SOLVABILITY CONDITIONS FOR SOME NON FREDHOLM OPERATORS IN HIGHER DIMENSIONS

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Abstract: We establish solvability in $H^2(\mathbb{R}^d)$ of certain linear elliptic equations involving the sums of second order differential operators without Fredholm property using the spectral and scattering theory for Schrödinger type operators, generalizing the results obtained in the earlier work [13].

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1. Introduction

Consider the equation

$$-\Delta u + V(x)u - au = f \tag{1.1}$$

with $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ tending to 0 at infinity. When $a \geq 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of equation (1.1) contains the origin. As a consequence, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimensions of its kernel and the codimension of its image are not finite. The present note deals with the studies of some properties of such operators. Let us note that elliptic problems involving non Fredholm operators were studied extensively in recent years (see [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], also [6]) along with their potential applications to the theory of reaction-diffusion equations (see [7], [8]). Non Fredholm operators are also used when studying wave systems with an infinite number of localized traveling waves (see [1]). In the particular case of $a = 0$ the operator A satisfies the Fredholm property in certain properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is essentially different and the approach developed in these articles cannot be applied.

One of the important issues about problems with non-Fredholm operators concerns their solvability. In the first part of the note we consider the equation

$$-\Delta_x u + V(x)u - \Delta_y u + U(y)u = f(x,y), \quad x,y \in \mathbb{R}^3. \tag{1.2}$$

The scalar potential functions involved in (1.2) are assumed to be shallow and short-range, satisfying the assumptions analogous to ones in [12] and [13].

Assumption 1. *The potential functions $V(x), U(y) : \mathbb{R}^3 \to \mathbb{R}$ satisfy the estimates*

$$|V(x)| \le \frac{C}{1+|x|^{3.5+\varepsilon}}, \quad |U(y)| \le \frac{C}{1+|y|^{3.5+\varepsilon}}$$

with some $\varepsilon > 0$ and and $x, y \in \mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1, \quad 4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|U\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|U\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1$$

and

$$\sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi, \quad \sqrt{c_{HLS}}\|U\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

Here C stands for a finite positive constant and c_{HLS} given on p.98 of [9] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\left| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f_1(x)f_1(y)}{|x-y|^2}dxdy \right| \le c_{HLS}\|f_1\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^2, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

Here and further down the norm of a function $f_1 \in L^p(\mathbb{R}^d),\ 1 \le p \le \infty,\ d \in \mathbb{N}$ is denoted as $\|f_1\|_{L^p(\mathbb{R}^d)}$. Let us denote the inner product of two functions as

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx, \tag{1.3}$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x)$ is bounded like, for instance the functions of the continuos spectrum of the Schrödinger operators discussed below, then the integral in the right side of (1.3) makes sense. By virtue of Lemma 2.3 of [12] the Schrödinger operator in the left side of (1.2) is self-adjoint and unitarily equivalent to $-\Delta_x - \Delta_y$ on $L^2(\mathbb{R}^6)$ via the wave operators. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$ and therefore such operator fails to satisfy the Fredholm property. The functions of the continuos spectrum satisfy the Schrödinger equation

$$[-\Delta_x + V(x)]\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3,$$

in the integral form the Lippmann-Schwinger equation

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{1.4}$$

and the orthogonality conditions $(\varphi_k(x), \varphi_{k_1}(x))_{L^2(\mathbb{R}^3)} = \delta(k-k_1),\ k, k_1 \in \mathbb{R}^3$. The integral operator involved in (1.4)

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi(x) \in L^\infty(\mathbb{R}^3).$$

We consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ and its norm $\|Q\|_\infty < 1$ under Assumption 1 via Lemma 2.1 of [12]. In fact, this norm is bounded above by the k-independent quantity. Similarly, for the second operator involved in (1.2) the functions of its continuous spectrum solve

$$[-\Delta_y + U(y)]\eta_q(y) = q^2\eta_q(y), \quad q \in \mathbb{R}^3,$$

in the integral formulation

$$\eta_q(y) = \frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|q||y-z|}}{|y-z|}(U\eta_q)(z)dz, \tag{1.5}$$

such that the the orthogonality relations $(\eta_q(y), \eta_{q_1}(y))_{L^2(\mathbb{R}^3)} = \delta(q-q_1),\ q, q_1 \in \mathbb{R}^3$ hold. The integral operator involved in (1.5) is

$$(P\eta)(y) := -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|q||y-z|}}{|y-z|}(U\eta)(z)dz, \quad \eta(y) \in L^\infty(\mathbb{R}^3).$$

For $P : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ its norm $\|P\|_\infty < 1$ under Assumption 1 by virtue of Lemma 2.1 of [12]. As before, this norm can be estimated above by the q-independent quantity. The product of these functions of the continuous spectrum $\varphi_k(x)\eta_q(y)$ form a complete system in $L^2(\mathbb{R}^6)$. Let us denote by the double tilde sign the generalized Fourier transform

$$\tilde{\tilde{f}}(k,q) := (f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}, \quad k, q \in \mathbb{R}^3. \tag{1.6}$$

We will be using the Sobolev space

$$H^2(\mathbb{R}^d) = \{u(x) : \mathbb{R}^d \to \mathbb{C} \mid u(x) \in L^2(\mathbb{R}^d),\ \Delta u \in L^2(\mathbb{R}^d)\}$$

equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^d)} = \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta u\|^2_{L^2(\mathbb{R}^d)}, \quad d \in \mathbb{N}.$$

Our first main statement is as follows.

Theorem 2. *Let Assumption 1 hold and $f(x,y) \in L^1(\mathbb{R}^6) \cap L^2(\mathbb{R}^6)$. Then equation (1.2) admits a unique solution $u(x,y) \in H^2(\mathbb{R}^6)$.*

Note that the solvability of problem (1.2) was treated before in Theorem 3 of [13] under the assumption that $|x|f(x,y),\ |y|f(x,y) \in L^1(\mathbb{R}^6)$.

In the second part of the note we consider the equation

$$-\Delta_x u - \Delta_y u + U(y)u = f(x,y), \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^3 \tag{1.7}$$

with $n \in \mathbb{N},\ n \geq 2$ and the scalar potential function involved in (1.7) is shallow and short-range under analogous assumptions as before such that the operator involved in the left side of (1.7) is self-adjoint and unitarily equivalent $-\Delta_x - \Delta_y$ on $L^2(\mathbb{R}^{n+3})$. Therefore, its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$ and such operator does not satisfy the Fredholm property. The products of the standard Fourier harmonics $\frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}$ and the perturbed plane waves $\eta_q(y)$ satisfying (1.5) form a complete system in $L^2(\mathbb{R}^{n+3})$. We consider another generalized Fourier transform

$$\tilde{\tilde{f}}(k,q) := (f(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y))_{L^2(\mathbb{R}^{n+3})}, \quad k \in \mathbb{R}^n, \quad q \in \mathbb{R}^3. \tag{1.8}$$

Our second main statement is as follows.

Theorem 3. *Let the potential function $U(y)$ satisfy Assumption 1 and $f(x,y) \in L^1(\mathbb{R}^{n+3}) \cap L^2(\mathbb{R}^{n+3}),\ n \geq 2$. Then problem (1.7) possesses a unique solution $u(x,y) \in H^2(\mathbb{R}^{n+3})$.*

Note that the solvability of equation (1.7) was established before in Theorem 6 of [13] under the assumption that $|x|f(x,y),\ |y|f(x,y) \in L^1(\mathbb{R}^{n+3})$.

Let us proceed to the proof of our statements.

2. The solvability of the non Fredholm problems

Proof of Theorem 2. Let us first suppose that problem (1.2) admits two solutions $u_{1,2}(x,y) \in H^2(\mathbb{R}^6)$. Then their difference $w(x,y) := u_1(x,y) - u_2(x,y) \in H^2(\mathbb{R}^6)$ solves the homogeneous equation

$$-\Delta_x w + V(x)w - \Delta_y w + U(y)w = 0.$$

Since the operator involved in the left side of this problem does not have any square integrable zero modes, just the essential spectrum, $w(x,y)$ vanishes a.e. on $\mathbb{R}^6$. By applying the generalized Fourier transform (1.6) to both sides of equation (1.2), we easily arrive at

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{k^2+q^2}, \quad k, q \in \mathbb{R}^3.$$

This enables us to express the norm as

$$\|u\|^2_{L^2(\mathbb{R}^6)} = \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dq \frac{|\tilde{\tilde{f}}(k,q)|^2}{(k^2+q^2)^2},$$

which can be easily written as

$$\int_{k^2+q^2\leq 1} dkdq \frac{|\tilde{\tilde{f}}(k,q)|^2}{(k^2+q^2)^2} + \int_{k^2+q^2>1} dkdq \frac{|\tilde{\tilde{f}}(k,q)|^2}{(k^2+q^2)^2}. \tag{2.9}$$

The second term in (2.9) can be easily estimated from above by

$$\int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dq |\tilde{\tilde{f}}(k,q)|^2 = \|f\|^2_{L^2(\mathbb{R}^6)} < \infty$$

due to one of our assumptions. By virtue of Corollary 2.2 of [12] we have

$$|\tilde{\tilde{f}}(k,q)| \leq \frac{1}{1-\|Q\|_\infty} \frac{1}{1-\|P\|_\infty} \frac{1}{(2\pi)^3} \|f\|_{L^1(\mathbb{R}^6)},$$

such that the first term in (2.9) can be easily bounded above by

$$\frac{1}{2} \frac{1}{(2\pi)^6} |S_6| \frac{1}{(1-\|Q\|_\infty)^2} \frac{1}{(1-\|P\|_\infty)^2} \|f\|^2_{L^1(\mathbb{R}^6)} < \infty$$

as assumed in the theorem. Here and below S_d stands for the unit sphere in $\mathbb{R}^d$ centered at the origin and $|S_d|$ for its Lebesgue measure. Hence the unique solution $u(x,y) \in L^2(\mathbb{R}^6)$, but since the right side of (1.2) is square integrable and the scalar potentials involved in it are bounded due to our assumptions, we have $u(x,y) \in H^2(\mathbb{R}^6)$ as well. ∎

Proof of Theorem 3. Assume first that equation (1.7) has two solutions $u_{1,2}(x,y) \in H^2(\mathbb{R}^{n+3})$, such that their difference $w(x,y) := u_1(x,y) - u_2(x,y) \in H^2(\mathbb{R}^{n+3})$ as well and is a solution of the homogeneous problem

$$-\Delta_x w - \Delta_y w + U(y) w = 0.$$

But the operator in the left side of the equation above has just the essential spectrum and no square integrable zero modes. Hence, $w(x,y) = 0$ a.e. in $\mathbb{R}^{n+3}$. Let us apply the generalized Fourier transform (1.8) to both sides of problem (1.7), which yields

$$\tilde{\hat{u}}(k,q) = \frac{\tilde{\hat{f}}(k,q)}{k^2+q^2}, \quad k \in \mathbb{R}^n, \quad q \in \mathbb{R}^3.$$

Hence we obtain

$$\|u\|^2_{L^2(\mathbb{R}^{n+3})} = \int_{\mathbb{R}^n} dk \int_{\mathbb{R}^3} dq \frac{|\tilde{\hat{f}}(k,q)|^2}{(k^2+q^2)^2},$$

which can be trivially expressed as

$$\int_{k^2+q^2\leq 1} dkdq \frac{|\tilde{\hat{f}}(k,q)|^2}{(k^2+q^2)^2} + \int_{k^2+q^2>1} dkdq \frac{|\tilde{\hat{f}}(k,q)|^2}{(k^2+q^2)^2}. \tag{2.10}$$

The second term in (2.10) can be trivially bounded above by

$$\int_{\mathbb{R}^n} dk \int_{\mathbb{R}^3} dq |\tilde{\hat{f}}(k,q)|^2 = \|f\|^2_{L^2(\mathbb{R}^{n+3})} < \infty$$

due to one of the assumptions of the theorem. By means of Corollary 2.2 of [12] we derive

$$|\tilde{\hat{f}}(k,q)| \leq \frac{1}{(2\pi)^{\frac{n+3}{2}}} \frac{1}{1 - \|P\|_\infty} \|f\|_{L^1(\mathbb{R}^{n+3})}.$$

This enables us to estimate the first term in (2.10) from above by

$$\frac{1}{(2\pi)^{n+3}} \frac{1}{n-1} \frac{1}{(1 - \|P\|_\infty)^2} |S_{n+3}| \|f\|^2_{L^1(\mathbb{R}^{n+3})} < \infty$$

as assumed in the theorem. Therefore, our unique solution $u(x,y) \in L^2(\mathbb{R}^{n+3})$. By virtue of our assumptions, the right side of (1.7) is square integrable and the scalar potential $U(y)$ in it is bounded, such that $u(x,y) \in H^2(\mathbb{R}^{n+3})$ as well. ■

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