Invariant curves for exact symplectic twist maps of the cylinder with Bryuno rotation numbers

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Abstract

Since Moser's seminal work it is well known that the invariant curves of smooth nearly integrable twist maps of the cylinder with Diophantine rotation number are preserved under perturbation. In this paper we show that, in the analytic class, the result extends to Bryuno rotation numbers. First, we will show that the series expansion for the invariant curves in powers of the perturbation parameter can be formally defined, then we shall prove that the series converges absolutely in a neighbourhood of the origin. This will be achieved using multiscale analysis and renormalisation group techniques to express the coefficients of the series as sums of values which are represented graphically as tree diagrams and then exploit cancellations between terms contributing to the same perturbation order. As a byproduct we shall see that, when perturbing linear maps, the series expansion for an analytic invariant curve converges for all perturbations if and only if the corresponding rotation number satisfies the Bryuno condition.

1 Introduction

Consider an exact symplectic twist map Φ on the cylinder $\mathbb{T}\times\mathbb{R}$

$$\begin{cases} x' = x + a(y) + \varepsilon\, f(x,y), \\ y' = y + \varepsilon\, g(x,y), \end{cases} \tag{1.1}$$

where $\varepsilon\in\mathbb{R}$ and the functions a,f,g are analytic, with $a'(y)>0$ (twist condition), and 2π-periodic in x. To fix notations we shall assume that the functions can be analytically extended to a complex domain $\mathcal{D}:=\mathbb{T}_\xi\times\mathcal{A}$, where $\mathbb{T}_\xi=\{x\in\mathbb{C}/2\pi\mathbb{Z} : |\mathrm{Im}\,x|\le\xi\}$ and $\mathcal{A}\subset\mathbb{C}$ is a complex neighbourhood of an interval of the real axis. We are interested in the case where the supremum norms of f and g satisfy $\max\{|f|,|g|\}=1$ in $\mathcal{D}$ and $|\varepsilon|\le\varepsilon_0$, with ε_0 small enough. For that reason, ε is called the perturbation parameter.

For $\varepsilon=0$ the map is integrable: y is kept constant, $y=y_0$, and the dynamics is a rotation of the variable x by an angle $2\pi\omega=a(y_0)$. Below, following Moser's example [20], we will consider initially the system

$$\begin{cases} x' = x + y + \varepsilon\, f(x,y), \\ y' = y + \varepsilon\, g(x,y) \end{cases} \tag{1.2}$$

and eventually discuss how the analysis has to be changed to deal with the general case (1.1).

Since we are assuming that Φ is exact symplectic, there exists a generating function $S(x,x') = S_0(x,x') + \varepsilon\sigma(x,x')$ such that (1.1) can be written in the form

$$y = -\frac{\partial S(x,x')}{\partial x}, \qquad y' = \frac{\partial S(x,x')}{\partial x'}. \tag{1.3}$$

One has

$$S_0(x,x') = \frac{1}{2}(x'-x)^2$$

for the system (1.2). In that case, if we take $\sigma(x,x') = 1-\cos x$, so that $f(x,y) = g(x,y) = \sin x$ in (1.2), we recover the *standard map*, a map widely studied in the literature since the original papers by Chirikov and Greene [4, 17, 19].

Requiring that the map Φ is exact symplectic is equivalent to require that it is area-preserving and has zero flux: in particular the latter condition is automatically satisfied if Φ is area preserving and preserves the boundaries of an annulus [16]. In the following it will be more convenient to study the system (1.1) in the form (1.3) to better exploit the symplectic nature of the map.

We say that the map (1.1) admits an invariant curve if there exist two functions h, H of period 2π such that $x = \psi + h(\psi)$, $y = y_0 + H(\psi)$ and the dynamics induced on the curve is given by $\psi' = \psi + 2\pi\omega$, with $2\pi\omega = a(y_0)$. The number ω will be called the *rotation number* of the curve. If the functions h and H are analytic, we say that the invariant curve is analytic. The existence of invariant curves for exact symplectic twist maps on the cylinder was first proved by Moser in the differentiable class by assuming a standard Diophantine condition on ω [20]. Moreover the result extends to more general maps, satisfying the intersection property [25]: the latter means that every smooth curve wrapping around the cylinder intersects itself under the action of the map. However, in the analytic class the Diophantine condition is not expected to be optimal. In the present paper we show that one can be take a weaker Diophantine condition on the rotation number ω: the *Bryuno condition* [2], defined as $\mathfrak{B}(\omega) < \infty$, where

$$\mathfrak{B}(\omega) := \sum_{k=0}^{\infty} \frac{1}{q_k} \log q_{k+1} \tag{1.4}$$

and $\{q_k\}$ are the denominators of the continued fraction expansion of ω [24]. In fact, we prove the following result.

Theorem 1.1. *Consider an exact symplectic map on the cylinder of the form (1.1), with f and g analytic. If ω satisfies the Bryuno condition then there exists $\varepsilon_0 > 0$ such that the map admits an analytic invariant curve with rotation number ω analytic in ε for $|\varepsilon| < \varepsilon_0$.*

This will be proved in Section 2 to 5 for maps of the form (1.2) and then extended to any maps of the form (1.1) in Section 6. As a byproduct of the proof we show directly that the perturbation series for the invariant curve converges absolutely. This requires some work, since the standard Lindstedt algorithm produces the coefficients as sums of many terms which can grow proportionally to factorials, so one has to show that there are compensations between such terms. We shall explicitly exhibit the cancellations which make the sum of all terms contributing to the same perturbation order k bounded by some constant to the power k. The cancellation mechanisms is similar to that found in the case of Hamiltonian flows [8, 9, 15], even though the implementation is sligthly different: unlike the case of flows we shall be able to obtain a closed equation involving only the conjugation function of the angle variable, on the other hand this will make a bit more delicate the cancellation analysis. Moreover, in the general case (1.1), the cancellations will be showed to involve remarkable identities that can be of interest in their own; see in particular Appendix B.

Recently the Bryuno condition received a lot of attention in the literature on small divisor problems arising in the context of analytic dynamical systems, including circle diffeomorphisms [27], holomorphic maps on the plane [26, 3], maximal and lower-dimensional tori in quasi-integrable systems [13, 18], skew-product systems [12, 6]. The Bryuno condition is known to be optimal in the case of the Siegel problem [27] and the circle diffeomorphisms [26]. Here we prove that, in the case of exact symplectic twist maps on the cylinder of the form (1.2), i.e. perturbations of the linear map, the Bryuno condition is optimal in order to have an analytic invariant curve depending analytically on ε. More precisely we obtain the following result.

Theorem 1.2. *Given the integrable map Φ_0 on $\mathbb{T}\times\mathbb{R}$*

$$\begin{cases} x' = x + y, \\ y' = y, \end{cases}$$

let Φ be an analytic exact symplectic map of the form $\Phi = \Phi_0 + \varepsilon\Phi_1$, with $\Phi_1 = (f, g)$ analytic in a domain $\mathcal{D}$ and such that $|\Phi_1| := \max\{|f|, |g|\} = 1$ in $\mathcal{D}$. Given $\omega \in \mathbb{R}$, there exists $\varepsilon_0 > 0$ such that any Φ admits an invariant curve with rotation number ω analytic in ε for $|\varepsilon| < \varepsilon_0$ if and only if ω satisfies the Bryuno condition.

The proof, to be given in Section 5, which we refer to for more details, follows from Theorem 1.1 and some results available in the literature [7, 1].

2 Formal power series and tree expansion

We start by considering exact symplectic maps of the form (1.2) and defer to Section 6 how to extend the analysis to (1.1). So, we prove first the following result, which is a spacial case of Theorem 1.1.

Theorem 2.1. *Consider an exact symplectic map on the cylinder of the form (1.2), with f and g analytic. If ω satisfies the Bryuno condition then there exists $\varepsilon_0 > 0$ such that the map admits an analytic invariant curve with rotation number ω analytic in ε for $|\varepsilon| < \varepsilon_0$.*

As said in Section 1 we rewrite (1.2) in the form

$$\begin{cases} y' = x' - x + \varepsilon \dfrac{\partial \sigma(x,x')}{\partial x'}, \\ y = x' - x - \varepsilon \dfrac{\partial \sigma(x,x')}{\partial x}, \end{cases} \tag{2.1}$$

where $\sigma(x,x')$ is an analytic 2π-periodic function of both its arguments. By the assumption of analyticity one has

$$\sigma(x,x') = \sum_{\nu,\mu\in\mathbb{Z}} \mathrm{e}^{i\nu x + i\mu x'} \sigma_{\nu,\mu}, \qquad |\sigma_{\nu,\mu}| \le \Xi\, \mathrm{e}^{-\xi(|\nu|+|\mu|)} \quad \forall \nu,\mu\in\mathbb{Z}, \tag{2.2}$$

for suitable positive constants Ξ and with ξ as after (1.1).

We look for two analytic functions h and H (conjugation) such that

$$x = \psi + h(\psi), \qquad y = 2\pi\omega + H(\psi) \tag{2.3}$$

and the dynamics in terms of ψ becomes $\psi' = \psi + 2\pi\omega$. By introducing (2.3) into (2.1) and imposing $\psi' = \psi + 2\pi\omega$ one finds the functional equation for h

$$\begin{aligned} &h(\psi+2\pi\omega) + h(\psi-2\pi\omega) - 2h(\psi) \\ &\quad = \varepsilon\, \partial_1\sigma(\psi+h(\psi), \psi+2\pi\omega+h(\psi+2\pi\omega)) \\ &\qquad + \varepsilon\, \partial_2\sigma(\psi-2\pi\omega+h(\psi-2\pi\omega), \psi+h(\psi)), \end{aligned} \tag{2.4}$$

where ∂_j denotes derivative with respect to the j-th argument. If the equation (2.4) can be solved then the function H can be obtained from

$$H(\psi) = h(\psi+2\pi\omega) - h(\psi) - \varepsilon\, \partial_1\sigma(\psi+h(\psi), \psi+2\pi\omega+h(\psi+2\pi\omega)). \tag{2.5}$$

Moreover H inherits the same regularity properties of h, so that we can confine ourselves to the functional equation (2.4).

If there exists a solution to (2.4) we say that the system admits an *invariant curve* with rotation number ω; if the functions are analytic in ψ and ε we say that the invariant curve is analytic in ψ and ε. In the following we will be interested in invariant curves of this kind.

By taking the formal Fourier-Taylor expansion of h,

$$h(\psi) = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\nu\in\mathbb{Z}} \mathrm{e}^{i\nu\psi} h_\nu^{(k)}, \tag{2.6}$$

one can rewrite (2.4) as infinitely many equations

$$\begin{aligned} \delta(\omega\nu)\, h_\nu^{(k)} = &\sum_{p,q\ge 0} \sum_{\substack{\nu_0+\mu_0+\nu_1+\ldots+\nu_p \\ +\mu_1+\ldots+\mu_q=\nu}} \sum_{\substack{k_1+\ldots+k_p \\ +k'_1+\ldots+k'_q=k-1}} \frac{(\mathrm{i}\nu_0)^{p+1}}{p!} \frac{(\mathrm{i}\mu_0)^q}{q!} \sigma_{\nu_0,\mu_0} \\ &\qquad \mathrm{e}^{2\pi\mathrm{i}(\mu_0+\mu_1+\ldots+\mu_q)\omega} h_{\nu_1}^{(k_1)} \ldots h_{\nu_p}^{(k_p)} h_{\mu_1}^{(k'_1)} \ldots h_{\mu_q}^{(k'_q)} \\ + &\sum_{p,q\ge 0} \sum_{\substack{\nu_0+\mu_0+\nu_1+\ldots+\nu_p \\ +\mu_1+\ldots+\mu_q=\nu}} \sum_{\substack{k_1+\ldots+k_p \\ +k'_1+\ldots+k'_q=k-1}} \frac{(\mathrm{i}\nu_0)^{p}}{p!} \frac{(\mathrm{i}\mu_0)^{q+1}}{q!} \sigma_{\nu_0,\mu_0} \\ &\qquad \mathrm{e}^{-2\pi\mathrm{i}(\nu_0+\nu_1+\ldots+\nu_p)\omega} h_{\nu_1}^{(k_1)} \ldots h_{\nu_p}^{(k_p)} h_{\mu_1}^{(k'_1)} \ldots h_{\mu_q}^{(k'_q)} \end{aligned} \tag{2.7}$$

where $\delta(u) := 2(\cos 2\pi u - 1)$. By analogy with the case of flows, we shall call *Lindtsedt series* the series (2.6).

Definition 2.2. *We say that the formal series (2.6) is a* formal solution *to equation (2.4) if there exist well-defined coefficients* $h_\nu^{(k)}$ *that solve (2.7) to any order* $k \in \mathbb{N}$.

As (2.7) shows, to compute the coefficients $h_\nu^{(k)}$ one has to deal with infinitely many sums (over the Fourier labels). Therefore (2.7) can be formally solved only if the sums can be performed and the right hand side vanishes whenever $\delta(\omega\nu) = 0$. In this section we want to show that this occurs if we impose a mild Diophantine condition on ω.

We require for ω to satisfy the Bryuno condition $\mathfrak{B}(\omega) < \infty$, with $\mathfrak{B}(\omega)$ defined in (1.4). In particular this implies that one has $\delta(\omega\nu) = 0$ if and only if $\nu = 0$. The following result holds.

Proposition 2.3. *Assume that* ω *satisfies the Bryuno condition. Then there exists a formal solution to (2.4) of the form (2.6). The coefficients* $h_\nu^{(k)}$ *of the formal solution are uniquely defined by requiring for the formal solution to have zero average, i.e.* $h_0^{(k)} = 0$ *for all* $k \geq 1$. *Moreover for any* $\xi_1, \xi_2 \geq 0$, *with* $\xi_1 + \xi_2 < \xi$, *and all* $k \in \mathbb{N}$ *there exists a positive constant* $C(k, \xi, \xi_1, \xi_2)$ *such that one has* $|h_\nu^{(k)}| \leq \xi_2^{-2k} C(k, \xi, \xi_1, \xi_2)\, \mathrm{e}^{-\xi_1|\nu|}$ *for all* $\nu \in \mathbb{Z}$.

We note since now that, in order to prove that a formal solution of the form (2.6) exists, a condition even weaker than the Bryuno condition can be taken; see Remark 2.9 below. Of course Proposition 2.3 is not enough to prove the existence of the conjugation, because one still has to control that the constants $C(k, \xi, \xi_1, \xi_2)$ do not grow too fast in k. To check this a more detailed analysis is needed, as will be performed in Section 3.

To prove Proposition 2.3 we shall use a tree expansion for the conjugation; see also [11, 10, 14] for an introduction to the subject. A graph is a set of points and lines connecting them. A tree θ is a graph with no cycle, such that all the lines are oriented toward a single point (*root*) which has only one incident line ℓ_0 (*root line*). All the points in a tree except the root are called *nodes* or *vertices*. The orientation of the lines in a tree induces a partial ordering relation ($\preceq$) between the nodes and the lines: we can imagine that each line carries an arrow pointing toward the root. Given two nodes v and w, we shall write $w \prec v$ every time v is along the oriented path of lines which connects w to the root. We denote by $V(\theta)$ and $L(\theta)$ the sets of nodes and lines in θ, respectively. Since a line $\ell \in L(\theta)$ is uniquely identified by the node v which it leaves, we may write $\ell = \ell_v$ and say that ℓ exits v. We write $\ell_w \prec \ell_v$ if $w \prec v$, and $w \prec \ell$ if $\ell = \ell_v$, with $w \preceq v$. For any line $\ell \in L(\theta)$ the set of nodes $v' \prec \ell$ and the set of lines $\ell' \preceq \ell$ form a tree θ': we shall say that θ' is a subtree of θ with root line ℓ; if ℓ enters the node v we say that θ' enters v.

If ℓ and ℓ' are two comparable lines with $\ell' \prec \ell$, we denote by $P(\ell, \ell')$ the oriented path of lines and nodes connecting ℓ' to ℓ, with ℓ and ℓ' not included (in particular $P(\ell, \ell') = \emptyset$ if ℓ' enters the node ℓ exits). If $w \prec v$ we denote by $\mathcal{P}(v, w)$ the oriented path connecting w to v, with w and v being included.

With each node $v \in V(\theta)$ we associate a *mode* label $(\nu_v, \mu_v) \in \mathbb{Z}^2$ and with each line $\ell \in L(\theta)$ we associate a *momentum* label ν_ℓ, with the constraint

$$\nu_{\ell_v} = \sum_{w \preceq v} (\nu_v + \mu_v)\,. \tag{2.8}$$

With each line $\ell \prec \ell_0$ we associate also a further label $\beta_\ell = \pm$, and we call $L_v^\beta(\theta)$ the set of lines ℓ entering the node $v \in V(\theta)$ such that $\beta_\ell = \beta$; set $p_v = |L_v^+(\theta)|$ and $q_v = |L_v^-(\theta)|$.

We say that two trees are equivalent if they can be transformed into each other by continuously deforming the lines in such a way that these do not cross each other and all the labels match. This provides an equivalence relation on the set of the trees. From now on we shall call trees tout court such equivalence classes.

We define the *propagator* associated with the line ℓ as

$$G_\ell := \begin{cases} \dfrac{1}{\delta(\omega\nu_\ell)} = \dfrac{1}{2(\cos 2\pi\omega\nu_\ell - 1)}, & \nu_\ell \neq 0, \\ 1 & \nu_\ell = 0, \end{cases} \tag{2.9}$$

and the *node factor* associated with the node v as

$$\begin{aligned} A_v \; &:= \frac{(\mathrm{i}\nu_v)^{p_v}}{p_v!}\frac{(\mathrm{i}\mu_v)^{q_v}}{q_v!}\sigma_{\nu_v,\mu_v}\Big((\mathrm{i}\nu_v)\,\mathrm{e}^{2\pi\mathrm{i}\mu_v\omega}\prod_{\ell\in L_v^-(\theta)}\mathrm{e}^{2\pi\mathrm{i}\nu_\ell\omega} + (\mathrm{i}\mu_v)\,\mathrm{e}^{-2\pi\mathrm{i}\nu_v\omega}\prod_{\ell\in L_v^+(\theta)}\mathrm{e}^{-2\pi\mathrm{i}\nu_\ell\omega}\Big) \\ &= \frac{(\mathrm{i}\nu_v)^{p_v}}{p_v!}\frac{(\mathrm{i}\mu_v)^{q_v}}{q_v!}\sigma_{\nu_v,\mu_v}\mathrm{e}^{2\pi\mathrm{i}\mu_v\omega}\Big(\prod_{\ell\in L_v^-(\theta)}\mathrm{e}^{2\pi\mathrm{i}\nu_\ell\omega}\Big)\Big((\mathrm{i}\nu_v) + (\mathrm{i}\mu_v)\,\mathrm{e}^{-2\pi\mathrm{i}\nu_{\ell_v}\omega}\Big), \end{aligned} \tag{2.10}$$

where we have used that

$$\nu_{\ell_v} = \nu_v + \mu_v + \sum_{\ell\in L_v^+(\theta)}\nu_\ell + \sum_{\ell\in L_v^-(\theta)}\nu_\ell. \tag{2.11}$$

Then we define the value of the tree θ as

$$\mathscr{V}(\theta) = \Big(\prod_{v\in V(\theta)} A_v\Big)\Big(\prod_{\ell\in L(\theta)} G_\ell\Big) \tag{2.12}$$

and call $\mathcal{T}_{k,\nu}$ the set of all trees of order k (that is with k nodes) and with momentum ν associated with the root line. Note that (2.12) is well-defined for ω satisfying $\mathfrak{B}(\omega) < \infty$.

Lemma 2.4. *Set $h_0^{(k)} = 0$ for all $k \geq 1$. Assume that $\sum_{\theta\in\mathcal{T}_{k,0}}\mathscr{V}(\theta) = 0$ $\forall k \leq k_0$ for some $k_0 \in \mathbb{N}$. Then, by setting*

$$h_\nu^{(k)} = \sum_{\theta\in\mathcal{T}_{k,\nu}}\mathscr{V}(\theta), \quad \nu \neq 0,$$

for all $k \leq k_0$ and all $\nu \in \mathbb{Z}$, one obtains a solution up to order k_0 to (2.7).

Proof. Equation (2.7) allows us to express the coefficients $h_\nu^{(k)}$, for $\nu \neq 0$, in terms of coefficients $h_{\nu'}^{(k')}$, with $k' < k$ and $\nu' \neq 0$ (since we are assuming $h_0^{(k)} = 0$ for all $k \geq 1$). By iterating, all the coefficients are eventually given by expressions involving only coefficients $h_{\nu'}^{(1)}$, with $\nu' \neq 0$. It is straightforward to see that such coefficients can be expressed as

$$h_{\nu'}^{(k)} = \sum_{\theta\in\mathcal{T}_{k,\nu'}}\mathscr{V}(\theta), \quad \nu' \neq 0, \tag{2.13}$$

with $k = 1$, provided $\mathscr{V}(\theta)$ is defined as in (2.12). Then one easily checks by induction that $h_{\nu'}^{(k)}$ is given by (2.13) for all $k \geq 1$ and all $\nu' \neq 0$. To see that the definition of the coefficients through (2.13) makes sense, one uses that sum over the mode labels can be performed thanks to (2.2) and the Bryuno condition $\mathfrak{B}(\omega) < \infty$.

As a consequence, the equations (2.7) are satisfied for all $k \in \mathbb{N}$ and all $\nu \neq 0$. The assumption ensures that, for all $k \leq k_0$, the right hand side of (2.7) vanishes for $\nu = 0$. Therefore the equations (2.7) are satisfied for all $k \leq k_0$ and all $\nu \in \mathbb{Z}$. □

Remark 2.5. The proof of Lemma 2.4 shows that, since any line ℓ of any tree θ can be considered as the root line of a subtree θ_ℓ, then, by construction, $\mathscr{V}(\theta_\ell)$ is a contribution to $h_{\nu_\ell}^{(k_\ell)}$, where k_ℓ is the order of θ_ℓ. In particular, the assumption $h_0^{(k)} = 0$ implies that, for any tree θ and any line $\ell \in L(\theta)$, one has $\nu_\ell \neq 0$.

Lemma 2.6. *One has* $\sum_{\theta \in \mathcal{T}_{k,0}} \mathscr{V}(\theta) = 0$ *for all* $k \geq 1$.

The proof of Lemma 2.6 is given in Appendix A.

Remark 2.7. Lemma 2.6, together with Lemma 2.4, implies that one can define the coefficients $h_\nu^{(k)}$ as in Lemma 2.4 for all $k \in \mathbb{N}$. Moreover, by Remark 2.5, if $\theta \in \mathcal{T}_{k,\nu}$ and ℓ_0 is the root line of θ, then $\nu_\ell \neq 0$ for all $\ell \in L(\theta) \setminus \{\ell_0\}$.

The following result yields immediately Proposition 2.3.

Lemma 2.8. *For any* $\xi_1, \xi_2 \geq 0$, *with* $\xi_1 + \xi_2 < \xi$, *and any* $k \geq 1$ *there exists a constant* $C_0(k, \xi, \xi_1, \xi_2)$ *such that one has*

$$\sum_{\theta \in \mathcal{T}_{k,\nu}} |\mathscr{V}(\theta)| \leq \xi_2^{-2k} C_0(k, \xi, \xi_1, \xi_2)\, \mathrm{e}^{-\xi_1 |\nu|}$$

for any $\nu \in \mathbb{Z} \setminus \{0\}$.

Proof. With the notations in (2.2), for any $\xi_2 \in (0, \xi)$ each node factor A_v can be bounded by $2\Xi\, (2\,\xi_2^{-1})^{(p_v+q_v+1)} \mathrm{e}^{-(\xi-\xi_2)(|\nu_v|+|\mu_v|)}$. By writing $\xi - \xi_2 = \xi_1 + (\xi - \xi_1 - \xi_2)$ in $\mathrm{e}^{-(\xi-\xi_2)(|\nu_v|+|\mu_v|)}$, one can extract a factor $\mathrm{e}^{-\xi_1(|\nu_v|+|\mu_v|)}$ per each node. Moreover one has

$$\sum_{v \in V(\theta)} (p_v + q_v + 1) = 2k - 1.$$

Then one uses the Bryuno condition in order to bound the sum over all the mode labels of the product of factors $\mathrm{e}^{-(\xi-\xi_1-\xi_2)(|\nu_v|+|\mu_v|)}$ times the product of propagators by a constant $C_1(k, \xi, \xi_1, \xi_2)$; one reasons as in Appendix H in [5], which we refer to for more details. Finally one has to sum over all the other labels and this produces a further factor C_2^k for a suitable constant C_2. Then the assertion follows with $C_0(k, \xi, \xi_1, \xi_2) = (4\Xi\, C_2)^k C_1(k, \xi, \xi_1, \xi_2)$. □

Remark 2.9. As a matter of fact, the existence of the formal solution (2.6) follows by only assuming that $q_n^{-1} \log q_{n+1} \to 0$ as $n \to \infty$; see also [5] for a similar analysis.

Remark 2.10. The formal solution to (2.4) found through the construction above has been obtained by arbitrarily imposing that $h_0^{(k)} = 0$ for all $k \in \mathbb{N}$. Therefore we have no uniqueness of the solution. However, the solution is unique if we require for its average to vanish. In principle one could fix arbitrarily the constants $h_0^{(k)}$ and the existence of the solution could still be proved by reasoning as above: this would simply shift the origin of the parametrisation of the invariant curve. However the choice we made is a natural one and simplifies the analysis.

3 Analyticity of the conjugation: multiscale analysis

To prove convergence of the power series in ε for the conjugation we need a more careful analysis, which requires multiscale techniques.

We introduce a C^∞ partition of unity as follows. For $b > a > 0$ let $\chi_{a,b}$ be a C^∞ non-increasing function defined on $(0, +\infty)$ such that

$$\chi(u) = \begin{cases} 1, & u \le a, \\ 0, & u \ge b. \end{cases} \tag{3.1}$$

For $n \ge 1$ define

$$\begin{aligned} &\delta_n^- := \frac{1}{4}\min\left\{\frac{q_{n+1}-q_n}{32\, q_n q_{n+1}}, \frac{1}{32 q_n}\right\}, \quad \delta_n^+ := \frac{1}{4}\min\left\{\frac{q_n - q_{n-1}}{32\, q_{n-1} q_n}, \frac{1}{32 q_n}\right\}, \\ &a_n := \frac{1}{32 q_n} - \delta_n^-, \qquad b_n := \frac{1}{32 q_n} + \delta_n^+. \end{aligned} \tag{3.2}$$

and set $\chi_n = \chi_{a_n,b_n}$. Then we define

$$\Psi_0(u) = 1 - \chi_1(u), \qquad \Psi_n(u) = \chi_n(u) - \chi_{n+1}(u), \quad n \ge 1. \tag{3.3}$$

Remark 3.1. Consider the partition of unity of $(0, +\infty)$ consisting of the characteristic functions of the intervals $(1/32q_{n+1}, 1/32q_n]$, $n \ge 0$, and of the half-line $(1/32q_1, +\infty)$. Then $\{\Psi_n\}_{n\ge 0}$ is a smoothed version of such a partition.

We associate with each line $\ell \in L(\theta)$ a further label $n_\ell \in \mathbb{Z}_+$, called the *scale label*, and redefine the propagator of the line ℓ as

$$\mathcal{G}_\ell = \Psi_{n_\ell}(\|\omega\nu_\ell\|)\, G_\ell = \frac{\Psi_{n_\ell}(\|\omega\nu_\ell\|)}{\delta(\omega\nu_\ell)}, \tag{3.4}$$

where

$$\|u\| := \inf_{\nu\in\mathbb{Z}} |u - \nu| \,. \tag{3.5}$$

Lemma 3.2. *If $\mathcal{G}_\ell \ne 0$, then $1/64q_{n_\ell+1} \le \|\omega\nu_\ell\| \le 1/6q_{n_\ell}$ if $n_\ell \ge 1$ and $1/64q_1 \le \|\omega\nu_\ell\|$ if $n_\ell = 0$.*

Proof. By definition of the functions Ψ_n. □

Definition 3.3. *For any $\ell \in L(\theta)$ we call*

$$\zeta_\ell = \min\{n \in \mathbb{Z}_+ : \Psi_n(\|\omega\nu_\ell\|) \ne 0\}$$

the minimum scale *of ℓ.*

Remark 3.4. For any line $\ell \in L(\theta)$ such that $\mathcal{G}_\ell \ne 0$ one has either $n_\ell = \zeta_\ell$ or $n_\ell = \zeta_\ell + 1$.

We denote by $\Theta_{k,\nu}$ the set of all labelled trees with such extra labels n_ℓ, ζ_ℓ for all $\ell \in L(\theta)$, with k nodes and momentum ν associated with the root line. Given a tree $\theta \in \Theta_{k,\nu}$ redefine the value of θ as

$$\mathscr{V}(\theta) = \Big(\prod_{v\in V(\theta)} A_v\Big)\Big(\prod_{\ell\in L(\theta)} \mathcal{G}_\ell\Big). \tag{3.6}$$

Definition 3.5. *We say that a line ℓ with scale $n_\ell = n$ satisfies the* support property *if* $1/128q_{n+1} \le \|\omega\nu_\ell\| \le 1/8q_n$.

Remark 3.6. Lemma 3.2 implies that if $\mathscr{V}(\theta) \neq 0$ then all lines $\ell \in L(\theta)$ satisfy the support property. However, as it will be shown later, it is convenient to consider also trees with zero value, but still satisfying the support property. This will be due to the fact that in the renormalisation procedure described in Section 4 the momenta ν_ℓ will be replaced by new momenta, to be denoted $\nu_\ell(\boldsymbol{t})$, so that the scales n_ℓ will be determined by these new momenta through the conditions $\Psi_{n_\ell}(\|\omega\nu_\ell(\boldsymbol{t})\|) \neq 0$. Nevertheless the old momenta ν_ℓ will be still found to satisfy the support property in terms of the scales n_ℓ.

The following result is proved as Lemma 2.4 and Lemma 2.6.

Lemma 3.7. *The coeffiecients*

$$h_0^{(k)} = 0 \qquad \text{and} \qquad h_\nu^{(k)} = \sum_{\theta \in \Theta_{k,\nu}} \mathscr{V}(\theta), \quad \nu \neq 0,$$

for all $k \ge 1$ and all $\nu \in \mathbb{Z}$, formally solve (2.7) to all orders $k \in \mathbb{N}$.

Definition 3.8. *We define a* cluster *T on scale n as a maximal connected set of nodes and lines connecting them such that all lines have scale $\le n$ and at least one of them has scale n.*

Given any subgraph T of a tree θ, denote by $L(T)$ and $V(T)$ the set of lines and the set of nodes, respectively, in T. If T is a cluster, $L(T)$ will be called the set of *internal lines* of T, while the lines which connect one node $v \in V(T)$ to a node $w \notin V(T)$ will be called the *external lines* of T: they will be said to enter T if the line is oriented toward to the node belonging to $V(T)$ and to exit T otherwise. A cluster can have only either one or no exiting line.

If T has one exiting line ℓ_T and only one entering line ℓ'_T call $\mathcal{P}_T$ the oriented path of lines and nodes connecting ℓ'_T to ℓ_T and define $n_T := \min\{n_{\ell_T}, n_{\ell'_T}\}$. If T is a subgraph of θ, consisting of all nodes and lines preceding the line ℓ_1 but not the line ℓ_2, then we shall still denote by $\mathcal{P}_T$ the oriented path connecting ℓ_2 to ℓ_1. Finally, for $\ell \in \mathcal{P}_T$, call $w(\ell)$ and $w'(\ell)$ the nodes along $\mathcal{P}_T$ which ℓ enters and exits, respectively.

Remark 3.9. If T is a cluster on scale n with one entering line and one exiting line, one has $n_T \ge n+1$.

Set also

$$M(\theta) = \sum_{v \in V(\theta)} (|\nu_v| + |\mu_v|)\,, \qquad M(T) = \sum_{v \in V(T)} (|\nu_v| + |\mu_v|)\,, \tag{3.7}$$

where T can be any subgraph of θ.

Definition 3.10. *A cluster T on scale n will be called a* self-energy cluster *if*
(1) it has one exiting line ℓ_T and only one entering line ℓ'_T,
(2) $\nu_{\ell'_T} = \nu_{\ell_T}$,
(3) $\nu_\ell \neq \nu_{\ell_T}$ for all $\ell \in \mathcal{P}_T$,
(4) $M(T) < q_{n_T}$.

Remark 3.11. The condition (2) in Definition 3.10 implies that the sum of the mode labels of the nodes of a self-energy cluster T is zero, that is

$$\sum_{v\in V(T)} (\nu_v + \mu_v) = 0.$$

We say that $\ell \in L(\theta)$ is a *resonant line* if ℓ exits a self-energy cluster. Let us denote by $N_n^{\bullet}(\theta)$ the number of non-resonant lines $\ell \in L(\theta)$ with $\zeta_\ell = n$ and by $P_n(\theta)$ the number of self-energy clusters on scale n in θ.

Lemma 3.12. *Given $\nu \in \mathbb{Z}$, if $\|\omega\nu\| \le 1/4q_n$ then either $\nu = 0$ or $|\nu| \ge q_n$.*

The proof of Lemma 3.12 can be found in [7].

Lemma 3.13. *Assume that all lines $\ell \in L(\theta)$ satisfies the support property. Then one has*

$$K_n(\theta) := N_n^{\bullet}(\theta) + P_n(\theta) \le \frac{2M(\theta)}{q_n}.$$

Proof. We prove by induction that for any tree θ one has

$$K_n(\theta) = 0, \qquad\qquad M(\theta) < q_n, \tag{3.8a}$$

$$K_n(\theta) \le \frac{2M(\theta)}{q_n} - 1, \qquad M(\theta) \ge q_n, \tag{3.8b}$$

First of all note that if $M(\theta) < q_n$ one has $|\nu_\ell| < q_n$ and hence, by Lemma 3.12, $n_\ell < n$ for all $\ell \in L(\theta)$: therefore $K_n(\theta) \ge 1$ requires $M(\theta) \ge q_n$. In particular this proves (3.8a). To prove (3.8b) we proceed by induction on the order k of the tree. For $k = 1$ the bound holds by the previous argument. Let assume that (3.8b) holds for any tree of order $k < k_0$ and consider a tree of order k_0. Let us denote by ℓ_0 the root line of θ and call $\ell_1, \ldots, \ell_p$ the lines with minimum scale $\ge n$ (if any) which are closest to ℓ_0 and $\theta_1, \ldots, \theta_p$ the subtrees with root lines $\ell_1, \ldots, \ell_p$, respectively. Note that $n_{\ell_0} \ge \zeta_{\ell_0}$, so ℓ_0 can exit a self-energy cluster on scale n only if $\zeta_{\ell_0} \ge n$ and $n_{\ell_0} \ge n + 1$.

1. If $\zeta_{\ell_0} \neq n$ and ℓ_0 does not exit a self-energy cluster on scale n, then one has $K_n(\theta) = 0$ if $p = 0$ and $K_n(\theta) = K_n(\theta_1) + \ldots + K_n(\theta_p)$ if $p \ge 1$. In the first case the bound (3.8b) trivially holds, while in the second case it follows from the inductive hypothesis.

2. If $\zeta_{\ell_0} = n$ and ℓ_0 is non-resonant, then one has $K_n(\theta) = 1 + K_n(\theta_1) + \ldots + K_n(\theta_p)$. If $p = 0$ the bound is trivially satisfied and if $p \ge 2$ it follows immediately from the inductive hypothesis. If $p = 1$ let T be the subgraph formed by all lines and nodes of θ which precede ℓ_0 but not ℓ_1. If T is not a cluster then it must contain at least one line ℓ with $n_\ell = n$ and $\zeta_\ell = n - 1$. Then $\|\omega\nu_\ell\| \le 1/8q_n$. If ℓ is not along the path $\mathcal{P}_T$ connecting ℓ_1 to ℓ_0 then $M(T) \ge |\nu_\ell| \ge q_n$; if ℓ is along the path $\mathcal{P}_T$ then $\nu_\ell \neq \nu_{\ell_1}$ (because $\zeta_\ell = n - 1$, while $\zeta_{\ell_1} \ge n$), so that $\|\omega(\nu_\ell - \nu_{\ell_1})\| \le \|\omega\nu_\ell\| + \|\omega\nu_{\ell_1}\| \le 1/4q_n$ implies $|\nu_\ell - \nu_{\ell_1}| \ge q_n$ and hence $M(T) \ge q_n$. If T is a cluster then either $M(T) \ge q_n$ or $\nu_{\ell_0} \neq \nu_{\ell_1}$ (as ℓ_0 is non-resonant, T cannot be a self-energy cluster and hence $M(T) < q_n$ would require $\nu_{\ell_0} \neq \nu_{\ell_1}$, because otherwise there should be at least one line $\ell \in \mathcal{P}_T$

such that $\nu_\ell = \nu_{\ell_1}$, which is not possible since $\zeta_\ell \le n-1$ and $\zeta_{\ell_1} = \zeta_{\ell_0} = n$). On the other hand $\nu_{\ell_0} \neq \nu_{\ell_1}$ would imply $\|\omega\,(\nu_{\ell_0} - \nu_{\ell_1})\,\| \le \|\omega\nu_{\ell_0}\| + \|\omega\nu_{\ell_1}\| \le 1/4q_n$ and hence $M(T) \ge q_n$ once more. Therefore, in all cases one has $M(\theta) - M(\theta_1) = M(T) \ge q_n$, so that $K_n(\theta) = 1 + K(\theta_1) \le 2M(\theta_1)/q_n \le 2M(\theta)/q_n - 1$ and the bound (3.8b) follows.

3. If $\zeta_{\ell_0} = n$ and ℓ_0 exits a self-energy cluster T on scale $\neq n$, then one has $K_n(\theta) = K_n(\theta_1) + \ldots + K_n(\theta_p)$ and once more (3.8b) is trivial if $p = 0$ and follows from the inductive hypothesis if $p \ge 1$.

4. If ℓ_0 exits a self-energy cluster T on scale n, then $p \ge 1$ and $K_n(\theta) = 1+K(\theta_1)+\ldots+K(\theta_p)$. If $p \ge 2$ the bound follows by the inductive hypothesis. If $p = 1$ then either $\ell_1 = \ell'_T$ or $\ell_1 \in \mathcal{P}_T$ (because ℓ'_T has $n_{\ell'_T} \ge n+1$ and hence $\zeta_{\ell'_T} \ge n$). If $\ell_1 = \ell'_T$ then T must contain at least one line ℓ such that $n_\ell = n$ and $\zeta_\ell = n-1$: if $\ell \notin \mathcal{P}_T$ then $M(T) \ge |\nu_\ell| \ge q_n$, while if $\ell \in \mathcal{P}_T$ then either $M(T) \ge q_n$ or $\nu_\ell \neq \nu_{\ell_1}$ and hence again $M(T) \ge |\nu_{\ell_1} - \nu_\ell| \ge q_n$. If $\ell_1 \in \mathcal{P}_T$ then $n_{\ell_1} = n$ and either $M(T) \ge q_n$ or $\nu_{\ell_1} \neq \nu_{\ell'_T}$: the latter case yields again $M(T) \ge |\nu_{\ell_1} - \nu_{\ell'_T}| \ge q_n$. In all cases one finds $K_n(\theta) \le 2M_n(\theta_1)/q_n \le 2M_n(\theta)/q_n - 1$, so that (3.8b) follows. □

As we shall see in Section 4, when bounding a tree value (3.6), the product of node factors is easily controlled, while Lemma 3.13 guarantees that the product of propagators would also be bounded proportionally to a constant to the power k, if only the resonant lines could be neglected: in other words, the small divisors can accumulate only in the presence of self-energy clusters. Then, we have to make sure that when this happens one has compensations between the tree values. We end this section by providing the basic algebraic cancellations which underlie such compensations: the proof is an extension of the very argument used in the proof of Lemma 2.6 and is easily described in terms of operations on trees. Finally, in Section 4 we will show show how to implement iteratively the cancellations in order to control the product of all propagators, including those of the resonant lines, and hence complete the proof of convergence of the series (2.6).

We denote by $\mathfrak{S}_{k,n}$ the set of all self-energy clusters T on scale n of order k, that is with $|V(T)| = k$. Given a self-energy cluster T define the value of T as

$$\mathscr{V}_T(\omega\nu_{\ell'_T}) = \Big(\prod_{v\in V(T)} A_v\Big)\Big(\prod_{\ell\in L(T)} \mathcal{G}_\ell\Big). \tag{3.9}$$

Remark 3.14. Given a self-energy cluster T, if we set $\nu = \nu_{\ell'_T}$ and define

$$\nu^0_\ell = \sum_{\substack{v\in V(T)\\ v\prec\ell}} (\nu_v + \mu_v)\,, \tag{3.10}$$

we have $\nu_\ell = \nu^0_\ell + \nu$ if $\ell \in \mathcal{P}_T$ and $\nu_\ell = \nu^0_\ell$ if $\ell \notin \mathcal{P}_T$. Therefore $\mathscr{V}_T(\omega\nu)$ depends on $\omega\nu$ only through the propagators and the node factors of the lines and of the nodes, respectively, along the path $\mathcal{P}_T$.

A self-energy cluster T can contain other self-energy clusters: call $\mathring{T}$ the set of lines and nodes in T which are outside any self-energy clusters contained inside T, and denote by $L(\mathring{T})$ and $V(\mathring{T})$ the set of lines and of nodes, respectively, in $\mathring{T}$.

Remark 3.15. By definition of self-energy cluster one has

$$\sum_{v\in V(\mathring{T})} (\nu_v + \mu_v) = 0$$

for any self-energy cluster T.

Given a self-energy cluster T, call $\mathfrak{F}(T)$ the set of all self-energy clusters $T' \in \mathfrak{S}_{k,n}$, where k and n are the order and scale, respectively, of T, with $\nu_{\ell'_{T'}} = \nu_{\ell'_T}$.

Lemma 3.16. *For any self-energy cluster T one has*

$$\sum_{T'\in\mathfrak{F}(T)} \mathscr{V}_{T'}(0) = 0, \qquad \sum_{T'\in\mathfrak{F}(T)} \partial_u \mathscr{V}_{T'}(0) = 0.$$

The proof is in Appendix A.

4 Analyticity of the conjugation: renormalisation procedure

The modified tree expansion envisaged in Section 3 will allow us to refine the bounds on the coefficients of Proposition 2.3 into the following result.

Proposition 4.1. *Assume that ω satisfies the Bryuno condition. Then there exists a solution $h(\psi)$ to (2.4) of the form (2.6). For any $\xi_1, \xi_2, \xi_3 \geq 0$, with $\xi_1 + \xi_2 + \xi_3 < \xi$, there exists a positive constant $C_0(\xi_3)$ such that for all $k \geq 1$ and $\nu \in \mathbb{Z}$ the coefficients $h_\nu^{(k)}$ of the solution $h(\psi)$ satisfy the bounds $|h_\nu^{(k)}| \leq (\xi - \xi_1 - \xi_2)^{-2k} \xi_2^{-2k} C_0^k(\xi_3) \mathrm{e}^{-\xi_1|\nu|}$ and hence the series (2.6) converges absolutely for ε small enough.*

The rest of this section is devoted to proving Proposition 4.1, which in turn yields immediately Theorem 2.1.

We define $\mathfrak{T}(\theta)$ as the set of self-energy clusters in θ and $\mathfrak{T}_1(\theta)$ as the set of maximal self-energy clusters T contained in θ, i.e. such that there is no self-energy cluster in θ containing T. Generally, for any self-energy cluster T' in θ, we denote by $\mathfrak{T}_1(T')$ the set of maximal self-energy clusters (strictly) contained in T'. Given a line $\ell \in L(\theta)$, either there is no self-energy cluster containing ℓ or there exist $p = p(\ell) \geq 1$ self-energy clusters $T_1, \ldots, T_p$ such that T_p is the minimal self-energy cluster containing ℓ and $T_1 \supset T_2 \supset \ldots \supset T_p$, with $T_j \in \mathfrak{T}_1(T_{j-1})$ for $j = 2, \ldots, p$ and $T_1 \in \mathfrak{T}_1(\theta)$. We call $\mathfrak{C}_\ell(\theta) := \{T_1, \ldots, T_p\}$ the *cloud* of ℓ in θ.

Define $\mathring{\theta}$ as the set of nodes and lines in θ which are outside any self-energy cluster $T \in \mathfrak{T}_1(\theta)$ and set

$$\mathscr{V}(\mathring{\theta}) = \Big(\prod_{v\in V(\mathring{\theta})} A_v\Big)\Big(\prod_{\ell\in L(\mathring{\theta})} \mathcal{G}_\ell\Big). \tag{4.1}$$

We can write $\mathscr{V}(\theta)$ in (3.6) as

$$\mathscr{V}(\theta) = \mathscr{V}(\mathring{\theta}) \prod_{T\in\mathfrak{T}_1(\theta)} \mathscr{V}_T(\omega\nu_{\ell'_T}). \tag{4.2}$$

Define the *localised value* of a self-energy cluster $T \in \mathfrak{T}_1(\theta)$ as

$$\mathcal{L}\mathscr{V}_T(u) = \mathscr{V}_T(0) + u\,\partial_u \mathscr{V}_T(0) \tag{4.3}$$

and the *regularised value* of T as $\mathcal{R}\mathscr{V}_T(u) = \mathscr{V}_T(u) - \mathcal{L}\mathscr{V}_T(u)$. One can write

$$\mathcal{R}\mathscr{V}_T(u) = u^2 \int_0^1 \mathrm{d}t_T\,(1-t_T)\,\partial_u^2 \mathscr{V}_T(t_T u), \tag{4.4}$$

where $t_T \in [0,1]$ will be called the *interpolation parameter* associated with the self-energy cluster T. Then we associate with any $T \in \mathfrak{T}_1(\theta)$ a label $\delta_T \in \{\mathcal{L}, \mathcal{R}\}$ and rewrite (4.2) as a sum of contributions

$$\mathscr{V}(\mathring{\theta})\Big(\prod_{\substack{T\in\mathfrak{T}_1(\theta)\\ \delta_T=\mathcal{R}}} \mathcal{R}\mathscr{V}_T(\omega\nu_{\ell'_T})\Big)\Big(\prod_{\substack{T\in\mathfrak{T}_1(\theta)\\ \delta_T=\mathcal{L}}} \mathcal{L}\mathscr{V}_T(\omega\nu_{\ell'_T})\Big). \tag{4.5}$$

Recall the definition of the nodes $w(\ell)$ and $w'(\ell)$ before Remark 3.9. Call $A^1_{w(\ell)}$ the node factor obtained from $A_{w(\ell)}$ by replacing the factor $\mathrm{e}^{2\pi \mathrm{i}\nu_\ell\omega}$ with $2\pi\mathrm{i}\mathrm{e}^{2\pi\mathrm{i}\nu_\ell\omega}$ if $\beta_\ell = -$ and with 0 if $\beta_\ell = +$, and call $A^2_{w'(\ell)}$ the node factor obtained from $A_{w'(\ell)}$ by replacing the factor $(\mathrm{i}\nu_{w'(\ell)} + \mathrm{i}\mu_{w'(\ell)}\mathrm{e}^{-2\pi\mathrm{i}\nu_\ell\omega})$ with $2\pi\mu_{w'(\ell)}\mathrm{e}^{-2\pi\mathrm{i}\nu_\ell\omega}$. The derivative of $\mathscr{V}_T(u)$ with respect to $u = \omega\nu_{\ell'_T}$ produce several contributions: for any line $\ell \in \mathcal{P}_T$ there are three terms which are obtained from $\mathscr{V}_T(u)$, respectively, (1) by replacing $\mathcal{G}_\ell$ with $\partial_u\mathcal{G}_\ell$, (2) by replacing $A_{w(\ell)}$ with $A^1_{w(\ell)}$, (3) by replacing $A_{w'(\ell)}$ with $A^2_{w'(\ell)}$, To distinguish the three contributions we can introduce a label $\gamma = 1,2,3$, so that all the contributions produced by differentiation will be indexed by two labels (ℓ,γ), with $\ell \in \mathcal{P}_T$ and $\gamma \in \{1,2,3\}$ and denoted by $\mathcal{U}_T(\omega\nu;\ell,\gamma)$.

When considering $\partial_u^2\mathscr{V}_T(u)$ in (4.4), one has a sum of contributions which can be identified by four labels $(\ell_1,\ell_2,\gamma_1,\gamma_2)$, with $\ell_1,\ell_2 \in \mathcal{P}_T$ and $\gamma_1,\gamma_2 \in \{1,2,3\}$, and denoted by $\mathcal{U}_T(u;\ell_1,\gamma,\ell_2,\gamma_2)$; for $\ell_1 = \ell_2$ we shall impose the constraint $\gamma_1 \le \gamma_2$ to avoid overcountings. More precisely $\mathcal{U}_T(u;\ell_1,\gamma_1,\ell_2,\gamma_1)$ is obtained from $\mathscr{V}_T(u)$ by the following replacements

$$\begin{array}{lll}
\ell_1 = \ell_2, & \gamma_1 = \gamma_2 = 1, & \mathcal{G}_{\ell_1} \to \partial_u^2\mathcal{G}_{\ell_1} \\
\ell_1 = \ell_2, & \gamma_1 = 1, \gamma_2 = 2, & \mathcal{G}_{\ell_1}A_{w(\ell_1)} \to \partial_u\mathcal{G}_{\ell_1}A^1_{w(\ell_1)} \\
\ell_1 = \ell_2, & \gamma_1 = 1, \gamma_2 = 3, & \mathcal{G}_{\ell_1}A_{w'(\ell_1)} \to \partial_u\mathcal{G}_{\ell_1}A^2_{w'(\ell_1)} \\
\ell_1 = \ell_2, & \gamma_1 = \gamma_2 = 2, & A_{w(\ell_1)} \to A^3_{w(\ell_1)} \\
\ell_1 = \ell_2, & \gamma_1 = 2, \gamma_2 = 3, & A_{w(\ell_1)}A_{w'(\ell_1)} \to A^1_{w(\ell_1)}A^2_{w'(\ell_1)} \\
\ell_1 = \ell_2, & \gamma_1 = \gamma_2 = 3, & A_{w'(\ell_1)} \to A^4_{w'(\ell_1)} \\
\ell_1 \ne \ell_2, & \gamma_1 = \gamma_2 = 1, & \mathcal{G}_{\ell_1}\mathcal{G}_{\ell_2} \to \partial_u\mathcal{G}_{\ell_1}\partial_u\mathcal{G}_{\ell_2} \\
\ell_1 \ne \ell_2, & \gamma_1 = 1, \gamma_2 = 2, & \mathcal{G}_{\ell_1}A_{w(\ell_2)} \to \partial_u\mathcal{G}_{\ell_1}A^1_{w(\ell_2)} \\
\ell_1 \ne \ell_2, & \gamma_1 = 2, \gamma_2 = 1, & \mathcal{G}_{\ell_2}A_{w(\ell_1)} \to \partial_u\mathcal{G}_{\ell_2}A^1_{w(\ell_1)} \\
\ell_1 \ne \ell_2, & \gamma_1 = 1, \gamma_2 = 3, & \mathcal{G}_{\ell_1}A_{w'(\ell_2)} \to \partial_u\mathcal{G}_{\ell_1}A^2_{w'(\ell_2)} \\
\ell_1 \ne \ell_2, & \gamma_1 = 3, \gamma_2 = 1, & \mathcal{G}_{\ell_2}A_{w'(\ell_1)} \to \partial_u\mathcal{G}_{\ell_2}A^2_{w'(\ell_1)} \\
\ell_1 \ne \ell_2, & \gamma_1 = \gamma_2 = 2, & A_{w(\ell_1)}A_{w(\ell_2)} \to A^1_{w(\ell_1)}A^1_{w(\ell_2)} \\
\ell_1 \ne \ell_2, & \gamma_1 = 2, \gamma_2 = 3, & A_{w(\ell_1)}A_{w'(\ell_2)} \to A^1_{w(\ell_1)}A^2_{w'(\ell_2)} \\
\ell_1 \ne \ell_2, & \gamma_1 = 3, \gamma_2 = 2, & A_{w(\ell_2)}A_{w'(\ell_1)} \to A^1_{w(\ell_2)}A^2_{w'(\ell_1)} \\
\ell_1 \ne \ell_2, & \gamma_1 = \gamma_2 = 3, & A_{w'(\ell_1)}A_{w'(\ell_2)} \to A^2_{w'(\ell_1)}A^2_{w'(\ell_2)}
\end{array}$$

where the node factor $A^3_{w(\ell)}$ is obtained from $A_{w(\ell)}$ by replacing the factor $\mathrm{e}^{2\pi \mathrm{i}\nu_\ell\omega}$ with $(2\pi\mathrm{i})^2\mathrm{e}^{2\pi\mathrm{i}\nu_\ell\omega}$ if $\beta_\ell=-$ and with 0 if $\beta_\ell=+$, and the node factor $A^4_{w'(\ell)}$ is obtained from $A_{w'(\ell)}$ by replacing the factor $(\mathrm{i}\nu_{w'(\ell)}+\mathrm{i}\mu_{w'(\ell)}\mathrm{e}^{-2\pi\mathrm{i}\nu_\ell\omega})$ with $-(2\pi)^2\mathrm{i}\mu_{w'(\ell)}\mathrm{e}^{2\pi\mathrm{i}\nu_\ell\omega}$. The list of values above may look a bit tricky: however, the labels $\ell_1,\ell_2,\gamma_1,\gamma_2$ simply identify the quantities that have to be differentiated in $\mathscr{V}_T(u)$. Therefore we have

$$\partial_u \mathscr{V}_T(u)=\sum_{\ell_1\in\mathcal{P}_T}\sum_{\gamma_1=1,2,3}\mathscr{U}_T(u;\ell_1,\gamma_1), \tag{4.6a}$$

$$\partial_u^2 \mathscr{V}_T(u)=\sum_{\ell_1,\ell_2\in\mathcal{P}_T}\sum_{\gamma_1,\gamma_2=1,2,3}{}^{'}\,\mathscr{U}_T(u;\ell_1,\gamma_1,\ell_2,\gamma_2). \tag{4.6b}$$

where, here and henceforth, the prime in the last sum recalls the constraint $\gamma_1\le\gamma_2$ when $\ell_1=\ell_2$. In (4.6a) we write

$$\mathscr{U}_T(u;\ell_1,\gamma_1)=\Big(\prod_{v\in V(T)}\overline{A}_v\Big)\Big(\prod_{\ell\in L(T)}\overline{\mathcal{G}}_\ell\Big), \tag{4.7}$$

where

$$\overline{\mathcal{G}}_\ell:=\begin{cases}\partial_u\mathcal{G}_\ell, & \ell=\ell_1,\gamma_1=1,\\ \mathcal{G}_\ell, & \text{otherwise,}\end{cases}\qquad \overline{A}_v:=\begin{cases}A_v^1, & v=w(\ell),\gamma=2,\\ A_v^2, & v=w'(\ell),\gamma=3,\\ A_v, & \text{otherwise.}\end{cases}$$

Analogously we can write in (4.6b)

$$\mathscr{U}_T(t_Tu;\ell_1,\gamma_1,\ell_2,\gamma_2)=\Big(\prod_{v\in V(T)}\overline{A}_v\Big)\Big(\prod_{\ell\in L(T)}\overline{\mathcal{G}}_\ell\Big), \tag{4.8}$$

where

$$\overline{\mathcal{G}}_\ell:=\begin{cases}\partial_u^2\mathcal{G}_\ell, & \ell=\ell_1=\ell_2,\gamma_1=\gamma_2=1,\\ \partial_u\mathcal{G}_\ell, & \ell=\ell_1,\gamma_1=1 \text{ or } \ell=\ell_2,\gamma_2=1, \text{ with } \ell_1\neq\ell_2,\\ \mathcal{G}_\ell, & \text{otherwise,}\end{cases} \tag{4.9}$$

and

$$\overline{A}_v:=\begin{cases}A_v^3, & v=w(\ell_1)=w(\ell_2),\gamma_1=\gamma_2=2,\\ A_v^4, & v=w'(\ell_1)=w'(\ell_2),\gamma_1=\gamma_2=3,\\ A_v^1, & v=w(\ell_1),\gamma_1=2 \text{ or } v=w(\ell_2),\gamma_1=2, \text{ with } \ell_1\neq\ell_2,\\ A_v^2, & v=w'(\ell_1),\gamma_1=3 \text{ or } v=w'(\ell_2),\gamma_1=3, \text{ with } \ell_1\neq\ell_2,\\ A_v, & \text{otherwise.}\end{cases} \tag{4.10}$$

Remark 4.2. The value of both $\overline{\mathcal{G}}_\ell$ and $\overline{A}_v$ is well-defined as it depends only on the number of derivatives acting on $\mathcal{G}_\ell$ and A_v, and not to the fact that it appears in a contribution to the first or second derivative of $\mathscr{V}_T(u)$.

Given a self-energy cluster $T\in\mathfrak{T}_1(\theta)$ with $\delta_T=\mathcal{L}$, as in Section 3 we call $\mathfrak{F}(T)$ the set of all self-energy clusters $T'\in\mathfrak{S}_{k,n}$, where k and n are the order and scale, respectively, of T, with $\nu_{\ell'_{T'}}=\nu_{\ell'_T}$. The following result allows us to get rid of all contributions (4.5) in which at least one self-energy cluster $T\in\mathfrak{T}_1(\theta)$ has $\delta_T=\mathcal{L}$.

Lemma 4.3. *Let T be a maximal self-energy cluster of θ of order $\overline{k}$ and scale $\overline{n}$. The sum of $\mathscr{V}(\theta')$ over all $\theta' \in \Theta_{k,\nu}$ obtained from θ by inserting any $T' \in \mathfrak{F}(T)$ instead of T and replacing $\mathscr{V}_{T'}(\omega\nu_{\ell'_{T'}})$ with $\mathcal{L}\mathscr{V}_{T'}(\omega\nu_{\ell'_{T'}})$ gives zero.*

Proof. The result follows from Lemma 3.16. □

So we are left only with the contribution (4.5) with all self-energy clusters $T \in \mathfrak{T}_1(\theta)$ having $\delta_T = \mathcal{R}$. We can write each factor $\mathcal{R}\mathscr{V}_T(\omega\nu_{\ell'_T})$ in (4.5) according to (4.4), (4.6b) and (4.8), that is

$$\mathcal{R}\mathscr{V}_T(\omega\nu_{\ell'_T}) = \sum_{\ell_1,\ell_2\in\mathcal{P}_T}\sum_{\gamma_1,\gamma_2=1,2,3}{}' (\omega\nu_{\ell'_T})^2 \int_0^1 \mathrm{d}t_T\,(1-t_T)\Big(\prod_{v\in V(T)} \overline{A}_v\Big)\Big(\prod_{\ell\in L(T)} \overline{\mathcal{G}}_\ell\Big). \tag{4.11}$$

Given a line $\ell \in T$, define the *cloud* of ℓ in T as the set $\mathfrak{C}_\ell(T) = \{T_1,\dots,T_p\}$ such that T_p is the minimal self-energy cluster containing ℓ and $T_1 \supset T_2 \supset \ldots \supset T_p$, with $T_j \in \mathcal{T}_1(T_{j-1})$ for $j = 2,\dots,p$ and $T_1 \in \mathfrak{T}_1(T)$. For fixed $\ell_1,\ell_2 \in \mathcal{P}_T$ let $\mathfrak{C}_{\ell_1}(T)$ and $\mathfrak{C}_{\ell_2}(T)$ the clouds of the two lines in T (if the two lines coincide there is only one cloud). We associate a label $\zeta_{T'} = 0$ with each self-energy cluster $T' \in \mathfrak{C}_{\ell_1}(T)\cap\mathfrak{C}_{\ell_2}(T)$. We denote by $\mathfrak{T}_0(T)$ the set of such self-energy clusters and set $\mathfrak{T}^*(T) = \mathfrak{T}(T)\setminus\mathfrak{T}_0(T)$. Call $\mathfrak{T}_1^*(T)$ the set of maximal self-energy clusters in $\mathfrak{T}^*(T)$. We associate with each $T' \in \mathfrak{T}_1^*(T)$ a label $\zeta_{T'} = 1$ if T' contains one of the two lines ℓ_1 and ℓ_2 (by construction it cannot contain both of them) and a label $\zeta_{T'} = 2$ if it does not contain any of them. Finally denote by T^* the set of nodes and lines in T which are outside the self-energy clusters $T' \in \mathfrak{T}_1^*(T)$.

Remark 4.4. Note that both $\mathfrak{T}_1^*(T)$ and T^* depend on ℓ_1 and ℓ_2. One can think of $\mathfrak{T}_1^*(T)$ as the set of self-energy clusters which become maximal in T when ignoring the self-energy clusters which belong to both the clouds of ℓ_1 and ℓ_2.

Remark 4.5. If $T' \in \mathfrak{T}_1^*(T)$ and $\zeta_{T'} = 2$, one has $\overline{\mathcal{G}}_\ell = \mathcal{G}_\ell$ for all lines $\ell \in L(T')$. If $T' \in \mathfrak{T}_1^*(T)$ and $\zeta_{T'} = 1$, there is one line $\ell_1 \in L(T')$ such that $\overline{\mathcal{G}}_{\ell_1} = \partial_u\mathcal{G}_{\ell_1}$, while $\overline{\mathcal{G}}_\ell = \mathcal{G}_\ell$ for $\ell \in L(T')\setminus\{\ell_1\}$.

By defining

$$\overline{\mathscr{V}}_{T^*}(\omega\nu_{\ell'_T}(t_T)) = \Big(\prod_{v\in V(T^*)} \overline{A}_v\Big)\Big(\prod_{\ell\in L(T^*)} \overline{\mathcal{G}}_\ell\Big), \tag{4.12a}$$

$$\overline{\mathscr{V}}_{T'}(\omega\nu_{\ell'_{T'}}(t_T)) = \Big(\prod_{v\in V(T')} \overline{A}_v\Big)\Big(\prod_{\ell\in L(T')} \overline{\mathcal{G}}_\ell\Big), \qquad T' \in \mathfrak{T}_1^*(T), \tag{4.12b}$$

where $\nu_\ell(t_T) = \nu_\ell$ if $\ell \notin \mathcal{P}_T$ and $\nu_\ell(t_T) = \nu_\ell + t_T\nu_{\ell'_T}$ if $\ell \in \mathcal{P}_T$, we can rewrite (4.11) as

$$\mathcal{R}\mathscr{V}_T(\omega\nu_{\ell'_T}) = \sum_{\ell_1,\ell_2\in\mathcal{P}_T}\sum_{\gamma_1,\gamma_2=1,2,3}{}' (\omega\nu_{\ell'_T})^2 \int_0^1 \mathrm{d}t_T\,(1-t_T)\,\overline{\mathscr{V}}_{T^*}(\omega\nu_{\ell'_T}(t_T)) \prod_{T'\in\mathfrak{T}_1^*(T)} \overline{\mathscr{V}}_{T'}(\omega\nu_{\ell'_{T'}}(t_T)). \tag{4.13}$$

For $T' \in \mathfrak{T}_1^*(T)$ we define

$$\mathcal{L}\overline{\mathscr{V}}_{T'}(u) = \begin{cases} \overline{\mathscr{V}}_{T'}(0) + u\,\partial_u \overline{\mathscr{V}}_{T'}(0), & \zeta_{T'} = 2, \\ \overline{\mathscr{V}}_{T'}(0), & \zeta_{T'} = 1, \end{cases} \tag{4.14}$$

and set $\mathcal{R}\overline{\mathscr{V}}_{T'}(u) = \overline{\mathscr{V}}_{T'}(u) - \mathcal{L}\overline{\mathscr{V}}_{T'}(u)$, so that

$$\begin{cases} \mathcal{R}\overline{\mathscr{V}}_{T'}(u) = u^2 \displaystyle\int_0^1 \mathrm{d}t_{T'}\,(1-t_{T'})\,\partial_u^2 \overline{\mathscr{V}}_{T'}(t_{T'}u), & \zeta_{T'} = 2, \\ \mathcal{R}\overline{\mathscr{V}}_{T'}(u) = u \displaystyle\int_0^1 \mathrm{d}t_{T'}\,\partial_u \overline{\mathscr{V}}_{T'}(tu), & \zeta_{T'} = 1. \end{cases} \tag{4.15}$$

Then we repeat the construction by associating a label $\delta_{T'} \in \{\mathcal{L}, \mathcal{R}\}$ with each $T' \in \mathfrak{T}_1^*(T)$ and write $\mathcal{R}\,\mathscr{V}_T(\omega\nu_{\ell'_T})$ in (4.13) as sum of contributions

$$\sum_{\ell_1,\ell_2 \in \mathcal{P}_T} \sideset{}{'}\sum_{\gamma_1,\gamma_2=1,2,3} (\omega\nu_{\ell'_T})^2 \int_0^1 \mathrm{d}t_T\,(1-t_T) \tag{4.16}$$
$$\overline{\mathscr{V}}_{T^*}(\omega\nu_{\ell'_T}(t_T)) \Big(\prod_{\substack{T' \in \mathfrak{T}_1^*(T) \\ \delta_T = \mathcal{R}}} \mathcal{R}\overline{\mathscr{V}}_{T'}(\omega\nu_{\ell'_{T'}}(t_T)) \Big) \Big(\prod_{\substack{T' \in \mathfrak{T}_1^*(T) \\ \delta_T = \mathcal{L}}} \mathcal{L}\overline{\mathscr{V}}_{T'}(\omega\nu_{\ell'_{T'}}(t_T)) \Big),$$

where each $\mathcal{R}\overline{\mathscr{V}}_{T'}(\omega\nu_{\ell'_{T'}}(t_T))$ is expressed as in (4.15), with $\partial_u^2 \overline{\mathscr{V}}_{T'}(t_{T'}u)$ and $\partial_u \overline{\mathscr{V}}_{T'}(t_{T'}u)$ written as in (4.6) to (4.8). In particular the momentum of any line $\ell \in L(T')$ becomes

$$\nu_\ell(t_{T'}, t_T) := \begin{cases} \nu_\ell^0, & \ell \notin \mathcal{P}_{T'}, \\ \nu_\ell^0 + t_{T'}\nu^0_{\ell'_{T'}}, & \ell \in \mathcal{P}_{T'} \text{ and } \ell'_{T'} \notin \mathcal{P}_T, \\ \nu_\ell^0 + t_{T'}\,(\nu^0_{\ell'_{T'}} + t_T \nu_{\ell_T}), & \ell \in \mathcal{P}_{T'} \text{ and } \ell'_{T'} \notin \mathcal{P}_T. \end{cases} \tag{4.17}$$

The contributions (4.16) in which at least one self-energy cluster $T' \in \mathfrak{T}_1^*(T)$ has $\delta_{T'} = \mathcal{L}$ vanish when summed together. This follows from the following result.

Lemma 4.6. *Let $\overline{k}$ and $\overline{n}$ be the order and the scale, respectively, of $T' \in \mathfrak{T}_1^*(T)$. The sum of $\mathscr{V}(\theta')$ over all $\theta' \in \Theta_{k,\nu}$ obtained from θ by inserting any self-energy cluster $T'' \in \mathfrak{F}(T')$ instead of T' and replacing $\mathscr{V}_{T''}(\omega\nu_{\ell'_{T''}})$ with $\mathcal{L}\,\mathscr{V}_{T''}(\omega\nu_{\ell'_{T''}})$ gives zero.*

Proof. One reasons as for Lemma 4.3. □

Therefore we have

$$\sum_{\theta \in \Theta_{k,n}} \mathscr{V}(\theta) = \sum_{\theta \in \Theta_{k,n}} \mathscr{V}(\mathring{\theta}) \prod_{T \in \mathfrak{T}_1(\theta)} \sum_{\ell_{T,1},\ell_{T,2} \in \mathcal{P}_T} \sideset{}{'}\sum_{\gamma_{T,1},\gamma_{T,2}=1,2,3} (\omega\nu_{\ell'_T})^2$$
$$\int_0^1 \mathrm{d}t_T\,(1-t_T)\,\overline{\mathscr{V}}_{T^*}(\omega\nu_{\ell'_T}(t_T)) \prod_{T' \in \mathfrak{T}_1^*(T)} \mathcal{R}\overline{\mathscr{V}}_{T'}(\omega\nu_{\ell'_{T'}}(t_T)). \tag{4.18}$$

The construction can be iterated further. At each step the order of the self-energy clusters has decreased, so that eventually the procedure stops. Moreover, every time we split the value of the a self-energy cluster into the sum of the localised value plus the regularised value, we can neglect the localised value. Indeed the following extension of Lemma 4.6 holds.

Lemma 4.7. *Given any $T \in \mathfrak{T}(T)$ denote by $\overline{k}$ and $\overline{n}$ the order and the scale, respectively, of T. The sum of $\mathscr{V}(\theta')$ over all $\theta' \in \Theta_{k,\nu}$ obtained from θ by inserting any $T' \in \mathfrak{F}(T)$ instead of T and replacing $\mathscr{V}_{T'}(\omega\nu_{\ell'_{T'}})$ with $\mathcal{L}\mathscr{V}_{T'}(\omega\nu_{\ell'_{T'}})$ gives zero.*

Proof. Simply note that, when computing the localised value of a self-energy cluster T, one puts $\nu_\ell = \nu_\ell^0$ for all lines $\ell \in L(T)$. Therefore the cancellation mechanisms underlying Lemma 3.16 apply to any self-energy cluster, indepenedently of the fact that it is maximal or inside other self-energy clusters. □

With any self-energy cluster T we associate a label $\zeta_T \in \{0, 1, 2\}$, an interpolation parameter $t_T \in [0, 1]$ and a measure $\pi_{\zeta_T}(t_T)\,\mathrm{d}t_T$, where

$$\pi_\zeta(t) = \begin{cases} (1-t)\,, & \zeta = 2, \\ 1, & \zeta = 1, \\ \delta(t-1), & \zeta = 0, \end{cases} \tag{4.19}$$

where δ is the Dirac delta.

Denote by $\boldsymbol{t} = \{t_T\}$ the set of all interpolation parameters and define recursively

$$\nu_\ell(\boldsymbol{t}) = \begin{cases} \nu_\ell^0, & \ell \notin \mathcal{P}_T, \\ \nu_\ell^0 + t_T \nu_{\ell'_T}(\boldsymbol{t}), & \ell \in \mathcal{P}_T, \end{cases} \tag{4.20}$$

where T is the minimal self-energy cluster containing ℓ (if any).

Remark 4.8. For any $\ell \in L(\theta)$ the momentum $\nu_\ell(\boldsymbol{t})$ depends only on the interpolation parameters of the self-energy clusters T' such that $\ell \in \mathcal{P}_{T'}$.

Eventually one obtains that

$$\sum_{\theta\in\Theta_{k,\nu}} \mathscr{V}(\theta) = \sum_{\theta\in\Theta_{k,\nu}} \Big(\prod_{v\in V(\mathring{\theta})} \overline{A}_v \Big)\Big(\prod_{\ell\in L(\mathring{\theta})} \overline{\mathcal{G}}_\ell \Big) \tag{4.21}$$
$$\prod_{T\in\mathfrak{T}(\theta)} \sum\nolimits_T^{\zeta_T} (\omega\nu_{\ell'_T}(\boldsymbol{t}))^{\zeta_T} \int_0^1 \pi_{\zeta_T}(t_T)\,\mathrm{d}t_T \Big(\prod_{v\in V(T^*)} \overline{A}_v \Big)\Big(\prod_{\ell\in L(T^*)} \overline{\mathcal{G}}_\ell \Big)$$

where $\overline{\mathcal{G}}_\ell$ and $\overline{A}_v$ are defined in (4.9) and (4.10), respectively, and we have introduced the shorthand notation

$$\sum\nolimits_T^{\zeta_T} = \begin{cases} \displaystyle\sum_{\ell_{T,1},\ell_{T,2}\in\mathcal{P}_T} \sideset{}{'}\sum_{\gamma_{T,1},\gamma_{T,2}=1,2,3}, & \zeta_T = 2, \\ \displaystyle\sum_{\ell_{T,1}\in\mathcal{P}_T} \sum_{\gamma_{T,1}=1,2,3}, & \zeta_T = 1. \end{cases}$$

Remark 4.9. The propagator of each line is differentiated at most twice.

In (4.21) we can bound, for any $\xi_2 \in (0, \xi)$,

$$\Big(\prod_{v\in V(\mathring{\theta})} |\overline{A}_v| \Big) \prod_{T\in\mathfrak{T}(\theta)} \Big(\prod_{v\in V(T^*)} |\overline{A}_v| \Big) \le (2\Xi)^k \, (2\xi_2^{-1})^{2k} \prod_{v\in V(\theta)} \mathrm{e}^{-(\xi-\xi_2)(|\nu_v|+|\mu_v|)} \tag{4.22}$$

and, if we denote by $L^\bullet(\theta)$ the set of non-resonant lines in θ,

$$
\begin{aligned}
&\Big(\prod_{\ell\in L(\mathring{\theta})}|\overline{\mathcal{G}}_\ell|\Big)\prod_{T\in\mathfrak{T}(\theta)}\Big(\prod_{\ell\in L(T^*)}|\overline{\mathcal{G}}_\ell|\Big)\\
&\quad\le C_1^k\Big(\prod_{\ell\in L^\bullet(\theta)}\|\omega\nu_\ell(\boldsymbol{t})\|^{-2}\Big)\Big(\prod_{T\in\mathfrak{T}(\theta)}\|\omega\nu_{\ell_T}(\boldsymbol{t})\|^{-2}\Big) \qquad (4.23)\\
&\quad\Big(\prod_{\substack{T\in\mathfrak{T}(\theta)\\ \zeta_T=2}}\|\omega\nu_{\ell'_T}(\boldsymbol{t})\|^2\|\omega\nu_{\ell_{T,1}}(\boldsymbol{t})\|^{-1}\|\omega\nu_{\ell_{T,2}}(\boldsymbol{t})\|^{-1}\Big)\Big(\prod_{\substack{T\in\mathfrak{T}(\theta)\\ \zeta_T=1}}\|\omega\nu_{\ell'_T}(\boldsymbol{t})\|\|\omega\nu_{\ell_{T,1}}(\boldsymbol{t})\|^{-1}\Big)
\end{aligned}
$$

fot a suitable constant C_1. For each line $\ell_{T,i}$ in (4.23) we can write

$$
\|\omega\nu_{\ell_{T,1}}(\boldsymbol{t})\|^{-1}=\|\omega\nu_{\ell_{T,1}}(\boldsymbol{t})\|^{-1}\prod_{T\in\mathfrak{C}_\ell(T_\ell)}\|\omega\nu_{\ell'_T}(\boldsymbol{t})\|^{-1}\|\omega\nu_{\ell'_T}(\boldsymbol{t})\|, \qquad (4.24)
$$

where T_ℓ the minimal self-energy cluster T' containing $\ell_{T,i}$ with $\zeta_{T'}\neq 0$.

Remark 4.10. Consider any summand in the right hand side of (4.21). For each line ℓ with $n_\ell\ge 1$ the propagator $\overline{\mathcal{G}}_\ell$ can be bounded proportionally to $\|\omega\nu_\ell(\boldsymbol{t})\|^{-2-p}$, if $\overline{\mathcal{G}}_\ell=\partial_u^p\mathcal{G}_\ell$ for $p\in\{0,1,2\}$, where $1/16q_{n_\ell+1}\le\|\omega\nu_\ell(\boldsymbol{t})\|\le 1/64q_{n_\ell}$ by Lemma 3.2.

In the light of Remark 4.10, if we use (4.24) and the fact that $n_\ell\le n_T$ for all $\ell\in L(T)$, we can bound in (4.23)

$$
\begin{aligned}
&\Big(\prod_{\substack{T\in\mathfrak{T}(\theta)\\ \zeta_T=2}}\|\omega\nu_{\ell'_T}(\boldsymbol{t})\|^2\|\omega\nu_{\ell_{T,1}}(\boldsymbol{t})\|^{-1}\|\omega\nu_{\ell_{T,2}}(\boldsymbol{t})\|^{-1}\Big)\Big(\prod_{\substack{T\in\mathfrak{T}(\theta)\\ \zeta_T=1}}\|\omega\nu_{\ell'_T}(\boldsymbol{t})\|\|\omega\nu_{\ell_{T,1}}(\boldsymbol{t})\|^{-1}\Big)\\
&\qquad\le C_2^k\Big(\prod_{T\in\mathfrak{T}(\theta)}\|\omega\nu_{\ell'_T}(\boldsymbol{t})\|^2\Big)\Big(\prod_{T\in\mathfrak{T}(\theta)}q_{n_T+1}^2\Big), \qquad (4.25)
\end{aligned}
$$

for a suitable constant C_2, so as to obtain

$$
\Big(\prod_{\ell\in L(\mathring{\theta})}|\overline{\mathcal{G}}_\ell|\Big)\prod_{T\in\mathfrak{T}(\theta)}\Big(\prod_{\ell\in L(T^*)}|\overline{\mathcal{G}}_\ell|\Big)\le C_1^kC_2^k\Big(\prod_{\ell\in L^\bullet(\theta)}\|\omega\nu_\ell(\boldsymbol{t})\|^{-2}\Big)\Big(\prod_{T\in\mathfrak{T}(\theta)}q_{n_T+1}^2\Big). \qquad (4.26)
$$

Therefore one has

$$
\Big(\prod_{\ell\in L(\mathring{\theta})}|\overline{\mathcal{G}}_\ell|\Big)\prod_{T\in\mathfrak{T}(\theta)}\Big(\prod_{\ell\in L(T^*)}|\overline{\mathcal{G}}_\ell|\Big)\le C_3^k\prod_{n\ge 0}q_{n+1}^{2(N_n^\bullet(\theta)+P_n(\theta))}. \qquad (4.27)
$$

To bound $N^\bullet(\theta)+P_n(\theta)$ in (4.27) we shall use Lemma 3.13. However, we have first to check that the lines satisfy the support properties. If the momentum of each line ℓ were ν_ℓ this would follow immediately from Lemma 3.2 (see Remark 3.6). Though, the momenta are $\nu_\ell(\boldsymbol{t})$, and the compact support functions Ψ_{n_ℓ} only assure that $\Psi_{n_\ell}(\|\omega\nu_\ell(\boldsymbol{t})\|)\neq 0$, that is $1/64q_{n_\ell+1}\le\|\omega\nu_\ell(\boldsymbol{t})\|\le 1/16q_{n_\ell}$. We want to show that the latter property implies the following result.

Lemma 4.11. *In each summand on the right hand side of (4.21), for all lines $\ell \in L(\theta)$ one has $1/128 q_{n_\ell+1} \le \|\omega\nu_\ell\| \le 1/8 q_{n_\ell}$.*

Proof. We say that a line ℓ has depth 0 if it is outside any self-energy cluster and depth $p \ge 1$ if there are p self-energy clusters $T_1 \supset T_2 \supset \ldots \supset T_p$, such that $T_i \in \mathfrak{T}_1^*(T_{i-1})$ for $i = 2, \ldots, p$, $T_1 \in \mathfrak{T}_1(\theta)$ and $\ell \in T_p^*$. In such a case one has $\nu_\ell(\boldsymbol{t}) = \nu_\ell^0$ if $\ell \notin \mathcal{P}_{T_p}$ and $\nu_\ell(\boldsymbol{t}) = \nu_\ell^0 + t_{T_p}\nu_{\ell'_{T_p}}(\boldsymbol{t})$ if $\ell \in \mathcal{P}_{T_p}$.

We want to prove the bound by induction on the depth of the lines.

For any line ℓ of depth 0 one has $\nu_\ell(\boldsymbol{t}) = \nu_\ell$, so that the bound trivially holds.

Let ℓ be a line with depth 1 and let T be the self-energy cluster containing ℓ. Again if $\nu_\ell^0 = \nu_\ell$ there is nothing to prove, so we have to consider explicitly only the case $\nu_\ell(\boldsymbol{t}) = \nu_\ell^0 + t_T\omega\nu_{\ell'_T}$. Let n be the scale of ℓ'_T. By definition of self-energy cluster one has $|\nu_\ell^0| \le M(T) < q_n$ and hence $\|\omega\nu_\ell^0\| > 1/4q_n$. On the other hand one has $\|\omega\nu_{\ell'_T}\| \le 1/16q_n$, so that

$$\frac{1}{2}\,\|\omega\nu_\ell^0\| \le \|\omega\nu_\ell^0\| - \|\omega\nu_{\ell'_T}\| \le \|\omega\nu_\ell\| \le \|\omega\nu_\ell^0\| + \|\omega\nu_{\ell'_T}\| \le \frac{3}{2}\|\omega\nu_\ell^0\|. \tag{4.28}$$

Let n_ℓ be the scale of the line ℓ: then $1/64q_{n_\ell+1} \le \|\omega\nu_\ell(\boldsymbol{t})\| \le 1/16q_{n_\ell}$ because $\Psi_{n_\ell}(\|\omega\nu_\ell(\boldsymbol{t})\|) \neq 0$ (see also Lemma 3.2). The quantity $\|\omega\nu_\ell(\boldsymbol{t})\|$ is between $\|\omega\nu_\ell^0\|$ and $\|\omega\nu_\ell\|$. If $\|\omega\nu_\ell^0\| < \|\omega\nu_\ell\|$ one has $\|\omega\nu_\ell\| \ge \|\omega\nu_\ell(\boldsymbol{t})\| \ge 1/64q_{n_\ell+1} \ge 1/128q_{n_\ell+1}$ and $\|\omega\nu_\ell\| \le 2\|\omega\nu_\ell^0\| \le 2\|\omega\nu_\ell(\boldsymbol{t})\| \le 1/8q_{n_\ell}$; if $\|\omega\nu_\ell^0\| \ge \|\omega\nu_\ell\|$ one has $\|\omega\nu_\ell\| \le \|\omega\nu_\ell(\boldsymbol{t})\| \le 1/16q_{n_\ell} \le 1/8q_{n_\ell}$ and $\|\omega\nu_\ell\| \ge \|\omega\nu_\ell^0\|/2 \ge \|\omega\nu_\ell(\boldsymbol{t})\|/2 \ge 1/128q_{n_\ell}$. Therefore in both cases one has $1/8q_{n_\ell} \ge \|\omega\nu_\ell\| \ge 1/128q_{n_\ell+1}$.

Now let us assume that the bound holds for all lines of depth $\le k$ and show that then it holds also for lines of depth $k+1$. Let ℓ be one of such lines and let T be the minimal self-energy cluster with $\zeta_T \neq 0$ containing ℓ. Once more if $\nu_\ell(\boldsymbol{t}) = \nu_\ell^0$ the bound holds trivially. If instead one has $\nu_\ell(\boldsymbol{t}) = \nu_\ell^0 + t_T\nu_{\ell'_T}(\boldsymbol{t})$ one can reason as in the previous case. One has $\nu_\ell = \nu_\ell^0 + \nu_{\ell'_T}$ and ℓ'_T has depth k, so that, by the inductive hypothesis, the bound $1/8q_n \ge \|\omega\nu_{\ell'_T}\| \ge 1/128q_{n+1}$ holds, if n is the scale of ℓ'_T. On the other hand $|\nu_\ell^0| \le M(T) < q_n$ and hence $\|\omega\nu_\ell^0\| > 1/4q_n$, so that (4.28) holds also for the line ℓ. Moreover $\|\omega\nu_\ell(\boldsymbol{t})\|$, for all values of the interpolation parameters $\boldsymbol{t}$, is between $\|\omega\nu_\ell^0\|$ and $\|\omega\nu_\ell\|$. Therefore one can apply the previous argument and bound $\|\omega\nu_\ell\|$ in terms of the scale n_ℓ of the line ℓ so as to obtain the desired bound. □

By combining Lemma 3.13 with Lemma 4.11 we obtain, for arbitrary $n_0 \in \mathbb{N}$,

$$C_3^k \prod_{n\ge 0} q_{n+1}^{2(N_n^\bullet(\theta)+P_n(\theta))} \le C_3^k q_{n_0}^{2k} \Big(\prod_{n\ge n_0} q_{n+1}^{4M(\theta)/q_n} \Big) \le C_3^k q_{n_0}^{2k} \exp\Big(4M(\theta) \sum_{n\ge n_0} \frac{1}{q_n} \log q_{n+1}\Big).$$

The sum converges by the assumption that $\mathfrak{B}(\omega) < \infty$, so that for any $\xi_3 \in (0,\xi_2)$ one can choose $n_0 = n_0(\xi_3)$ suitably large so that

$$\Big(\prod_{\ell\in L(\mathring{\theta})} |\overline{\mathcal{G}}_\ell| \Big) \prod_{T\in\mathfrak{T}(\theta)} \Big(\prod_{\ell\in L(T^*)} |\overline{\mathcal{G}}_\ell| \Big) \le C_3^k C_4^k(\xi_3)\, \mathrm{e}^{\xi_3 M(\theta)}, \tag{4.29}$$

with $C_4(\xi_3) = q_{n_0(\xi_3)}^2$.

By collecting together the bounds (4.22) and (4.29), and taking $\xi_1 \in (0, \xi - \xi_2 - \xi_3)$, we obtain that each summand in the right hand side of (4.21) is bounded by

$$(32\, \Xi\, C_3 C_4(\xi_3))^k \,\mathrm{e}^{-\xi_1 |\nu|} \prod_{v \in V(\theta)} \mathrm{e}^{-(\xi - \xi_1 - \xi_2)(|\nu_v| + |\mu_v|)}. \tag{4.30}$$

We have still to perform the sum over the tree labels. This is controlled by the following result.

Lemma 4.12. *There exists a positive constant C_5 such that the the sum of the quantities (4.30) over the tree labels is bounded by $C_5^k C_4(\xi_3)(\xi - \xi_1 - \xi_2)^{-2k} \xi_2^{-2k} \mathrm{e}^{-\xi_1|\nu|}$.*

Proof. The number of unlabelled trees of order k is bounded by 4^k. Then we have to sum over all the labels. The sum over the labels ζ_T and the choices of the lines $\ell_{T,1}, \ell_{T,2}$ if $\zeta_T = 2$ and $\ell_{T,1}$ if $\zeta_T = 1$ is bounded by C_6^k for some positive constant C_6. The sum over the Fourier labels can be bounded thanks to the factors $\mathrm{e}^{-(\xi - \xi_1 - \xi_2)(|\nu_v| + |\mu_v|)}$ associated with the nodes and produces a further factor $C_7^k (\xi - \xi_1 - \xi_2)^{-2k}$, for a suitable constant C_7. Then the assertion follows with $C_5 = 64 \Xi C_3 C_6 C_7$. □

5 Some comments

The results of Sections 4 can be summarised in the statement of Theorem 2.1. The result can be strengthened into Theorem 1.2, as the following argument shows.

Proof of Theorem 1.2. First note that 2.1 ensures that if ω satisfies the Bryuno condition then for ε small enough there exists an invariant curve with rotation number ω. So, to complete the proof we have to show that if ω does not satisfy the Bryuno condition, then there exists a symplectic map Φ arbitrarily close to Φ_0 such that no analytic invariant curve with rotation number ω and analytic in ε exists. One can take as Φ the standard map.

$$\begin{cases} x' = x + y + \varepsilon\, K \,\sin x, \\ y' = y + \varepsilon\, K \,\sin x, \end{cases} \tag{5.1}$$

where a suitable constant K has been introduced in order to make $|\Phi_1| = 1$ in $\mathcal{D}$. Indeed, for such a map, if one denotes by $\rho_0(\omega)$ the radius of analyticity of the conjugation, one has [7, 1]

$$|\log \rho_0(\omega) + 2\mathfrak{B}(\omega)| \le C, \tag{5.2}$$

for some universal constant C. This means that the radius of convergence vanishes when ω does not satisfy the Bryuno condition: hence no analytic invariant curve analytic in ε exists in such a case. □

Remark 5.1. Of course, for the existence of an analytic invariant curve analytic in ε, the Bryuno condition cannot be optimal for every perturbation. A trivial counterexample is the null perturbation for which every invariant curve exists independently of its rotation number. More generally one can consider a perturbation which vanishes on a fixed unperturbed curve. Simply take Φ given by (1.2) with $f(x,y) = (y - y_0)\, F(x,y)$ and $g(x,y) = (y - y_0)\, G(x,y)$, with F, G arbitrary analytic functions: the invariant curve $x = x + y_0, y = y_0$ persists for any F, G, without any further assumption on the rotation number $\omega = y_0$.

Remark 5.2. An interesting problem is whether the Bryuno condition is optimal for the existence of an analytic invariant curve analytic in ε for generic perturbations of Φ_0.

Remark 5.3. Another interesting problem is whether a result like Theorem 1.2 extends to the more general systems of the form (1.1), which will be considered in Section 6.

Remark 5.4. Theorem 1.2 does not exclude the existence of analytic invariant curves which are not analytic in the perturbation parameter ε_0. It is an open problem whether analytic invariant curves may exist under weaker conditions than Bryuno's, i.e. whether the Bryuno condition is optimal for the system to be analytically conjugated to a rotation.

A final comment concerns the Bryuno condition as formulated in Section 1. Usually the condition is stated, in any dimension d, by requiring for $\boldsymbol{\omega} \in \mathbb{R}^d$ to satisfy

$$\mathfrak{B}_1(\boldsymbol{\omega}) = \sum_{n=0}^{\infty} \frac{1}{2^n} \log \frac{1}{\alpha_n(\boldsymbol{\omega})} < \infty, \qquad \alpha(\boldsymbol{\omega}) = \inf_{\substack{\boldsymbol{\nu} \in \mathbb{Z}^d \\ 0<|\boldsymbol{\nu}|\le 2^n}} |\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \,. \tag{5.3}$$

It is straightforward to check that for $d = 1$ and $\boldsymbol{\omega} = (1, \omega)$ the condition (6.3) is equivalent to requiring the Bryuno condition, that is $\mathfrak{B}_1(1, \omega) < \infty$ if and only if $\mathfrak{B}(\omega) < \infty$ [27, 13, 6].

The *Bryuno function* is the solution of the functional equation [26, 27]

$$B(\alpha) = -\log \alpha + \alpha\, B(1/\alpha), \qquad \alpha \in (0, 1). \tag{5.4}$$

Again it is not difficult to show that, if $[\cdot]$ denotes the integer part, one has $\mathfrak{B}(\omega) < \infty$ if and only if $B(\omega - [\omega]) < \infty$; see for instance [27].

A somewhat similar Diophantine condition has been proposed by Rüssmann [22, 21] and is known as the *Rüssmann condition*. Such a condition can be shown to imply the Bryuno condition [23]. However, in dimension $d = 2$, is equivalent to the Bryuno condition, in the sense that, if $\boldsymbol{\omega} = (1, \omega)$, then ω satisfies the Rüssmann condition if and only if $\mathfrak{B}(\omega) < \infty$.

6 Generalisations

In the general case (1.1) the generating function has the form $S(x, x') = S_0(x' - x) + \varepsilon\sigma(x, x')$, with S_0 a primitive of the inverse function of a, which exists under the hypothesis that a satisfies the twist condition. Denoting $b = a^{-1}$ and writing the map as

$$\begin{cases} y' = b(x' - x) + \varepsilon \, \dfrac{\partial\sigma(x, x')}{\partial x'}, \\ y = b(x' - x) - \varepsilon \, \dfrac{\partial\sigma(x, x')}{\partial x}, \end{cases} \tag{6.1}$$

if we look for a conjugation of the form

$$x = \psi + h(\psi), \qquad y = b(2\pi\omega) + H(\psi),$$

we arrive at the functional equation

$$\begin{aligned} & b\,(2\pi\omega + h(\psi) - h(\psi - 2\pi\omega)) + \varepsilon\, \partial_2\sigma(\psi - 2\pi\omega + h(\psi - 2\pi\omega), \psi + h(\psi)), \\ & \quad = b\,(2\pi\omega + h(\psi + 2\pi\omega) - h(\psi)) - \varepsilon\, \partial_1\sigma(\psi + h(\psi), \psi + 2\pi\omega + h(\psi + 2\pi\omega)), \end{aligned} \tag{6.2}$$

which replaces (2.4). If we expand

$$b(2\pi\omega + h) = \sum_{k=0}^{\infty} B_k h^k, \qquad B_k = \frac{1}{k!}\, \frac{\partial^k}{\partial h^k} b(2\pi\omega + h)\bigg|_{h=0}, \tag{6.3}$$

and use that $B_1 \neq 0$, we obtain

$$\begin{aligned} & h(\psi + 2\pi\omega) + h(\psi - 2\pi\omega) - 2h(\psi) \\ & \quad = \sum_{k=2}^{\infty} \frac{B_k}{B_1} \Big[(h(\psi) - h(\psi - 2\pi\omega))^k - (h(\psi + 2\pi\omega) - h(\psi))^k \Big] \\ & \qquad\qquad + \frac{\varepsilon}{B_1}\, \partial_1 \sigma(\psi + h(\psi), \psi + 2\pi\omega + h(\psi + 2\pi\omega)) \\ & \qquad\qquad\qquad + \frac{\varepsilon}{B_1}\, \partial_2 \sigma(\psi - 2\pi\omega + h(\psi - 2\pi\omega), \psi + h(\psi)). \end{aligned} \tag{6.4}$$

Then by writing h as in (2.6), we have, instead of (2.7),

$$\begin{aligned} \delta(\omega\nu)\, h_\nu^{(k)} = & \sum_{s=2}^{\infty} \bar{B}_s \sum_{\nu_1+\ldots+\nu_s=\nu} \sum_{k_1+\ldots+k_s=k} \prod_{i=1}^{s} \delta_-(\omega\nu_i) h_{\nu_i}^{(k_i)} \\ & - \sum_{s=2}^{\infty} \bar{B}_s \sum_{\nu_1+\ldots+\nu_s=\nu} \sum_{k_1+\ldots+k_s=k} \prod_{i=1}^{s} \delta_+(\omega\nu_i) h_{\nu_i}^{(k_i)} \\ & + \sum_{p,q\geq 0} \sum_{\substack{\nu_0+\mu_0+\nu_1+\ldots+\nu_p \\ +\mu_1+\ldots+\mu_q=\nu}} \sum_{\substack{k_1+\ldots+k_p \\ +k'_1+\ldots+k'_q=k-1}} \frac{(\mathrm{i}\nu_0)^{p+1}}{p!} \frac{(\mathrm{i}\mu_0)^q}{q!} \sigma_{\nu_0,\mu_0} \\ & \qquad \mathrm{e}^{2\pi\mathrm{i}(\mu_0+\mu_1+\ldots+\mu_q)\omega} h_{\nu_1}^{(k_1)} \ldots h_{\nu_p}^{(k_p)} h_{\mu_1}^{(k'_1)} \ldots h_{\mu_q}^{(k'_q)} \\ & + \sum_{p,q\geq 0} \sum_{\substack{\nu_0+\mu_0+\nu_1+\ldots+\nu_p \\ +\mu_1+\ldots+\mu_q=\nu}} \sum_{\substack{k_1+\ldots+k_p \\ +k'_1+\ldots+k'_q=k-1}} \frac{(\mathrm{i}\nu_0)^{p}}{p!} \frac{(\mathrm{i}\mu_0)^{q+1}}{q!} \sigma_{\nu_0,\mu_0} \\ & \qquad \mathrm{e}^{-2\pi\mathrm{i}(\nu_0+\nu_1+\ldots+\nu_p)\omega} h_{\nu_1}^{(k_1)} \ldots h_{\nu_p}^{(k_p)} h_{\mu_1}^{(k'_1)} \ldots h_{\mu_q}^{(k'_q)}, \end{aligned} \tag{6.5}$$

where we have set

$$\delta(u) := 2(\cos 2\pi u - 1), \quad \delta_\pm = \pm\left(\mathrm{e}^{\pm\mathrm{i}\omega\nu} - 1\right), \quad \bar{B}_k = \frac{B_k}{B_1}, \quad \bar{\sigma}(x,x') = \frac{\sigma(x,x')}{B_1}.$$

By comparing (6.5) with (2.7), one easily realise that a diagrammatic expansion as derived in Section 2 is still possible. The main difference is that now, to take into account the extra contributions in the first two lines of (6.5), we have to associate with each node an extra label $\alpha_v = 0, 1$ with the following properties. For $\alpha_v = 0$ the mode label is $(\nu_v, \mu_v) \in \mathbb{Z}^2$ and the corresponding node factor is given by (2.10), with σ_{ν_v,μ_v} replaced by $\bar{\sigma}_{\nu_v,\mu_v} = \sigma_{\nu_v,\mu_v}/B_1$. For $\alpha_v = 1$ the mode label is $(0,0)$ and either $p_v = 0$ or $q_v = 0$, while the node factor is given by

$$A_v = \begin{cases} \bar{B}_{q_v} \displaystyle\prod_{\ell\in L_v(\theta)^-} \delta_-(\omega\nu_\ell), & p_v = 0, \\ -\bar{B}_{p_v} \displaystyle\prod_{\ell\in L_v^+(\theta)} \delta_+(\omega\nu_\ell), & q_v = 0. \end{cases} \tag{6.6}$$

Then, if we set $V_\alpha(\theta)=\{v\in V(\theta):\alpha_v=\alpha\}$ and define the order of θ as $|V_0(\theta)|$, we can define the set $\mathcal{T}_{k,\nu}$ as the set of trees of order k and momentum ν associated with the root line. The following result is easily proven.

Lemma 6.1. *Let θ be a tree of order k. Then the number of nodes of θ is less than $2k-1$.*

Apart from these changes the discussion proceeds like in Section 2. In particular the analogous of Lemma 2.6 still holds, so that we have the following result.

Lemma 6.2. *One has* $\displaystyle\sum_{\theta\in\mathcal{T}_{k,0}}\mathscr{V}(\theta)=0$ *for all* $k\geq 1$.

Of course, the proof requires some extra work, because we have to take into account the new nodes: we defer it to Appendix B. Then, by reasoning as in Section 2 one proves that the perturbation series (2.6) is well defined to all orders.

Also the analysis in Sections 3 and 4 can be performed in the same way, the only difference being that now the derivatives act also on the factors $\delta_\pm(\omega\nu_\ell)$ appearing in (6.6). In particular, by defining the set $\mathfrak{F}(T)$ as in Section 3, one has that the following analogue of Lemma 3.16 holds true.

Lemma 6.3. *For any self-energy cluster T one has*

$$\sum_{T'\in\mathfrak{F}(T)}\mathscr{V}_{T'}(0)=0,\qquad \sum_{T'\in\mathfrak{F}(T)}\partial_u\mathscr{V}_{T'}(0)=0.$$

Again, the proof requires some adaptations with respect to that of Lemma 3.16, because one has to check that the new node factors do not destroy the cancellation mechanisms acting on the tree values. We refer once more to Appendix B for the details.

Therefore Theorem 1.1 follows. Indeed, once Lemma 6.3 has been proven, the analysis is essentially a repetition of the analysis performed in Section 5, up to minor adaptations, that are required to take into account the presence of the new factors. The only relevant novelty with respect to the case (1.2) is the discussion, presented in Appendix B, of the cancellations leading to Lemma 6.2 and Lemma 6.3.

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A Cancellations

Let v be the node of a tree θ: define $\mathfrak{S}_v$ as the group of permutations of the subtrees entering v, i.e. of the trees that have as the root line one of the lines entering the node v. Given any line $\ell\in L(\theta)$, define $\mathfrak{B}_\ell$ as the group of transformations which either leave β_ℓ unchanged or swap it with its opposite.

Proof of Lemma 2.6. When performing the sum over the trees we reason as follows. Given a tree θ define $\mathcal{F}(\theta)$ as the family of trees obtained through the following operations:

1. detach the root line of θ and re-attach to any other node of θ;
2. apply a transformation in the Cartesian product over all $\ell \in L(\theta)$ of the groups $\mathfrak{B}_\ell$;
3. apply a transformation in the Cartesian product over all $v \in V(\theta)$ of the groups $\mathfrak{G}_v$.

We can rewrite the sum over the trees in $\mathcal{T}_{k,0}$ as

$$\sum_{\theta\in\mathcal{T}_{k,0}} \mathscr{V}(\theta) = \sum_{\theta\in\mathcal{T}_{k,0}} \frac{1}{|\mathcal{F}(\theta)|} \sum_{\theta'\in\mathcal{F}(\theta)} \mathscr{V}(\theta').$$

We want to show that for any $\theta \in \mathcal{T}_{k,0}$ one has

$$\sum_{\theta'\in\mathcal{F}(\theta)} \mathscr{V}(\theta') = 0. \tag{A.1}$$

For $v' \neq v$ consider the tree θ' obtained from θ by detaching the root line ℓ_0 from the node v and re-attaching it to the node v'. Call $\mathcal{P} = \mathcal{P}(v, v')$ the path of lines and nodes connecting v to v': denote by $w_1, \ldots, w_r$, $r \geq 2$, the ordered nodes in $\mathcal{P}$, with $w_1 = v$ and $w_r = v'$, and by $\ell_1, \ldots, \ell_{r-1}$ the lines connecting them. The node factors and propagators associated with the nodes and lines, respectively, which do not belong to $\mathcal{P}$ do not change.

The momenta of the lines $\ell \in \mathcal{P}$ revert their direction, so that if the line ℓ has momentum ν_ℓ in θ, then it acquires a momentum $\nu'_\ell = -\nu_\ell$ in θ' (we are using that the sum over all the mode labels is zero). In particular the propagators do not change (by parity). On the contrary, the node factors of the nodes along $\mathcal{P}$ change because of the change both of the momenta of the lines and of the combinatorial factors, as the number of lines entering the nodes may change.

Consider the trees obtained from θ by applying the transformations in the groups $\mathfrak{B}_\ell$ for all $\ell \in \mathcal{P}$, and sum together all the corresponding values. We obtain a quantity $A(\theta)$ of the form

$$A(\theta) = (\mathrm{i}\nu_v + \mathrm{i}\mu_v)\, A_0(\theta)\, B(\theta), \tag{A.2}$$

where

$$\begin{aligned} A_0(\theta) = \prod_{j=1}^{r-1} \left[\frac{(\mathrm{i}\nu_{w_j})^{P_j+1}}{(P_j+1)!} \frac{(\mathrm{i}\mu_{w_j})^{Q_j}}{Q_j!} + \frac{(\mathrm{i}\nu_{w_j})^{P_j}}{P_j!} \frac{(\mathrm{i}\mu_{w_j})^{Q_j+1}}{(Q_j+1)!} \mathrm{e}^{2\pi \mathrm{i}\nu_{\ell_j}\omega} \right] \times \\ \times \left[\frac{(\mathrm{i}\nu_{w_r})^{P_r}}{P_r!} \frac{(\mathrm{i}\mu_{w_r})^{Q_r}}{Q_r!} \right] \prod_{j=2}^{r} \left(\mathrm{i}\nu_{w_j} + \mathrm{i}\mu_{w_j} \mathrm{e}^{-2\pi\mathrm{i}\nu_{\ell_{j-1}}\omega} \right), \end{aligned} \tag{A.3}$$

and $B(\theta)$ takes into account all the other factors. In (A.3) we have called P_j and Q_j the number of lines entering the node w_j and not lying on the path $\mathcal{P}$ which carry a label $+$ and $-$, respectively. This means that, if we set $p_j = p_{w_j}$ and $q_j = q_{w_j}$, one has $(p_j, q_j) = (P_j + 1, Q_j)$ if $\beta_{\ell_j} = +$ and $(p_j, q_j) = (P_j, Q_j + 1)$ if $\beta_{\ell_j} = -$, while $(p_r, q_r) = (P_r, Q_r)$. The factor $(\mathrm{i}\nu_v + \mathrm{i}\mu_v)$ in (A.2) is due to the fact that ℓ_v is the root line of θ and hence $\nu_{\ell_v} = 0$ as $\theta \in \mathcal{T}_{k,0}$. When considering the analogous quantity $A(\theta')$ for θ' one finds $A(\theta') = (\mathrm{i}\nu_{v'} + \mathrm{i}\mu_{v'}) A_0(\theta') B(\theta')$, where we have used that $\nu_{\ell_{v'}} = 0$ in θ', with $A_0(\theta')$ and $B(\theta')$ defined in a similar way, by taking into account the different orientation of the lines along $\mathcal{P}$. However, one has $B(\theta') = B(\theta)$ by construction. The quantity $A_0(\theta')$ differs from $A_0(\theta)$ because the sets $L^{\pm}_{w_j}(\theta')$ may have changed with respect to $L^{\pm}_{w_j}(\theta)$ as a consequence of the shifting of the root line from v to v'. Consider the group $\mathfrak{G}_{w_1}$ and identifies the trees which are obtained from each other by the action of a transformation of the group. Suppose that $\beta_{\ell_1} = +$. If θ_1 is the subtree with root line ℓ_1 and $s_1 \geq 1$ is the number of subtrees equivalent to θ_1 entering w_1, the corresponding combinatorial factor is modified into

$$\frac{1}{p_1!}\binom{p_1}{s_1} = \frac{1}{s_1!(p_1-s_1)!}, \tag{A.4}$$

while the corresponding factor in θ' is

$$\frac{1}{(p_1-1)!}\binom{p_1-1}{s_1-1} = \frac{1}{(s_1-1)!(p_1-s_1)!},$$

which we have to divide by s_1 to avoid overcounting when considering all trees in $\mathcal{F}(\theta)$: indeed, there are in θ other s_1 nodes in the s_1 trees equivalent to θ_1 which produce trees to be identified with θ'. So that a combinatorial factor (A.4) is obtained for the node w_1 in both cases. Similarly one deal with the case in which $\beta_{\ell_1} = -$ and $r_j \geq 0$ is the number of subtrees entering w_1 equivalent to θ_1 (of course either r_1 or s_1 must be 0), so that we conclude that the overall combinatorial factor associated to the node w_1 is

$$\frac{1}{s_1!(p_1-s_1)!}\frac{1}{r_1!(q_1-r_1)!} \tag{A.5}$$

for both sets of trees obtained from θ and θ'.

Analogous considerations hold for the node w_r, with the roles of θ and θ' exchanged. Also the combinatorial factors of the other nodes along $\mathcal{P}$ can be dealt with in a similar way: it may happen that the number of subtrees equivalent to θ_j entering the node w_j, with $1 < j < r$, changes either from s_j to s_j+1 or from r_j to r_j+1, but then one has to divide the corresponding combinatorial factor by s_j+1 or r_j+1, in the respective cases, to avoid overcounting. On the other hand if the combinatorial factor changes either from s_j to s_j-1 or from r_j to r_j-1, then it is the combinatorial factor in θ which must be divided by s_j or r_j, respectively.

By summing together the values of all the inequivalent trees obtained by following the prescription above, we find that (A.3) is replaced by

$$\begin{aligned} A_1(\theta) := \prod_{j=1}^{r} \left[\frac{(\mathrm{i}\nu_{w_j})^{P_j}}{s_j!(P_j-s_j)!}\frac{(\mathrm{i}\mu_{w_j})^{Q_j}}{r_j!(Q_j-r_j)!} \right] \times \\ \times \left[\prod_{j=1}^{r-1} \left(\mathrm{i}\nu_{w_j} + \mathrm{i}\mu_{w_j} \mathrm{e}^{2\pi\mathrm{i}\nu_{\ell_j}\omega} \right) \right] \left[\prod_{j=2}^{r} \left(\mathrm{i}\nu_{w_j} + \mathrm{i}\mu_{w_j} \mathrm{e}^{-2\pi\mathrm{i}\nu_{\ell_{j-1}}\omega} \right) \right]. \end{aligned} \tag{A.6}$$

As noted before, the ordering of the nodes along $\mathcal{P}$ changes and each momentum ν_{ℓ_j} is replaced with $\nu'_{\ell_j} = -\nu_{\ell_j}$: thus, one finds $A_1(\theta') = A_1(\theta)$ too. In conclusion, if one defines $\mathscr{V}_0(\theta) = A_1(\theta)\, B(\theta)$, one has

$$\sum_{\theta' \in \mathcal{F}(\theta)} \mathscr{V}(\theta') = \mathscr{V}_0(\theta) \sum_{v \in V(\theta)} (\mathrm{i}\nu_v + \mathrm{i}\mu_v),$$

which yields (A.1). □

Proof of Lemma 3.16. The argument is very close to that used in the proof of Lemma 2.6. Given a self-energy cluster T call $\mathcal{F}(T)$ the set of all self-energy clusters T' obtained from T by the following operations:

1. detach the exiting line of T and re-attach it to any $v \in V(\mathring{T})$.
2. apply a transformation in the Cartesian product over all $\ell \in L(\mathring{T})$ of the groups $\mathfrak{B}_\ell$;
3. apply a transformation in the Cartesian product over all $v \in V(\mathring{T})$ of the groups $\mathfrak{G}_v$.

Any line $\ell \in \mathcal{P}_T$ divides T into two disjoint sets $T_1(\ell)$ and $T_2(\ell)$ such that $T_2(\ell)$ contains all the nodes and lines which precede ℓ and $T_1(\ell) = T \setminus (\{\ell\} \cup T_2(\ell))$. Define $L(T_j(\ell))$ and $V(T_j(\ell))$, $j = 1, 2$, in the obvious way. Call $\mathcal{F}_1(T;\ell)$ the family of self-energy clusters T' obtained from T by the following operations:

1. attach the exiting line to any node $v \in V(\mathring{T}_1(\ell))$ and the entering line to any node $v \in V(\mathring{T}_2(\ell))$,
2. apply a transformation in the Cartesian product over all $\ell \in L(\mathring{T})$ of the groups $\mathfrak{B}_\ell$;
3. apply a transformation in the Cartesian product over all $v \in V(\mathring{T})$ of the groups $\mathfrak{G}_v$.

Analogously, call $\mathcal{F}_2(T;\ell)$ the family of self-energy clusters T' obtained from T by the same operations as $\mathcal{F}_1(T;\ell)$, but by attaching the exiting line to any node $v \in V(\mathring{T}_2(\ell))$ and the entering line to any node $v \in V(\mathring{T}_1(\ell))$, instead of applying the operations 1.

We start by proving the first identity of Lemma 3.16. Consider the self-energy cluster $T' \in \mathcal{F}(T)$ obtained from T by detaching the root line from the node $v \in V(T)$ and re-attaching to the node $v' \in V(\mathring{T})$. Call $\mathcal{P} = \mathcal{P}(v, v')$ the path connecting v' to v: $\mathcal{P}$ contains r nodes $w_1 = v \succ w_2 \succ \ldots \succ w_r = v'$, connected by lines $\ell_1, \ldots, \ell_{r-1}$.

The propagators and the node factors associated with the lines and nodes, respectively, not belonging to $\mathcal{P}$ do not change. If we apply the transformations in the groups $\mathfrak{B}_\ell$ to all lines $\ell \in \mathcal{P}$ and we sum together the values of all the self-energy clusters we get, we realise that the sum of the products of the node factors associated with the nodes in $\mathcal{P}$, is of the form $A(T) = \left(\mathrm{i}\nu_{w_1} + \mathrm{i}\mu_{w_1} \mathrm{e}^{-2\pi \mathrm{i}\nu\omega}\right) A_0(T)\, B(T)$, where

$$
\begin{aligned}
A_0(T) = \prod_{j=1}^{r-1} \left[\frac{(\mathrm{i}\nu_{w_j})^{P_j+1}}{(P_j+1)!} \frac{(\mathrm{i}\mu_{w_j})^{Q_j}}{Q_j!} + \frac{(\mathrm{i}\nu_{w_j})^{P_j}}{P_j!} \frac{(\mathrm{i}\mu_{w_j})^{Q_j+1}}{(Q_j+1)!} \mathrm{e}^{2\pi \mathrm{i}\nu_{\ell_j}\omega} \right] \times \\
\times \left[\frac{(\mathrm{i}\nu_{w_r})^{P_r}}{P_r!} \frac{(\mathrm{i}\mu_{w_r})^{Q_r}}{Q_r!} \right] \prod_{j=2}^{r} \left(\mathrm{i}\nu_{w_j} + \mathrm{i}\mu_{w_j} \mathrm{e}^{-2\pi \mathrm{i}\nu_{\ell_{j-1}}\omega} \right),
\end{aligned}
$$

with $\nu_{\ell_j} = \nu^0_{\ell_j}$ if $\ell_j \notin \mathcal{P}_T$ and $\nu_{\ell_j} = \nu^0_{\ell_j} + \nu_{\ell'_T}$ otherwise, and $B(T)$ takes into account all the other factors. Here we are using the same notations as described after (2.3). Then we apply the transformations in the groups $\mathfrak{G}_{w_j}$ for $j = 1, \ldots, r$, and reason as in the proof of Lemma 2.6: summing together all the inequivalent self-energy clusters produces a quantity of the form

$\left(\mathrm{i}\nu_{w_1} + \mathrm{i}\mu_{w_1}\mathrm{e}^{-2\pi\mathrm{i}\nu\omega}\right) A_1(T)\, B(T)$, with

$$A_1(T) := \prod_{j=1}^{r} \left[\frac{(\mathrm{i}\nu_{w_j})^{P_j}}{s_j!(P_j - s_j)!} \frac{(\mathrm{i}\mu_{w_j})^{Q_j}}{r_j!(Q_j - r_j)!} \right] \times \\ \times \left[\prod_{j=1}^{r-1} \left(\mathrm{i}\nu_{w_j} + \mathrm{i}\mu_{w_j} \mathrm{e}^{2\pi\mathrm{i}\nu_{\ell_j}\omega} \right) \right] \left[\prod_{j=2}^{r} \left(\mathrm{i}\nu_{w_j} + \mathrm{i}\mu_{w_j} \mathrm{e}^{-2\pi\mathrm{i}\nu_{\ell_{j-1}}\omega} \right) \right],$$

The same procedure as above, when applied to the self-energy cluster T', produces a factor $\left(\mathrm{i}\nu_{w_1} + \mathrm{i}\mu_{w_1}\mathrm{e}^{-2\pi\mathrm{i}\nu\omega}\right) A_1(T')\, B(T')$, where $B(T') = B(T)$ and

$$A_1(T') := \prod_{j=1}^{r} \left[\frac{(\mathrm{i}\nu_{w_j})^{P_j}}{s_j!(P_j - s_j)!} \frac{(\mathrm{i}\mu_{w_j})^{Q_j}}{r_j!(Q_j - r_j)!} \right] \times \\ \times \left[\prod_{j=2}^{r} \left(\mathrm{i}\nu_{w_j} + \mathrm{i}\mu_{w_j} \mathrm{e}^{2\pi\mathrm{i}\nu'_{\ell_{j-1}}\omega} \right) \right] \left[\prod_{j=1}^{r-1} \left(\mathrm{i}\nu_{w_j} + \mathrm{i}\mu_{w_j} \mathrm{e}^{-2\pi\mathrm{i}\nu'_{\ell_j}\omega} \right) \right],$$

where $\nu'_{\ell_j} = -\nu^0_{\ell_j}$ if $\ell_j \notin \mathcal{P}_T$ and $\nu'_{\ell_j} = -\nu^0_{\ell_j} + \nu_{\ell'_T}$ otherwise. Therefore for $\omega\nu_{\ell'_T} = 0$ (so that $\omega\nu = 0$ as well) one has $A_1(T) = A_2(T')$ and hence

$$\sum_{T' \in \mathcal{F}(T)} \mathscr{V}_{T'}(0) = A_1(T)\, B(T) \sum_{v \in V(\mathring{T})} (\mathrm{i}\nu_v + \mathrm{i}\mu_v) = 0,$$

and since all the self-energy clusters in $\mathcal{F}(T)$ belong to the set $\mathfrak{F}(T)$ the first identity follows.

Now we pass to the second identity of Lemma 3.16. First of all, we note that the derivative ∂_u acts on the propagators $\delta(\omega\nu_\ell)$ and on the factors $\mathrm{e}^{\pm 2\pi\mathrm{i}\omega\nu_\ell}$ of the node factors corresponding to lines $\ell \in \mathcal{P}_T$.

Given a self-energy cluster T let ℓ be a line along the path $\mathcal{P}_T$. Call $\mathcal{A}_T(0;\ell)$ the contribution to $\partial_u \mathscr{V}_T(0)$ obtained by differentiating the propagator of the line ℓ. The differentiated propagator is

$$\partial_u \mathcal{G}_\ell|_{u=0} = \partial_u \left(1/\delta(\omega\nu_\ell + u) \right)|_{u=0} = \frac{2\pi \sin 2\pi\omega\nu_\ell}{\delta^2(\omega\nu_\ell)} = \frac{\pi \sin 2\pi\omega\nu_\ell}{2(\cos 2\pi\omega\nu_\ell - 1)^2}.$$

Consider all the self-energy clusters $T' \in \mathcal{F}_1(T;\ell)$: by reasoning as in the proof of the first identity one finds that

$$\sum_{T' \in \mathcal{F}_1(T;\ell)} \mathcal{A}_{T'}(0;\ell) = \partial_u \mathcal{G}_\ell|_{u=0}\, C_0(T) \sum_{v \in V(\mathring{T}_1)} (\mathrm{i}\nu_v + \mathrm{i}\mu_v) \sum_{w \in V(\mathring{T}_2)} (\mathrm{i}\nu_w + \mathrm{i}\mu_w)\,, \tag{A.7}$$

for some common factor $C_0(T)$. Analogously one finds

$$\sum_{T' \in \mathcal{F}_2(T;\ell)} \mathcal{A}_{T'}(0;\ell) = \partial_u \mathcal{G}_\ell|_{u=0}\, C_0(T) \sum_{v \in V(\mathring{T}_2)} (\mathrm{i}\nu_v + \mathrm{i}\mu_v) \sum_{w \in V(\mathring{T}_1)} (\mathrm{i}\nu_w + \mathrm{i}\mu_w)\,, \tag{A.8}$$

with the same factor $C_0(T)$ as in (A.7); we have used that if $\nu = \nu_{\ell'_T}$ and $\nu_{\ell'} + \sigma\nu$, with $\sigma \in \{0,1\}$, is the momentum associated with the line $\ell' \in \mathcal{P}_{T'}$ in $T' \in \mathcal{F}_1(T;\ell)$, then the momentum associated with the same line ℓ' in $T'' \in \mathcal{F}_2(T;\ell)$ is either $\nu_{\ell'} + \sigma'\nu$ or $-\nu_{\ell'} + \sigma''\nu$, with $\sigma'\,\sigma'' \in \{0,1\}$; moreover all propagators and node factors have to computed at $u = \omega\nu = 0$. Therefore if we sum together the two contributions (A.7) and (A.8) they cancel out.

Call $\mathcal{B}_T(0;\ell)$ and $\mathcal{C}_T(0;\ell)$ the contributions obtained by differentiating the factor $\mathrm{i}\mu_{w(\ell)}\mathrm{e}^{2\pi\mathrm{i}\nu_\ell\omega}$ and the factor $\mathrm{i}\mu_{w'(\ell)}\mathrm{e}^{-2\pi\mathrm{i}\nu_\ell\omega}$, respectively. The derivative, computed at $\omega\nu = 0$, gives an extra constant factor $2\pi\mathrm{i}$ and $-2\pi\mathrm{i}$, respectively. Therefore, with respect to $\mathscr{V}_T(0)$, $\mathcal{B}_T(0;\ell)$ and $\mathcal{C}_T(0;\ell)$ contain a factor

$$-2\pi\mathrm{i}\left(\mu_{w(\ell)}\nu_{w'(\ell)}\mathrm{e}^{2\pi\mathrm{i}\nu_\ell\omega} + \mu_{w(\ell)}\mu_{w'(\ell)}\right) \quad \text{and} \quad 2\pi\mathrm{i}\left(\nu_{w(\ell)}\mu_{w'(\ell)}\mathrm{e}^{-2\pi\mathrm{i}\nu_\ell\omega} + \mu_{w(\ell)}\mu_{w'(\ell)}\right)$$

respectively, instead of $(\mathrm{i}\nu_{w(\ell)}+\mathrm{i}\mu_{w(\ell)}\mathrm{e}^{-2\pi\mathrm{i}\nu_\ell})\,(\mathrm{i}\nu_{w'(\ell)}+\mathrm{i}\mu_{w'(\ell)}\mathrm{e}^{2\pi\mathrm{i}\nu_\ell})$. Therefore the contributions containing the terms $\pm 2\pi\mathrm{i}\mu_{w(\ell)}\mu_{w'(\ell)}$ cancel out. When summing the remaining contributions arising from $\mathcal{B}_{T'}(0;\ell)$ and $\mathcal{C}_{T'}(0;\ell)$, for $T' \in \mathcal{F}_1(T,\ell)$ we obtain a quantity which is, up to the sign, the same as the quantity obtained by summing all contributions $\mathcal{B}_{T''}(0;\ell)$ and $\mathcal{C}_{T''}(0;\ell)$ for $T'' \in \mathcal{F}_2(T,\ell)$. The argument is the same as before: again one uses the parity of the propagator and the fact that ν_ℓ becomes $-\nu_\ell$ when passing from T' to T''. □

B Cancellations in the general case

We use the same notations as in Appendix A. Moreover we need the following identities.

Lemma B.1. *Let $x_1,\ldots,x_n \in \mathbb{R}$ such that $x_1+\ldots+x_n = 0$. Then*

$$\prod_{i=1}^{n}\left(1-\mathrm{e}^{-x_i}\right) = \prod_{i=1}^{n}\left(\mathrm{e}^{x_i}-1\right).$$

Proof. Write $\mathrm{e}^{x_i}-1 = \mathrm{e}^{x_i}(1-\mathrm{e}^{-x_i})$ in each factor of the second product. □

Lemma B.2. *Let $x_1,\ldots,x_n \in \mathbb{R}$ such that $x_1+\ldots+x_n = 0$. Then*

$$\sum_{i=1}^{n} x_i \prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\mathrm{e}^{-x_j}\right) = \sum_{i=1}^{n} x_i \prod_{\substack{j=1\\ j\neq i}}^{n}\left(\mathrm{e}^{x_j}-1\right).$$

Proof. We can rewrite

$$\sum_{i=1}^{n} x_i \prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\mathrm{e}^{-x_j}\right) = \sum_{i=1}^{n} x_i \left(1-\mathrm{e}^{-x_i}\right)^{-1} \prod_{j=1}^{n}\left(1-\mathrm{e}^{-x_j}\right)$$

and, analogously, using that $x_1+\ldots+x_n = 0$,

$$\sum_{i=1}^{n} x_i \prod_{\substack{j=1\\ j\neq i}}^{n}\left(\mathrm{e}^{x_j}-1\right) = \sum_{i=1}^{n} x_i \left(\mathrm{e}^{x_i}-1\right)^{-1} \prod_{j=1}^{n}\left(\mathrm{e}^{x_j}-1\right) = \sum_{i=1}^{n} x_i \left(\mathrm{e}^{x_i}-1\right)^{-1} \prod_{j=1}^{n}\left(1-\mathrm{e}^{-x_j}\right),$$

so that the identity follows if we prove that

$$\sum_{i=1}^{n} x_i \left(1 - \mathrm{e}^{-x_i}\right)^{-1} = \sum_{i=1}^{n} x_i \left(\mathrm{e}^{x_i} - 1\right)^{-1}$$

for all $x_1, \ldots, x_n$ such that $x_1 + \ldots + x_n = 0$. By writing $\left(1 - \mathrm{e}^{-x_i}\right)^{-1} = \mathrm{e}^{x_i} \left(\mathrm{e}^{x_i} - 1\right)^{-1}$, we arrive at

$$\sum_{i=1}^{n} x_i \mathrm{e}^{x_i} \left(\mathrm{e}^{x_i} - 1\right)^{-1} = \sum_{i=1}^{n} x_i \left(\mathrm{e}^{x_i} - 1\right)^{-1} \qquad \Longrightarrow \qquad \sum_{i=1}^{n} x_i \left(\mathrm{e}^{x_i} - 1\right) \left(\mathrm{e}^{x_i} - 1\right)^{-1} = 0$$

which is trivially satisfied if $x_1 + \ldots + x_n = 0$. $\square$

Proof of Lemma 6.2. Let θ be a tree in $\mathcal{T}_{k,0}$. If the root line exits a node v with $\alpha_v = 1$ the corresponding value vanishes by Lemma B.1. For any node $v \in V(\theta)$ with $\alpha_v = 1$ call Λ_v the set of lines incident with the nodes v (i.e. either exiting or entering v) and, for $\ell \in \Lambda_v$, set $\Lambda_v(\ell) = \Lambda_v \setminus \{\ell\}$ and

$$\Delta(v, \ell) := \prod_{\ell' \in \Lambda_v(\ell)} \delta_-(\omega \nu_{\ell'}) - \prod_{\ell' \in \Lambda_v(\ell)} \delta_+(\omega \nu_{\ell'}).$$

For $\theta \in \mathcal{T}_{k,0}$ any $\ell \in L(\theta)$ divides θ into two distinct sets θ_ℓ^1 and θ_ℓ^2, where θ_ℓ^1 is the set of lines and nodes in θ which precede ℓ and θ_ℓ^2 is the set of lines and nodes which do not precede ℓ. If $V(\theta_\ell^i)$ and $L(\theta_\ell^i)$ denote, respectively, the set of nodes and the set of lines in θ_ℓ^i, one has $V(\theta) = V(\theta_\ell^1) \cup V(\theta_\ell^2)$ and $L(\theta_\ell^1) \cup \{\ell\} \cup L(\theta_\ell^2)$. For $i = 1, 2$, if $\{v \in V(\theta_\ell^i) : \alpha_v = 1\} = \emptyset$ set $S_\ell(\theta_\ell^i) = 1$, otherwise set

$$S_\ell(\theta_\ell^i) := \prod_{\substack{v \in V(\theta_\ell^i) \\ \alpha_v = 1}} \Delta(v, \ell(v)), \tag{B.1}$$

where $\ell(v) \in \Lambda_v$ is the line incident with v along the path connecting v with ℓ. If θ' is a subset of θ_ℓ^i we define $S_\ell(\theta')$ as in (B.1), with the product restricted to the nodes $v \in V(\theta')$.

Define the family $\mathcal{F}(\theta)$ as in the proof of Lemma 2.6 in Appendix A, with the further operations of replacing p_v with q_v for each node $v \in V(\theta)$. Define also the family $\mathcal{F}_\ell^i(\theta)$, $i = 1, 2$, as the family obtained through the same operations considered for $\mathcal{F}(\theta)$, but with the operation 1 applied only to the nodes $v \in V(\theta_\ell^i)$. We want to prove that

$$\sum_{\theta' \in \mathcal{F}_\ell^i(\theta)} \mathscr{V}(\theta') = \nu(\theta_\ell^i)\, S_\ell(\theta_\ell^1)\, S_\ell(\theta_\ell^2)\, B(\theta), \qquad i = 1, 2, \tag{B.2}$$

where we have defined

$$\nu(\theta_\ell^i) := \sum_{v \in V(\theta_\ell^i)} \left(\nu_v + \mu_v\right),$$

and $B(\theta)$ is a suitable common factor independent of θ'. Note that $\nu(\theta_\ell^1) = -\nu(\theta_\ell^2) = \nu_\ell$.

The identity (B.2) immediately implies Lemma 6.2. Indeed, assuming (B.2), one can write, for any line $\ell \in L(\theta)$,

$$\sum_{\theta' \in \mathcal{F}(\theta)} \mathscr{V}(\theta') = \sum_{\theta' \in \mathcal{F}^1_\ell(\theta)} \mathscr{V}(\theta') + \sum_{\theta' \in \mathcal{F}^2_\ell(\theta)} \mathscr{V}(\theta') = \left(\nu(\theta^1_\ell) + \nu(\theta^2_\ell)\right) B(\theta) S_\ell(\theta^1_\ell) S_\ell(\theta^2_\ell),$$

which yields the assertion since $\nu(\theta^1_\ell) + \nu(\theta^2_\ell) = 0$.

Hence, we are left with the proof of (B.1). This is done is by induction on the number of nodes v with $\alpha_v = 1$ contained in θ^i_ℓ. If θ^i_ℓ does not contain any node v with $\alpha_v = 1$, then the assertion follows immediately from the same argument used in proving Lemma 2.6. Otherwise let $N \geq 1$ be the number of such nodes contained, say, in θ^2_ℓ. Let v be the node in $V(\theta^2_\ell)$ such that $\alpha_v = 1$ and $\alpha_w = 0$ for all nodes $w \in V(\theta_0)$, where θ_0 is the set of nodes and lines which precede ℓ and do not precede ℓ_v If $\ell_1, \ldots, \ell_s$, with $s \geq 2$, is the number of lines entering v, call $\theta_1, \ldots, \theta_s$ the subtrees with root lines $\ell_1, \ldots, \ell_s$, respectively. If we sum the values of all trees in $\mathcal{F}(\theta)$ obtained by attaching the root line to a node $v \in V(\theta_0)$ we obtain a contribution containing a common factor $B_1(\theta)$ times

$$\nu(\theta_0)\, \Delta(v, \ell_v) \prod_{j=1}^{s} S_\ell(\theta_j)$$

by Lemma B.2. Analogously, if we sum the values of all trees in $\mathcal{F}(\theta)$ obtained by attaching the root line to a node $v \in V(\theta_i)$, we obtain a contribution containing the same common factor $B_1(\theta)$ times

$$\nu(\theta_i)\, \Delta(v, \ell_i) \prod_{j=1}^{s} S_\ell(\theta_j).$$

This follows from the the inductive hypothesis, noting that $S_\ell(\theta_i) = S_{\ell_i}(\theta_i)$. Therefore, by summing together all such contributions, we arrive at a common factor times

$$\left(\nu(\theta_0)\Delta(v, \ell_v) + \sum_{i=1}^{s} \nu(\theta_i)\Delta(v, \ell_i) \right) \prod_{j=1}^{s} S_\ell(\theta_j).$$

Since $\theta \in \mathcal{T}_{k,0}$, when the root line is attached to a node $v \in \theta_i$, the momentum associated with the line ℓ_v is $-\nu_{\ell_v}$. Then, using that $\nu(\theta_0) = \nu_\ell - \nu_{\ell_v}$ and that

$$-\nu_{\ell_v}\Delta(v, \ell_v) + \sum_{i=1}^{s} \nu(\theta_i)\Delta(v, \ell_i) = 0$$

by Lemma 6.2, we obtain

$$\sum_{\theta' \in \mathcal{F}^2_\ell(\theta)} \mathscr{V}(\theta') = \nu_\ell\, B_1(\theta)\, \Delta(v, \ell(v)) \prod_{i=1}^{s} S_\ell(\theta_i),$$

where we have used that $\ell_v = \ell(v)$. The inductive hypothesis also implies that $B_1(\theta)$ contains a factor $\Delta(w, \ell(w))$ for each node w with $\alpha_w = 1$ in both $V(\theta^1_\ell)$ and $\cup_{i=1}^{s} V(\theta_i)$, so that the identity (B.2) is proven for $i = 2$. The case $i = 1$ can be discussed in the same way. □

Proof of Lemma 6.3. One combines the proof of Lemma 3.16 with the proof of Lemma 6.2. The proof of the first identity is similar to the proof of Lemma 6.2, as well as the first identity of Lemma 3.16 was obtained by following closely the proof of Lemma 2.6. To prove the second identity one reasons again as done for Lemma 3.16, by using the identity (B.2), with ℓ being the line which is differentiated when the derivative acts on the propagator $\mathcal{G}_\ell$. □

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