THE SHARP CORNER FORMATION IN 2D EULER DYNAMICS OF PATCHES: INFINITE DOUBLE EXPONENTIAL RATE OF MERGING

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ABSTRACT. For the 2d Euler dynamics of patches, we investigate the convergence to the singular stationary solution in the presence of a regular strain. The rate of merging as well as the growth of curvature are shown to be double exponential.

1. Introduction and statement of results

In this paper, we study the 2d Euler dynamics of patches. This problem attracted a lot of attention in the both physics and mathematical literature in the last several decades and became a classical one. The existence of weak solutions for Euler equation is due to Yudovich [13]; his result, in particular, ensures that the dynamics of patches is well-defined. In [3], Chemin proved that if the boundary of the patch is sufficiently regular then it will retain the same regularity forever; another proof of that fact was given later by Bertozzi and Constantin [1]. For closely related models (e.g., SQG), there were attempts recently to prove that a singularity can occur in finite time (see, e.g., [5] and related [6, 8, 10]). We recommend the wonderful books [2, 4] for introduction to the subject and for simplified proofs.

Although we have global regularity for the 2d Euler dynamics of contours, very little was known about the lower bounds on the curvature growth or on the distance between two interacting patches. In this paper, we show that some known bounds are sharp and, perhaps more importantly, explain the mechanism of the singularity formation.

In $\mathbb{R}^2 \sim \mathbb{C}$, we consider the 2d Euler dynamics of two identical patches that are symmetric with respect to the origin. These patches will be infinitely smooth and separated from each other for all times. The areas of the patches, the distance between them, the value of vorticity, the curvature of the boundary– all these quantities are of order one as $t = 0$.

Let $\Omega' = \{-z, z \in \Omega\}$ be the image of Ω under the central symmetry. The main result of the paper is

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Theorem 1.1. *Let positive δ be sufficiently small. Then, there is a simply connected domain $\Omega(0)$ with smooth boundary $\Gamma(0)$, $\mathrm{dist}(\Omega(0),\Omega'(0))\sim 1$, and a time-dependent incompressible odd strain*

$$S(z,t)=(P(z,t),Q(z,t))$$

such that

$$\mathrm{dist}(\Omega(t),0)\lesssim e^{-e^{\delta t}}$$

where $\Omega(t)$ is the Euler dynamics of $\Omega(0)$ in the presence of the strain $S(z,t)$. Moreover, $S(z,t)$ can be taken Lipschitz regular in z and

$$\sup_{z,t}\frac{|S(z,t)|}{|z|}<\infty \tag{1}$$

Remark 1. These contours will touch each other at $t=+\infty$ and the touching point is at the origin. In the local coordinates around the origin the functions parameterizing the contours converge to $\pm|x|$ in a self-similar way such that the curvature grows in the double exponential rate. In the lemma 2.1 below, we show that under the S–strain alone no point can approach the origin in the rate faster than exponential as long as assumption (1) is made and so it is the nonlinear term that produces the "double exponentially" fast singularity formation. On the PDE level, we will construct the approximate solution to the 2d Euler dynamics of contours where the error can be interpreted as the strain. It is important to mention here that two centrally symmetric patches can not approach the origin in the rate faster than double exponential even if one places them in the strain S from the theorem 1.1. In other words, the following estimate holds true

$$\mathrm{dist}(\Omega(t),0)\gtrsim e^{-e^{Dt}}$$

which some positive constant D. This is an immediate corollary of the lemma 8.1 from [2], page 315. The constant δ from the theorem (as well as D above) can be changed by a simple rescaling (e.g., multiplying the value of vorticity by a constant or by scaling the patches around the origin) thus the size of δ is small only when compared to the parameters of the problem.

The interaction of two vortices was extensively studied in the physics literature (see, e.g., [12, 11]). For example, the merger mechanism was discussed in [11] where some justifications (both numerical and analytical) were given. In our paper, we provide rigorous analysis of that process and obtain the sharp bounds.

In [7], the authors study an interesting question of the "sharp front" formation. Loosely speaking, the sharp front forms if, for example, two level sets of vorticity, each represented by a smooth time-dependent curve, converge to a fixed smooth curve as $t\to\infty$. Let the "thickness" of the front be denoted by $\delta(t)$. In [7], the following estimate for 2d Euler dynamics is given (see theorem 3, p. 4312)

$$\delta(t)>e^{-(At+B)}$$

with constants A and B depending only on the geometry of the front. The scenario considered in our paper is different as the singularity forms at a point.

The idea of the proof comes from the following very natural question. Consider an active-scalar dynamics

$$\dot\theta=\nabla\theta\cdot\nabla^{\perp}(A\theta)$$

where A is the convolution with a kernel $K(\xi)$ and $\nabla^\perp$ denotes $(\partial_y, -\partial_x)$. In many cases, $A = \Delta^{-\alpha}$ with $\alpha > 0$ so K is positive near the origin, smooth away from the origin, and obeys some symmetries inherited from its symbol on the Fourier side, e.g., is radially symmetric. The physically motivated cases are $\alpha = 1$ (2d Euler) which will be treated here and $\alpha = 1/2$ (SQG) which is somewhat less established in physics literature but still very interesting mathematically. If one considers the problem on the 2d torus $\mathbb{T}^2 = [-\pi, \pi]^2$, then there is a stationary singular weak solution (the author learned about this solution from [10]), a "cross"

$$\theta_s = \chi_E + \chi_{E'} - \chi_J - \chi_{J'}$$

where E is the first quadrant and J is the second one (see [9]). One can think about two patches touching each other at the origin and each forming the right angle. This picture is also centrally symmetric. Now, the question is: is this configuration stable? In other words, can we perturb these patches a little so that they will converge to the stationary solution at least around the origin? The flow generated by θ_s is hyperbolic and so is unstable. However, a suitably chosen curve placed into the stationary hyperbolic flow will converge to a sharp corner as time evolves. The problem of course is that the actual flow is induced by the patch itself and so it will be changing in time. That suggests that one has to be very careful with the choice of the initial patch to guarantee that this process is self-sustaining. Nevertheless, that seems possible and thus the mechanism of singularity formation through the hyperbolic flow can probably be justified. We do it here by neglecting the smaller order terms. In general, the application of some sort of fixed point argument seems to be needed. Either way, this scenario is a zero probability event if the "random" initial condition is chosen. However, if one wants the steep growth of the curvature for a long time, this can be achieved for the open set of the initial data so from that perspective our construction is realistic.

We will handle the case $\alpha = 1$ only. For $\alpha > 1$, the same argument shows that the curvature can grow exponentially in time and the strain can be taken exponentially decaying. In this case the process can be made self-similar with a rich family of explicit curves that appear as the scaling limits. However, in general the process does not have to be self-similar. For $\alpha < 1$, we expect our technique to show that contours can touch each other in finite time thus proving the blow-up. This, however, will require some serious refinement of our method and will be addressed in a separate publication.

There are many examples of singular contours that are stationary under the Euler dynamics provided that certain smooth strain is present (e.g., a rotation). It would be interesting to perform the stability analysis for each of them.

2. Definition and properties of $\Psi(z,t)$ and $\Gamma(t)$

We start with brief explanations. In $\mathbb{R}^2$, in contrast to $\mathbb{T}^2$, the kernel of Δ^{-1} is easier to write and Δ^{-1} can be defined on compactly supported L^2 functions so we will address the problem on the whole plane rather than on $\mathbb{T}^2$. On the 2d torus, similar results hold.

We first consider incompressible strain Ψ which satisfies the following properties (for more details and the picture, see Appendix, Figure 1):

1. Ψ is odd and is compactly supported.

2. Around the points $(4,4)$ and $(-2,2)$ it is the standard hyperbolic time-independent flow. Say, at $(-2,2)$, we choose the separatrices to be $y_1=-x$ and $y_2=2+(x+2)$. The flow is attracting along y_2 and is repelling along y_1. This flow will produce (see lemma 2.2) the needed initial condition for the flow around the origin which we discuss next.

3. Around the origin, we choose $\Psi=-(\psi_1,\psi_2)$ such that

$$\psi_1=-y\log y,\ \psi_2=-x\log y \tag{2}$$

if $x\sim 0, y>0$. The important point is that $\Psi\sim\nabla^\perp\Delta^{-1}(\chi_N+\chi_{N'})$ where $N=\{(x,y),x\in[-1,1],|x|<y<1\}$ and the errors are Lipschitz and can be neglected (analogous calculation in case of $\mathbb{T}^2$ was done in [9], formula (7). In this formula, do the change of variables $u=y-x, v=y+x$ and collect the errors).

Then, we carefully choose the domain $\Omega(0)$ with $\Gamma(0)=\partial\Omega(0)$ and let it evolve under the flow Ψ producing $\Omega(t)$ and $\Gamma(t)$.

Formally, we have

$$\dot\theta=\nabla\theta\cdot\Psi(z,t)$$

and $\theta(z,t)=\chi_{\Omega(t)}(z)$. However, the explicit calculation will show that

$$\left(\nabla^\perp\Delta^{-1}\theta\right)_n=\Psi(z,t)-R(z,t)$$

when restricted to $\Gamma(t)$ and the normal component is taken in the left hand side. The error $R(z,t)$ is more regular than $\Psi(z,t)$ (and odd as θ is even) and it is also divergence free as the difference of two divergence free vector fields. Thus, we have

$$\dot\theta=\nabla\theta\cdot\left(\nabla^\perp\Delta^{-1}\theta+R(z,t)\right) \tag{3}$$

on the boundary. Now, the point is that the dynamics of $\Omega(t)$ is very explicit and one has

$$\mathrm{dist}(\Omega(t),\Omega'(t))=2\mathrm{dist}(\Omega(t),0)\sim e^{-e^{\delta t}}\to 0$$

where δ is small. The curvature of $\Gamma(t)$ grows as double exponential.

The next lemma shows that the double exponential rate of convergence to zero can not be reached merely by the $R(z,t)$ term in the transport equation (3).

Lemma 2.1. *Let $R(z,t)$ be an odd vector field satisfying (1) and globally Lipschitz regular in z. Consider $f(z,t)=\chi_{\Omega(t)}(z)$ that solves*

$$\dot f=\nabla f\cdot R(z,t),\quad f(z,0)=\chi_{\Omega(0)}(z)$$

and $\mathrm{dist}(\Omega(0),0)>0$. *Then,*

$$\mathrm{dist}(\Omega(t),0)\gtrsim e^{-Ct}$$

Proof. For the characteristics, we have an equation

$$\dot z=-R(z,t),\quad z(0)=z_0$$

As R is odd, $R(0,t)=0$ and (1) yields

$$|R(z,t)|<L|z|,\quad\forall t\ge 0$$

Therefore,

$$|\dot r|\le Lr,\quad r=|z|^2$$

and

$$r(0)e^{-Lt}\le r(t)$$

□

The purpose of the next lemma is to show how to choose the initial contour around the point $(-2,2)$ to later guarantee necessary initial conditions for the dynamics around $(0,0)$. Let the local coordinates near $(-2,2)$ be denoted by (ξ,η).

Lemma 2.2. *Fix any $\delta>0$ and consider the standard hyperbolic dynamics around the origin*

$$\begin{cases} \dot\xi=\xi, & \xi(0)=\xi_0 \\ \dot\eta=-\eta, & \eta(0)=\eta_0 \end{cases}$$

Let $f(\xi)=0$ for $\xi\le 0$ and $f(\xi)=\xi^{-1}\exp(-\xi^{-\delta})$ for $\xi>0$. Consider the evolution of the smooth curve $\Gamma(0)=\{(\xi,f(\xi)),|\xi|<1\}$ under this flow. Call it $\Gamma(t)=\{(\xi,f(\xi,t)),|\xi|<1\}$. Then,

$$f(1,t)=e^{-e^{\delta t}}$$

Proof. This is a straightforward calculation. The point (ξ_0,η_0) moves to (ξ_0e^t,η_0e^{-t}) in time t. Thus, the point $(e^{-t},f(e^{-t}))\in\Gamma(0)$ will move to $(1,e^{-e^{\delta t}})$ in time t. □

Remark 2. Notice that the part of $\Gamma(t)$ that belongs to the left half-plane does not change in time. Within the window $|\xi|<1$, the curve $\Gamma(t)$ is always smooth and converges to the coordinate axis.

The next calculations describe the evolution of $\Gamma(0)$ in the stationary flow generated by Ψ which is given by (2) around the origin. Consider the following dynamical system

$$\begin{cases} \dot x=-y\log y, & x(0)=x_0 \\ \dot y=-x\log y, & y(0)=y_0\ge|x_0| \end{cases} \tag{4}$$

within the window $\{|x|<0.5,\ 0<y<1\}$. Let us consider the smooth curve $\Gamma(t)=\{y=g(x,t),|x|<0.5\}$ evolving under this flow. To define its dynamics within the window $(-0.5,0.5)$ it is sufficient to say what the value of $g(-0.5,t)$ is at every moment t and we also need to prescribe $g(x,0)$ (however this is not so important since these points will quickly leave the domain of interest). We will make the following choice

$$g(-0.5,t)=\sqrt{0.25+\epsilon^2(t)}$$

where

$$\epsilon(t)=\exp(-\exp(\delta t))$$

The lemma 2.2 above shows that this regime can be reached in the natural way. Now, some properties of the dynamics (4) are in order.

(A) This dynamics is hyperbolic, the separatrices are $y_{1(2)}=\pm|x|$. In particular, $g(x,t)>|x|$ all the time. The origin is stable under this dynamics.

(B) The invariant sets that belong to $\{y>|x|\}$ are given by the family of hyperbolas $y_a=\sqrt{x^2+a^2}$. Let us call $a>0$ the "tag" of the hyperbola. We will be interested in the small values of parameter a. Had we chosen $g(-0.5,t)$ to be a constant, $\Gamma(t)$ would have eventually became the corresponding hyperbola.

(C) The quantity $y^2(t)-x^2(t)$ is an invariant of the motion so each point starting at time $t=T$ in its original position $(-0.5,g(-0.5,T))$ will slide along the hyperbola with changing speed until it leaves the window $|x|<0.5$. This hyperbola will be tagged by $a(T)=\sqrt{g^2(-0.5,T)-0.25}=\epsilon(T)$.

(D) Due to the last observation, we can integrate (4) to get

$$2\int_{-0.5}^{x(t)} \frac{d\xi}{\sqrt{\xi^2+\epsilon^2(T)}\log(\xi^2+\epsilon^2(T))} = -(t-T) \tag{5}$$
$$\int_{\sqrt{0.25+\epsilon^2(T)}}^{y(t)} \frac{d\eta}{\sqrt{\eta^2-\epsilon^2(T)}\log\eta} = t-T$$

in the left half-plane. We need the following

Proposition 2.1. *We have*

$$\int_\beta^{0.5} \frac{dx}{|\log(x^2+\epsilon^2)|\sqrt{x^2+\epsilon^2}} = \frac{1}{2}\begin{cases} \log\log\beta^{-1}, & \beta\in(\epsilon,0.5) \\ \log\log\epsilon^{-1}, & \beta\in(0,\epsilon] \end{cases}$$
$$-\frac{1}{2}\log\log 2 + O(|\log\epsilon|^{-1})$$

Proof. First, take $\beta=0$ and split the integral into two. The first one gives

$$\left|\int_0^\epsilon \frac{dx}{|\log(x^2+\epsilon^2)|\sqrt{x^2+\epsilon^2}}\right| \lesssim \frac{1}{|\log\epsilon|}\int_0^\epsilon \frac{dx}{\sqrt{x^2+\epsilon^2}} = \frac{C}{|\log\epsilon|}$$

and for the second one

$$\int_\epsilon^{0.5} \frac{dx}{|\log(x^2+\epsilon^2)|\sqrt{x^2+\epsilon^2}} = \int_1^{0.5\epsilon^{-1}} \frac{dx_1}{|2\log\epsilon+\log(x_1^2+1)|\sqrt{x_1^2+1}}$$

$$= \int_1^{0.5\epsilon^{-1}} \frac{dx_1}{x_1|2\log\epsilon+\log(x_1^2+1)|} + \int_1^{0.5\epsilon^{-1}} \frac{dx_1}{x_1|2\log\epsilon+\log(x_1^2+1)|}\left(\frac{1}{\sqrt{x_1^2+1}}-\frac{1}{x_1}\right)$$

The second integral is bounded by $C|\log\epsilon|^{-1}$. For the first one, we obtain similarly

$$\int_1^{0.5\epsilon^{-1}} \frac{dx_1}{x_1|2\log\epsilon+\log(x_1^2)|} + O(|\log\epsilon|^{-1}) = \frac{1}{2}\int_\epsilon^{0.5}\frac{dx}{x|\log x|} + O(|\log\epsilon|^{-1})$$
$$= \frac{1}{2}\log\log\frac{1}{\epsilon} - \frac{1}{2}\log\log 2 + O(|\log\epsilon|^{-1})$$

For any $\beta>0$, one can repeat the same calculations to obtain the statement of the lemma. ∎

Part 1: negative x**.** Now, fix some $x\in(-0.5,0)$ and apply the proposition with $t\gg 1$ to find $T(x,t)$, i.e. the starting time of the trajectory which ends at $(x,\sqrt{x^2+\epsilon^2(T(x,t))})$ at time t. We first apply lemma to find asymptotics for $T(0,t)$ through solving

$$(1+\delta)T(0,t) = t + \log\log 2 + O(e^{-\delta T(0,t)}) \tag{6}$$

This defines the precise asymptotics for $T(0,t)$ in terms of t. Denote $\widehat{T}(t)=T(0,t)$, $\widehat{\epsilon}(t)=\epsilon(T(0,t))$. This quantity, $\widehat{\epsilon}(t)$, has a simple geometric interpretation: this is the distance from the origin to the intersection of $\Gamma(t)$ with OY. Indeed, $g(0,t)=\widehat{\epsilon}(t)$.

For $x\in(-\widehat{\epsilon}(t),0)$, the asymptotics is the same, i.e.

$$T(x,t) = t-\delta\widehat{T}+\log\log 2+O(e^{-\delta\widehat{T}})$$

If $x \in (-0.5, -\widehat{\epsilon}(t))$, then

$$T(x,t) = t - \log\log\frac{1}{|x|} + \log\log 2 + O(e^{-\delta T})$$

This analysis gives us the necessary asymptotical bounds for the curve. We will need to work with rescaled functions later on so consider

$$\widehat{g}(x,t) = \frac{g(x\widehat{\epsilon}(t),t)}{\widehat{\epsilon}(t)} = \sqrt{x^2 + \frac{\epsilon^2(T(x\widehat{\epsilon}(t),t))}{\widehat{\epsilon}^2(t)}}$$

Now we always have $\widehat{g}(0,t) = 1$. For the function

$$h(x,t) = \frac{\epsilon(T(x\widehat{\epsilon}(t),t))}{\widehat{\epsilon}(t)} \tag{7}$$

we get

1. $h(x,t)$ increases in x and $h(0,t) = 1$. This follows from the definition.
2. The asymptotical formula for T leads to (if $x < -2$)

$$\log h(x,t) = e^{\delta\widehat{T}} - e^{\delta T} =$$

$$\exp(\delta\widehat{T}) - \exp\left[\delta\left(t - \log\log\frac{1}{\widehat{\epsilon}(t)|x|} + \log\log 2\right)\right] + O(\delta) \tag{8}$$

This formula is not hard to analyze. Indeed, take $x \in [-0.5\widehat{\epsilon}^{-1}(t), -2]$. Then, we have

$$\log\log\frac{1}{\widehat{\epsilon}(t)|x|} = \log\log\frac{e^{e^{\delta\widehat{T}}}}{|x|} = \delta\widehat{T} + O\Big(e^{-\delta\widehat{T}}\log|x|\Big)$$

So,

$$\delta\left(t - \log\log\frac{1}{\widehat{\epsilon}(t)|x|} + \log\log 2\right) = \delta\Big(t - \delta\widehat{T} + \log\log 2\Big) + \delta O\Big(e^{-\delta\widehat{T}}\log|x|\Big)$$

and, substituting into (8), one gets

$$\log h(x,t) = \delta O(\log|x|) + O(\delta) \tag{9}$$

Thus, we have a bound

$$|h(x,t)| \gtrsim |x|^{-C\delta} \tag{10}$$

uniformly in time.

Remark 3. We could have chosen

$$\epsilon(t) = \exp\left(-\exp a(t)\right) \tag{11}$$

where $a(t) = \delta t + C + \overline{o}(1), a'(t) = \delta + \overline{o}(1),\ \delta \ll 1$, and $a(t)$ is smooth. Then, repeating the estimates above one obtains the same bound (10).

Remark 4. Notice that the function $\delta\widehat{T}(t)$ can be written as $a(t)$ in the previous remark. Indeed, $\widehat{T} = (1+\delta)^{-1}t + C + O(e^{-\delta_1 t})$ follows from (6). Moreover, differentiating

$$2\int_{-0.5}^{0}\frac{dx}{\sqrt{x^2+\epsilon^2(\widehat{T}(t))}\log(x^2+\epsilon^2(\widehat{T}(t)))} = -(t - \widehat{T}(t))$$

in t and performing elementary estimates, one gets

$$\widehat{T}'(t) = C + O(e^{-\delta_2 t})$$

which implies the necessary bound on the derivative.

Part 2: positive x**.** For the positive $x \in (0, 0.5)$, the analysis is similar. For fixed $t \gg 1$, take any point with $x > 0$ and define $T_1(x,t)$ as the time in the past at which this point crossed the vertical axis OY. Obviously $T_1 < t$. If we find $T_1(x,t)$, then

$$g(x,t) = \sqrt{x^2 + \widehat{\epsilon}^2(T_1(x,t))}$$

we can rescale and repeat our analysis.

To find T_1, we have equation

$$2\int_0^x \frac{d\xi}{\sqrt{\xi^2 + \widehat{\epsilon}^2(T_1)}\log(\xi^2 + \widehat{\epsilon}^2(T_1))} = -(t - T_1) \tag{12}$$

Repeating the estimates from the proposition above, we get

$$t - T_1 \lesssim e^{-\delta T_1}$$

for $x \in (0, \widehat{\epsilon}(T_1)) \sim (0, \widehat{\epsilon}(t))$. Outside this interval, we have

$$\log\Big|\frac{\log \hat{\epsilon}(T_1)}{\log x}\Big| + O\Big(e^{-\delta \widehat{T}(T_1)}\Big) = t - T_1$$

Since we have the formula (6), this relation provides asymptotics for $T_1(x,t)$. For example, for $\widetilde{T}(t) = T_1(0.5, t)$

$$\widetilde{T}(t) = \frac{1+\delta}{1+2\delta}t + C + O\Big(\exp\Big(-\frac{\delta\widetilde{T}}{1+\delta}\Big)\Big) \tag{13}$$

Now that we know the hyperbolic tag of the point with $x = 0.5$ (call it $\epsilon_2(t)$, so $\epsilon_2(t) = \epsilon(T(0, T_1(0.5,t))))$ we can find the asymptotical shape of $g(x,t)$ for $x > 0$ by applying the same estimates but reversing the time.

Differentiating (12) in t for $x = 0.5$, one gets

$$\widetilde{T}'(t) = \frac{1+\delta}{1+2\delta} + O(e^{-\delta_3 t}) \tag{14}$$

Now, due to (13), (14) and the remarks 3 and 4 above, we can conclude that, after rescaling,

$$0 \le \log h(x,t) = \delta O(\log|x|), \quad x > 2$$

and so

$$1 \le h(x,t) \lesssim |x|^{C\delta}, \quad x > 2$$

uniformly in time. The function $h(x,t)$ grows monotonically in x.

We will also need to control $\widehat{g}'(x,t)$. For that we only have to bound $h'(x,t)$. Consider $x < 0$, the positive values can be treated similarly. From (7), we obtain

$$h'(x,t) = \epsilon'(T(x\widehat{\epsilon}(t),t))T_x(x\widehat{\epsilon}(t),t)$$

Differentiating the first equation in (5) in x, one gets

$$T'(x,t) = \frac{2}{\sqrt{x^2 + \epsilon^2(x,T)}\log(x^2 + \epsilon^2(T))}$$
$$-\int_{-0.5}^x 2\epsilon(T)\epsilon'(T)T'(x,t)\left(\frac{\log^{-1}(\xi^2 + \epsilon^2(T)) + 2\log^{-2}(\xi^2 + \epsilon^2(T))}{(\xi^2 + \epsilon^2(T))^{1.5}}\right)d\xi$$

Therefore,

$$|T'(x,t)| \lesssim \frac{1}{\epsilon|\log\epsilon|}, \quad |h'(x,t)| \lesssim \delta$$

It is left to say that the smoothness of $\epsilon(t)$ guarantees smoothness of $\widehat{g}(x,t)$. Thus, we can summarize the calculations done above in the following lemma

Lemma 2.3. *The function* $\widehat{g}(x,t)$ *is smooth and*

$$\begin{aligned} |x| < \widehat{g}(x,t) < \sqrt{x^2+1}, &\quad x \le 0 \\ \sqrt{x^2+1} < \widehat{g}(x,t) < x + Cx^{C\delta-1}, &\quad x > 1 \\ |\widehat{g}'(x,t) - \operatorname{sgn}(x)| \lesssim \delta|x|^{C\delta-1}, &\quad |x| > 1 \end{aligned}$$

These estimates are uniform in time.

3. $\Gamma(t)$ is an approximate solution to 2D Euler contour dynamics

In this section we will show that the time-dependent contour studied in the previous section is an approximate solution to the 2d Euler contour dynamics. It will be an approximate solution in a sense that it will satisfy the equation up to some terms that are more regular than the leading ones. From now on we assume that δ is small enough.

Assume that the upper part of $\Gamma(t)$ is parameterized around the origin (e.g., $|x| < 0.5, 0 < y < 0.5$) by the smooth function $y(x,t)$. Then, as the problem is centrally-symmetric, the equation for 2d Euler evolution is (e.g., [5], formula (4) for SQG and α-patches with $\alpha < 1$, the Euler case is similar)

$$\dot{y}(x,t) = \int_{-0.5}^{0.5} (y'(x,t) - y'(\xi,t)) \log\left(\frac{(x-\xi)^2 + (y(x,t)-y(\xi,t))^2}{(x+\xi)^2 + (y(x,t)+y(\xi,t))^2}\right) d\xi + R(x,y,t) \tag{15}$$

where $R(x,y,t)$ is produced by the parts of the curve outside the window $(x,y) \in [-0.5,0.5] \times [-0.5,0.5]$.

We prefer to write this equation in a different form (after simple time rescaling)

$$\dot{y}(x,t) = -\int_{-0.5}^{0.5} (y'(x,t) - y'(\xi,t))(x\xi + y(x,t)y(\xi,t))K(x,\xi)d\xi + R(x,y,t)$$

with

$$K(x,\xi) = \frac{1}{b-a}\int_a^b \frac{d\eta}{\eta},$$
$$a = (x-\xi)^2 + (y(x,t)-y(\xi,t))^2, \quad b = (x+\xi)^2 + (y(x,t)+y(\xi,t))^2$$

and we can continue as

$$\dot{y}(x,t) = y'(x,t)xk_1(x,t) + y'(x,t)y(x,t)k_2(x,t) + xk_3(x,t) + y(x,t)k_4(x,t) + R(x,y,t) \tag{16}$$

where the coefficients k_j are defined correspondingly.

This form of equation for the curve's evolution is canonical in a certain way. Indeed, assume that we want to trace evolution of the curve $\gamma(t)$ parameterized by $(x, u(x,t)), x \in [-0.5, 0.5]$ if it is evolving under the flow $\Phi = (P(x,y,t), Q(x,y,t))$. The tangent to the curve at any point is $(1, u'(x,t))$ and we can subtract from Φ

any multiple of this tangent direction without changing the evolution of $\gamma(t)$. Thus, we end up with equation

$$\dot{u}(x,t) = Q(x,u,t) - u'(x,t)P(x,u,t)$$

We will be concerned with the behavior of the curve around the origin and so we need more precise information about the field Φ around that point. In this paper, all vector-fields involved in the dynamics are odd so $P(0,0,t) = Q(0,0,t) = 0$ and we write $P(x,y,t) = xp_1(x,y,t) + yp_2(x,y,t), Q(x,y,t) = xq_1(x,y,t) + yq_2(x,y,t)$ and substitute into the equation to get

$$\dot{u}(x,t) = xq_1(x,u,t) + uq_2(x,u,t) - xu'(x,t)p_1(x,u,t) - u(x,t)u'(x,t)p_2(x,u,t) \tag{17}$$

Comparing (16) and (17), we notice that in our case the coefficients p_j and q_j depend nonlocally and nonlinearly on the shape of the curve itself.

Our next goal will be to show that the function $g(x,t)$ constructed in the previous section satisfies (16) up to more regular terms. Thus, the dynamics induced by $\Gamma(t)$ around the origin will be approximately equal to the one generated by the "cross". This is what we want to show in the calculations done below.

We will need to compute coefficients k_j generated by this $g(x,t)$. The key ingredient will be the statement on the rescaled function $\widehat{g}(x,t)$ obtained in lemma 2.3. We will be interested mostly in two terms: k_2 and k_3 as they will contain the logarithmic terms. For k_3, we have

$$k_3 = \int_{-0.5}^{0.5} \xi g'(\xi,t) K(x,\xi) d\xi$$

and for k_2

$$-k_2 = \int_{-0.5}^{0.5} g(\xi,t) K(x,\xi) d\xi$$

We are going to exploit the fact that $g(x,t)$ is asymptotically $|x|$ around the origin and if one substitutes $|x|$ into the formulas for $k_{2(3)}$, then the necessary logarithmic singularities will appear, the same singularities that defined the dynamics of $g(x,t)$ itself.

We will handle k_3, the analysis for k_2 is similar. It is more convenient to rescale the variables $x = x_1\widehat{\epsilon}(t), \xi = \xi_1\widehat{\epsilon}(t)$. After the substitution, we get

$$k_3(x) = \frac{1}{4}\int_{-0.5\widehat{\epsilon}^{-1}}^{0.5\widehat{\epsilon}^{-1}} \frac{\xi_1\widehat{g}'(\xi_1,t)}{\xi_1 x_1 + \widehat{g}(\xi_1,t)\widehat{g}(x_1,t)} \left(\int_{(x_1-\xi_1)^2+(\widehat{g}(x_1,t)-\widehat{g}(\xi_1,t))^2}^{(x_1+\xi_1)^2+(\widehat{g}(x_1,t)+\widehat{g}(\xi_1,t))^2} \frac{d\eta}{\eta}\right) d\xi_1 \tag{18}$$

Consider $x \in [1, 0.5\widehat{\epsilon}^{-1}(t)]$, the other values can be treated similarly. We split the integral into two parts and then handle them differently. By lemma 2.3, we can write $\widehat{g}(x,t) = |x| + \rho(x,t)$ where $|x| > 1$ and $|\rho(x,t)| < |x|^{-\gamma}$, $|\rho'(x,t)| < |x|^{-\gamma}, \gamma = 1-$ with estimates being time-independent. Around the origin, $\widehat{g}(x,t)$ is smooth and is uniformly bounded away from zero.

Then, for

$$I_1 = \frac{1}{4}\int_{0}^{0.5\widehat{\epsilon}^{-1}} \frac{\xi_1\widehat{g}'(\xi_1,t)}{\xi_1 x_1 + \widehat{g}(\xi_1,t)\widehat{g}(x_1,t)} \left(\int_{(x_1-\xi_1)^2+(\widehat{g}(x_1,t)-\widehat{g}(\xi_1,t))^2}^{(x_1+\xi_1)^2+(\widehat{g}(x_1,t)+\widehat{g}(\xi_1,t))^2} \frac{d\eta}{\eta}\right) d\xi_1$$

we have

$$I_1 = \int_0^{0.5(x_1\widehat{\epsilon})^{-1}} \frac{\xi_2(1+\rho'(\xi_2 x_1,t))}{\xi_2+(1+\rho(x_1,t)/x_1)(\xi_2+\rho(x_1\xi_2,t)/x_1)}\cdot A\, d\xi_2$$

$$A = \frac{1}{4}\int_{(\xi_2-1)^2+(1-\xi_2+\rho(x_1,t)/x_1-\rho(x_1\xi_2,t)/x_1)^2}^{(1+\xi_2)^2+(1+\xi_2+\rho(x_1,t)/x_1+\rho(\xi_2 x_1,t)/x_1)^2}\frac{d\eta}{\eta}$$

For A, we have a representation

$$A = \frac{1}{\xi_2}+\frac{\rho(x_1,t)}{2x_1\xi_2}+O(\xi_2^{-2}), \quad \xi_2>1$$

so

$$I_1 = \frac{1}{2}\int_1^{0.5(x_1\widehat{\epsilon})^{-1}} \frac{d\xi_2}{\xi_2}\Big(1+O\left(\frac{\rho(x_1\xi_2,t)}{x_1\xi_2}\right)\Big)\Big(1+\rho'(\xi_2 x_1,t)\Big)+\ldots$$
$$= -0.5\log(x)+r(x,t)$$

where $r(x,t)$ in this formula and in the text below will denote the error term uniformly bounded in x and t.

For the other integral, changing the sign in integration

$$I_2 = \int_0^{0.5\widehat{\epsilon}^{-1}} \xi_1(1+\rho'(-\xi_1,t))\frac{1}{b-a}\int_a^b\frac{d\eta}{\eta}d\xi$$

Here, we have

$$b = (x_1-\xi_1)^2+(x_1+\xi_1+\rho(x_1,t)+\rho(-\xi_1,t))^2$$
$$a = (x_1+\xi_1)^2+(x_1-\xi_1+\rho(x_1,t)-\rho(-\xi_1,t))^2$$

As both x_1 and ξ_1 are large in the interesting regime, we are in the situation when

$$a,b>(x_1^2+\xi_1^2)/2$$

so we can use the mean-value formula

$$\frac{1}{b-a}\int_a^b\frac{d\eta}{\eta}=\frac{1}{b}+\frac{b-a}{2\eta_1^2},\ \eta_1\in(a,b)$$

Substituting, we have two terms: $I_2=T_1+T_2$.

$$T_1=\int_0^{0.5(\widehat{\epsilon}x_1)^{-1}}\xi_2(1+\rho'(-x_1\xi_2,t))B^{-1}d\xi_2$$

where

$$B = 2(1+\xi_2^2)+2(1+\xi_2)(\rho(x_1,t)/x_1+\rho(-x_1\xi_2,t)/x_1)$$
$$+(\rho(x_1,t)/x_1+\rho(-x_1\xi_2,t)/x_1)^2$$

Thus,

$$T_1=\int_0^{0.5(\widehat{\epsilon}x_1)^{-1}}\frac{\xi_2(1+\rho'(-x_1\xi_2,t))}{2(1+\xi_2^2)}d\xi_2+r(x,t)=-0.5\log x+r(x,t)$$

For the other term, we have

$$|T_2|\lesssim\int_0^{0.5\widehat{\epsilon}^{-1}}\xi_1\frac{x_1|\rho(-\xi_1)|+\xi_1|\rho(x_1)|+|\rho(x_1)\rho(-\xi_1)|}{(x_1^2+\xi_1^2)^2}d\xi_1$$

$$\lesssim \int_1^{0.5\epsilon^{-1}} \frac{\xi_1 d\xi_1}{(x_1^2+\xi_1^2)^2}\left(\frac{x_1}{\sqrt{\xi_1}}+\frac{\xi_1}{\sqrt{x_1}}\right)d\xi_1 < C$$

Combining all the terms, we have

$$k_3 = \log x + r(x,t), \quad x > \widehat{\epsilon}(t)$$

For $x_1 \sim 0$, we get $I_{1(2)} = -0.5\log\widehat{\epsilon} + r(x,t)$. These calculations show that

$$k_3 = -\begin{cases} \log|x| + r(x,t), & |x| > \widehat{\epsilon} \\ \log\widehat{\epsilon} + r(x,t), & |x| < \widehat{\epsilon} \end{cases} \tag{19}$$

Analogous estimates can be obtained for k_2, they yield

$$k_2 = \begin{cases} \log|x| + r(x,t), & |x| > \widehat{\epsilon} \\ \log\widehat{\epsilon} + r(x,t), & |x| < \widehat{\epsilon} \end{cases} \tag{20}$$

The estimates for other terms are

$$|k_{1(4)}| < C$$

uniformly in x and t. Indeed,

$$k_1 = -\int_{-0.5}^{0.5} \xi K(x,\xi)d\xi, \quad k_3 = \int_{-0.5}^{0.5} g'(\xi,t)g(\xi,t)K(x,\xi)d\xi$$

and if one does the same analysis as we did for k_3 in (18), we will get the sum of two integrals: one over positive ξ_1 and the other one over negative ξ_1. Each will have the same large logarithmic leading term but they will come with different signs now and so will cancel each other in the sum leaving us with the uniformly bounded error terms.

On the other hand, by construction $g(x,t)$ satisfies the following equation

$$\dot{g}(x,t) = \log g(x,t)(g(x,t)g'(x,t) - x), \quad |x| < 0.5 \tag{21}$$

Indeed, solving (21) by the method of characteristics, we see that $g(x,t)$ is moved by the flow (2).

The asymptotics of $g(x,t)$ implies that

$$\log g(x,t) = \log|x| + r(x,t), \quad |x| > \widehat{\epsilon}$$

and

$$\log g(x,t) = \log\widehat{\epsilon}(t) + r(x,t), \quad |x| < \widehat{\epsilon}$$

These calculations ensure that $g(x,t)$ satisfies

$$\dot{g}(x,t) = \int_{-0.5}^{0.5}(g'(x,t)-g'(\xi,t))\log\left(\frac{(x-\xi)^2+(g(x,t)-g(\xi,t))^2}{(x+\xi)^2+(g(x,t)+g(\xi,t))^2}\right)d\xi + \tag{22}$$
$$g'(x,t)x\widehat{k}_1(x,t) + g'(x,t)g(x,t)\widehat{k}_2(x,t) + x\widehat{k}_3(x,t) + g(x,t)\widehat{k}_4(x,t)$$

at any time with $\widehat{k}_j$ being uniformly bounded in x and t. They absorbed k_j's along with terms generated by the distant part of $\Gamma(t)$. Indeed, the patches are centrally symmetric so $R(x,y,t)$ in (15) is smooth and $R(0,0,t)=0$.

Now, we are ready to prove our main result, theorem 1.1.

Proof. The behavior of contour $\Gamma(t)$ is explicit for all t. In particular, outside a ball of radius 0.5 around the origin, $\Gamma(t)$ is infinitely smooth so the field $\nabla^\perp\Delta^{-1}\chi_{\Omega_t\backslash B_{0.5}(0)}$ will have smooth normal components on $\Gamma(t)$. Its contribution to the strain is innocuous. As for the part $\nabla^\perp\Delta^{-1}\chi_{\Omega_t\cap B_{0.5}(0)}$, we computed correction in the local coordinates in (22). Thus, in accordance with (17), we can define the strain around the origin by first letting

$$P(x,y,t) = -x\widehat{k}_1(x,t) + y\widehat{k}_2(x,t),\ Q(x,y,t) = x\widehat{k}_3(x,t) + y\widehat{k}_4(x,t)$$

on $\Gamma(t)$ itself and then continuing these functions to all of $B_{0.5}(0)$ in a natural way by first extending $\widehat{k}_j$'s.

This calculation shows that the strain $(P(x,y,t), Q(x,y,t))$ satisfies

$$|(P,Q)| \lesssim \sqrt{x^2+y^2}$$

around the origin uniformly in t and that proves (1). The Lipschitz continuity at arbitrary point is an obvious corollary of smoothness of $\Gamma(t)$. In our arguments, we need this regularity just to be able to uniquely solve the ordinary differential equations of the dynamics. We do not try to control the growth in time of the global Lipschitz seminorm although this analysis is possible. □

4. Appendix: the flow $\Psi(z,t)$ and the curve $\Gamma(0)$

In this Appendix we introduce the flow $\Psi(z,t)$ and $\Gamma(0)$– the flow and the initial curve, respectively.

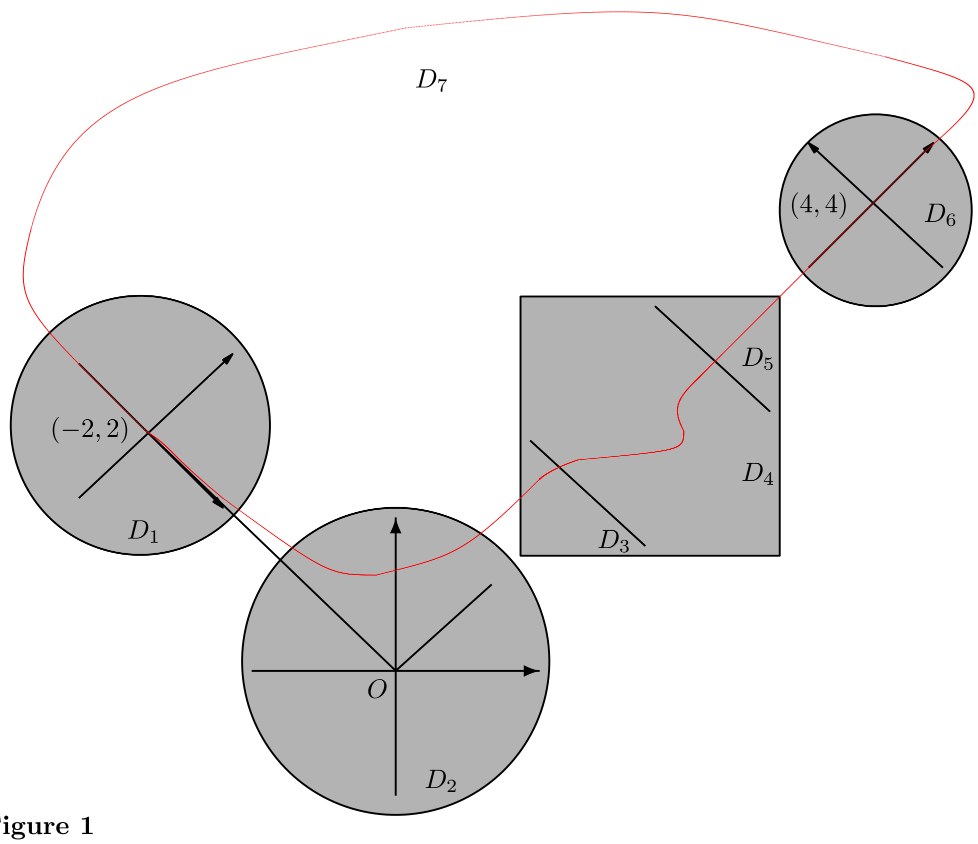

Figure 1

The picture (Figure 1) describes the choice of the upper part of the initial contour $\Gamma(0)$ (red ink) and the way the incompressible flow Ψ is constructed. Within D_1 and D_6 the dynamics is standard hyperbolic with separatrices along the axes. In

D_2, the flow is also hyperbolic and generated by (2) but separatrices are rotated by $\pm\pi/4$ with respect to coordinate axes. Between D_1 and D_2 the potential can be smoothly interpolated. In $D_{3(5)}$, the flow is laminar with direction perpendicular to the black segments and in the north-eastern direction. The potential between zones D_2 and D_3 can be smoothly interpolated, as well as the potential between D_5 and D_6. In the zone D_7, the potential is zero so the curve is frozen. This zone again is interpolated smoothly between D_1 and D_6. In the zone D_4, we construct non-stationary potential in the following way (only in this zone the flow is time-dependent!):

The argument below allows an interpolation between two laminar flows and guarantees the prescribed evolution of the curve $\Gamma(t)$ in these laminar zones. What we want is to define dynamics in the regions D_3, D_4, D_5 right after the flow leaves D_2. We need to define this dynamics in such a way that the motion of $\Gamma(t)$ is localized to these regions and, moreover, that it does not move in D_5. Once again, in D_3 and D_5 we postulate the flow to be laminar and then we want to define it in D_4. We will do that in the local coordinates.

Assume that potential $\Lambda(z) = -y$ in $B = \{z : -1 < x < 0\} \cup \{z : 1 < x < 2\} \sim D_3 \cup D_5$. This potential generates the laminar flow

$$\dot{\theta} = \nabla\theta \cdot \nabla^{\perp}\Lambda$$

where $\nabla^{\perp}\Lambda(z) = (-1, 0)$. We want to define smooth $\Lambda(z,t)$ in $D_4 = \{z : 0 < x < 1\}$ such that the resulting $\Lambda(z,t)$ is smooth globally on $D_3 \cup D_4 \cup D_5$. Moreover, given smooth decaying $\delta(t)$ (e.g., $\delta \in L^1(\mathbb{R}^+)$ is enough for decay condition), we need to define a curve $\Gamma(0) = \{(x, \gamma(x, 0))\}$ that evolves under this flow $\Gamma(t) = \{(x, \gamma(x,t))\}$ such that $\gamma(0,t) = \delta(t)$ and $\gamma(1,t) = 0$. This function $\delta(t)$ is determined by $\Gamma(t)$ in the zone D_2 where it approaches the separatrix in the superexponential rate. To be more precise, δ is proportional to the distance from $\Gamma(t)$ to this separatrix in the area where D_2 and D_3 meet.

We will look for

$$\Lambda(z,t) = -y - g_1(x) g_2(x - t)$$

where $g_{1(2)}$ are smooth. Then, to guarantee the global smoothness, we need $g_1(x) = 0$ around $x = 0$ and $x = 1$. Now, take a point $(0, \delta(T))$ and trace its trajectory for $t > T$. We have

$$x(t,T) = t - T,\; t \in [T, T+1]$$

and

$$y(t,T) = \delta(T) - \int_T^t \Big(g_1'(\tau - T) g_2(\tau - T - \tau) + g_1(\tau - T) g_2'(\tau - T - \tau) \Big) d\tau, \quad t \in [T, T+1]$$

Since we want $y(T+1, T) = 0$ and g_1 vanishes on the boundary,

$$\delta(T) = g_2'(-T) \int_T^{T+1} g_1(\tau - T) d\tau = g_2'(-T) \int_0^1 g_1(x) dx$$

and this identity should hold for all $T > 0$. Take any g_1 with mean one, this defines g_2 on the negative half-line as long as we set $g_2(-\infty) = 0$. We can continue it now to the whole line in a smooth fashion to have g_2 globally defined.

How do we define the initial curve at $t = 0$? We extend smooth $\delta(t)$ to $t \in [-1, 0]$ arbitrarily and apply the procedure explained above to $t \in [-1, \infty)$. The curve that we see at $t = 0$ will be the needed initial value for the dynamics that starts at $t = 0$.

It is only left to mention that to localize the picture in the vertical direction we can multiply $\Lambda(z,t)$ be a suitable cut-off in the y direction.

The part of the curve that is in D_5, D_6, D_7, and the north-western part of D_1 is stationary, it does not move– this is easy to ensure by making this part of the curve the level set of the stationary potential $\Lambda(z)$. For the rest of the curve, it does change in time and the flow is directed along it in the anti-clockwise direction. The main point however is that at the origin O the sharp corner will be formed with double exponential rate as long as a regular exterior strain is imposed on the whole system. We want to reiterate that this phenomenon is purely nonlinear as the strain itself is not capable of providing the double exponential attraction to the origin.

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