# BIFURCATION AND MULTIPLICITY RESULTS FOR CRITICAL NONLOCAL FRACTIONAL LAPLACIAN PROBLEMS

ALESSIO FISCELLA, GIOVANNI MOLICA BISCI, AND RAFFAELLA SERVADEI

Abstract. In this paper we consider the following critical nonlocal problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \,, \end{cases}$$

where $s \in (0,1)$, $\Omega$ is an open bounded subset of $\mathbb{R}^n$, $n > 2s$, with Lipschitz boundary, $\lambda$ is a positive real parameter, $2^* := 2n/(n-2s)$ is the fractional critical Sobolev exponent, while $\mathcal{L}_K$ is the nonlocal integrodifferential operator

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \Big( u(x+y) + u(x-y) - 2u(x) \Big) K(y)\, dy \,, \quad x \in \mathbb{R}^n \,,$$

whose model is given by the fractional Laplacian $-(-\Delta)^s$.

Along the paper, we prove a multiplicity and bifurcation result for this problem, using a classical theorem in critical points theory. Precisely, we show that in a suitable left neighborhood of any eigenvalue of $-\mathcal{L}_K$ (with Dirichlet boundary data) the number of nontrivial solutions for the problem under consideration is at least twice the multiplicity of the eigenvalue. Hence, we extend the result got by Cerami, Fortunato and Struwe in [12] for classical elliptic equations, to the case of nonlocal fractional operators.

## Contents



## 1. Introduction

In recent years, nonlocal problems and operators have been widely studied in the literature and have attracted the attention of lot of mathematicians coming from different research areas. The interest towards equations involving nonlocal operators has grown more and more, thanks to their intriguing analytical structure and in view of several applications in a wide range of contexts. Indeed, fractional and nonlocal operators appear in concrete applications in many fields such as, among the others, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science,

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water waves, thin obstacle problem, optimal transport, image reconstruction, through a new and fascinating scientific approach (see, e.g., the papers [2, 7, 10, 16, 21, 34, 35, 36] and references therein).

After the seminal paper [9] by Brezis and Nirenberg, the critical problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2_*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

has been widely studied in the literature (see, e.g. [11, 12, 13, 14, 15, 17, 18, 20, 23, 32, 37] just to name a few), also due to its relevant relations with problems arising in differential geometry and in physics, where a lack of compactness occurs (see, for instance, the famous Yamabe problem). Here $\Omega$ is an open bounded subset of $\mathbb{R}^n$, $n > 2$, and $2_* := 2n/(n-2)$ is the critical Sobolev exponent.

The first multiplicity result for problem (1.1) was proved by Cerami, Fortunato and Struwe in [12], where it was shown that in a suitable left neighborhood of any eigenvalue of $-\Delta$ (with Dirichlet boundary data) the number of solutions is at least twice the multiplicity of the eigenvalue. The authors also gave an estimate of the length of this neighborhood, which depends on the best critical Sobolev constant, on the Lebesgue measure of the set where the problem is set and on the space dimension.

Later, in [13] the authors proved that in dimension $n \geqslant 6$ and for $\lambda > 0$ less than the first eigenvalue of $-\Delta$ (with homogeneous Dirichlet boundary conditions), problem (1.1) has at least two nontrivial solutions, one of which is a changing sign solution (for other results on changing sign solutions see, for instance, [15, 23, 32]). More recently, in [18] the authors improved the result got in [13], while in [17] Devillanova and Solimini proved the existence of infinitely many solutions for (1.1), provided the dimension $n \geqslant 7$ and the parameter $\lambda$ is positive. Finally, in [15] the authors showed that for $n \geqslant 4$ problem (1.1) has at least $(n+1)/2$ pairs of nontrivial solutions, provided $\lambda > 0$ is not an eigenvalue of $-\Delta$, and $(n+1-m)/2$ pairs of nontrivial solutions, if $\lambda$ is an eigenvalue of $-\Delta$ with multiplicity $m < n+2$. When $m \geqslant n+2$, [15] gave no information about the multiplicity of solutions for (1.1), when $\lambda$ is an eigenvalue of $-\Delta$. A partial answer to this question was given in [14], where the authors showed that when $n \geqslant 5$ and $\lambda \geqslant \lambda_1$, then problem (1.1) has at least $(n+1)/2$ pairs of nontrivial solutions.

A natural question is whether all these results can be extended to the fractional nonlocal counterpart of (1.1), i.e. to the following problem

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.2}$$

where $s \in (0,1)$ is fixed, $\Omega$ is an open bounded subset of $\mathbb{R}^n$, $n > 2s$, with Lipschitz boundary, $2^*$ is the fractional critical Sobolev exponent given by

$$2^* := 2n/(n-2s)$$

and $-(-\Delta)^s$ is the fractional Laplace operator which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy$$

for $x \in \mathbb{R}^n$ (see [19] and references therein for further details on the fractional Laplacian). Note that, in the nonlocal setting, the condition $u = 0$ in $\mathbb{R}^n \setminus \Omega$ is the natural counterpart of the homogeneous Dirichlet boundary data $u = 0$ on $\partial\Omega$, due to the nonlocal character of the problem.

If we deal with the existence of nontrivial solutions for problem (1.2), a positive answer has been given in the recent papers [24, 25, 30], also in presence of a perturbation of the critical term (for this see, for instance, [29, 31]): in all these papers the classical and well known existence results for (1.1) were extended to the nonlocal fractional setting. Other interesting existence results for nonlocal problems driven by fractional operators in a critical setting can be found in [4, 22, 33] and references therein.

Aim of this paper is to focus the attention on the multiplicity of solutions for (1.2). In particular, our starting point will be the paper [12] and our goal will be to extend the result obtained there to the nonlocal fractional setting.

Precisely, along this work we consider a generalization of (1.2), given by the following nonlocal critical problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{1.3}$$

Here $\mathcal{L}_K$ is the nonlocal operator defined as follows:

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy\,, \qquad x \in \mathbb{R}^n\,,$$

where the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is a function such that

$$mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\{|x|^2, 1\}\,; \tag{1.4}$$

$$\text{there exists } \theta > 0 \text{ such that } K(x) \geqslant \theta |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,; \tag{1.5}$$

$$K(x) = K(-x) \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,. \tag{1.6}$$

The integro-differential operator $\mathcal{L}_K$ is a generalization of the fractional Laplacian, since, taking $K(x) = |x|^{-(n+2s)}$, we get $\mathcal{L}_K = -(-\Delta)^s$.

Of course, the trivial function $u \equiv 0$ is a solution of problem (1.3). Here we are interested in nontrivial solutions. Before stating our main results, we need to introduce the functional space we will work in, the variational formulation of the problem under consideration, as well as the spectrum of the operator $-\mathcal{L}_K$.

1.1. **Notations.** Problem (1.3) has a variational character and the natural space where finding weak solutions for it is the functional space $X_0$, defined as follows (for more details we refer to [26, 27], where this space was introduced and some properties of this space were proved).

Along this paper the space $X$ denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to $\Omega$ of any function $g$ in $X$ belongs to $L^2(\Omega)$ and

$$\text{the map } (x, y) \mapsto (g(x) - g(y))\sqrt{K(x-y)} \text{ is in } L^2\big((\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy\big)\,,$$

where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$, while

$$X_0 := \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}\,.$$

We note that $X$ and $X_0$ are non-empty, since $C^2_0(\Omega) \subseteq X_0$ by [26, Lemma 5.1]: for this we need condition (1.4).

The space $X$ is endowed with the norm defined as

$$\|g\|_X := \|g\|_{L^2(\Omega)} + \Big(\int_Q |g(x) - g(y)|^2 K(x-y) dx\, dy\Big)^{1/2}\,, \tag{1.7}$$

where $Q := (\mathbb{R}^n \times \mathbb{R}^n) \setminus \mathcal{O}$ and $\mathcal{O} = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^n \times \mathbb{R}^n$ (see, for instance, [27] for a proof). Moreover, by [27, Lemma 6] as a norm on $X_0$ we can take the function

$$X_0 \ni g \mapsto \|g\|_{X_0} := \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |g(x) - g(y)|^2 K(x-y)\, dx\, dy\right)^{1/2}\,. \tag{1.8}$$

Also, $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space, as proved in [27, Lemma 7], with scalar product given by

$$X_0 \times X_0 \ni (u, v) \mapsto \langle u, v\rangle_{X_0} := \int_{\mathbb{R}^n \times \mathbb{R}^n} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\, K(x-y)\, dx\, dy\,.$$

The usual fractional Sobolev space $H^s(\Omega)$ is endowed with the so-called *Gagliardo norm* (see, for instance [1, 19]) given by

$$\|g\|_{H^s(\Omega)} := \|g\|_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}. \tag{1.9}$$

Hence, even in the model case in which $K(x) = |x|^{-(n+2s)}$, the norms in (1.7) and (1.9) are not the same, because $\Omega\times\Omega$ is strictly contained in $Q$: this makes the space $X_0$ not equivalent to the usual fractional Sobolev spaces and the classical fractional Sobolev space approach not sufficient for studying our problem from a variational point of view.

The weak formulation of problem (1.3) is given by (here we use the symmetry of the kernel $K$)

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx \\ \qquad\qquad\qquad\qquad\displaystyle = \int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)dx \quad \forall\,\varphi\in X_0 \\ u\in X_0\,, \end{cases} \tag{1.10}$$

which represents the Euler–Lagrange equation of the functional $\mathcal{J}_{K,\lambda}: X_0\to\mathbb{R}$ defined as

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u) := \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx \\ - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}\,dx\,. \end{aligned} \tag{1.11}$$

Along the present paper $\{\lambda_k\}_{k\in\mathbb{N}}$ denotes the sequence of the eigenvalues of the following problem

$$\begin{cases} -\mathcal{L}_K u = \lambda\,u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n\setminus\Omega\,, \end{cases} \tag{1.12}$$

with

$$0<\lambda_1<\lambda_2\leqslant\ldots\leqslant\lambda_k\leqslant\lambda_{k+1}\leqslant\ldots \tag{1.13}$$

$$\lambda_k\to+\infty \text{ as } k\to+\infty\,,$$

and with $e_k$ as eigenfunction corresponding to $\lambda_k$. Also, we choose $\{e_k\}_{k\in\mathbb{N}}$ normalized in such a way that this sequence provides an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $X_0$. For a complete study of the spectrum of the integrodifferential operator $-\mathcal{L}_K$ we refer to [24, Proposition 2.3], [28, Proposition 9 and Appendix A] and [30, Proposition 4]. Finally, we say that the eigenvalue $\lambda_k$, $k\geqslant 2$, has multiplicity $m\in\mathbb{N}$ if

$$\lambda_{k-1}<\lambda_k=\cdots=\lambda_{k+m-1}<\lambda_{k+m}\,.$$

In this case the set of all the eigenvalues corresponding to $\lambda_k$ agrees with

$$\operatorname{span}\{e_k,\ldots,e_{k+m-1}\}\,.$$

In the sequel we also refer to the best fractional critical Sobolev constant $S_K$ defined as follows

$$S_K := \inf_{v\in X_0\setminus\{0\}} S_K(v) \tag{1.14}$$

where for any $v\in X_0\setminus\{0\}$

$$S_K(v) := \frac{\displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n} |v(x)-v(y)|^2K(x-y)\,dx\,dy}{\Big(\displaystyle\int_\Omega |v(x)|^{2^*}dx\Big)^{2/2^*}}\,. \tag{1.15}$$

Finally, in what follows, $|\Omega|$ denotes the Lebesgue measure of the set $\Omega$. Now, we are able to state the main result of the present paper.

1.2. **Main result of the paper.** As we said here above, the main feature of this paper concerns the existence of multiple solutions for problems (1.2) and (1.3). Precisely, with the notations introduced in Subsection 1.1, our main result reads as follows:

**Theorem 1.** *Let $s \in (0,1)$, $n > 2s$, $\Omega$ be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary, and let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying assumptions* (1.4)–(1.6). *Let $\lambda \in \mathbb{R}$ and let $\lambda^\star$ be the eigenvalue of problem* (1.10) *given by*

$$\lambda^\star := \min\left\{\lambda_k : \ \lambda < \lambda_k\right\} . \tag{1.16}$$

*and let $m \in \mathbb{N}$ be its multiplicity. Assume that*

$$\lambda \in (\lambda^\star - S_K|\Omega|^{-\frac{2s}{n}}, \lambda^\star)\,, \tag{1.17}$$

*where $S_K$ is the best fractional critical Sobolev constant defined in* (1.14).

*Then, problem* (1.3) *admits at least $m$ pairs of nontrivial solutions $\{-u_i(\lambda),\, u_i(\lambda)\}$ such that*

$$\|u_i(\lambda)\|_{X_0} \to 0 \quad \textit{as } \lambda \to \lambda^\star$$

*for any $i = 1, \dots, m$.*

This theorem represents the fractional nonlocal counterpart of the famous multiplicity result got by Cerami, Fortunato and Struwe in [12], using essentially an abstract critical point theorem due to Bartolo, Benci and Fortunato in [5], whose main tool is a pseudo-index theory introduced in [6] for studying indefinite functionals.

Problem (1.3) is variational in nature. The first difficulty in treating this problem consists in writing its variational formulation, which has to take into account also the 'boundary' condition $u = 0$ in $\mathbb{R}^n \setminus \Omega$. For this we set the weak problem in a suitable functional space $X_0$, whose definition is inspired, but not equivalent, to the one of the fractional Sobolev spaces.

The weak solutions of (1.3) can be found as critical points of the Euler-Lagrange functional $\mathcal{J}_{K,\lambda}$ associated with the equation. As usual when dealing with critical problems, one of the main difficulty in treating the problem is due to the lack of compactness that occurs. Indeed, the effect of presence of the critical term $|u|^{2^*-2}u$ is that Palais-Smale condition for $\mathcal{J}_{K,\lambda}$ does not hold at any level, but just under a suitable threshold, which, in our case, depends on the best fractional critical Sobolev constant related to the compact embedding $X_0 \hookrightarrow L^{2^*}(\Omega)$, on $s$ and $n$.

In Theorem 1 we prove that, in a suitable left neighborhood of any eigenvalue of the integrodifferential operator $-\mathcal{L}_K$ (with homogeneous Dirichlet boundary condition) the number of nontrivial solutions for problem (1.3) is, at least, twice the multiplicity of the eigenvalue. Hence, we show that there is a bifurcation from any eigenvalue of the operator $-\mathcal{L}_K$. In addition, we give an estimate of the length of this left neighborhood, in which the existence of multiple solutions occurs. This estimates depends on the best fractional critical Sobolev constant $S_K$, on the Lebesgue measure of $\Omega$, on $n$ and $s$, as stated in (1.17). We would like to point out that this condition is crucial in order to show that the energy functional associated with (1.3) satisfies all the geometric assumptions required by the abstract critical point theorem used along the present paper.

## 2. A multiplicity and bifurcation result

This section is devoted to the proof of Theorem 1, which is mainly based on variational and topological methods. Precisely, here we will perform the following result due to Bartolo, Benci and Fortunato (see [12, Theorem 2.5] and [5, Theorem 2.4]), which, as usual for abstract critical points theorems, gives the existence of critical points for a functional (sufficiently smooth), provided it satisfies suitable geometric and compactness conditions.

**Theorem 2** (Abstract critical point theorem)**.** *Let $H$ be a real Hilbert space with norm $\|\cdot\|$ and suppose that $\mathcal{I} \in C^1(H, \mathbb{R})$ is a functional on $H$ satisfying the following conditions:*

($I_1$) $\mathcal{I}(u) = \mathcal{I}(-u)$ *and* $\mathcal{I}(0) = 0$*;*
($I_2$) *there exists a constant $\beta > 0$ such that the Palais-Smale condition for $\mathcal{I}$ holds in $(0, \beta)$;*
($I_3$) *there exist two closed subspaces $V, W \subset H$ and positive constants $\rho$, $\delta$, $\beta'$, with $\delta < \beta' < \beta$, such that*
  *i)* $\mathcal{I}(u) \leqslant \beta'$ *for any* $u \in W$
  *ii)* $\mathcal{I}(u) \geqslant \delta$ *for any* $u \in V$ *with* $\|u\| = \rho$
  *iii)* $\mathit{codim}\, V < +\infty$ *and* $\mathit{dim}\, W \geqslant \mathit{codim}\, V$.

*Then, there exist at least $\mathit{dim}\, W - \mathit{codim}\, V$ pairs of critical points of $\mathcal{I}$, with critical values belonging to the interval $[\delta, \beta']$.*

The abstract result got in [5, Theorem 2.4] is a generalization of [3, Theorem 2.13], obtained using a pseudo-index theory, introduced in [6] for exploiting the existence of multiple critical points of functionals which are neither bounded above nor below on a Hilbert space.

2.1. **Proof of Theorem 1.** The idea consists in applying Theorem 2 to the functional $\mathcal{J}_{K,\lambda}$, defined in (1.11). It is easily seen that $\mathcal{J}_{K,\lambda}$ is well defined thanks to [27, Lemma 6]. Moreover, $\mathcal{J}_{K,\lambda} \in C^1(X_0)$ and

$$\begin{aligned}\langle \mathcal{J}'_{K,\lambda}(u), \varphi\rangle = &\int_{\mathbb{R}^n\times\mathbb{R}^n} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx\\ &-\int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)\,dx\end{aligned}$$

for any $u, \varphi \in X_0$. Thus, critical points of $\mathcal{J}_{K,\lambda}$ are solutions to problem (1.10), that is weak solutions for (1.3). Note also that $\mathcal{J}_{K,\lambda}$ is even and $\mathcal{J}_{K,\lambda}(0) = 0$, so that condition ($I_1$) of Theorem 2 is verified by $\mathcal{J}_{K,\lambda}$.

It remains to prove that $\mathcal{J}_{K,\lambda}$ satisfies assumptions ($I_2$) and ($I_3$) of Theorem 2. At this purpose, let us proceed by steps.

**Step 1** (**Compactness of the functional $\mathcal{J}_{K,\lambda}$**)**.** In the sequel we prove that, for suitable values of $c$, say

$$c < \frac{s}{n} S_K^{n/(2s)}\,, \tag{2.1}$$

the functional $\mathcal{J}_{K,\lambda}$ satisfies the following Palais-Smale condition at level $c \in \mathbb{R}$, i.e.

every sequence $\{u_j\}_{j\in\mathbb{N}}$ in $X_0$ such that

$$\mathcal{J}_{K,\lambda}(u_j) \to c \quad \text{as } j \to +\infty \tag{2.2}$$

and

$$\sup\Big\{\big|\langle\, \mathcal{J}'_{K,\lambda}(u_j), \varphi\,\rangle\big| \,:\, \varphi \in X_0\,, \|\varphi\|_{X_0} = 1\Big\} \to 0 \quad \text{as } j \to +\infty \tag{2.3}$$

admits a subsequence strongly convergent in $X_0$.

*Proof.* Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $X_0$ such that (2.2) and (2.3) are satisfied.

First of all we prove that

$$\text{the sequence } \{u_j\}_{j\in\mathbb{N}} \text{ is bounded in } X_0\,. \tag{2.4}$$

For any $j \in \mathbb{N}$ by (2.2) and (2.3) it easily follows that there exists $\kappa > 0$ such that

$$|\mathcal{J}_{K,\lambda}(u_j)| \leqslant \kappa\,, \tag{2.5}$$

and

$$\Big|\langle \mathcal{J}'_{K,\lambda}(u_j), \frac{u_j}{\|u_j\|_{X_0}}\rangle\Big| \leqslant \kappa\,, \tag{2.6}$$

and so

$$\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{2}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle \leqslant \kappa\left(1 + \|u_j\|_{X_0}\right). \tag{2.7}$$

Furthermore,

$$\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{2}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle = \left(\frac{1}{2} - \frac{1}{2^*}\right)\|u_j\|^{2^*}_{L^{2^*}(\Omega)} = \frac{s}{n}\|u_j\|^{2^*}_{L^{2^*}(\Omega)},$$

so that, thanks to (2.7), we get that for any $j \in \mathbb{N}$

$$\|u_j\|^{2^*}_{L^{2^*}(\Omega)} \leqslant \kappa_*\left(1 + \|u_j\|_{X_0}\right) \tag{2.8}$$

for a suitable positive constant $\kappa_*$. Consequently, recalling that $2^* > 2$ and using the Hölder inequality, we get

$$\|u_j\|^2_{L^2(\Omega)} \leqslant |\Omega|^{2s/n}\|u_j\|^2_{L^{2^*}(\Omega)} \leqslant \kappa_*^{2/2^*}|\Omega|^{2s/n}\left(1 + \|u_j\|_{X_0}\right)^{2/2^*},$$

that is

$$\|u_j\|^2_{L^2(\Omega)} \leqslant \widetilde{\kappa}\left(1 + \|u_j\|_{X_0}\right), \tag{2.9}$$

for a suitable $\widetilde{\kappa} > 0$ not depending on $j$. By (2.5), (2.8) and (2.9), we have that

$$\begin{aligned}\kappa \geqslant \mathcal{J}_{K,\lambda}(u_j) &= \frac{1}{2}\|u_j\|^2_{X_0} - \frac{\lambda}{2}\|u_j\|^2_{L^2(\Omega)} - \frac{1}{2^*}\|u_j\|^{2^*}_{L^{2^*}(\Omega)}\\ &\geqslant \frac{1}{2}\|u_j\|^2_{X_0} - \underline{\kappa}\left(1 + \|u_j\|_{X_0}\right),\end{aligned}$$

with $\underline{\kappa} > 0$ independent of $j$, and so (2.4) is proved.

Now, let us show that

$$\text{problem (1.10) admits a solution } u_\infty \in X_0. \tag{2.10}$$

Since the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $X_0$ by (2.4), and $X_0$ is a Hilbert space, then, up to a subsequence, still denoted by $\{u_j\}_{j\in\mathbb{N}}$, there exists $u_\infty \in X_0$ such that $u_j \to u_\infty$ weakly in $X_0$, that is

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^n} (u_j(x) - u_j(y))(\varphi(x) - \varphi(y))K(x-y)\,dx\,dy \to\\ &\qquad\int_{\mathbb{R}^n\times\mathbb{R}^n} (u_\infty(x) - u_\infty(y))(\varphi(x) - \varphi(y))K(x-y)\,dx\,dy\end{aligned} \tag{2.11}$$

for any $\varphi \in X_0$, as $j \to +\infty$. Moreover, by (2.4), (2.8), the embedding properties of $X_0$ into the classical Lebesgue spaces (see [27, Lemma 8] and [29, Lemma 9]) and the fact that $L^{2^*}(\mathbb{R}^n)$ is a reflexive space we have that, up to a subsequence

$$u_j \to u_\infty \quad \text{weakly in } L^{2^*}(\mathbb{R}^n) \tag{2.12}$$

$$u_j \to u_\infty \quad \text{in } L^2(\mathbb{R}^n) \tag{2.13}$$

and

$$u_j \to u_\infty \quad \text{a.e. in } \mathbb{R}^n \tag{2.14}$$

as $j \to +\infty$.

As a consequence of (2.4) and (2.8), we have that $\|u_j\|_{L^{2^*}(\Omega)}$ is bounded uniformly in $j$, hence the sequence $\{|u_j|^{2^*-2}u_j\}_{j\in\mathbb{N}}$ is bounded in $L^{2^*/(2^*-1)}(\Omega)$, uniformly in $j$. Thus, (2.12) yields

$$|u_j|^{2^*-2}u_j \to |u_\infty|^{2^*-2}u_\infty \quad \text{weakly in } L^{2^*/(2^*-1)}(\Omega) \tag{2.15}$$

as $j \to +\infty$, and so

$$\int_\Omega |u_j(x)|^{2^*-2}u_j(x)\varphi(x)dx \to \int_\Omega |u_\infty(x)|^{2^*-2}u_\infty(x)\varphi(x)dx \quad \forall\, \varphi \in L^{2^*}(\Omega) \tag{2.16}$$

as $j \to +\infty$. Hence, in particular, we have that

$$\int_\Omega |u_j(x)|^{2^*-2}u_j(x)\varphi(x)\,dx \to \int_\Omega |u_\infty(x)|^{2^*-2}u_\infty(x)\varphi(x)\,dx \quad \forall\, \varphi \in X_0 \tag{2.17}$$

as $j \to +\infty$, being $X_0 \subseteq L^{2^*}(\Omega)$.

By (2.3), for any $\varphi \in X_0$ as $j \to +\infty$

$$\begin{aligned} 0 \leftarrow \langle \mathcal{J}'_{K,\lambda}(u_j), \varphi\rangle = &\int_{\mathbb{R}^n\times\mathbb{R}^n} (u_j(x)-u_j(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy - \lambda\int_\Omega u_j(x)\varphi(x)\,dx \\ &- \int_\Omega |u_j(x)|^{2^*-2}u_j(x)\varphi(x)\,dx\,, \end{aligned}$$

so that, passing to the limit as $j \to +\infty$ and taking into account (2.11), (2.13) and (2.17) we get

$$\begin{aligned} &\int_{\mathbb{R}^n\times\mathbb{R}^n} (u_\infty(x)-u_\infty(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy - \lambda\int_\Omega u_\infty(x)\varphi(x)\,dx \\ &\qquad - \int_\Omega |u_\infty(x)|^{2^*-2}u_\infty(x)\varphi(x)\,dx = 0\,. \end{aligned}$$

Then, $u_\infty$ is a solution of problem (1.10) and this proves (2.10).

As it was proved in [24, Proposition 4.1], the function $u_\infty$ in (2.10) satisfies the following three relations[1], which be useful in carried on our proof:

$$\mathcal{J}_{K,\lambda}(u_\infty) = \frac{s}{n}\int_\Omega |u_\infty(x)|^{2^*}\,dx \geqslant 0\,, \tag{2.18}$$

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u_j) =&\mathcal{J}_{K,\lambda}(u_\infty) - \frac{1}{2^*}\int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx \\ &+\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy + o(1) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy \\ &\qquad\qquad = \int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx + o(1) \end{aligned} \tag{2.20}$$

as $j \to +\infty$.

Now, by (2.20) we get that

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy - \frac{1}{2^*}\int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx \\ &\quad = \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy + o(1) \\ &\quad = \frac{s}{n}\int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy + o(1) \end{aligned}$$

as $j \to +\infty$. This relation, combined with (2.19), gives

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u_j) = &\mathcal{J}_{K,\lambda}(u_\infty) \\ &+\frac{s}{n}\int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy + o(1) \end{aligned} \tag{2.21}$$

[1]We will recall the proof of these relations in the Appendix A for reader's convenience.

as $j\to+\infty$. Taking into account (2.2) and (2.21) we conclude that

$$\begin{aligned}c &= \mathcal{J}_{K,\lambda}(u_\infty)\\ &\quad+\frac{s}{n}\int_{\mathbb{R}^n\times\mathbb{R}^n}|u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy+o(1)\end{aligned}\tag{2.22}$$

as $j\to+\infty$.

Since the sequence $\{\|u_j\|_{X_0}\}_{j\in\mathbb{N}}$ is bounded in $\mathbb{R}$ by (2.4), then we can assume that, up to a subsequence, if necessary,

$$\|u_j-u_\infty\|^2_{X_0}=\int_{\mathbb{R}^n\times\mathbb{R}^n}|u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy\to L\tag{2.23}$$

where $L\in[0,+\infty)$, and so, by (2.20), we have

$$\int_\Omega|u_j(x)-u_\infty(x)|^{2^*}\,dx\to L\tag{2.24}$$

as $j\to+\infty$. By (2.23), (2.24) and the definition of $S_K$, we get

$$L^{2/2^*}S_K\leqslant L\,.$$

Then,

$$\text{either}\quad L=0\quad\text{or}\quad L\geqslant S_K^{n/(2s)}\,.$$

By (2.18), (2.22) and (2.23), we obtain

$$c=\mathcal{J}_{K,\lambda}(u_\infty)+\frac{s}{n}\,L\geqslant\frac{s}{n}\,L\,.\tag{2.25}$$

If $L\geqslant S_K^{n/(2s)}$, then, by (2.25), we would get

$$c\geqslant\frac{s}{n}\,L\geqslant\frac{s}{n}S_K^{n/(2s)}$$

which contradicts (2.1). Thus, $L=0$. As a consequence of this and of (2.23), we have that

$$\|u_j-u_\infty\|_{X_0}\to0$$

as $j\to+\infty$. This shows that the sequence $\{u_j\}_{j\in\mathbb{N}}$ has a (strongly) convergent subsequence. Hence, $\mathcal{J}_{K,\lambda}$ satisfies the Palais-Smale condition at any level $c$, provided (2.1) is satisfied. This ends the proof of Step 1. □

Now, let $\lambda^\star$ be as in (1.16). Then,

$$\lambda^\star=\lambda_k\ \text{ for some } k\in\mathbb{N}\,.$$

Since $\lambda^\star$ has multiplicity $m\in\mathbb{N}$ by assumption, we have that

$$\begin{aligned}&\lambda^\star=\lambda_1<\lambda_2 &&\text{if } k=1\\ &\lambda_{k-1}<\lambda^\star=\lambda_k=\dots=\lambda_{k+m-1}<\lambda_{k+m} &&\text{if } k\geqslant2\,.\end{aligned}\tag{2.26}$$

Also, before going on with the proof of Theorem 1, we would note that, under condition (1.17), the parameter $\lambda$ is such that

$$\lambda>0\,.\tag{2.27}$$

Indeed, by definition of $\lambda^\star$ and taking into account (1.13), it is easily seen that

$$\lambda^\star\geqslant\lambda_1\,.\tag{2.28}$$

In addition, the variational characterization of the first eigenvalue $\lambda_1$ (see [28, Proposition 9 and Appendix A]) gives that

$$\lambda_1=\min_{u\in X_0\setminus\{0\}}\frac{\displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n}|u(x)-u(y)|^2K(x-y)dx\,dy}{\displaystyle\int_\Omega|u(x)|^2\,dx}\,.\tag{2.29}$$

Since, by Hölder inequality, it holds true that

$$\int_\Omega |u(x)|^2\,dx \leqslant |\Omega|^{2s/n}\left(\int_\Omega |u(x)|^{2^*}\,dx\right)^{2/2^*},$$

by this and (2.29), we get

$$\lambda_1 \geqslant S_K|\Omega|^{-2s/n},$$

which combined with (2.28) yields

$$\lambda^\star \geqslant S_K|\Omega|^{-2s/n}.$$

Hence, as a consequence of this and of (1.17), we get (2.27).

**Step 2 (Geometric structure of the functional $\mathcal{J}_{K,\lambda}$).** With the notations of the abstract result stated in Theorem 2, we set

$$W = \mathrm{span}\,\{e_1,\dots,e_{k+m-1}\}$$

and

$$V = \begin{cases} X_0 & \text{if } k=1\\ \Big\{u\in X_0:\ \langle u,e_j\rangle_{X_0}=0\ \ \forall j=1,\dots,k-1\Big\} & \text{if } k\geqslant 2\,. \end{cases}$$

Note that both $W$ and $V$ are closed subset of $X_0$ and

$$\dim W = k+m-1 \qquad\qquad \operatorname{codim} V = k-1\,, \tag{2.30}$$

so that $(I_3)-iii)$ of Theorem 2 is satisfied. In what follows, we prove that the functional $\mathcal{J}_{K,\lambda}$ has the geometric features required by Theorem 2.

*Proof.* Let us show that the functional $\mathcal{J}_{K,\lambda}$ verifies assumption $(I_3)-i)$ and $ii)$ of Theorem 2 (here condition (1.17) will be crucial).

For this, let $u\in W$. Then,

$$u(x) = \sum_{i=1}^{k+m-1} u_i e_i(x)$$

with $u_i\in\mathbb{R}$, $i=1,\dots,k+m-1$.

Since $\{e_1,\dots,e_k,\dots\}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal one of $X_0$ (see [28, Proposition 9]), taking into account (2.26) we get

$$\|u\|^2_{X_0} = \sum_{i=1}^{k+m-1} u_i^2\|e_i\|^2_{X_0} = \sum_{i=1}^{k+m-1}\lambda_i u_i^2 \leqslant \lambda_k\sum_{i=1}^{k+m-1}u_i^2 = \lambda_k\|u\|^2_{L^2(\Omega)} = \lambda^\star\|u\|^2_{L^2(\Omega)}\,,$$

so that, by this and Hölder inequality, we have

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u) &= \frac12\int_{\mathbb{R}^n\times\mathbb{R}^n}|u(x)-u(y)|^2K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega|u(x)|^2dx\\ &\qquad\qquad - \frac{1}{2^*}\int_\Omega|u(x)|^{2^*}dx\\ &\leqslant \frac12(\lambda^\star-\lambda)\int_\Omega|u(x)|^2dx - \frac{1}{2^*}\int_\Omega|u(x)|^{2^*}dx\\ &\leqslant \frac12(\lambda^\star-\lambda)|\Omega|^{\frac{2s}{n}}\left(\int_\Omega|u(x)|^{2^*}dx\right)^{\frac{2}{2^*}} - \frac{1}{2^*}\int_\Omega|u(x)|^{2^*}dx.\end{aligned} \tag{2.31}$$

Now, for $t\geqslant 0$ let

$$g(t) = \frac12(\lambda^\star-\lambda)|\Omega|^{\frac{2s}{n}}t^2 - \frac{1}{2^*}t^{2^*}.$$

Note that the function $g$ is differentiable in $(0,+\infty)$ and

$$g'(t) = (\lambda^\star-\lambda)|\Omega|^{\frac{2s}{n}}t - t^{2^*-1}$$

so that $g'(t) \geqslant 0$ if and only if

$$t \leqslant \bar{t} = \left[(\lambda^\star - \lambda)|\Omega|^{\frac{2s}{n}}\right]^{1/(2^*-2)}.$$

As a consequence of this, $\bar{t}$ is a maximum point for $g$ and so for any $t \geqslant 0$

$$g(t) \leqslant \max_{t\geqslant 0} g(t) = g(\bar{t}) = \frac{s}{n}(\lambda^\star - \lambda)^{\frac{n}{2s}}|\Omega|\,. \tag{2.32}$$

By (2.31) and (2.32) we get

$$\sup_{u\in W} \mathcal{J}_{K,\lambda}(u) \leqslant \max_{t\geqslant 0} g(t) = \frac{s}{n}(\lambda^\star - \lambda)^{\frac{n}{2s}}|\Omega|\,. \tag{2.33}$$

We observe that

$$\frac{s}{n}(\lambda^\star - \lambda)^{\frac{n}{2s}}|\Omega| > 0$$

since $\lambda < \lambda^\star$ by (1.17).

Finally, let $u \in V$. Then,

$$\|u\|^2_{X_0} \geqslant \lambda^\star \|u\|^2_{L^2(\Omega)} \tag{2.34}$$

Indeed, if $u \equiv 0$, then the assertion is trivial, while if $u \in V \setminus \{0\}$ it follows from the variational characterization of $\lambda^\star = \lambda_k$ given by

$$\lambda_k = \min_{u\in V\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)\,dx\,dy}{\displaystyle\int_\Omega |u(x)|^2\,dx}\,,$$

as proved in [28, Proposition 9].

Thus, by (1.14), (2.34) and taking into account that $\lambda > 0$ (see (2.27)), it follows that

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u) &\geqslant \frac{1}{2}\left(1-\frac{\lambda}{\lambda^\star}\right)\|u\|^2_{X_0} - \frac{1}{2^* S_K^{1/2}}\|u\|^{2^*}_{X_0}\\ &= \|u\|^2_{X_0}\left(\frac{1}{2}\left(1-\frac{\lambda}{\lambda^\star}\right) - \frac{1}{2^* S_K^{1/2}}\|u\|^{2^*-2}_{X_0}\right).\end{aligned} \tag{2.35}$$

Now, let $u \in V$ be such that $\|u\|_{X_0} = \rho > 0$. Since $2^* > 2$, we can choose $\rho$ sufficiently small, say $\rho \leqslant \bar{\rho}$ with $\bar{\rho} > 0$, so that

$$\frac{1}{2}\left(1-\frac{\lambda}{\lambda^\star}\right) - \frac{1}{2^* S_K^{1/2}}\rho^{2^*-2} > 0 \tag{2.36}$$

and

$$\rho^2\left(\frac{1}{2}\left(1-\frac{\lambda}{\lambda^\star}\right) - \frac{1}{2^* S_K^{1/2}}\rho^{2^*-2}\right) < \frac{\rho^2}{2}\left(1-\frac{\lambda}{\lambda^\star}\right) < \frac{s}{n}(\lambda^\star-\lambda)^{\frac{n}{2s}}|\Omega|\,. \tag{2.37}$$

□

Now, we can conclude the proof of Theorem 1. By Step 1 it easily follows that $\mathcal{J}_{K,\lambda}$ satisfies $(I_2)$ with

$$\beta = \frac{s}{n} S_K^{n/(2s)} > 0\,.$$

Also, by Step 2 (see (2.33)–(2.37)) we get that $\mathcal{J}_{K,\lambda}$ verifies $(I_3)$ with

$$\rho = \bar{\rho}\,,$$

$$\beta' = \frac{s}{n}(\lambda^\star-\lambda)^{\frac{n}{2s}}|\Omega|$$

and

$$\delta = \bar{\rho}^2\left(\frac{1}{2}\left(1-\frac{\lambda}{\lambda^\star}\right) - \frac{1}{2^* S_K^{1/2}}\bar{\rho}^{2^*-2}\right).$$

Note that

$$0<\delta<\beta'<\beta$$

thanks to (2.36), (2.37) and assumption (1.17).

All in all, the functional $\mathcal{J}_{K,\lambda}$ satisfies both the compactness assumption and the geometric features required by the abstract critical points theorem stated in Theorem 2. As a consequence, $\mathcal{J}_{K,\lambda}$ has $m$ pairs $\{-u_i(\lambda),\, u_i(\lambda)\}$ of critical points whose critical value $\mathcal{J}_{K,\lambda}(\pm u_i(\lambda))$ is such that

$$0<\bar\rho^2\left(\frac{1}{2}\left(1-\frac{\lambda}{\lambda^\star}\right)-\frac{1}{2^*S_K^{1/2}}\bar\rho_{X_0}^{2^*-2}\right)\leqslant \mathcal{J}_{K,\lambda}(\pm u_i(\lambda))\leqslant \frac{s}{n}(\lambda^\star-\lambda)^{\frac{n}{2s}}|\Omega| \tag{2.38}$$

for any $i=1,\dots,m$.

Since $\mathcal{J}_{K,\lambda}(0)=0$ and (2.38) holds true, it is easy to see that these critical points are all different from the trivial function. Hence, problem (1.3) admits $m$ pairs of nontrivial weak solutions.

Now, fix $i\in\{1,\dots,m\}$. By (2.38) we obtain

$$\begin{aligned}(\lambda^\star-\lambda)^{n/2s}|\Omega|\frac{s}{n}&\geqslant \mathcal{J}_{K,\lambda}(u_i(\lambda))\\ &=\mathcal{J}_{K,\lambda}(u_i(\lambda))-\frac{1}{2}\left\langle \mathcal{J}'_\lambda(u_i(\lambda)),u_i(\lambda)\right\rangle\\ &=\left(\frac{1}{2}-\frac{1}{2^*}\right)\|u_i(\lambda)\|^{2^*}_{L^{2^*}(\Omega)}\\ &=\frac{s}{n}\,\|u_i(\lambda)\|^{2^*}_{L^{2^*}(\Omega)}\,,\end{aligned} \tag{2.39}$$

so that, passing to the limit as $\lambda\to\lambda^\star$ in (2.39), it follows that

$$\|u_i(\lambda)\|^{2^*}_{L^{2^*}(\Omega)}\to 0\quad\text{as } \lambda\to\lambda^\star\,. \tag{2.40}$$

Then, by (2.40), since $L^{2^*}(\Omega)\hookrightarrow L^2(\Omega)$ continuously (being $\Omega$ bounded), we also get

$$\|u_i(\lambda)\|^{2}_{L^{2}(\Omega)}\to 0\quad\text{as } \lambda\to\lambda^\star\,. \tag{2.41}$$

So, arguing as above, we have

$$(\lambda^\star-\lambda)^{n/2s}|\Omega|\frac{s}{n}\geqslant \mathcal{J}_{K,\lambda}(u_i(\lambda))=\frac{1}{2}\|u_i(\lambda)\|^2_{X_0}-\frac{\lambda}{2}\|u_i(\lambda)\|^2_{L^2(\Omega)}-\frac{1}{2^*}\|u_i(\lambda)\|^{2^*}_{L^{2^*}(\Omega)},$$

which combined with (2.40) and (2.41) gives

$$\|u_i(\lambda)\|_{X_0}\to 0\quad\text{as } \lambda\to\lambda^\star\,.$$

This concludes the proof of Theorem 1.

## Appendix A

Here we give the details of the proof of relations (2.18)–(2.20), which were proved in [24, Proposition 4.1], in order to make the present paper self-contained and for reader's convenience.

*Proof of* (2.18). By (2.10), taking $\varphi=u_\infty\in X_0$ as a test function in (1.10), we get

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}|u_\infty(x)-u_\infty(y)|^2K(x-y)dx\,dy-\lambda\int_\Omega |u_\infty(x)|^2\,dx=\int_\Omega |u_\infty(x)|^{2^*}dx\,,$$

so that

$$\mathcal{J}_{K,\lambda}(u_\infty)=\left(\frac{1}{2}-\frac{1}{2^*}\right)\int_\Omega |u_\infty(x)|^{2^*}dx=\frac{s}{n}\int_\Omega |u_\infty(x)|^{2^*}dx\geqslant 0\,.$$

This concludes the proof of (2.18). □

*Proof of* (2.19). At this purpose, by (2.4) and taking into account the embedding properties of $X_0$ into the Lebesgue spaces (see [27, Lemma 8] and [29, Lemma 9]), the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $X_0$ and in $L^{2^*}(\Omega)$. Thus, since (2.14) holds true, by the Brezis-Lieb Lemma (see [8, Theorem 1]), we get

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} &|u_j(x)-u_j(y)|^2K(x-y)\,dx\,dy\\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy\\ &\quad+\int_{\mathbb{R}^n\times\mathbb{R}^n}|u_\infty(x)-u_\infty(y)|^2K(x-y)\,dx\,dy+o(1)\end{aligned}\tag{A.1}$$

and

$$\int_\Omega |u_j(x)|^{2^*}\,dx=\int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx+\int_\Omega|u_\infty(x)|^{2^*}\,dx+o(1)\tag{A.2}$$

as $j\to+\infty$.

Now, by (2.13), (A.1) and (A.2) we deduce that

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u_j)&=\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}|u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy\\ &\quad+\frac12\int_{\mathbb{R}^n\times\mathbb{R}^n}|u_\infty(x)-u_\infty(y)|^2K(x-y)\,dx\,dy-\frac{\lambda}{2}\int_\Omega|u_\infty(x)|^2dx\\ &\quad-\frac{1}{2^*}\int_\Omega|u_j(x)-u_\infty(x)|^{2^*}\,dx-\frac{1}{2^*}\int_\Omega|u_\infty(x)|^{2^*}\,dx+o(1)\\ &=\mathcal{J}_{K,\lambda}(u_\infty)+\frac12\int_{\mathbb{R}^n\times\mathbb{R}^n}|u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy\\ &\quad-\frac{1}{2^*}\int_\Omega|u_j(x)-u_\infty(x)|^{2^*}\,dx+o(1)\end{aligned}$$

as $j\to+\infty$, which gives (2.19). □

*Proof of* (2.20). For this, note that, as a consequence of (2.12), (2.15) and (A.2), we get

$$\begin{aligned}\int_\Omega&\Big(|u_j(x)|^{2^*-2}u_j(x)-|u_\infty(x)|^{2^*-2}u_\infty(x)\Big)\Big(u_j(x)-u_\infty(x)\Big)\,dx\\ &=\int_\Omega|u_j(x)|^{2^*}\,dx-\int_\Omega|u_\infty(x)|^{2^*-2}u_\infty(x)u_j(x)\,dx\\ &\quad-\int_\Omega|u_j(x)|^{2^*-2}u_j(x)u_\infty(x)\,dx+\int_\Omega|u_\infty(x)|^{2^*}\,dx\\ &=\int_\Omega|u_j(x)|^{2^*}\,dx-\int_\Omega|u_\infty(x)|^{2^*}\,dx+o(1)\\ &=\int_\Omega|u_j(x)-u_\infty(x)|^{2^*}\,dx+o(1)\end{aligned}\tag{A.3}$$

as $j\to+\infty$.

By (2.3), (2.4) and (2.10), we have that

$$o(1)=\langle\mathcal{J}'_{K,\lambda}(u_j),u_j-u_\infty\rangle=\langle\mathcal{J}'_{K,\lambda}(u_j)-\mathcal{J}'_{K,\lambda}(u_\infty),u_j-u_\infty\rangle\tag{A.4}$$

as $j \to +\infty$. On the other hand, by (2.13) and (A.3), it holds true that

$$
\begin{aligned}
\langle \mathcal{J}'_{K,\lambda}(u_j) - \mathcal{J}'_{K,\lambda}(u_\infty), u_j - u_\infty \rangle \\
= \int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy \\
- \lambda \int_\Omega |u_j(x) - u_\infty(x)|^2\,dx \\
- \int_\Omega \left(|u_j(x)|^{2^*-2}u_j(x) - |u_\infty(x)|^{2^*-2}u_\infty(x)\right)(u_j(x) - u_\infty(x))\,dx \\
= \int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy \\
- \int_\Omega |u_j(x) - u_\infty(x)|^{2^*}\,dx + o(1)
\end{aligned}
\tag{A.5}
$$

as $j \to +\infty$. Then, combining (A.4) and (A.5), we get (2.20). □

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Dipartimento di Matematica, Università di Milano, Via Cesare Saldini 50, 20133 Milano, Italy

*E-mail address*: `alessio.fiscella@unimi.it`

Dipartimento PAU, Università 'Mediterranea' di Reggio Calabria, Via Melissari 24, 89124 Reggio Calabria, Italy

*E-mail address*: `gmolica@unirc.it`

Dipartimento di Matematica e Informatica, Università della Calabria, Ponte Pietro Bucci 31 B, 87036 Arcavacata di Rende (Cosenza), Italy

*E-mail address*: `servadei@mat.unical.it`