psi_k\}_{k=1}^{\infty} \subset H^1_{0,\ A}(\Omega) \cap H^2_A(\Omega)$ complete orthonormal sequences from $L^2(\Omega)$. The function Φ has an equivalent definition given in Lemma 11, and we use it in the statement of the theorem below. We consider only non-negative potential functions and define

$$H^1_{0,+}(\Omega) := \{V \in H^1_0(\Omega) \mid V \geq 0\}.$$

Our second main statement deals with the existence of stationary states.

Theorem 2. *Let* $f \in \mathcal{C}$ *and* $\Lambda > 0$ *be fixed. The functional* Φ

$$(V, \sigma) \in H^1_{0,+}(\Omega) \times \mathbb{R} \to -\frac{1}{2}\int_{\Omega}|\nabla V|^2 dx - Tr[F((-i\nabla + A)^2 + V + \sigma)] - \sigma\Lambda$$

is continuous, strictly concave, bounded from above and $-\Phi(V, \sigma)$ *is coercive. There exists a unique maximizer* (V_0, σ_0) *of* $\Phi(V, \sigma)$. *Let* $\{\psi_{0,k}\}_{k=1}^{\infty}$ *be the orthonormal sequence of eigenfunctions of the Hamiltonian* $(-i\nabla + A)^2 + V_0$ *corresponding to the eigenvalues* $\{\mu_{0,k}\}_{k=1}^{\infty}$ *and let* $\lambda_{0,k} := f(\mu_{0,k} + \sigma_0)$. *Then* $(\Psi_0, \underline{\lambda}_0, \mu_0, V_0)$ *is a stationary state of the magnetic Schrödinger-Poisson system, where* $\sum_{k=1}^{\infty}\lambda_{0,k} = \Lambda$ *and* $(\Psi_0, \underline{\lambda}_0) \in \mathcal{L}$.

We will prove Theorem 1 in Section 4, and Theorem 2 in Section 6, restricting our attention to classes of systems most relevant to plasma physics, namely, quantum particles in a magnetic field in the 3-dimensional space.

3 Preliminaries

Let us establish the following trivial statement.

Lemma 3. *For $(\Psi, \underline{\lambda}) \in \mathcal{L}$ we have*

$$n_{\psi,\lambda} := \sum_{k\in\mathbb{N}} \lambda_k |\psi_k|^2 \in L^2(\Omega).$$

Let $V_{\psi,\lambda}$ denote the Coulomb potential induced by $n_{\psi,\lambda}$, such that

$$-\Delta V_{\psi,\lambda}(x) = n_{\psi,\lambda}(x), \quad x \in \Omega; \quad V_{\psi,\lambda}(x) = 0, \quad x \in \partial\Omega.$$

Then $V_{\psi,\lambda} \in H_0^1(\Omega) \cap H^2(\Omega)$.

Proof. We will use the Sobolev embedding

$$\|\phi\|_{L^\infty(\Omega)} \le c_e \|\phi\|_{H^2(\Omega)}, \tag{3.1}$$

where $c_e > 0$ is the constant of the embedding. Thus, we estimate for $n_{\psi,\lambda}(x) \ge 0, \ x \in \Omega$

$$n_{\psi,\lambda}(x) \le \sum_{k=1}^{\infty} \lambda_k \|\psi_k\|^2_{L^\infty(\Omega)} \le c_e^2 \|\Psi\|^2_{Z_\Omega},$$

where the norm $\|.\|_{Z_\Omega}$ is defined in the Appendix by (6.7) picking $A = 0$. By means of the equivalence of magnetic and non magnetic norms established in Lemma 14 of the Appendix, the right side of this inequality can be bounded above by

$$C\|\Psi\|^2_{\dot{Z}_{\Omega,A}} = C\sum_{k=1}^{\infty} \lambda_k \int_\Omega |(-i\nabla + A)^2 \psi_k|^2 dx < \infty.$$

Therefore, $n_{\psi,\lambda}(x) \in L^\infty(\Omega)$ in our bounded domain, which yields $n_{\psi,\lambda}(x) \in L^2(\Omega)$. Note that the particle density $n_{\psi,\lambda}$ vanishes on the boundary of the set Ω by means of formula (1.10) and boundary conditions (1.9). Hence, $\Delta V_{\psi,\lambda} \in L^2(\Omega)$. Let $\{\mu_k^0\}_{k\in\mathbb{N}}$ designate the eigenvalues of the Dirichlet Laplacian on $L^2(\Omega)$ and μ_1^0 is the lowest one of them. Obviously,

$$\mu_k^0 > 0, \quad k \in \mathbb{N}.$$

Due to the fact that

$$V_{\psi,\lambda} = (-\Delta)^{-1} n_{\psi,\lambda},$$

we obtain $\|V_{\psi,\lambda}\|_{L^2(\Omega)} \le \frac{1}{\mu_1^0} \|n_{\psi,\lambda}\|_{L^2(\Omega)} < \infty$. Moreover, since $V_{\psi,\lambda}$ vanishes on the Lipschitz boundary of the bounded set Ω via (1.9), V is a trace-zero function in $H^1(\Omega)$. □

By virtue of the result of Lemma 13 of the Appendix, for every initial state $(\Psi(0), \underline{\lambda}) \in \mathcal{L}$, there exists a unique strong solution of system (1.6)-(1.9), where $(\Psi(t), \underline{\lambda}) \in \mathcal{L}$ for all $t \ge 0$.

The energy $\mathcal{H}(\Psi, \underline{\lambda})$ of a state $(\Psi, \underline{\lambda}) \in \mathcal{L}$, defined by (2.9), is a conserved quantity along solutions of the magnetic Schrödinger-Poisson system (see Lemma 19 of the Appendix). Let us assume that $\lambda_k > 0$ via density arguments. To prove the nonlinear stability for a specified stationary state, we will use the following auxiliary lemmata.

Lemma 4. *Let $f \in \mathcal{C}$.*

a) For every $\beta > 1$ there exists $C = C(\beta) \in \mathbb{R}$, such that for $s \le 0$ we have

$$F(s) \ge -\beta s + C$$

b) Let $V \in H_0^1(\Omega)$ and $V(x) \ge 0$ for $x \in \Omega$. Then both operators $f((-i\nabla + A)^2 + V)$ and $F((-i\nabla + A)^2 + V)$ are trace class.

Proof. The part a) of the lemma follows from the fact that function $F(s)$ is smooth with the slope varying from $-\infty$ to 0, and convex; hence, its graph is situated above a tangent line to it.

Let us denote the magnetic Dirichlet eigenvalues of $(-i\nabla + A)^2$ on $L^2(\Omega)$, $\Omega \subset \mathbb{R}^3$ as $\{\mu_k^A\}_{k=1}^{\infty}$ and the corresponding orthonormal sequence of eigenfunctions as $\{\varphi_k^A(x)\}_{k=1}^{\infty}$. By means of the equivalence of the appropriate magnetic and non magnetic norms established in Lemma 14, we have the lower bound

$$\sum_{k=1}^{N} \mu_k^A = \sum_{k=1}^{N} \|(-i\nabla + A)\varphi_k^A\|_{L^2(\Omega)}^2 \geq C \sum_{k=1}^{N} \|\nabla \varphi_k^A\|_{L^2(\Omega)}^2.$$

By means of the sharp semiclassical result of [13], the right side of this inequality can be estimated below by $CN^{\frac{5}{3}}$, such that for each eigenvalue we have

$$\mu_N^A \geq CN^{\frac{2}{3}}, \ N \in \mathbb{N},$$

with a constant here dependent on $|\Omega| < \infty$ (see e.g. [13]). The sharp semiclassical lower bound on the sum of magnetic Dirichlet eigenvalues when the magnetic field is constant was established in [10]. Due to the fact that the potential function $V(x) \geq 0$ for $x \in \Omega$ as assumed, we easily estimate from below the eigenvalues μ_k of the Hamiltonian $(-i\nabla + A)^2 + V$ for $k \in \mathbb{N}$ as

$$\mu_k \geq \mu_k^A \geq Ck^{\frac{2}{3}}. \tag{3.2}$$

Let us express

$$\mathrm{Tr}\left(F((-i\nabla + A)^2 + V)\right) = \sum_{k=1}^{\infty} F(\mu_k) < \infty,$$

because $F(s)$ is decreasing, satisfies bound (2.6), and the series with a general term $(1 + Ck^{\frac{2}{3}})^{-\frac{5}{2}-\varepsilon}$ converges. Clearly,

$$\mathrm{Tr}\left(f((-i\nabla + A)^2 + V)\right) = \sum_{k=1}^{\infty} f(\mu_k) < \infty,$$

since $f(s)$ decreases, obeys estimate (2.1) and the series with a general term $(1 + Ck^{\frac{2}{3}})^{-\frac{7}{2}-\varepsilon}$ is convergent. □

Lemma 5. *Let $\psi \in H^1_{0,\ A}(\Omega) \cap H^2_A(\Omega)$ with $\|\psi\|_{L^2(\Omega)} = 1$, the potential function $V \in H^1_0(\Omega)$ and $V(x) \geq 0$ for $x \in \Omega$. Then,*

$$F((\psi, ((-i\nabla + A)^2 + V)\psi)_{L^2(\Omega)}) \leq (\psi, F((-i\nabla + A)^2 + V)\psi)_{L^2(\Omega)} \tag{3.3}$$

holds with equality if ψ is an eigenstate of the Hamiltonian $(-i\nabla + A)^2 + V$.

Proof. The Spectral Theorem gives us

$$(-i\nabla + A)^2 + V = \sum_{k=1}^{\infty} \mu_k P_k.$$

Here the operators $\{P_k\}_{k=1}^{\infty}$ are the orthogonal projections onto the bound states corresponding to the eigenvalues $\{\mu_k\}_{k=1}^{\infty}$. Thus

$$F((\psi, ((-i\nabla + A)^2 + V)\psi)_{L^2(\Omega)}) = F\left(\sum_{k=1}^{\infty} \mu_k \|P_k\psi\|_{L^2(\Omega)}^2\right).$$

We bound above the right side of (3.3) by

$$\sum_{k=1}^{\infty} F(\mu_k)\|P_k\psi\|^2_{L^2(\Omega)}.$$

Bound (3.3) comes from Jensen's inequality. When ψ is an eigenstate of the operator $(-i\nabla+A)^2+V$ which corresponds to an eigenvalue μ_k, for some $k \in \mathbb{N}$, both sides of (3.3) are equal to $F(\mu_k)$. Note that the converse of this statement is not true in general. Indeed, if we consider as ψ a linear combination of more than one eigenstate of the Hamiltonian with corresponding eigenvalues μ_k situated outside the support of $F(s)$, then both sides of (3.3) will vanish. □

In the statement below we prove that a stationary solution belongs to the state space for our magnetic Schrödinger-Poisson system.

Lemma 6. *Let the quadruple* $(\Psi_0, \underline{\lambda}_0, \mu_0, V_0)$ *satisfy equations (2.2), (2.3) and (2.4), where* Ψ_0 *is a complete orthonormal system in* $L^2(\Omega)$ *and the distribution* $f \in \mathcal{C}$. *Then, we have*

$$\sum_{k=1}^{\infty} \lambda_{0,k} \int_\Omega |(-i\nabla + A)^2\psi_{0,k}|^2 dx < \infty,$$

such that $(\Psi_0, \underline{\lambda}_0) \in \mathcal{L}$.

Proof. Let us express the following quantity using identities (2.2) and (2.4) as

$$\begin{aligned} &\sum_{k=1}^{\infty} \lambda_{0,k}\|(-i\nabla + A)\psi_{0,k}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla V_0|^2 dx \\ &= \sum_{k=1}^{\infty} \lambda_{0,k}((-i\nabla + A)^2 + V_0)\psi_{0,k}, \psi_{0,k})_{L^2(\Omega)} = \sum_{k=1}^{\infty} f(\mu_{0,k})\mu_{0,k}. \end{aligned} \tag{3.4}$$

The potential function $V_0(x) \geq 0$ in Ω since it is superharmonic by means of (2.3), and vanishes on the boundary of Ω. Thus, $\mu_{0,k} > 0,\ k \in \mathbb{N}$ and via (2.1) the right side of (3.4) can be estimated from above by

$$\sum_{k=1}^{\infty} C(1 + \mu_{0,k})^{-\frac{7}{2}-\varepsilon}\mu_{0,k} < \infty,$$

by means of the eigenvalue estimate (3.2). Hence, we also have

$$\nabla V_0 \in L^2(\Omega), \quad (-i\nabla + A)\psi_{0,k} \in L^2(\Omega),\ k \in \mathbb{N}, \tag{3.5}$$

such that $\psi_{0,k} \in H^1_{0,\ A}(\Omega)$ for $k \in \mathbb{N}$. Note that the standard requirement $V_0 \in L^1(\Omega)$ (see e.g. p.234 of [16]) holds here as well. By virtue of Hölder's inequality, we obtain

$$\int_\Omega |V_0|^2|\psi_{0,k}|^2 dx \leq \left(\int_\Omega |V_0|^6 dx\right)^{\frac{1}{3}} \left(\int_\Omega |\psi_{0,k}|^3 dx\right)^{\frac{2}{3}} < \infty$$

due to the Sobolev inequality (see e.g. [16]) along with the equivalence of magnetic and non magnetic norms proved in Lemma 14 of the Appendix below. Thus, $V_0\psi_{0,k} \in L^2(\Omega),\ k \in \mathbb{N}$. Equation (2.2) yields that $(-i\nabla + A)^2\psi_{0,k} \in L^2(\Omega)$ as well, such that $\psi_{0,k} \in H^2_A(\Omega),\ k \in \mathbb{N}$. By virtue of (2.4), we have $\lambda_{0,k} \geq 0,\ k \in \mathbb{N}$. Convergence of the series on the right side of (3.4) yields

$$\sum_{k=1}^{\infty} \lambda_{0,k} = \sum_{k=1}^{\infty} f(\mu_{0,k}) < \infty, \tag{3.6}$$

such that $\underline{\lambda}_0 = \{\lambda_{0,k}\}_{k=1}^{\infty} \in l^1$. By means of (2.2), we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}\lambda_{0,k}\int_{\Omega}|(-i\nabla+A)^2\psi_{0,k}|^2dx\\&=\sum_{k=1}^{\infty}\lambda_{0,k}\mu_{0,k}^2-2\sum_{k=1}^{\infty}\lambda_{0,k}\mu_{0,k}\int_{\Omega}V_0|\psi_{0,k}|^2dx+\sum_{k=1}^{\infty}\lambda_{0,k}\|V_0\psi_{0,k}\|_{L^2(\Omega)}^2.\end{aligned} \tag{3.7}$$

Let us prove that the first term in the right side of (3.7) is convergent. Indeed, (2.1) yields

$$\sum_{k=1}^{\infty}\lambda_{0,k}\mu_{0,k}^2\le C\sum_{k=1}^{\infty}(1+\mu_{0,k})^{-\frac{7}{2}-\varepsilon}\mu_{0,k}^2<\infty,$$

via the eigenvalue estimate (3.2). For the third term in the right side of (3.7) via Hölder's inequality we obtain

$$\sum_{k=1}^{\infty}\lambda_{0,k}\|V_0\psi_{0,k}\|_{L^2(\Omega)}^2\le\left(\int_{\Omega}|V_0|^6dx\right)^{\frac{1}{3}}\sum_{k=1}^{\infty}\lambda_{0,k}\left(\int_{\Omega}|\psi_{0,k}|^3dx\right)^{\frac{2}{3}},$$

with $V_0(x)\in L^6(\Omega)$ as discussed above. By means of the Schwarz inequality

$$\int_{\Omega}|\psi_{0,k}|^3dx\le\sqrt{\int_{\Omega}|\psi_{0,k}|^6dx}\sqrt{|\Omega|}.$$

The Sobolev inequality (see e.g. [16]) along with the equivalence of magnetic and nonmagnetic norms (see Lemma 14) yield

$$\sum_{k=1}^{\infty}\lambda_{0,k}\|\psi_{0,k}\|_{L^6(\Omega)}^2\le C\sum_{k=1}^{\infty}\lambda_{0,k}\|\nabla\psi_{0,k}\|_{L^2(\Omega)}^2\le C\sum_{k=1}^{\infty}\lambda_{0,k}\|(-i\nabla+A)\psi_{0,k}\|_{L^2(\Omega)}^2<\infty$$

via the estimate (3.4). The second term in the right side of (3.7) can be estimated above by applying the Schwarz inequality to it twice, namely

$$\int_{\Omega}V_0|\psi_{0,k}|^2dx\le\|V_0\psi_{0,k}\|_{L^2(\Omega)}$$

and

$$\sum_{k=1}^{\infty}\lambda_{0,k}\mu_{0,k}\|V_0\psi_{0,k}\|_{L^2(\Omega)}\le\sqrt{\sum_{k=1}^{\infty}\lambda_{0,k}\mu_{0,k}^2}\sqrt{\sum_{s=1}^{\infty}\lambda_{0,s}\|V_0\psi_{0,s}\|_{L^2(\Omega)}^2}<\infty$$

as it was proven above. □

Remark. *In the stationary situation, our magnetic Schrödinger-Poisson problem can be easily expressed as*

$$-\Delta V_0=f((-i\nabla+A)^2+V_0)(x,x),\quad x\in\Omega,$$
$$V_0(x)=0,\quad x\in\partial\Omega.$$

We turn our attention to defining the corresponding Casimir functional for a fixed $f\in\mathcal{C}$. The following trivial lemma proved in [3] (see also [18]) yields the alternative representation for the Legendre transform of our integrated distribution function. Evidently, $f\in\mathcal{C}$ considered on the $(-\infty,s_0]$ semi- axis has an inverse f^{-1}.

Lemma 7. *For the function $F(s)$ defined in (2.5) and $s\le 0$ we have*

$$F^*(s)=\int_{-s}^{0}f^{-1}(\sigma)d\sigma. \tag{3.8}$$

In the following section, we establish the nonlinear stability of stationary states, by virtue of the energy-Casimir functional defined above.

4 Stability of stationary states

In the present section, we prove Theorem 1, which gives us the lower bound in terms of the electrostatic field. The technical lemma below is crucial for establishing this nonlinear stability result.

Lemma 8. *Let $V \in H_0^1(\Omega)$ and $V \geq 0$.*

(i) Then, for $(\Psi, \underline{\lambda}) \in \mathcal{L}$, the lower bound

$$\sum_{k=1}^{\infty} \left\{ F^*(-\lambda_k) + \lambda_k \int_\Omega [|(-i\nabla + A)\psi_k|^2 + V|\psi_k|^2] dx \right\} \geq -Tr[F((-i\nabla + A)^2 + V)] \tag{4.1}$$

holds.

(ii) Equality is attained for $(\Psi, \underline{\lambda}) = (\Psi_V, \underline{\lambda}_V)$, where $\psi_{V,k} \in H^1_{0,\ A}(\Omega) \cap H^2_A(\Omega)$, $k \in \mathbb{N}$ denotes the orthonormal sequence of eigenfunctions of the Hamiltonian $(-i\nabla + A)^2 + V$ with corresponding eigenvalues $\mu_{V,k}$ and $\lambda_{V,k} = f(\mu_{V,k})$, $k \in \mathbb{N}$.

Proof. By means of definition (2.7), we have

$$F^*(s) \geq \mu s - F(\mu), \quad \mu \in \mathbb{R}, \quad s \leq 0,$$

which yields

$$F^*(-\lambda_k) + \lambda_k \mu_k \geq -F(\mu_k), \quad k \in \mathbb{N}. \tag{4.2}$$

Then let

$$\mu_k := \int_\Omega \left\{ |(-i\nabla + A)\psi_k|^2 + V|\psi_k|^2 \right\} dx = (\psi_k, ((-i\nabla + A)^2 + V)\psi_k)_{L^2(\Omega)}, \quad k \in \mathbb{N}.$$

Let us note that after summation, using Lemma 5 we obtain

$$\begin{aligned} &\sum_{k=1}^{\infty} \left\{ F^*(-\lambda_k) + \lambda_k \int_\Omega \left\{ |(-i\nabla + A)\psi_k|^2 + V|\psi_k|^2 \right\} dx \right\} \\ &\geq -\sum_{k=1}^{\infty} F((\psi_k, ((-i\nabla + A)^2 + V)\psi_k)_{L^2(\Omega)}) \\ &\geq -\sum_{k=1}^{\infty} (\psi_k, F((-i\nabla + A)^2 + V)\psi_k)_{L^2(\Omega)}. \end{aligned}$$

The definition of trace yields that the right side of the inequality above is given by

$$-\mathrm{Tr}\left(F((-i\nabla + A)^2 + V)\right),$$

which completes the proof of part (i) of the lemma. To establish part (ii), we suppose that $(\Psi, \underline{\lambda}) = (\Psi_V, \underline{\lambda}_V)$, where $\psi_{V,k}$ are eigenfunctions of the Hamiltonian $(-i\nabla + A)^2 + V$ and μ_k defined above are the corresponding eigenvalues $\mu_{V,k}$, $k \in \mathbb{N}$. Thus, on the right side of lower bound (4.1) we have

$$-\mathrm{Tr}\left(F((-i\nabla + A)^2 + V)\right) = -\sum_{k=1}^{\infty} F(\mu_{V,k}).$$

Next, let us use the identity $\lambda_{V,k} = f(\mu_{V,k}) = -F'(\mu_{V,k})$. Then, via Lemma 7, $F^{*\prime}(-\lambda_{V,k}) = f^{-1}(\lambda_{V,k}) = \mu_{V,k}$, $k \in \mathbb{N}$. With the argument of Lemma 7 of [3], we arrive at

$$F^*(-\lambda_{V,k}) = \sup_{\lambda \in \mathbb{R}}(-\lambda\lambda_{V,k} - F(\lambda)) = -f^{-1}(\lambda_{V,k})\lambda_{V,k} - F(f^{-1}(\lambda_{V,k})) = -\lambda_{V,k}\mu_{V,k} - F(\mu_{V,k}).$$

Thus, the left side of (4.1) will be equal to $-\sum_{k=1}^{\infty} F(\mu_{V,k})$ as well. □

Armed with the technical statement above, we proceed to prove our first main result.

Proof of Theorem 1. Let $(\Psi, \underline{\lambda}) \in \mathcal{L}$ and the potential function $V = V_{\Psi,\lambda}$ induced by this state. We will use the following identity for the energy of the electrostatic field

$$\frac{1}{2}\|\nabla V - \nabla V_0\|^2_{L^2(\Omega)} = \frac{1}{2}\int_\Omega |\nabla V|^2 dx + \frac{1}{2}\int_\Omega |\nabla V_0|^2 dx + \int_\Omega V_0 \Delta V dx.$$

By virtue of the definition of the energy-Casimir functional, this can be expressed as

$$\mathcal{H}_C(\Psi, \underline{\lambda}) - \Big\{ \sum_{k=1}^{\infty} \Big(F^*(-\lambda_k) + \lambda_k \int_\Omega |(-i\nabla + A)\psi_k|^2 dx \Big) - \frac{1}{2}\int_\Omega |\nabla V_0|^2 dx - \int_\Omega V_0 \Delta V dx \Big\},$$

which equals to

$$\mathcal{H}_C(\Psi, \underline{\lambda}) - \Big\{ \sum_{k=1}^{\infty} \Big[F^*(-\lambda_k) + \lambda_k \int_\Omega (|(-i\nabla + A)\psi_k|^2 + V_0|\psi_k|^2) dx \Big] - \frac{1}{2}\int_\Omega |\nabla V_0|^2 dx \Big\}.$$

Applying first Lemma 8 part (i), and then Lemma 8 part (ii), we obtain that the expression above is estimated from above by

$$\begin{aligned}
&\mathcal{H}_C(\Psi, \underline{\lambda}) - \Big\{ -\mathrm{Tr}[F((-i\nabla + A)^2 + V_0)] - \frac{1}{2}\int_\Omega |\nabla V_0|^2 dx \Big\} \\
&= \mathcal{H}_C(\Psi, \underline{\lambda}) - \Big\{ \sum_{k=1}^{\infty} \Big[F^*(-\lambda_{0,k}) + \lambda_{0,k} \int_\Omega (|(-i\nabla + A)\psi_{0,k}|^2 + V_0|\psi_{0,k}|^2) dx \Big] - \frac{1}{2}\int_\Omega |\nabla V_0|^2 dx \Big\} \\
&= \mathcal{H}_C(\Psi, \underline{\lambda}) - \Big\{ \sum_{k=1}^{\infty} \Big[F^*(-\lambda_{0,k}) + \lambda_{0,k} \int_\Omega |(-i\nabla + A)\psi_{0,k}|^2 dx \Big] + \frac{1}{2}\int_\Omega |\nabla V_0|^2 dx \Big\} \\
&= \mathcal{H}_C(\Psi, \underline{\lambda}) - \mathcal{H}_C(\Psi_0, \underline{\lambda}_0).
\end{aligned}$$

Due to the fact that the Casimir functional is constant along the solutions of our magnetic Schrödinger-Poisson system, which is globally well-posed as proved in Lemma 13 of the Appendix below, for an initial condition $(\Psi(0), \underline{\lambda}) \in \mathcal{L}$, we can use $\mathcal{H}_C(\Psi(0), \underline{\lambda})$ in the estimate above instead of $\mathcal{H}_C(\Psi(t), \underline{\lambda})$. □

Having proved the nonlinear stability of the stationary states of the magnetic Schrödinger-Poisson system, our main goal is to establish the existence of such states which satisfy the assumptions of the stability theorem.

5 Dual functionals

For each distribution function $f \in \mathcal{C}$ we will obtain a corresponding stationary state as the unique maximizer of a functional defined below. We use the energy-Casimir functional from the stability result to derive such a dual functional. The tool below will be the saddle point principle. Let us recall that, for $\Lambda > 0$ fixed $\mathcal{G}(\Psi, \underline{\lambda}, V, \sigma)$ is defined as

$$\sum_{k=1}^{\infty} [F^*(-\lambda_k) + \lambda_k \int_\Omega [|(-i\nabla + A)\psi_k|^2 + V|\psi_k|^2] dx] - \frac{1}{2}\int_\Omega |\nabla V|^2 dx + \sigma\Big[\sum_{k=1}^{\infty} \lambda_k - \Lambda\Big].$$

Here as above $\Psi = \{\psi_k\}_{k=1}^{\infty} \subset H^1_{0,\,A}(\Omega) \cap H^2_A(\Omega)$ is a complete orthonormal system in $L^2(\Omega)$ and $\underline{\lambda} \in l^1_+ = \{(\lambda_k) \in l^1 \mid \lambda_k \geq 0,\ k \in \mathbb{N}\}$. Now we allow the function $V \in H^1_0(\Omega)$ to vary independently of Ψ and $\underline{\lambda}$. The parameter $\sigma \in \mathbb{R}$ here plays the role of a Lagrange multiplier. The following lemma shows how the functional defined above is related to our energy-Casimir functional.

Lemma 9. *For arbitrary* $\Psi, \underline{\lambda}, \sigma$,

$$sup_V \mathcal{G}(\Psi, \underline{\lambda}, V, \sigma) = \mathcal{H}_C(\Psi, \underline{\lambda}) + \sigma\Big[\sum_{k=1}^{\infty} \lambda_k - \Lambda\Big].$$

The supremum is attained at $V = V_{\psi,\lambda}$.

Proof. Let us express the functional defined above as

$$\mathcal{G}(\Psi, \underline{\lambda}, V, \sigma) = \sum_{k=1}^{\infty}[F^*(-\lambda_k) + \lambda_k \int_\Omega |(-i\nabla + A)\psi_k|^2 dx + \frac{1}{2}\lambda_k \int_\Omega |\psi_k|^2 V_{\psi,\lambda} dx]$$
$$+ \sum_{k=1}^{\infty} \lambda_k \int_\Omega V|\psi_k|^2 dx - \frac{1}{2}\int_\Omega |\nabla V_{\psi,\lambda}|^2 dx - \frac{1}{2}\int_\Omega |\nabla V|^2 dx + \sigma\Big[\sum_{k=1}^{\infty} \lambda_k - \Lambda\Big].$$

By virtue of the definition of the energy-Casimir functional (2.8) we obtain

$$\mathcal{H}_C(\Psi, \underline{\lambda}) - \int_\Omega V\Delta V_{\psi,\lambda} dx - \frac{1}{2}\int_\Omega |\nabla V_{\psi,\lambda}|^2 dx - \frac{1}{2}\int_\Omega |\nabla V|^2 dx + \sigma\Big[\sum_{k=1}^{\infty} \lambda_k - \Lambda\Big].$$

To complete the proof of the lemma, we write the expression above as

$$\mathcal{H}_C(\Psi, \underline{\lambda}) - \frac{1}{2}\|\nabla V_{\psi,\lambda} - \nabla V\|^2_{L^2(\Omega)} + \sigma\Big[\sum_{k=1}^{\infty} \lambda_k - \Lambda\Big].$$

□

In the following Section, we will prove that the functional $\Phi(V, \sigma)$ defined in (2.11) admits a unique maximizer, which is a stationary state of our magnetic Schrödinger-Poisson system. We first prove the following technical statement, which is the generalization of Lemma 8 above.

Lemma 10. *Let* $V \in H_0^1(\Omega)$ *and* $V \geq 0$. *Then for* $(\Psi, \underline{\lambda}) \in \mathcal{L}$ *and* $\sigma \in \mathbb{R}$, *the estimate from below*

$$\sum_{k=1}^{\infty}\Big[F^*(-\lambda_k) + \lambda_k\Big(\int_\Omega [|(-i\nabla + A)\psi_k|^2 + V|\psi_k|^2]dx + \sigma\Big)\Big] \geq -Tr[F((-i\nabla + A)^2 + V + \sigma)] \quad (5.1)$$

is valid. Equality in it is achieved when $(\Psi, \underline{\lambda}) = (\Psi_V, \underline{\lambda}_V)$, *where* $\psi_{V,k} \in H^1_{0,A}(\Omega) \cap H^2_A(\Omega),\ k \in \mathbb{N}$ *is the orthonormal sequence of eigenfunctions of the operator* $(-i\nabla + A)^2 + V$ *which correspond to eigenvalues* $\mu_{V,k}$. *Furthermore,* $\lambda_{V,k} = f(\mu_{V,k} + \sigma),\ k \in \mathbb{N}$.

Proof. Let us use inequality (4.2) with

$$\mu_k := \int_\Omega \Big(|(-i\nabla + A)\psi_k|^2 + V|\psi_k|^2\Big)dx + \sigma = (\psi_k, ((-i\nabla + A)^2 + V + \sigma)\psi_k)_{L^2(\Omega)}, \quad k \in \mathbb{N}.$$

Hence,

$$F^*(-\lambda_k) + \lambda_k\Big(\int_\Omega \Big[|(-i\nabla + A)\psi_k|^2 + V|\psi_k|^2\Big]dx + \sigma\Big)$$
$$\geq -F((\psi_k, ((-i\nabla + A)^2 + V + \sigma)\psi_k)_{L^2(\Omega)}), \quad k \in \mathbb{N}. \quad (5.2)$$

Obviously,

$$(-i\nabla + A)^2 + V + \sigma = \int_0^\infty (\lambda + \sigma) dE_\lambda,$$

where E_λ denotes the spectral family associated with the Hamiltonian $(-i\nabla + A)^2 + V$. Hence $d\nu_k(\lambda) := (\psi_k, dE_\lambda\psi_k)_{L^2(\Omega)}$ is a probability measure for $k \in \mathbb{N}$. Jensen's inequality yields

$$F((\psi_k, ((-i\nabla + A)^2 + V + \sigma)\psi_k)_{L^2(\Omega)}) = F\Big(\int_0^\infty (\lambda + \sigma)d\nu_k(\lambda)\Big)$$
$$\leq \int_0^\infty F(\lambda + \sigma)d\nu_k(\lambda) = (\psi_k, F((-i\nabla + A)^2 + V + \sigma)\psi_k)_{L^2(\Omega)}.$$

This estimate from above along with (5.2) and summation over $k \in \mathbb{N}$ yield the desired inequality (5.1).

Then let us consider $\{\psi_{V,k}\}_{k=1}^\infty \subset H^1_{0,\ A}(\Omega) \cap H^2_A(\Omega)$, which form a complete orthonormal system in $L^2(\Omega)$. Thus $((-i\nabla + A)^2 + V)\psi_{V,k} = \mu_{V,k}\psi_{V,k}$ and $\lambda_{V,k} = f(\mu_{V,k} + \sigma),\ k \in \mathbb{N}$. In such case the right side of (5.1) equals to

$$-\sum_{k=1}^\infty (F((-i\nabla + A)^2 + V + \sigma)\psi_{V,k}, \psi_{V,k})_{L^2(\Omega)} = -\sum_{k=1}^\infty F(\mu_{V,k} + \sigma).$$

We have for $k \in \mathbb{N}$

$$F^*(-\lambda_{V,k}) = \sup_{\lambda\in\mathbb{R}}(-\lambda\lambda_{V,k} - F(\lambda)) = -f^{-1}(\lambda_{V,k})\lambda_{V,k} - F(f^{-1}(\lambda_{V,k})),$$

since the supremum above is achieved at the maximal point $\lambda^* := f^{-1}(\lambda_{V,k})$. The equality $\lambda_{V,k} = f(\mu_{V,k} + \sigma)$ gives us $f^{-1}(\lambda_{V,k}) = \mu_{V,k} + \sigma$. Hence

$$F^*(-\lambda_{V,k}) = -(\mu_{V,k} + \sigma)\lambda_{V,k} - F(\mu_{V,k} + \sigma).$$

An easy calculation yields that the left side of (5.1) is equal to $-\sum_{k=1}^\infty F(\mu_{V,k} + \sigma)$. □

Having proved the technical lemma above, we are able to obtain the expression for the dual functional for our problem.

Lemma 11. *The infimum in definition (2.11) is achieved at* $\Psi = \{\psi_{V,k}\}_{k=1}^\infty$*, an orthonormal sequence of eigenfunctions of the Hamiltonian* $(-i\nabla + A)^2 + V,\ V \geq 0$*, which correspond to the eigenvalues* $\mu_{V,k}$ *with* $\lambda_{V,k} = f(\mu_{V,k} + \sigma)$ *for* $k \in \mathbb{N}$*. Moreover, the dual functional is given by*

$$\Phi(V, \sigma) = -\frac{1}{2}\int_\Omega |\nabla V|^2 dx - Tr[F((-i\nabla + A)^2 + V + \sigma)] - \sigma\Lambda. \tag{5.3}$$

Proof. We prove that the operator $F((-i\nabla + A)^2 + V + \sigma)$ is trace class. We have

$$\mathrm{Tr}[F((-i\nabla + A)^2 + V + \sigma)] = \sum_{k=1}^\infty F(\mu_{V,k} + \sigma).$$

Because the potential function $V \geq 0$ as assumed, we use bounds (3.2) and (2.6) and obtain the series with the general term $(1 + Ck^{\frac{2}{3}} + \sigma)^{-\frac{5}{2}-\varepsilon}$. This series is clearly convergent. Let us conclude the proof of the lemma by referring to the statement of Lemma 10 above. □

6 Existence of stationary states

In the present section we establish, for each distribution function $f \in \mathcal{C}$ and each value of $\Lambda > 0$, the existence of a unique maximizer of our functional Φ, which will be a stationary state of our magnetic Schrödinger-Poisson system.

Proof of Theorem 2. We first prove that the inequality

$$\begin{aligned}&\operatorname{Tr}[F((-i\nabla + A)^2 + \alpha(V_1 + \sigma_1) + (1 - \alpha)(V_2 + \sigma_2))]\\ &\le \alpha \operatorname{Tr}[F((-i\nabla + A)^2 + V_1 + \sigma_1)] + (1 - \alpha)\operatorname{Tr}[F((-i\nabla + A)^2 + V_2 + \sigma_2)]\end{aligned} \tag{6.1}$$

is valid for any $\alpha \in (0, 1)$ and $(V_j, \sigma_j) \in H^1_{0,+}(\Omega) \times \mathbb{R}$, $j = 1, 2$. Let $\phi \in H^1_{0,\ A}(\Omega) \cap H^2_A(\Omega)$ and $\|\phi\|_{L^2(\Omega)} = 1$. Let us make use of the spectral decompositions

$$(-i\nabla + A)^2 + V_1 = \int_0^\infty \gamma dP_\gamma, \quad (-i\nabla + A)^2 + V_2 = \int_0^\infty \beta dQ_\beta,$$

where P_γ and Q_β are the spectral families associated with the operators $(-i\nabla + A)^2 + V_1$ and $(-i\nabla + A)^2 + V_2$ respectively. Therefore, we are in position to introduce the probability measures

$$d\nu(\gamma) := (\phi, dP_\gamma \phi)_{L^2(\Omega)}, \quad d\mu(\beta) := (\phi, dQ_\beta \phi)_{L^2(\Omega)} \tag{6.2}$$

and express

$$\begin{aligned}&F((\phi, [(-i\nabla + A)^2 + \alpha(V_1 + \sigma_1) + (1 - \alpha)(V_2 + \sigma_2)]\phi)_{L^2(\Omega)})\\ &= F\Big(\alpha \int_0^\infty (\gamma + \sigma_1)d\nu(\gamma) + (1 - \alpha)\int_0^\infty (\beta + \sigma_2)d\mu(\beta)\Big).\end{aligned}$$

Due to the fact that F is strictly convex on its support, we derive the upper bound for the expression above via Jensen's inequality as

$$\alpha \int_0^\infty F(\gamma + \sigma_1)d\nu(\gamma) + (1 - \alpha)\int_0^\infty F(\beta + \sigma_2)d\mu(\beta).$$

By virtue of definition (6.2) we obtain

$$\alpha(\phi, F((-i\nabla + A)^2 + V_1 + \sigma_1)\phi)_{L^2(\Omega)} + (1 - \alpha)(\phi, F((-i\nabla + A)^2 + V_2 + \sigma_2)\phi)_{L^2(\Omega)}.$$

Assume $\{\psi_k\}_{k=1}^\infty$ to be the set of eigenfunctions of the operator $(-i\nabla+A)^2+\alpha(V_1+\sigma_1)+(1-\alpha)(V_2+\sigma_2)$ forming a complete orthonormal system in $L^2(\Omega)$. Then the argument above yields

$$\begin{aligned}&\sum_{k=1}^\infty F((\psi_k, [(-i\nabla + A)^2 + \alpha(V_1 + \sigma_1) + (1 - \alpha)(V_2 + \sigma_2)]\psi_k)_{L^2(\Omega)})\\ &\le \alpha \sum_{k=1}^\infty (\psi_k, F((-i\nabla + A)^2 + V_1 + \sigma_1)\psi_k)_{L^2(\Omega)}\\ &\quad + (1 - \alpha)\sum_{k=1}^\infty (\psi_k, F((-i\nabla + A)^2 + V_2 + \sigma_2)\psi_k)_{L^2(\Omega)}.\end{aligned}$$

Hence we obtain inequality (6.1). Suppose equality here holds. Since the function F is strictly convex on its support, we conclude that the operators $(-i\nabla + A)^2 + V_1 + \sigma_1$ and $(-i\nabla + A)^2 + V_2 + \sigma_2$ with potential functions V_1 and V_2, which vanish on the boundary of Ω, have the same set of eigenvalues and the corresponding eigenfunctions are $\{\psi_k\}_{k=1}^\infty$. Thus, $V_1(x) = V_2(x)$ in Ω and $\sigma_1 = \sigma_2$, and $\operatorname{Tr}[F((-i\nabla + A)^2 + V + \sigma)]$ is strictly convex. Because $-\frac{1}{2}\int_\Omega |\nabla V|^2 dx$ and $-\sigma\Lambda$ are concave, we obtain that our functional given by (5.3) is strictly concave.

Now we proceed to the proof of its boundedness from above and coercivity. Evidently, the Poincaré inequality implies that

$$\frac{1}{2}\int_\Omega |\nabla V|^2 dx \ge \frac{C_1}{2}\|V\|^2_{H^1_0(\Omega)}$$

with a constant $C_1 > 0$. Denote as μ_V the lowest eigenvalue of the Hamiltonian $(-i\nabla + A)^2 + V$. Obviously, we have the estimate with a trial function $\tilde{\phi}$ as

$$\mu_V \leq \int_{\Omega} \{|(-i\nabla + A)\tilde{\phi}|^2 + V|\tilde{\phi}|^2\} dx, \quad \|\tilde{\phi}\|_{L^2(\Omega)} = 1.$$

We fix $\tilde{\phi}$ as the ground state of the magnetic Dirichlet Laplacian $(-i\nabla + A)^2$ on $L^2(\Omega)$. Note that the lowest eigenvalue of such operator can be compared with the smallest eigenvalue of the negative Dirichlet Laplacian on $L^2(\Omega)$ by means of the comparison of magnetic non magnetic norms proved in Lemma 14 of the Appendix. Thus

$$\int_{\Omega} |(-i\nabla + A)\tilde{\phi}|^2 dx = \mu_1^A.$$

Let us introduce

$$C_2 := \sqrt{\int_{\Omega} |\tilde{\phi}|^4 dx} > 0.$$

Such constant is finite. Indeed, the comparison of magnetic and non magnetic norms established in Lemma 14 of the Appendix along with the Sobolev inequality yield $\tilde{\phi} \in L^6(\Omega)$. We have $\tilde{\phi} \in L^4(\Omega)$ by virtue of the Hölder's inequality. The Schwarz inequality yields

$$\int_{\Omega} V|\tilde{\phi}|^2 dx \leq C_2 \|V\|_{L^2(\Omega)} \leq C_2 \|V\|_{H_0^1(\Omega)}.$$

Hence

$$\mu_V \leq \mu_1^A + C_2 \|V\|_{H_0^1(\Omega)}.$$

This gives us the estimate from above

$$\Phi(V, \sigma) \leq -\frac{C_1}{2} \|V\|_{H_0^1(\Omega)}^2 - F(\mu_1^A + C_2 \|V\|_{H_0^1(\Omega)} + \sigma) - \sigma\Lambda. \tag{6.3}$$

We use the convexity property, namely

$$F(x) \geq -\beta x + C_3,$$

where $\beta > \Lambda > 0$ is suffiently large. Thus we arrive at the inequality

$$\Phi(V, \sigma) \leq -\frac{C_1}{2} \|V\|_{H_0^1(\Omega)}^2 + (\beta - \Lambda)\sigma + \beta C_2 \|V\|_{H_0^1(\Omega)} + \beta\mu_1^A - C_3.$$

A straightforward computation gives us

$$\Phi(V, \sigma) \leq -\frac{C_1}{4} \|V\|_{H_0^1(\Omega)}^2 + C_4 + (\beta - \Lambda)\sigma + \beta\mu_1^A - C_3.$$

We choose $\beta = 2\Lambda$ and introduce the nonnegative constant $k := \max\{C_4 + 2\Lambda\mu_1^A - C_3, 0\}$. Thus

$$\Phi(V, \sigma) \leq -\frac{C_1}{4} \|V\|_{H_0^1(\Omega)}^2 + \Lambda\sigma + k. \tag{6.4}$$

Inequalities (6.3) and (6.4) yield

$$\Phi(V, \sigma) \leq -\frac{C_1}{4} \|V\|_{H_0^1(\Omega)}^2 - \Lambda|\sigma| + k.$$

This proves that our functional $\Phi(V, \sigma)$ is bounded above and $-\Phi(V, \sigma)$ is coercive. Hence, $\Phi(V, \sigma)$ has a unique maximizer (V_0, σ_0). Let the hamiltonian $(-i\nabla + A)^2 + V_0$ have the sequence of eigenvalues $\{\mu_{0,k}\}_{k=1}^{\infty}$ and the correspondent orthonormal sequence of eigenfunctions is $\{\psi_{0,k}\}_{k=1}^{\infty}$, namely

$$((-i\nabla + A)^2 + V_0)\psi_{0,k} = \mu_{0,k}\psi_{0,k}, \quad k \in \mathbb{N}$$

and denote $\lambda_{0,k} := f(\mu_{0,k} + \sigma_0)$. We arrive at

$$\Phi(V_0, \sigma) = -\frac{1}{2}\int_{\Omega} |\nabla V_0|^2 dx - \sum_{k=1}^{\infty} \int_{\mu_{0,k}+\sigma}^{\infty} f(\xi) d\xi - \sigma\Lambda,$$

and $\sigma = \sigma_0$ is its critical point. Thus,

$$0 = \frac{d\Phi}{d\sigma}(V_0, \sigma)|_{\sigma=\sigma_0} = -\Lambda + \sum_{k=1}^{\infty} f(\mu_{0,k} + \sigma_0) = \sum_{k=1}^{\infty} \lambda_{0,k} - \Lambda,$$

such that $\sum_{k=1}^{\infty} \lambda_{0,k} = \Lambda$. The first variation of $\Phi(V, \sigma_0)$ at $V = V_0$ vanishes as well. Hence, a trivial calculation yields

$$-\Delta V_0(x) = \sum_{k=1}^{\infty} \lambda_{0,k} |\psi_{0,k}(x)|^2.$$

By direct substitution, the functions $\psi_k(x,t) = e^{-i\mu_{0,k}t}\psi_{0,k}(x)$, $k \in \mathbb{N}$ satisfy the magnetic Schrödinger equation

$$i\frac{\partial \psi_k}{\partial t} = [(-i\nabla + A)^2 + V_0]\psi_k, \quad x \in \Omega, \quad t \geq 0.$$

The density matrix

$$\rho_0(t,x,y) = \sum_{k=1}^{\infty} \lambda_{0,k}\psi_k(x,t)\bar{\psi}_k(y,t) = \sum_{k=1}^{\infty} \lambda_{0,k}\psi_{0,k}(x)\overline{\psi_{0,k}(y)}.$$

Hence $\dfrac{\partial \rho_0}{\partial t} = 0$ and the particle concentration $n_0(t,x) = \rho_0(t,x,x)$.

Therefore, $(\Psi_0, \underline{\lambda}_0, \mu_0, V_0)$ is a stationary state of our magnetic Schrödinger-Poisson system. Finally, we are able to prove that $(\Psi_0, \underline{\lambda}_0) \in \mathcal{L}$, which can be established analogously to the proof of Lemma 6 above. □

We have the following statement relating the functionals Φ and $\mathcal{H}_C$.

Proposition 12. *Let the assumptions of Theorem 2 hold, such that* $(\Psi_0, \underline{\lambda}_0, \mu_0, V_0)$ *is the corresponding stationary state of our magnetic Schrödinger-Poisson system. Then* $\Phi(V_0, \sigma_0) = \mathcal{H}_C(\Psi_0, \underline{\lambda}_0)$.

Proof. We have, from Lemma 11,

$$\Phi(V_0, \sigma_0) = -\frac{1}{2}\int_{\Omega} |\nabla V_0|^2 dx - \mathrm{Tr}[F((-i\nabla + A)^2 + V_0 + \sigma_0)] - \sigma_0\Lambda$$

and

$$\mathcal{H}_C(\Psi_0, \underline{\lambda}_0) = \sum_{k=1}^{\infty} F^*(-\lambda_{0,k}) + \sum_{k=1}^{\infty} \lambda_{0,k} \int_{\Omega} |(-i\nabla + A)\psi_{0,k}|^2 dx + \frac{1}{2}\int_{\Omega} |\nabla V_0|^2 dx.$$

By virtue of Lemma 10,

$$\sum_{k=1}^{\infty} \Big[F^*(-\lambda_{0,k}) + \lambda_{0,k}\Big(\int_{\Omega} [|(-i\nabla + A)\psi_{0,k}|^2 + V_0|\psi_{0,k}|^2] dx + \sigma_0\Big)\Big]$$
$$= -\mathrm{Tr}[F((-i\nabla + A)^2 + V_0 + \sigma_0)],$$

which gives us the statement of the proposition. □

Acknowledgements V.V. thanks C. Sulem for the support and stimulating discussions.

Appendix: Global well-posedness

In this Appendix, we extend the global well-posedness result of [9] proved for the Schrödinger-Poisson system in a bounded domain with the Dirichlet boundary conditions to the case when a smooth magnetic field is turned on.

We introduce the magnetic Sobolev norms for functions

$$\|f\|^2_{H^1_A(\Omega)} := \|f\|^2_{L^2(\Omega)} + \|(-i\nabla + A)f\|^2_{L^2(\Omega)}, \tag{6.5}$$

$$\|f\|^2_{H^2_A(\Omega)} := \|f\|^2_{L^2(\Omega)} + \|(-i\nabla + A)^2 f\|^2_{L^2(\Omega)}. \tag{6.6}$$

The usual Sobolev norms $\|f\|_{H^1(\Omega)}$ and $\|f\|_{H^2(\Omega)}$ will be used when the magnetic vector potential vanishes. Let us define the inner product for fixed $\underline{\lambda} \in l^1$, $\lambda_k > 0$, and for sequences of square integrable functions $\Phi := \{\phi_k\}_{k=1}^{\infty}$ and $\Psi := \{\psi_k\}_{k=1}^{\infty}$ as

$$(\Phi, \Psi)_{X_\Omega} := \sum_{k=1}^{\infty} \lambda_k (\phi_k, \psi_k)_{L^2(\Omega)}.$$

Clearly, it induces the norm

$$\|\Phi\|_{X_\Omega} := (\sum_{k=1}^{\infty} \lambda_k \|\phi_k\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Let us introduce the corresponding Hilbert space

$$X_\Omega := \{\Phi = \{\phi_k\}_{k=1}^{\infty} \mid \phi_k \in L^2(\Omega),\ \forall\, k \in \mathbb{N},\ \|\Phi\|_{X_\Omega} < \infty\}.$$

We have the following result.

Lemma 13. *For every initial state* $(\Psi(x,0), \underline{\lambda}) \in \mathcal{L}$, *there exists a unique mild solution* $\Psi(x,t)$, $t \in [0,\infty)$, *of (1.6)-(1.9) with* $(\Psi(x,t), \underline{\lambda}) \in \mathcal{L}$. *This is also a unique strong global solution in* X_Ω.

Proving the global well-posedness of the Schrödinger-Poisson system plays a critical role in establishing the existence and nonlinear stability of stationary states, i.e. the nonlinear bound states of the Schrödinger-Poisson system, which was done in the non magnetic case in [9, 18]. These issues in the semi-relativistic regime were addressed recently in [1], [2], [3]. The corresponding one dimensional problem was studied in [21]. The existence of solutions for a single Nonlinear Schrödinger (NLS) equation with a magnetic field was established in [15], see also [14].

Let us make a fixed choice of $\underline{\lambda} = \{\lambda_k\}_{k=1}^{\infty} \in \ell^1$, with $\lambda_k \geq 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$, denoting the sequence of coefficients determined by the initial data ρ_0 of the Hartree-von Neumann equation (1.1) via (1.5), for $t = 0$. Let us introduce the inner products $(\cdot,\cdot)_{Y_{\Omega,A}}$ and $(\cdot,\cdot)_{Z_{\Omega,A}}$ inducing the generalized inhomogeneous magnetic Sobolev norms

$$\|\Phi\|_{Y_{\Omega,A}} := \left(\sum_{k=1}^{\infty} \lambda_k \|\phi_k\|^2_{H^1_A(\Omega)}\right)^{\frac{1}{2}} \quad \text{and} \quad \|\Phi\|_{Z_{\Omega,A}} := \left(\sum_{k=1}^{\infty} \lambda_k \|\phi_k\|^2_{H^2_A(\Omega)}\right)^{\frac{1}{2}}. \tag{6.7}$$

We define the corresponding Hilbert spaces

$$Y_{\Omega,A} := \{\Phi = \{\phi_k\}_{k=1}^{\infty} \mid \phi_k \in H^1_{0,\ A}(\Omega),\ \forall\, k \in \mathbb{N},\ \|\Phi\|_{Y_{\Omega,A}} < \infty\}$$

and

$$Z_{\Omega,A} := \{\Phi = \{\phi_k\}_{k=1}^{\infty} \mid \phi_k \in H^1_{0,\ A}(\Omega) \cap H^2_A(\Omega),\ \forall\, k \in \mathbb{N},\ \|\Phi\|_{Z_{\Omega,A}} < \infty\}$$

respectively. Let us also introduce the generalized homogenous magnetic Sobolev norms

$$\|\Phi\|_{\dot{Y}_{\Omega,A}} := \left(\sum_{k=1}^{\infty} \lambda_k \|(-i\nabla + A)\phi_k\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}, \tag{6.8}$$

$$\|\Phi\|_{\dot{Z}_{\Omega,A}} := \left(\sum_{k=1}^{\infty} \lambda_k \|(-i\nabla + A)^2\phi_k\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}. \tag{6.9}$$

The notations $\|\Phi\|_{Y_\Omega}$, $\|\Phi\|_{\dot{Y}_\Omega}$, $\|\Phi\|_{Z_\Omega}$, $\|\Phi\|_{\dot{Z}_\Omega}$ will be used when the magnetic vector potential vanishes, similarly to Section 3 of [9]. We have the following equivalence of magnetic and non magnetic norms.

Lemma 14. *Assume that the vector potential $A(x) \in C^1(\bar{\Omega}, \mathbb{R}^3)$ and the Coulomb gauge is chosen, namely*

$$divA = 0. \tag{6.10}$$

a) Let $f(x) \in H^1_{0,\ A}(\Omega)$. Then the norms

$$\|(-i\nabla + A)f\|_{L^2(\Omega)}, \quad \|\nabla f\|_{L^2(\Omega)}, \quad \|f\|_{H^1(\Omega)}, \quad \|f\|_{H^1_A(\Omega)}$$

are equivalent.

b) Let $f(x) \in H^2_A(\Omega)$. Then the norms

$$\|(-i\nabla + A)^2 f\|_{L^2(\Omega)}, \quad \|\Delta f\|_{L^2(\Omega)}, \quad \|f\|_{H^2(\Omega)}, \quad \|f\|_{H^2_A(\Omega)}$$

are equivalent.

c) Let $\Phi(x) \in Y_{\Omega,A}$. Then the norms

$$\|\Phi\|_{Y_{\Omega,A}}, \quad \|\Phi\|_{\dot{Y}_{\Omega,A}}, \quad \|\Phi\|_{Y_\Omega}, \quad \|\Phi\|_{\dot{Y}_\Omega}$$

are equivalent.

d) Let $\Phi(x) \in Z_{\Omega,A}$. Then the norms

$$\|\Phi\|_{Z_{\Omega,A}}, \quad \|\Phi\|_{\dot{Z}_{\Omega,A}}, \quad \|\Phi\|_{Z_\Omega}, \quad \|\Phi\|_{\dot{Z}_\Omega}$$

are equivalent.

Proof. We will make use of the diamagnetic inequality (see e.g. p.179 of [16])

$$\int_\Omega |(-i\nabla + A)f|^2 dx \geq \int_\Omega |\nabla |f||^2 dx \tag{6.11}$$

along with the Poincaré inequality

$$\int_\Omega |\nabla g(x)|^2 dx \geq c_p \int_\Omega |g(x)|^2 dx, \tag{6.12}$$

where the constant $c_p > 0$ depends upon our domain Ω with Dirichlet boundary conditions. In the argument below, with a slight abuse of notations C will denote a finite, positive constant. Since the vector potential $A(x)$ is bounded in Ω, as assumed, we easily obtain

$$\|(-i\nabla + A)f\|_{L^2(\Omega)} \leq \|\nabla f\|_{L^2(\Omega)} + C\|f\|_{L^2(\Omega)},$$

which can be trivially bounded above by $C\|\nabla f\|_{L^2(\Omega)}$ by virtue of inequality (6.12). Evidently,

$$\|\nabla f\|_{L^2(\Omega)} \leq \|(-i\nabla + A)f\|_{L^2(\Omega)} + \|Af\|_{L^2(\Omega)} \leq \|(-i\nabla + A)f\|_{L^2(\Omega)} + C\|f\|_{L^2(\Omega)},$$

which can be easily estimated from above by virtue of inequalities (6.11) and (6.12) by $C\|(-i\nabla + A)f\|_{L^2(\Omega)}$. Using the definition of the norm (6.5), we obtain

$$\|f\|_{H^1_A(\Omega)} \leq C\|f\|_{H^1(\Omega)}, \quad \|f\|_{H^1(\Omega)} \leq C\|f\|_{H^1_A(\Omega)}.$$

Obviously, $\|\nabla f\|_{L^2(\Omega)} \leq \|f\|_{H^1(\Omega)}$. Inequality (6.12) yields $\|f\|_{H^1(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}$, which completes the proof of the part a) of the lemma.

Since the vector potential $A(x)$ satisfies (6.10), we have

$$(-i\nabla + A)^2 = -\Delta - 2iA\nabla + A^2,$$

such that

$$\|(-i\nabla + A)^2 f\|_{L^2(\Omega)} \leq \|\Delta f\|_{L^2(\Omega)} + 2\|A\nabla f\|_{L^2(\Omega)} + \|A^2 f\|_{L^2(\Omega)}.$$

Using the Schwarz inequality along with (6.12) we estimate the right side of the inequality above by $C\|\Delta f\|_{L^2(\Omega)}$. Evidently,

$$\begin{aligned}\|\Delta f\|_{L^2(\Omega)} &\leq \|(-i\nabla + A)^2 f\|_{L^2(\Omega)} + \|2iA\nabla f - A^2 f\|_{L^2(\Omega)} \\ &\leq \|(-i\nabla + A)^2 f\|_{L^2(\Omega)} + 2\|A\nabla f\|_{L^2(\Omega)} + \|A^2 f\|_{L^2(\Omega)}.\end{aligned}$$

The result of the part a) of the lemma yields

$$\|A\nabla f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)} \leq C\|(-i\nabla + A)f\|_{L^2(\Omega)},$$

which can be bounded above by means of the Schwarz inequality along with (6.11) and (6.12) by $C\|(-i\nabla + A)^2 f\|_{L^2(\Omega)}$. Similarly,

$$\|A^2 f\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} \leq C\|(-i\nabla + A)f\|_{L^2(\Omega)} \leq C\|(-i\nabla + A)^2 f\|_{L^2(\Omega)}.$$

Hence,

$$\|\Delta f\|_{L^2(\Omega)} \leq C\|(-i\nabla + A)^2 f\|_{L^2(\Omega)}. \tag{6.13}$$

Clearly, $\|\Delta f\|_{L^2(\Omega)} \leq \|f\|_{H^2(\Omega)}$. By means of (6.12), we have $\|f\|_{H^2(\Omega)} \leq C\|\Delta f\|_{L^2(\Omega)}$. Evidently, inequality (6.12) yields

$$\|f\|_{H^2_A(\Omega)} \leq \sqrt{C\|\Delta f\|^2_{L^2(\Omega)} + \|(-i\nabla + A)^2 f\|^2_{L^2(\Omega)}},$$

which can be easily estimated from above by $C\|(-i\nabla + A)^2 f\|_{L^2(\Omega)}$. By means of definition (6.6), we have $\|(-i\nabla + A)^2 f\|_{L^2(\Omega)} \leq \|f\|_{H^2_A(\Omega)}$, which completes the proof of the part b) of the lemma. The results of parts c) and d) of the lemma follow easily from the definitions of the corresponding norms involved in (6.7), (6.8) and (6.9). □

Let $\Psi = \{\psi_m\}_{m=1}^{\infty}$ be a wave function and the magnetic kinetic energy operator acts on it $(-i\nabla + A)^2\Psi$ componentwise. We have the following two auxiliary lemmas.

Lemma 15. *The domain of the magnetic Dirichlet kinetic energy operator is given by*

$$D((-i\nabla + A)^2) = Z_{\Omega,A} \subseteq X_\Omega.$$

Proof. Let $\Psi \in Z_{\Omega,A}$. Hence

$$\|\Psi\|_{Z_{\Omega,A}} \geq \left(\sum_{k=1}^{\infty} \lambda_k \|\psi_k\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} = \|\Psi\|_{X_\Omega},$$

such that $\|\Psi\|_{X_\Omega} < \infty$ as well. □

Lemma 16. *The operator* $(-i\nabla + A)^2$ *generates the group* $e^{-i(-i\nabla+A)^2 t}$, $t \in \mathbb{R}$, *of unitary operators on* X_Ω.

Let us rewrite the Schrödinger-Poisson system for $x \in \Omega$ into the form

$$\frac{\partial \Psi}{\partial t} = -i(-i\nabla + A)^2\Psi + F[\Psi(x,t)], \text{ where } F[\Psi] := i^{-1}V[\Psi]\Psi, \tag{6.14}$$

$$-\Delta V[\Psi] = n[\Psi], \text{ where } V|_{\partial\Omega} = 0 \text{ and } \psi_k(t,.)|_{\partial\Omega} = 0,$$

and trivially obtain the following auxiliary result.

Lemma 17. *The map defined in (6.14)* $F : Z_{\Omega,A} \to Z_{\Omega,A}$ *is locally Lipschitz continuous.*

Proof. In the proof of Lemma 3.1 of [9] dealing with the non magnetic case it was proven that for $\Phi, \Psi \in Z_\Omega$ we have

$$\|F(\Phi) - F(\Psi)\|_{Z_\Omega} \leq C(\|\Phi\|^2_{Y_\Omega} + \|\Psi\|^2_{Y_\Omega})\|\Phi - \Psi\|_{Z_\Omega},$$

with $\Phi = \{\phi_k\}_{k=1}^\infty$, $\Psi = \{\psi_k\}_{k=1}^\infty$ and $t \in [0,T)$. By virtue of Lemma 14 above we have the equivalence of magnetic and non magnetic norms, such that

$$\|F(\Phi) - F(\Psi)\|_{Z_{\Omega,A}} \leq C(\|\Phi\|^2_{Y_{\Omega,A}} + \|\Psi\|^2_{Y_{\Omega,A}})\|\Phi - \Psi\|_{Z_{\Omega,A}}. \tag{6.15}$$

□

Standard arguments (see for instance [19, Theorem 1.7 §6]) yield, using Lemma 17, that the magnetic Schrödinger-Poisson system above possesses a unique mild solution Ψ in $Z_{\Omega,A}$ on a time interval $[0,T)$, with some $T > 0$, which satisfies the integral equation

$$\Psi(t) = e^{-i(-i\nabla+A)^2 t}\Psi(0) + \int_0^t e^{-i(-i\nabla+A)^2(t-s)}F[\Psi(s)]ds \tag{6.16}$$

in $Z_{\Omega,A}$. Furthermore,

$$\lim_{t\nearrow T}\|\Psi(t)\|_{Z_{\Omega,A}} = \infty$$

if T is finite. Let us also note that Ψ is a unique strong solution in X_Ω. Below we are going to prove that this solution is in fact global in time. First we establish the following lemma.

Lemma 18. *Suppose for the unique mild solution (6.16) of the magnetic Schrödinger-Poisson system (1.6)-(1.9) at* $t = 0$ *functions* $\{\psi_k(x,0)\}_{k=1}^\infty$ *form a complete orthonormal system in* $L^2(\Omega)$. *Then, for any* $t \in [0,T)$, *the set* $\{\psi_k(x,t)\}_{k=1}^\infty$ *remains a complete orthonormal system in* $L^2(\Omega)$. *Furthermore, the* X_Ω*-norm is preserved, such that* $\|\Psi(x,t)\|_{X_\Omega} = \|\Psi(x,0)\|_{X_\Omega}$, $t \in [0,T)$.

Proof. For the given solution $\Psi(t)$ of the magnetic Schrödinger-Poisson system on $[0,T)$, we obtain the time-dependent magnetic one-particle Hamiltonian

$$H_{A,V_\Psi}(t) = (-i\nabla + A)^2 + V_\Psi(t,x),$$

where the potential V_Ψ satisfies $-\Delta V_\Psi(t,x) = n[\Psi(t)]$ with Dirichlet boundary conditions, see (1.2).

Thus the components of $\Psi(t)$ solve the *non-autonomous* magnetic Schrödinger equation $i\partial_t\psi_k(t,x) = H_{A,V_\Psi}(t)\psi_k(t,x)$, for $k \in \mathbb{N}$, on the time interval $[0,T)$. Hence we obtain for $t \in [0,T)$,

$$\psi_k(x,t) = e^{-i\int_0^t H_{A,V_\Psi}(\tau)d\tau}\psi_k(x,0), \; k \in \mathbb{N}, \tag{6.17}$$

such that

$$
\begin{aligned}
(\psi_k(x,t), \psi_l(x,t))_{L^2(\Omega)} &= (e^{-i\int_0^t H_{A,V_\Psi}(\tau)d\tau}\psi_k(x,0), e^{-i\int_0^t H_{A,V_\Psi}(\tau)d\tau}\psi_l(x,0))_{L^2(\Omega)} \\
&= (\psi_k(x,0), \psi_l(x,0))_{L^2(\Omega)} = \delta_{k,l}, \quad k,l \in \mathbb{N},
\end{aligned}
$$

where $\delta_{k,l}$ denotes the Kronecker symbol. Therefore, for $k \in \mathbb{N}$,

$$
\|\psi_k(x,t)\|^2_{L^2(\Omega)} = \|\psi_k(x,0)\|^2_{L^2(\Omega)},
$$

such that for $t \in [0,T)$, the X_Ω-norm is preserved,

$$
\|\Psi(x,t)\|_{X_\Omega} = (\sum_{k=1}^{\infty} \lambda_k \|\psi_k(x,t)\|^2_{L^2(\Omega)})^{\frac{1}{2}} = (\sum_{k=1}^{\infty} \lambda_k \|\psi_k(x,0)\|^2_{L^2(\Omega)})^{\frac{1}{2}} = \|\Psi(x,0)\|_{X_\Omega}.
$$

We consider an arbitrary function $f(x) \in L^2(\Omega)$. Obviously, we have the expansion

$$
f(x) = \sum_{k=1}^{\infty} (f(y), \psi_k(y,0))_{L^2(\Omega)} \psi_k(x,0)
$$

and analogously

$$
e^{i\int_0^t H_{A,V_\Psi}(\tau)d\tau} f(x) = \sum_{k=1}^{\infty} (e^{i\int_0^t H_{A,V_\Psi}(\tau)d\tau} f(y), \psi_k(y,0))_{L^2(\Omega)} \psi_k(x,0).
$$

Thus, by virtue of (6.17) we arrive at the expansion

$$
f(x) = \sum_{k=1}^{\infty} (f(y), \psi_k(y,t))_{L^2(\Omega)} \psi_k(x,t)
$$

for $t \in [0,T)$. □

Below we derive the conservation of energy for the solutions to the magnetic Schrödinger-Poisson system in the following sense.

Lemma 19. *For the unique mild solution (6.16) of the magnetic Schrödinger-Poisson system (1.6)-(1.9) and for any value of time $t \in [0,T)$ we have the identity*

$$
\|\Psi(x,t)\|^2_{\dot{Y}_{\Omega,A}} + \frac{1}{2}\|\nabla V[\Psi(x,t)]\|^2_{L^2(\Omega)} = \|\Psi(x,0)\|^2_{\dot{Y}_{\Omega,A}} + \frac{1}{2}\|\nabla V[\Psi(x,0)]\|^2_{L^2(\Omega)}. \tag{6.18}
$$

Proof. Complex conjugation of the magnetic Schrödinger-Poisson system (1.6) gives us

$$
-i\frac{\partial \bar{\psi}_k}{\partial t} = (i\nabla + A)^2 \bar{\psi}_k + V[\Psi(x,t)]\bar{\psi}_k, \quad k \in \mathbb{N}. \tag{6.19}
$$

By adding the k-th equation of (1.6) multiplied by $\frac{\partial \bar{\psi}_k}{\partial t}$, and the k-th equation in (6.19) multiplied by $\frac{\partial \psi_k}{\partial t}$, we derive

$$
\frac{\partial}{\partial t}\|(-i\nabla + A)\psi_k\|^2_{L^2(\Omega)} + \int_\Omega V[\Psi(x,t)]\frac{\partial}{\partial t}|\psi_k|^2 dx = 0, \quad k \in \mathbb{N}.
$$

Multiplying by λ_k, and summing over k, we trivially obtain

$$
\frac{\partial}{\partial t}\|\Psi(x,t)\|^2_{\dot{Y}_{\Omega,A}} + \int_\Omega V[\Psi(x,t)]\frac{\partial}{\partial t} n[\Psi(x,t)]dx = 0. \tag{6.20}
$$

It can be easily verified that

$$\frac{\partial}{\partial t}\|\nabla V[\Psi(x,t)]\|_{L^2(\Omega)}^2 = 2\int_{\Omega} V[\Psi(x,t)]\frac{\partial}{\partial t}n[\Psi(x,t)]dx.$$

By substituting this identity in (6.20) we complete the proof of the lemma. □

Armed with the auxiliary statements established above, we now proceed to the proof of the main result of the Appendix.

Proof of Lemma 13. From (6.16) we easily obtain

$$(-i\nabla + A)^2\Psi(t) = e^{-i(-i\nabla+A)^2 t}(-i\nabla + A)^2\Psi(0) + \int_0^t e^{-i(-i\nabla+A)^2(t-s)}\{(-i\nabla + A)^2 F[\Psi(s)]\}ds.$$

Let us apply the norm $\|.\|_{X_\Omega}$ to both sides of the identity above, to arrive at

$$\|\Psi(t)\|_{\dot{Z}_{\Omega,A}} \leq \|\Psi(0)\|_{\dot{Z}_{\Omega,A}} + \int_0^t \|F[\Psi(s)]\|_{\dot{Z}_{\Omega,A}} ds.$$

By virtue of result (6.15) of Lemma 17 above, we have

$$\|F[\Psi]\|_{Z_{\Omega,A}} \leq C\|\Psi\|_{Y_{\Omega,A}}^2 \|\Psi\|_{Z_{\Omega,A}}.$$

Lemma 19 gives us the boundedness of the $\|\Psi\|_{\dot{Y}_{\Omega,A}}^2$ by the right side of identity (6.18), such that

$$\|F[\Psi(t)]\|_{\dot{Z}_{\Omega,A}} \leq C_0\|\Psi(t)\|_{\dot{Z}_{\Omega,A}},$$

with the constant C_0 proportional to the initial energy. Thus

$$\|\Psi(t)\|_{\dot{Z}_{\Omega,A}} \leq \|\Psi(0)\|_{\dot{Z}_{\Omega,A}} + \int_0^t C_0\|\Psi(s)\|_{\dot{Z}_{\Omega,A}} ds.$$

Gronwall's lemma implies that

$$\|\Psi(t)\|_{\dot{Z}_{\Omega,A}} \leq \|\Psi(0)\|_{\dot{Z}_{\Omega,A}} e^{C_0 t}, \quad t \in [0,T).$$

By virtue of the blow-up alternative, this yields that our magnetic Schrödinger-Poisson system is globally well-posed in $Z_{\Omega,A}$. □

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