ABSOLUTELY CONTINUOUS SPECTRUM OF A TYPICAL SCHRÖDINGER OPERATOR WITH A SLOWLY DECAYING POTENTIAL

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1. Main results

We study the absolutely continuous spectrum of a Schrödinger operator

$$H = -\Delta + \alpha V, \qquad \alpha \in \mathbb{R}, \tag{1}$$

acting in the space $L^2(\mathbb{R}^d)$. While the potential V involved in this definition is a function of $x \in \mathbb{R}^d$, we shall also study the dependence of V on the spherical coordinates $r = |x|$ and $\theta = x/|x|$. Therefore, sometimes the value of V at $x \in \mathbb{R}^d$ will be denoted by $V(r, \theta)$. Even less often the radial variable will be denoted by ρ. Our main result is the following

Theorem 1.1. *Let V be a real valued bounded potential on $\mathbb{R}^d$ and let*

$$W(r, \theta) = \int_0^r V(\rho, \theta) d\rho, \qquad \forall r > 0. \tag{2}$$

Assume that W belongs to the space $\mathcal{H}^1_{\rm loc}(\mathbb{R}^d)$ of functions having (generalized) locally square integrable derivatives. Suppose that

$$\int_{\mathbb{R}^d} \frac{|\nabla W|^2}{|x|^{d-1}} dx < \infty. \tag{3}$$

Then the absolutely continuous spectrum of the operator (1) is essentially supported by the interval $[0, \infty)$ for almost every $\alpha \in \mathbb{R}$. That is, the spectral projection $E(\Omega)$ corresponding to any set $\Omega \subset [0, \infty)$ is different from zero $E(\Omega) \neq 0$ as soon as the Lebesgue measure of Ω is positive.

Note that in $d = 1$, condition (3) turns into

$$\int_{\mathbb{R}} |V|^2 dx < \infty. \tag{4}$$

Operators with such potentials were studied in the work of Deift and Killip [1], the main result of which states that absolutely continuous spectrum of the operator $-d^2/dx^2 + V$ covers the positive half-line $[0, \infty)$, if V satisfies (4).

However, it is not clear what is the proper generalization of condition (4) in $d \geq 2$. Most likely, it should be replaced by (cf. [16])

$$\int_{\mathbb{R}^d} \frac{V^2}{|x|^{d-1}} dx < \infty, \tag{5}$$

but the problem of proving the existence of the absolutely continuous spectrum under this assumption turns out to be very hard. Usually, one assumes more than (5).

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For instance, the result of the article [8] by Galina Perelman says that the absolutely continuous spectrum of the Schrödinger operator

$$-\Delta + V$$

is essentially supported by $[0,\infty)$, if

$$|V(x)| + |\nabla_\theta V(x)| \le \frac{C}{(|x|+1)^{1/2+\epsilon}}, \qquad \epsilon > 0. \tag{6}$$

Here, the symbol $\nabla_\theta V$ denotes the vector of derivatives with respect to the angular variables. A more precise definition of ∇_θ is :

$$\nabla = \frac{x}{r}\frac{\partial}{\partial r} + \frac{1}{r}\nabla_\theta.$$

We see that, besides the decay of V at infinity, the result of [8] requires a decay of $\nabla_\theta V$. Theorem 1.1 has a similar assumption, however it deals with a wider class of potentials compared to the one considered in [8]. Indeed, (6) implies (3), but the converse is not true.

The class of functions described by the condition (3) is wider than the space of functions satisfying (6) not only because the function under the integral sign in (3) is allowed to have different behavior along different directions, but also because the definition of W involves some averaging. For instance, the potential

$$V(r,\theta) = \frac{1}{r^{1/2+\delta}}\Big(2 + \sin(r^\gamma \theta)\sin^n(\theta/2)\Big), \qquad d=2,\ \theta \in [0,2\pi),$$

fulfills the condition (3) but does not satisfy (6) if $\gamma > \delta$.

2. Auxiliary material

Notations. Throughout the text, $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary parts of a complex number z. The notation $\mathbb{S}$ stands for the unit sphere in $\mathbb{R}^d$. Its area is denoted by $|\mathbb{S}|$. For a selfadjoint operator $B = B^*$ and a vector g of a Hilbert space the expression $((B-k-i0)^{-1}g,g)$ is always understood as the limit

$$\Big((B-k-i0)^{-1}g,g\Big) = \lim_{\varepsilon\to 0}\Big((B-k-i\varepsilon)^{-1}g,g\Big), \qquad \varepsilon > 0,\ k \in \mathbb{R}.$$

This limit exists for almost every $k \in \mathbb{R}$.

The following simple and very well known statement plays very important role in our proof.

Lemma 2.1. *Let B be a self-adjoint operator in a separable Hilbert space $\mathfrak{H}$ and let $g \in \mathfrak{H}$. Then the function*

$$\eta(k) := \operatorname{Im}\Big((B-k-i0)^{-1}g,g\Big) \ge 0$$

is integrable over $\mathbb{R}$. Moreover,

$$\int_{-\infty}^{\infty} \frac{\eta(k)dk}{1+k^2} \le \pi\Big((B^2+I)^{-1}g,g\Big). \tag{7}$$

We will also need the following consequence of Hardy's inequality:

Lemma 2.2. *Let V be a real valued potential vanishing inside the unit ball and let W be the function defined in (2). Then*

$$\int_{|x|<R} \frac{|W|^2}{|x|^{d+1}}dx \le 4\int_{|x|<R} \frac{|\nabla W|^2}{|x|^{d-1}}dx,$$

for any $R > 1$.

In the beginning of the proof of Theorem 1.1 we will assume that V is compactly supported. We will obtain certain estimates on the derivative of the spectral measure of the operator $-\Delta + \alpha V$ for compactly supported potentials and then we will extend these estimates to the case of an arbitrary V satisfying the conditions of Theorem 1.1. We will approximate V by compactly supported functions. It is important not to destroy the inequalities obtained previously for "nice" V. Therefore the way we select approximations plays a very important role in our proof.

Let us describe our choice of compactly supported functions V_n approximating the given potential V. Let us choose a spherically symmetric function $\zeta \in \mathcal{H}^1(\mathbb{R}^d)$ such that

$$\zeta(x) = \begin{cases} 1, & \text{if} \quad |x| < 1; \\ 0, & \text{if} \quad |x| > 2. \end{cases}$$

Assume for simplicity that $0 \leq \zeta \leq 1$ and $|\nabla \zeta| \leq 1$. Define

$$\zeta_n(x) = \zeta(x/n).$$

Note that $\nabla \zeta_n \neq 0$ only in the shperical layer $\{ x : \ n \leq |x| \leq 2n\}$. Moreover $|\nabla \zeta_n| \leq 1/n$, which leads to the estimate

$$|\nabla \zeta_n(x)| \leq 2/|x|.$$

Our approximations of V will be the functions V_n defined as

$$V_n = \frac{\partial}{\partial r}\,(\zeta_n W), \tag{8}$$

where W is the function from (2). Thus, approximations of V by V_n correspond to approximations of W by

$$W_n = \zeta_n W. \tag{9}$$

Observe that, in this case,

$$\int_{\mathbb{R}^d} \frac{|\nabla W_n(x)|^2}{|x|^{d-1}} dx \leq \int_{|x|<2n} \frac{2|\nabla W(x)|^2 + 8|x|^{-2}|W(x)|^2}{|x|^{d-1}} dx \leq 34 \int_{\mathbb{R}^d} \frac{|\nabla W(x)|^2}{|x|^{d-1}} dx.$$

Therefore,

$$\sup_n \int_{\mathbb{R}^d} \frac{|\nabla W_n(x)|^2}{|x|^{d-1}} dx < \infty. \tag{10}$$

One can also easily show that

$$||V_n||_\infty \leq 3||V||_\infty. \tag{11}$$

3. Proof of Theorem 1.1

Our proof is based on the relation between the derivative of the spectral measure and the so called scattering amplitude. Both objects should be introduced properly. While the spectral measure can be defined for any linear self-adjoint mapping, the scattering amplitude will be introduced only for a differential operator. Let f be a vector in the Hilbert space $\mathfrak{H}$ and H be a self-adjoint operator in $\mathfrak{H}$. It turns out that the quadratic form of the resolvent of H can be written as a Cauchy integral

$$((H-z)^{-1}f, f) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{t-z}, \qquad \operatorname{Im} z \neq 0.$$

The measure μ in this representation is called the spectral measure of H corresponding to the element f.

Let us now introduce the scattering amplitude. Assume that the support of the potential V is compact and take any compactly supported function function f. Then

$$(H-z)^{-1}f = e^{ik|x|}\frac{A_f(k,\theta)}{|x|^{(d-1)/2}} + O(|x|^{-(d+\delta)/2}), \qquad \text{as } |x|\to\infty,\ \theta = \frac{x}{|x|},\ k^2 = z,\ \operatorname{Im} k \geq 0,\ \delta > 0,$$

with some $A_f(k,\theta)$. Clearly, the relation

$$\mu'(\lambda) = \pi^{-1}\lim_{z\to\lambda+i0}\operatorname{Im}\left((H-z)^{-1}f, f\right) = \pi^{-1}\lim_{z\to\lambda+i0}\operatorname{Im} z||(H-z)^{-1}f||^2$$

implies that

$$\pi\mu'(\lambda) = \sqrt{\lambda}\int_{\mathbb{S}}|A_f(k,\theta)|^2\,d\theta, \qquad k^2 = \lambda > 0. \tag{12}$$

Formula (12) is a very important estimate that relates the absolutely continuous spectrum to so-called extended states. The rest of the proof will be devoted to a lower estimate of $|A_f(k,\theta)|$.

Consider first the case $d = 3$. For our purposes, it is sufficient to assume that f is the characteristic function of the unit ball. In this case, f is a spherically symmetric function. Traditionally, H is viewed as an operator obtained by a perturbation of

$$H_0 = -\Delta.$$

In its turn, $(H-z)^{-1}$ can be viewed as an operator obtained by a perturbation of $(H_0-z)^{-1}$. The theory of such perturbations is often based on the second resolvent identity

$$(H-z)^{-1} = (H_0-z)^{-1} - (H-z)^{-1}\alpha V(H_0-z)^{-1}, \tag{13}$$

which turns out to be useful for our reasoning. As a consequence of (13), we obtain that

$$A_f(k,\theta) = F(k) + A_g(k,\theta), \qquad z = k^2 + i0,\ k > 0, \tag{14}$$

where $g(x) = \alpha V(x)(H_0-z)^{-1}f$ and $F(k)$ is defined by

$$(H_0-z)^{-1}f = e^{ik|x|}\frac{F(k)}{|x|^{(d-1)/2}}, \qquad \text{for } |x| > 1 \qquad (\text{recall that } d = 3). \tag{15}$$

Without loss of generality, one can assume that $V(x) = 0$ inside the unit ball. In this case,

$$g = F(k)h_k, \qquad \text{where} \quad h_k(x) = \alpha V(x)e^{ik|x|}|x|^{(1-d)/2}. \tag{16}$$

According to (12), (14) and (16), we obtain

$$\pi\mu'(\lambda) \geq |F(k)|^2\Big(|\mathbb{S}|\sqrt{\lambda} - \operatorname{Im}\Big((H-z)^{-1}h_k,\, h_k\Big)\Big). \tag{17}$$

Therefore, in order to establish the presence of the absolutely continuous spectrum, we need to show that the quantity $\operatorname{Im}\Big((H-z)^{-1}h_k,\, h_k\Big)$ is small. The chain of the arguments that led us to this conclusion has been suggested by Boris Vainberg. The method developed by the author in the previous version of the paper was much longer.

Let us define η_0 setting

$$\alpha^2k^{-2}\eta_0(k,\alpha) := \frac{1}{k}\operatorname{Im}\Big((H-z)^{-1}h_k,\, h_k\Big) \geq 0.$$

Obviously η_0 is positive for all real $k \neq 0$, because we agreed that $z = k^2 \pm i0$ if $\pm k > 0$. This is very convenient. Since $\eta_0 > 0$, we can conclude that η_0 is small on a rather large set if the integral of this function is small. That is why we will try to estimate

$$J(V) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\eta_0(k,\alpha)}{(\alpha^2+k^2)}\frac{|k|\,dk d\alpha}{(k^2+1)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\eta_0(k,tk)}{(k^2+1)(t^2+1)}\,dkdt. \tag{18}$$

Now, we employ a couple of tricks, one of which has an artificial character and will be appreciated not immediately but a bit later. Instead of dealing with the operator H, we will deal with $H+\varepsilon I$ where $\varepsilon>0$ is a small parameter. We will first obtain an integral estimate for the quantity

$$\eta_\varepsilon(k,\alpha)=\frac{k}{\alpha^2}\mathrm{Im}\Big((H+\varepsilon-z)^{-1}h_k,\,h_k\Big).$$

The latter estimate will be not uniform in ε, but we can still pass to the limit $\varepsilon\to 0$ according to Fatou's lemma, because

$$\eta_0(k,\alpha)=\lim_{\varepsilon\to0}\eta_\varepsilon(k,\alpha)\qquad \text{a.e. on } \mathbb{R}\times\mathbb{R}.$$

The second trick is to set $\alpha=kt$ and represent η_ε in the form

$$\eta_\varepsilon(k,kt)=\mathrm{Im}\Big((B+1/k)^{-1}H_\varepsilon^{-1/2}v,\,H_\varepsilon^{-1/2}v\Big) \tag{19}$$

where $v=V|x|^{(1-d)/2}$, $H_\varepsilon=H_0+\varepsilon I$ and B is the bounded selfadjoint operator defined by

$$B=H_\varepsilon^{-1/2}\Big(-2i\frac{\partial}{\partial r}-\frac{i(d-1)}{|x|}+tV\Big)H_\varepsilon^{-1/2}.$$

The symbol r in the latter formula denotes the radial variable $r=|x|$. The reader can easily establish that B is not only self-adjoint but bounded as well. Note that it is the parameter ε that makes B bounded.

In order to justify (19) at least formally, one has to introduce the operator U of multiplication by the function $\exp(ik|x|)$. Using this notation, we can represent η_ε in the following form

$$\eta_\varepsilon(k,tk)=k\mathrm{Im}\Big(U^{-1}(H+\varepsilon-z)^{-1}Uv,\,v\Big).$$

Since we deal with a unitary equivalence of operators, we can employ the formula

$$U^{-1}(H+\varepsilon-z)^{-1}U=(U^{-1}HU+\varepsilon-z)^{-1}.$$

On the other hand, since H is a differential operator and U is an operator of multiplication, the commutator $[H,U]:=HU-UH$ can be easily found

$$\Big[H,U\Big]=kU\Big(-2i\frac{\partial}{\partial r}-\frac{i(d-1)}{|x|}+k\Big).$$

The latter equality implies that

$$U^{-1}HU+\varepsilon-z=H_\varepsilon+k\Big(-2i\frac{\partial}{\partial r}-\frac{i(d-1)}{|x|}+tV\Big)=H_\varepsilon^{1/2}(I+kB)H_\varepsilon^{1/2}.$$

Consequently,

$$kU^{-1}(H+\varepsilon-z)^{-1}U=H_\varepsilon^{-1/2}(B+1/k)^{-1}H_\varepsilon^{-1/2}. \tag{20}$$

A more detailed proof of (20) will be given in the last section called "Appendix". These details do not have so much value for us at the moment. It is more important that, now, (19) follows from (20).

Let us have a look at the formula (19). If k belongs to the upper half plane then so does $-1/k$. Since B is a self-adjoint operator, $\pi^{-1}\eta_\varepsilon(k,kt)$ coincides with the derivative of the spectral measure of the operator B corresponding to the element $H_\varepsilon^{-1/2}v$. According to Lemma 2.1, the latter observation implies that

$$\int_{-\infty}^{\infty}\frac{\eta_\varepsilon(k,kt)}{(1+k^2)}dk\le\pi\Big((B^2+I)^{-1}H_\varepsilon^{-1/2}v,H_\varepsilon^{-1/2}v\Big),$$

which leads to

$$\int_{-\infty}^{\infty}\frac{\eta_\varepsilon(k,kt)}{(1+k^2)}dk\le\pi\Big(B^{-1}H_\varepsilon^{-1/2}v,B^{-1}H_\varepsilon^{-1/2}v\Big)=\pi||B^{-1}H_\varepsilon^{-1/2}v||^2. \tag{21}$$

Our further arguments will be related to the estimate of the quantity in the right hand side of (21). We will show now that

$$\lim_{\varepsilon\to 0} ||B^{-1}H_\varepsilon^{-1/2}v||^2 \le C\int_{\mathbb{R}^d} \frac{|\nabla W|^2}{|x|^{d-1}}dx. \tag{22}$$

In order to do that we use the representation

$$B^{-1}H_\varepsilon^{-1/2} = H_\varepsilon^{1/2}T^{-1}, \tag{23}$$

where $T\subset T^*$ is the first order differential operator, defined by

$$T = -2i\frac{\partial}{\partial r} - \frac{i(d-1)}{|x|} + tV, \qquad D(T) = D(H_\varepsilon^{1/2}).$$

The representation (23) is a simple consequence of the fact that $B = H_\varepsilon^{-1/2}TH_\varepsilon^{-1/2}$.

Let us discuss the basic properties of the operator T. The study of these properties is rather simple, because one can derive an explicit formula for the resolvent of T. For that purpose, one needs to recall the theory of ordinary differential equations, which says that the equation

$$y' + p(t)y = f(t), \qquad y = y(t),\ t\in\mathbb{R},$$

is equivalent to the relation

$$\Big(e^{\int p\,dt}y\Big)' = e^{\int p\,dt}f.$$

Put differently,

$$y' + p(t)y = e^{-\int p\,dt}\Big(e^{\int p\,dt}y\Big)'.$$

This gives us a clear idea of how to handle the operator T. Let U_0 and U_1 be the operators of multiplication by $|x|^{(d-1)/2}$ and by $\exp(2^{-1}itW)$, then

$$T = -2iU_1^{-1}U_0^{-1}\Big[\frac{\partial}{\partial r}\Big]U_0U_1, \qquad \text{and} \qquad T^{-1} = \frac{i}{2}U_1^{-1}U_0^{-1}\Big[\frac{\partial}{\partial r}\Big]^{-1}U_0U_1.$$

Since $[\frac{\partial}{\partial r}]^{-1}$ means just the simple integration with respect to r and $\partial W/\partial r = V$,

$$\begin{gathered} T^{-1}v = \frac{i}{2}e^{-2^{-1}itW}|x|^{-(d-1)/2}\int_0^r e^{2^{-1}itW}V\,dr = \\ \frac{1}{t}e^{-2^{-1}itW}|x|^{-(d-1)/2}(e^{2^{-1}itW}-1) = \frac{1}{t}|x|^{-(d-1)/2}(1-e^{-2^{-1}itW}). \end{gathered} \tag{24}$$

Note, that $T^{-1}v$ turns out to be compactly supported, which leaves no doubt about the relation $v\in D(T^{-1})$. Combining (23) with (24), we conclude that

$$\lim_{\varepsilon\to 0}||B^{-1}H_\varepsilon^{-1/2}v||^2 \le ||\nabla T^{-1}v||^2 \le C\Big(\int_{\mathbb{R}^d}\frac{|W|^2}{|x|^{d+1}}dx + \int_{\mathbb{R}^d}\frac{|\nabla W|^2}{|x|^{d-1}}dx\Big).$$

Now (22) follows from Lemma 2.2. We remind the reader that (21), (22) are needed to estimate the quantity $J(V)$ from (18). We can say now that

$$J(V) \le C\int_{\mathbb{R}^d}\frac{|\nabla W|^2}{|x|^{d-1}}dx.$$

Using Chebyshev's inequality, we derive from (17) that

$$\text{meas}\ \{(\lambda,\alpha)\in[\lambda_1,\lambda_2]\times[\alpha_1,\alpha_2]: \quad |\mathbb{S}|\sqrt{\lambda} - \frac{\pi\mu'(\lambda)}{|F(k)|^2} > s\} \le Cs^{-1}\int_{\mathbb{R}^d}\frac{|\nabla W|^2}{|x|^{d-1}}dx \tag{25}$$

for finite $\lambda_j > 0$ and $\alpha_j > 0$. The constant $C > 0$ in (25) depends on $||V||_\infty$, $\lambda_j > 0$ and $\alpha_j > 0$. If $s = 2^{-1}|\mathbb{S}|\sqrt{\lambda_1}$ then (25) turns into

$$\text{meas}\,\{(\lambda,\alpha) \in [\lambda_1,\lambda_2]\times[\alpha_1,\alpha_2] : \quad \frac{\pi\mu'(\lambda)}{|F(k)|^2} < |\mathbb{S}|(\sqrt{\lambda} - 2^{-1}|\sqrt{\lambda_1})\} \le C_0 \int_{\mathbb{R}^d} \frac{|\nabla W|^2}{|x|^{d-1}} dx. \tag{26}$$

We can say now that the proof is more or less completed, because the quantity in the right hand side can be made arbitrary small if we replace V by $V - V_n$, where V_n are defined in (8) and n is sufficiently large. Put differently, we keep the "tails" of V and remove only a compactly supported portion of it. The latter operation changes V only on a compact set. According to the Scattering Theory, this operation does not change the absolutely continuous spectrum of the Schrödinger operator $-\Delta + V$. This implies, that without loss of generality, we can assume that $\pi\mu'(\lambda) \ge 2^{-1}|\mathbb{S}|\sqrt{\lambda}|F(\sqrt{\lambda})|^2$on a set of a large measure.

4. Semi-continuity of the entropy

Let us complete the proof and mention the missing ingredients. First, in order to understand what we achieved, we summarize the results. We found such approximations of V by compactly supported potentials V_n that the corresponding spectral measures μ_n satisfy the estimate (see (26)):

$$\pi\mu_n'(\lambda) \ge \frac{|\mathbb{S}|\sqrt{\lambda}}{2}|F(\sqrt{\lambda})|^2 = \frac{|\mathbb{S}|}{2\lambda^{5/2}}\Big|\sqrt{\lambda}\cos(\sqrt{\lambda}) - \sin(\sqrt{\lambda})\Big|^2 \tag{27}$$

on a set of pairs (λ,α) of very large Lebesgue measure. Denote the characteristic function of the intersection of this set with the rectangle $[\lambda_1,\lambda_2]\times[\alpha_1,\alpha_2]$ by χ_n. Thus, inequality (27) holds on the support of χ_n. (By the way, we assume that $\lambda_1 > 0$ and $\alpha_1 > 0$ are positive.)

We will study the behavior of χ_n as $n \to \infty$. The difficulty of the situation is that χ_n might change with the growth of n. However, since the unit ball in any Hilbert space is compact in the weak topology, without loss of generality, we can assume that χ_n converges weakly in L^2 to a square integrable function χ. In a certain sense, we can say that χ_n does not change much if n is sufficiently large. Now the situation is less hopeless, because the limit χ preserves properties of the sequence χ_n. The necessary information about the limit χ can be easily obtained from the information about χ_n. It is clear that $0 \le \chi \le 1$ and $\chi > 0$ on a set of very large measure $(\lambda_2-\lambda_1)(\alpha_2-\alpha_1) - \varepsilon$. Indeed, let $\tilde{\chi}$ be the characteristic function of the set where $\chi > 1 + \varepsilon_0$. Since $\int\int \chi_n\tilde{\chi} d\lambda\, d\alpha \le \int\int \tilde{\chi} d\lambda\, d\alpha$, we obtain that

$$(1+\varepsilon_0)\int\int \tilde{\chi} d\lambda\, d\alpha \le \int\int \tilde{\chi} d\lambda\, d\alpha,$$

which is possible only in the case when $\tilde{\chi} = 0$ almost everywhere. Consequently, $\chi \le 1$ and therefore we can judge about the size of the set where $\chi > 0$ by the value of the integral $\int\int \chi\, d\lambda\, d\alpha = \lim_{n\to\infty}\int\int \chi_n\, d\lambda\, d\alpha$.

It is also known, that if V_n converges to $V \in L^\infty(\mathbb{R}^d)$ in L^2_{loc}, then

$$\mu_n \to \mu \qquad \text{as } n \to \infty \tag{28}$$

weakly for any fixed α. We see that both sequences μ_n and χ_n have a limit, however they converge in a weak sense, which brings additional difficulties. Therefore we have to find a quantity that not only depends on a pair of measures (semi-)continuously with respect to the weak topology, but is also infinite as soon as the derivative of one of the measures $\mu' = 0$ vanishes on a large set. Such a quantity is the entropy, defined by

$$S = \int_{\alpha_1}^{\alpha_2}\int_{\lambda_1}^{\lambda_2} \log\Big(\frac{\mu'(\lambda)}{\chi(\lambda,\alpha)}\Big)\chi(\lambda,\alpha)\, d\lambda d\alpha.$$

Its properties were thoroughly studied in [7]. It can diverge only to negative infinity, but if it is finite, then $\mu' > 0$ almost everywhere on the set $\{(\lambda,\alpha):\ \chi>0\}$. We can formulate a more general definition:

Definition. Let ρ, ν be finite Borel measures on a compact Hausdorff space, X. We define the entropy of ρ relative to ν by

$$S(\rho|\nu) = \begin{cases} -\infty, & \text{if } \rho \text{ is not } \nu-\text{ac} \\ -\int_X \log(\frac{d\rho}{d\nu})d\rho, & \text{if } \rho \text{ is } \nu-\text{ac}. \end{cases} \tag{29}$$

Theorem 4.1. (cf.[7]) *The entropy $S(\rho|\nu)$ is jointly upper semi-continuous in ρ and ν with respect to the weak topology. That is, if $\rho_n \to \rho$ and $\nu_n \to \nu$ as $n \to \infty$, then*

$$S(\rho|\nu) \geq \limsup_{n\to\infty} S(\rho_n|\nu_n).$$

Relation (28) literally means that

$$\int \phi(\lambda,\alpha)\, d\mu_n \to \int \phi(\lambda,\alpha)\, d\mu \qquad \text{as } n\to\infty,$$

for any fixed α and any continuous compactly supported function ϕ. By the Lebesgue dominated convergence theorem, we obtain that

$$\int\int \phi(\lambda,\alpha)\, d\mu_n d\alpha \to \int\int \phi(\lambda,\alpha)\, d\mu d\alpha \qquad \text{as } n\to\infty,$$

which means that the sequence of measures

$$\text{meas}_n\,(\Omega) := \int\int_{(\lambda,\alpha)\in\Omega} d\mu_n\, d\alpha$$

converges weakly as well. Now, Theorem 4.1 implies that

$$\int_{\alpha_1}^{\alpha_2}\int_{\lambda_1}^{\lambda_2} \log\Bigl(\frac{\mu'(\lambda)}{\chi(\lambda,\alpha)}\Bigr)\chi(\lambda,\alpha)\, d\lambda d\alpha \geq \liminf_{n\to\infty} \int_{\alpha_1}^{\alpha_2}\int_{\lambda_1}^{\lambda_2} \log\Bigl(\frac{\mu_n'(\lambda)}{\chi_n(\lambda,\alpha)}\Bigr)\chi_n(\lambda,\alpha)\, d\lambda d\alpha > -\infty,$$

because logarithmic integrals are semi-continuous with respect to weak convergence of measures. This proves that $\mu' > 0$ on the support of χ which is a subset of $[\lambda_1,\lambda_2]\times[\alpha_1,\alpha_2]$ whose measure is not smaller than $(\lambda_2-\lambda_1)(\alpha_2-\alpha_1)-\varepsilon$. It remains to observe that ε is arbitrary.

This proves our main result for $d=3$. Now, if $d\neq 3$, then equality of the form (15) is incorrect. We have to deal with the terms of smaller order that must appear in the right hand side of (15). However, one can avoid this difficulty replacing the operator H_0 by

$$H_0 = -\Delta - \frac{\kappa_d\tilde{\chi}}{|x|^2}P_0, \qquad \kappa_d = \Bigl(\frac{d-2}{2}\Bigr)^2 - \frac{1}{4},$$

where P_0 is the projection onto the space of spherically symmetric functions and $\tilde{\chi}$ is the characteristic function of the compliment of the unit ball (cf. [14]).

The semi-continuity of logarithmic integrals (29) was discovered for the broader audience by R.Killip and B.Simon in [7]. The reason why S is semi-continuous is that S is representable as an infimum of a difference of two integrals with respect to the measures ν and ρ:

$$S(\rho|\nu) = \inf_F\Bigl(\int F(x)d\nu - \int(1+\log F(x))d\rho\Bigr), \qquad \min_x F(x) > 0.$$

In conclusion of this section, we would like to draw your attention to the papers [2]-[6], [8]-[14] which contain an important work on the absolutely continuous spectrum of multi-dimensional Schrödinger operators. Two of these papers ([5], [14]) deal with families of Schrödinger operators $-\Delta+\alpha V$, where V is not only decaying but oscillating as well.

5. Appendix

Here we prove the relation (20). If k is not real, then U^{-1} is an unbounded operator. However, this fact does not bring additional difficulties, because we will apply the operator U^{-1} only to functions that decay at infinity sufficiently fast. Let us formulate now the statement which justifies (20).

Proposition 5.1. *Let V be a compactly supported real potential. Let k be a point in the upper half-plane, let $z=k^2$ and let $v\in L^2(\mathbb{R}^d)$ be a compactly supported function. Then*

$$u=k(H+\varepsilon-z)^{-1}Uv,\qquad \alpha=kt,\quad \operatorname{Im} z\neq 0.$$

is representable in the form

$$u=Uw$$

with $w\in L^2(\mathbb{R}^d)$. Moreover,

$$w=H_\varepsilon^{-1/2}(B+1/k)^{-1}H_\varepsilon^{-1/2}v,$$

where $H_\varepsilon=-\Delta+\varepsilon$ and

$$H_\varepsilon^{1/2}BH_\varepsilon^{1/2}v=-2i\frac{\partial v}{\partial r}-\frac{i(d-1)v}{|x|}+tVv. \tag{30}$$

Proof. Consider the function $w=e^{-ik|x|}u$. It is easy to see that w is a solution of the differential equation

$$-\Delta w+\varepsilon w+ktVw-2ik\frac{\partial w}{\partial r}-\frac{ik(d-1)w}{|x|}=k\,v.$$

Moreover, w decays at infinity as $O\Big(e^{-(\operatorname{Im}\sqrt{k^2-\varepsilon}-\operatorname{Im}k)|x|}\Big)$. Consequently, $w\in D(H_\varepsilon)$ and

$$H_\varepsilon w+kH_\varepsilon^{1/2}BH_\varepsilon^{1/2}w=k\,v$$

The proof is completed. $\square$

Another statement which might help the reader to understand our arguments, deals with analytic properties of the resolvent of H.

Proposition 5.2. *Assume that V is a bounded compactly supported potential. Let χ be the characteristic function of a compact set containing the support of V. Then the operator valued function*

$$T(k)=\chi(H-k^2)^{-1}\chi,\qquad \alpha=kt,$$

admits a meromorphic continuation into the plane with the cut along the half-line $\{\,z=iy,\ y\le 0\,\}$.

Proof. Indeed, if $V=0$, then the proof of the statement can be found in [17]. Note that

$$T_0(k)=\chi(H_0-k^2)^{-1}\chi,\qquad k\in\mathbb{C}\setminus\{\,z=iy,\ y\le 0\,\},$$

is an integral operator whose kernel depends on k analytically. Moreover, the results of [17] clearly say that $T_0(k)$ is compact. The relation

$$\chi(H-k^2)^{-1}\chi=\chi(H_0-k^2)^{-1}\chi-tk\chi(H_0-k^2)^{-1}V\chi(H-k^2)^{-1}\chi$$

implies that

$$T(k)=\Big(I+tkT_0(k)V\Big)^{-1}T_0(k).$$

The statement follows now from the analytic Fredholm alternative. □

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