EFFECTIVE DYNAMICS OF AN ELECTRON COUPLED TO AN EXTERNAL POTENTIAL IN NON-RELATIVISTIC QED

VOLKER BACH, THOMAS CHEN, JÉRÉMY FAUPIN, JÜRG FRÖHLICH, AND ISRAEL MICHAEL SIGAL

Abstract. In the framework of non-relativistic QED, we establish the relationship between the renormalized mass of the electron due to its interaction with the quantized electromagnetic field, and the kinematic mass appearing in its response to a slowly varying external force. Specifically, we study the dynamics of an electron in a slowly varying external potential and with slowly varying initial conditions and prove that, for a long time, it is accurately described by the effective dynamics of a Schrödinger electron in the same external potential and for the same initial data, with a kinetic energy operator determined by the renormalized dispersion law of the translation-invariant QED model.

1. Introduction

In this paper we establish the relationship between the renormalized mass of the electron due to its interaction with the quantized electromagnetic field, and the kinematic mass appearing in its response to a slowly varying external force. We work in the standard framework of non-relativistic quantum electrodynamics (QED). The renormalized electron mass is defined as the inverse curvature of the dispersion relation $E = E(p)$, the energy of a dressed electron as a function of its momentum p (no external potentials present), while the kinematic mass is defined in terms of the effective dynamics of the electron under the influence of an external force.

Our starting point is the dynamics generated by the Hamiltonian H^V describing a non-relativistic electron interacting with the quantized electromagnetic field and under the influence of a slowly varying potential V_ϵ. We consider the time evolution of states parametrized by wave functions $u_0^\epsilon \in H^1(\mathbb{R}^3)$, with $\|u_0^\epsilon\|_{L^2} = 1$ and $\|\nabla u_0^\epsilon\|_{L^2} \leq \epsilon$, and prove that this evolution is accurately approximated, on a long time interval, by an effective Schrödinger dynamics generated by the one-particle Schrödinger operator

$$H_{\text{eff}} := E(-i\nabla_x) + V_\epsilon(x)\,, \tag{1.1}$$

with the kinetic energy given by the dispersion relation $E(p)$. This result is in line with the idea that every physical dynamics is an effective one, derived from a more precise theory. While results of a similar nature were proven for massive bosons, [26], ours is the first result involving massless photons and the resulting renormalized mass of the electrons. (Similarly, one may analyze the effects of interactions with phonons and other massless bosons.)

In the framework of non-relativistic QED, the Hilbert space of states of the system is given by

$$\mathcal{H} := L^2(\mathbb{R}^3) \otimes \mathfrak{F}\,, \tag{1.2}$$

where $L^2(\mathbb{R}^3)$ denotes the Hilbert space associated with the electron degrees of freedom, neglecting spin (for notational convenience). The space $\mathfrak{F}$ is the Fock space of photons in the Coulomb gauge,

$\mathfrak{F} = \bigoplus_{n\geq 0} \mathfrak{F}_n$. Here, $\mathfrak{F}_n := \mathrm{Sym}(\, L^2(\mathbb{R}^3 \times \{+,-\}\,)\,)^{\otimes n}$ denotes the physical Hilbert space of states of n photons. The Hamiltonian is given on this space by the expression

$$H^V \;=\; H \,+\, V_\epsilon \otimes \mathbf{1}_f \,, \tag{1.3}$$

where H is the generator of the dynamics of a single, freely moving non-relativistic electron minimally coupled to the quantized electromagnetic field, i.e.,

$$H \;:=\; \frac{1}{2}(\,-i\nabla_x \otimes \mathbf{1}_f \,+\, \sqrt{\alpha} A(x)\,)^2 \,+\, \mathbf{1}_{el} \otimes H_f \,, \tag{1.4}$$

and where $V_\epsilon(x) := V(\epsilon x)$ is a slowly varying potential, with $\epsilon > 0$ small; its precise properties are formulated in Theorem 1.1 below. Furthermore,

$$A(x) \;:=\; \sum_\lambda \int_{|k|\leq 1} \frac{dk}{|k|^{1/2}} \,\{\, \epsilon_\lambda(k)\, e^{-ikx} \otimes a_\lambda(k) \,+\, h.c.\,\} \tag{1.5}$$

denotes the quantized electromagnetic vector potential with an ultraviolet cutoff imposed, $|k| \leq 1$, and

$$H_f \;=\; \sum_\lambda \int dk\, |k|\, a_\lambda^*(k)\, a_\lambda(k) \tag{1.6}$$

denotes the photon Hamiltonian. In (1.5) and (1.6), $a_\lambda^*(k)$, $a_\lambda(k)$ are the usual photon creation- and annihilation operators; $\lambda = \pm$ indicates photon helicity, and $\epsilon_\lambda(k)$ is a polarization vector perpendicular to k corresponding to helicity λ.

To give a precise formulation of our result, we observe that the Hamiltonian H is translation invariant. We thus represent the Hilbert space of the system as a direct integral,

$$\mathcal{H} \;=\; \int^{\oplus} dp\, \mathcal{H}_p \;, \tag{1.7}$$

over the spectrum of the total momentum operator

$$P_{tot} \;:=\; -i\nabla_x \otimes \mathbf{1}_f \,+\, \mathbf{1}_{el} \otimes P_f, \tag{1.8}$$

where $P_f = \sum_\lambda \int dk\, k\, a_\lambda^*(k) a_\lambda(k)$ is the momentum operator associated to the quantized radiation field, and each fiber Hilbert space $\mathcal{H}_p$ is isomorphic to $\mathfrak{F}$. We let $H(p) = H|_{\mathcal{H}_p}$ denote the fiber Hamiltonian corresponding to total momentum p. We define $E(p) = \inf \mathrm{spec} H(p)$, with $p \in \mathcal{S}$, where

$$\mathcal{S} \;:=\; \{\, p \in \mathbb{R}^3 \,\big|\, |p| \leq \frac{1}{3} \,\} \tag{1.9}$$

and continued suitably to $p \in \mathbb{R}^3 \setminus \mathcal{S}$. For $\rho > 0$, we introduce the family of maps $\mathcal{J}_0^\rho : L^2(\mathbb{R}^3) \mapsto \mathcal{H}$, from the quantum mechanical one-electron state space $L^2(\mathbb{R}^3)$ to a subset of *dressed one-electron states*, as

$$\mathcal{J}_0^\rho(\,u\,) \;:=\; \int_{\mathcal{S}} dp\, \widehat{u}(p)\; e^{-ix(p-P_f)}\, \chi_{\mathcal{S}_\mu}(p)\, \Phi^\rho(p)\,, \tag{1.10}$$

where x is the electron position, $\chi_{\mathcal{S}_\mu}$ is a smooth approximate characteristic function of the set $S_\mu := (1-\mu)\mathcal{S} \subset \mathcal{S} \subset \mathbb{R}^3$, $(0 < \mu < 1)$, and $\Phi^\rho(p)$ is an approximate ground state of $H(p)$ (dressed by a cloud of soft photons with frequencies $\leq \rho$).

In this paper we study the time evolution of one-electron states, $\mathcal{J}_0^\rho(u_0^\epsilon)$, with electron wave functions u_0^ϵ, dressed by an *infrared cloud* of photons with frequencies $\leq \rho$. More precisely, we study solutions of the Schrödinger equation

$$i\partial_t \Psi(t) \,=\, H^V\,\Psi(t)\,, \quad \text{with}\;\; \Psi(0) \,=\, \mathcal{J}_0^\rho(u_0^\epsilon)\,. \tag{1.11}$$

The key idea is to relate the solution $\Psi(t) \,=\, e^{-itH^V}\mathcal{J}_0^\rho(u_0^\epsilon)$ of this Schrödinger equation to the solution of the Schrödinger equation

$$i\partial_t u(t) \,=\, H_{\text{eff}}\,u(t)\,, \quad \text{with} \qquad u(t=0) = u_0^\epsilon\,, \tag{1.12}$$

corresponding to the one-particle Schrödinger operator (1.1), where $E(p)$ has been defined above. We consider the comparison state

$$\mathcal{J}_0^\rho(u(t)) \,\in\, \mathcal{H}\,, \tag{1.13}$$

where $u(t) := e^{-itH_{\text{eff}}}u_0^\epsilon$ is the solution of (1.12), and show that $\Psi(t)$ remains close to $\mathcal{J}_0^\rho(u(t))$, for a long time. The choice of initial data satisfying $\|\nabla u_0^\epsilon\|_{L^2} \leq \epsilon$ guarantees that $\widehat{u}(t)$ remains concentrated in $\mathcal{S}$ during the time scales relevant in this problem.

Theorem 1.1. *Let $0 < \mu < 1$ and $0 < \epsilon < \mu/3$, and define J_0^ρ by (1.10). Assume that $u_0^\epsilon \in L^2(\mathbb{R}^3)$ is a normalized vector obeying $\|\nabla u_0^\epsilon\|_{L^2(\mathbb{R}^3)} \leq \epsilon$. Furthermore, assume that $V \in L^\infty(\mathbb{R}^3;\mathbb{R})$ is such that $\widehat{V} \in L^1(\mathbb{R}^3)$ and that $\widehat{V}$ is supported in the unit ball,*

$$\text{supp}(\widehat{V}) \,\subset\, \big\{\, k \in \mathbb{R}^3 \,|\, |k| \leq 1\,\big\}\,. \tag{1.14}$$

Let $0 < \delta < 2/3$ and choose $\rho = \epsilon^{\frac{2}{3}-\delta} =: \rho_\epsilon$. There exists $\alpha_\delta > 0$ such that, for all $0 \leq \alpha \leq \alpha_\delta$, the bound

$$\|\, e^{-itH^V}\,\mathcal{J}_0^{\rho_\epsilon}(\,u_0^\epsilon\,) \,-\, \mathcal{J}_0^{\rho_\epsilon}(\,e^{-itH_{\text{eff}}}\,u_0^\epsilon\,)\,\|_{\mathcal{H}} \,\leq\, C_\delta\,\big(\,\epsilon^{2/3-\delta}t + \epsilon^{4/3-\delta/2}t^2\,\big)\,, \tag{1.15}$$

holds for all times t.

1.1. **Outline of proof strategy.** To prove Theorem 1.1, we introduce an *infrared regularized* version of the model defined by (1.3), (1.4), obtained by restricting the integration domain in the quantized electromagnetic vector potential (1.5) to the region $\{\sigma \leq |k| \leq 1\}$, for an arbitrary infrared cutoff $\sigma > 0$. Thereby, we obtain infrared regularized Hamiltonians H_σ^V and H_σ, as well as an infrared regularized family of maps $\mathcal{J}_\sigma^\rho$ corresponding to $\mathcal{J}_0^\rho$.

We note that, unlike $H(p)$, the infrared cut-off fiber Hamiltonian $H_\sigma(p)$ has a ground state $\Psi_\sigma(p) \in \mathcal{H}_p \cong \mathfrak{F}$, for every $p \in \mathcal{S}$ and for $\sigma > 0$, but $\Psi_\sigma(p)$ does not possess a limit in $\mathcal{H}_p \cong \mathfrak{F}$, as $\sigma \searrow 0$. In particular, we expect that the number of photons in the state $\Psi_\sigma(p)$ diverges, as $\sigma \searrow 0$, (thus the lack of convergence of $\Psi_\sigma(p)$ in $\mathfrak{F}$). This is a well-known aspect of the *infrared problem* in QED, [8, 9, 10, 11, 22]. It is remedied by applying a dressing transformation, $W^\rho_{\nabla E_\sigma(p)}$, defined in (2.9), below, to $\Psi_\sigma(p)$, where $E_\sigma(p) = \inf \text{spec} H_\sigma(p)$. The resulting vector, $\Phi_\sigma^\rho(p) := W^\rho_{\nabla E_\sigma(p)}\Psi_\sigma(p)$, describes an *infraparticle* (or *dressed electron) state* containing infrared photons with frequencies in $[\sigma, \rho]$. As $\sigma \searrow 0$, the limit

$$\Phi^\rho(p) = \lim_{\sigma\to 0}\Phi_\sigma^\rho(p) \tag{1.16}$$

exists in $\mathfrak{F}$, see Proposition 2.2. This allows us to construct the map $\mathcal{J}_0^\rho$ as the limit of the maps $\mathcal{J}_\sigma^\rho$, as $\sigma \searrow 0$.

We note that $\Phi^\rho_\sigma(p)$ is the ground state eigenvector of the fiber Hamiltonian

$$K^\rho_\sigma(p) \,:=\, W^\rho_{\nabla E_\sigma(p)}\, H_\sigma(p)\, (W^\rho_{\nabla E_\sigma(p)})^* \tag{1.17}$$

which is obtained by applying to $H_\sigma(p)$ the *Bogoliubov transformation* corresponding to the dressing transformation $W^\rho_{\nabla E_\sigma(p)}$.

In Theorem 2.3, below, we prove that an estimate similar to (1.15) holds for the infrared regularized model,

$$\|\, e^{-itH^V_\sigma}\, \mathcal{J}^{\rho_\epsilon}_\sigma(\, u^\epsilon_0\,) \,-\, \mathcal{J}^{\rho_\epsilon}_\sigma(\, e^{-itH_{\mathrm{eff},\sigma}}\, u^\epsilon_0\,)\,\|_{\mathcal{H}} \,\leq\, C_\delta\, \left(\, \epsilon^{2/3-\delta} t + \epsilon^{4/3-\delta/2} t^2\,\right), \tag{1.18}$$

uniformly in $\sigma \geq 0$, for $\rho_\epsilon = \epsilon^{\frac{2}{3}-\delta}$. This result crucially uses the regularity properties of the dressed electron states $\Phi^\rho_\sigma(p)$, which allow us to take advantage of the fact that V_ϵ is slowly varying. An additional key ingredient is the bound $\|(H_\sigma(p) - K^\rho_\sigma(p))\Phi^\rho_\sigma(p)\|_{\mathfrak{F}} \,\leq\, C\alpha^{\frac12}\,\rho^{\frac12}|p|$, for $p \in \mathcal{S}$, proven in Appendix C.

In section 3, we control the limit $\sigma \searrow 0$, thus concluding the proof of Theorem 1.1. This requires control of the radiation emitted by the electron due to its acceleration in the external potential V_ϵ, in the limit $\sigma \searrow 0$.

Remark 1.2. *Theorem 1.1 implies that, for all δ' such that $\delta < \delta' < 2/3$,*

$$\|\, e^{-itH^V}\, \mathcal{J}^{\rho_\epsilon}_0(\, u^\epsilon_0\,) \,-\, \mathcal{J}^{\rho_\epsilon}_0(\, e^{-itH_{\mathrm{eff}}}\, u^\epsilon_0\,)\,\|_{\mathcal{H}} \,\leq\, C_\delta\, \epsilon^{\delta'-\delta} \tag{1.19}$$

holds for all times t with $0 \leq t \leq \epsilon^{-\frac{2}{3}+\delta'}$. We note that the time scale of order $\mathcal{O}(\epsilon^{-\frac{2}{3}+\delta'})$ is not sharp; we leave further investigation of an optimal time scale to future work. Indeed, for a regularized model based on massive photons (see [26]), or nonzero infrared cutoff $\sigma > 0$, the dynamics can be controlled up to a time scale $\mathcal{O}(\epsilon^{-1})$. The main obstacle against passing beyond a time scale of order $\mathcal{O}(\epsilon^{-\frac{2}{3}+\delta'})$ within our current approach comes from the θ-Hölder continuity of $\Phi^\rho_\sigma(p)$ in p, only established for $\theta < 2/3$.

2. Infrared cut-off and construction of $\Phi^\rho(p)$

As noted in the introduction, we analyze the original dynamics by first imposing an infrared cut-off, and controlling the dynamics generated by the resulting Hamiltonian. Thus, we define the IR regularized Hamiltonian

$$H^V_\sigma \,=\, H_\sigma \,+\, V_\epsilon(x) \otimes \mathbf{1}_f\,, \tag{2.1}$$

where

$$H_\sigma \,:=\, \frac{1}{2}(\, -i\nabla_x \otimes \mathbf{1}_f \,+\, \sqrt{\alpha} A_\sigma(x)\,)^2 \,+\, \mathbf{1}_{el} \otimes H_f \tag{2.2}$$

is the generator of the dynamics of a single, freely moving non-relativistic electron minimally coupled to the electromagnetic radiation field. In (2.2),

$$A_\sigma(x) \,=\, \sum_\lambda \int_{\sigma\leq|k|\leq 1} \frac{dk}{|k|^{1/2}}\, \{\, \epsilon_\lambda(k)\, e^{-ikx} \otimes a_\lambda(k) \,+\, h.c.\,\} \tag{2.3}$$

denotes the quantized electromagnetic vector potential with an infrared and ultraviolet cutoff corresponding to $\sigma \leq |k| \leq 1$. Since $V \in L^\infty(\mathbb{R}^3)$ is a bounded operator, $D(H_\sigma^V) = D(H_\sigma) = D(-\Delta_x \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f)$.

The Hamiltonian H_σ is translation invariant. Representing the Hilbert space of the system as a direct integral, $\mathcal{H} = \int^\oplus dp\, \mathcal{H}_p$, over the spectrum of the total momentum operator, P_{tot} (see (1.7) and (1.8)), we let $H_\sigma(p) = H_\sigma|_{\mathcal{H}_p}$ denote the fiber Hamiltonian corresponding to total momentum p. While $H(p)$ has a ground state only for $p = 0$, it is proven in [2, 6] that, for $p \in \mathcal{S} := \{p \in \mathbb{R}^3 | |p| \leq 1/3\}$ and $\sigma > 0$,

$$E_\sigma(p) = \inf \mathrm{spec} H_\sigma(p) \tag{2.4}$$

is a non-degenerate eigenvalue (the fiber ground state energy) of $H_\sigma(p)$. This motivates the introduction of the cut-off. We let $\Psi_\sigma(p) \in \mathfrak{F}$, with $\|\Psi_\sigma(p)\|_{\mathfrak{F}} = 1$, denote the corresponding normalized fiber ground state,

$$H_\sigma(p)\Psi_\sigma(p) = E_\sigma(p)\, \Psi_\sigma(p)\,, \tag{2.5}$$

for $p \in \mathcal{S}$. Properties of the fiber ground state energy $E_\sigma(p)$, and of dressed electron states $\Phi_\sigma(p)$, for $p \in \mathcal{S}$, are given in the following proposition proven in [2, 6, 9, 10]:

Proposition 2.1. *The infimum of the spectrum of the fiber Hamiltonian, $E_\sigma(p) = \inf \mathrm{spec}(H_\sigma(p))$, satisfies:*

(1) *For any $\sigma > 0$, $E_\sigma \in C^2(\mathcal{S})$, and for all $p \in \mathcal{S} = \left\{ p \in \mathbb{R}^3 \,|\, |p| \leq \frac{1}{3} \right\}$, $E_\sigma(p)$ is a simple eigenvalue.*

(2) *For $\alpha > 0$ sufficiently small, there exists a constant c, with $0 < c < \infty$, such that, for any $p \in \mathcal{S}$,*

$$|\nabla_p E_\sigma(p) - p| \leq c\, \alpha\, |p|\,, \quad \text{and} \quad 1 - c\,\alpha \leq \partial^2_{|p|} E_\sigma(p) \leq 1\,, \tag{2.6}$$

hold uniformly in α ($0 < \alpha \ll 1$) and $\sigma \geq 0$.

(3) *The following limit exists in $C^2(\mathcal{S})$*

$$\lim_{\sigma \searrow 0} E_\sigma(\,\cdot\,) = E(\,\cdot\,). \tag{2.7}$$

Let $b^*_\lambda(k)$, $b_\lambda(k)$ denote creation- and annihilation operators on the fiber space, see Appendix A. For $0 < \sigma < \rho \leq 1$ and $p \in \mathcal{S}$, we introduce the Weyl operators

$$W^\rho_{\nabla E_\sigma(p)} := \exp\Big[\alpha^{\frac{1}{2}} \sum_\lambda \int_{\sigma \leq |k| \leq \rho} dk\, \frac{\nabla E_\sigma(p) \cdot \epsilon_\lambda(k) b_\lambda(k) - h.c.}{|k|^{1/2}(|k| - \nabla E_\sigma(p) \cdot k)}\Big]\,, \tag{2.8}$$

with $\nabla E_\sigma(p) \equiv \nabla_p E_\sigma(p)$, which are unitary on $\mathfrak{F}$, for $\sigma > 0$. Moreover, we define dressed electron states

$$\Phi^\rho_\sigma(p) := W^\rho_{\nabla E_\sigma(p)}\, \Psi_\sigma(p)\,. \tag{2.9}$$

The properties of these states are described in the following proposition

Proposition 2.2. *For any $p \in \mathcal{S}$, $0 < \rho \leq 1$, and for sufficiently small values of the finestructure constant $0 < \alpha \ll 1$, the ground state eigenvector $\Phi^\rho_\sigma(p)$ of $K^\rho_\sigma(p) := W^\rho_{\nabla E_\sigma(p)}\, H_\sigma(p)\, (W^\rho_{\nabla E_\sigma(p)})^*$ satisfies:*

(1) *The strong limit*

$$\Phi^\rho(p) \,:=\, \lim_{\sigma\to 0} \Phi^\rho_\sigma(p) \tag{2.10}$$

exists in $\mathfrak{F}$.

(2) *For* $\theta < \frac{2}{3}$, *the vectors* $\Phi^\rho_\sigma(p)$ *are* θ*-Hölder continuous in* p,

$$\sup_{p,q\in\mathcal{S}} \frac{\| \, \Phi^\rho_\sigma(p) - \Phi^\rho_\sigma(q) \, \|}{|p-q|^\theta} \,\leq\, C(\theta)\,, \tag{2.11}$$

uniformly in σ *and* ρ, *with* $0 \leq \sigma < \rho \leq 1$.

The proof of θ-Hölder continuity for $\theta < \frac{2}{3}$ is given in Appendix B; (see also [9, 10, 22] for earlier results covering the range $\theta < \frac{1}{4}$, in the case where $\rho = 1$).

For arbitrary $u \in L^2(\mathbb{R}^3)$ (with Fourier transform denoted by $\widehat{u}$), we define the linear map

$$\mathcal{J}^\rho_\sigma \,:\, u \;\;\mapsto\;\; \int_{\mathcal{S}} dp\, \widehat{u}(p)\; e^{-ix(p-P_f)}\, \chi_{\mathcal{S}_\mu}(p)\, \Phi^\rho_\sigma(p)\,, \tag{2.12}$$

where x is the electron position, $\chi_{\mathcal{S}_\mu}$ is a smooth approximate characteristic function of the set

$$\mathcal{S}_\mu \,:=\, (1-\mu)\,\mathcal{S} \subset \mathcal{S} \subset \mathbb{R}^3, \tag{2.13}$$

and $0 < \mu < 1$. Note that $\mathcal{J}^\rho_\sigma \,:\, L^2(\mathbb{R}^3) \to \mathcal{M} \,\subset\, \mathcal{H}$, where

$$\mathcal{M} \,:=\, \Big\{ \int_{\mathbb{R}^3} dp\, \widehat{u}(p)\, e^{-ix(p-P_f)}\, \chi_{\mathcal{S}_\mu}(p)\, \Phi^\rho_\sigma(p) \,\Big|\, u \in L^2(\mathbb{R}^3) \Big\}, \tag{2.14}$$

the space of vectors in $\mathcal{H}$ supported on the one-particle shell of the operator $\int_{\mathcal{S}}^{\oplus} dp\, K^\rho_\sigma(p)$.

We also note that in (2.14) we do not require that $\mathrm{supp}(\widehat{u}) \subset \mathcal{S}_\mu$; instead, we cutoff $\widehat{u}$ outside the region $\mathcal{S}_\mu$ by multiplying it by $\chi_{\mathcal{S}_\mu}$.

Furthermore, we consider an initial wave function $u_0^\epsilon(x)$ satisfying $\|u_0^\epsilon\|_{L^2} = 1$ and $\|\nabla u_0^\epsilon\|_{L^2} \leq \epsilon$. Our main goal in this paper is to study the solution of the Schrödinger equation

$$i\partial_t \Psi(t) \,=\, H^V_\sigma \, \Psi(t)\,, \quad \text{with } \; \Psi(0) \,=\, \mathcal{J}^\rho_\sigma(u_0^\epsilon)\,, \tag{2.15}$$

which is given by

$$\Psi(t) \,=\, e^{-itH^V_\sigma}\, \mathcal{J}^\rho_\sigma(u_0^\epsilon) \,\in\, \mathcal{H}\,, \tag{2.16}$$

and, in particular, to determine its properties for large t.

The key idea is to relate $\Psi(t)$ to the solution of the Schrödinger equation

$$i\partial_t u(t,x) \;\;=\;\; (H_{\mathrm{eff},\sigma}\, u)(t,x) \quad , \quad u(0,x) = u_0^\epsilon(x) \tag{2.17}$$

corresponding to the one-particle Schrödinger operator

$$H_{\mathrm{eff},\sigma} \,:=\, E_{\mathrm{eff},\sigma}(-i\nabla_x) \,+\, V_\epsilon(x)\,, \tag{2.18}$$

where $(t,x) \in \mathbb{R}\times\mathbb{R}^3$. The kinetic energy operator $E_{\mathrm{eff},\sigma}(p) \in C^2(\mathbb{R}^3)$ is defined by

$$E_{\mathrm{eff},\sigma}(p) \,:=\, E_\sigma(p)\,, \;\text{ if } p \in \mathcal{S}\,, \tag{2.19}$$

and suitably continued to $\mathbb{R}^3 \setminus \mathcal{S}$. We consider the comparison state

$$\mathcal{J}^\rho_\sigma(u(t)) \,\in\, \mathcal{H}\,, \tag{2.20}$$

where $u(t) := e^{-itH_{\mathrm{eff},\sigma}} u_0^\epsilon$ is the solution of (2.17), and show that $\Psi(t)$ remains close to $\mathcal{J}_\sigma^\rho(u(t))$, for a long time. The choice of initial data satisfying $\|\nabla u_0^\epsilon\|_{L^2} \leq \epsilon$ guarantees that $\widehat{u}(t)$ remains concentrated in $\mathcal{S}$ during the time scales relevant in this problem.

As a first step in proving Theorem 1.1, we will prove the following result.

Theorem 2.3. *Under the conditions of Theorem 1.1, there exists $\alpha_\delta > 0$ such that, for all $0 \leq \alpha \leq \alpha_\delta$, the bound*

$$\| e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon) - \mathcal{J}_\sigma^\rho(e^{-itH_{\mathrm{eff},\sigma}} u_0^\epsilon) \|_{\mathcal{H}} \leq C_\delta (1 + \ln(\rho^{-1})) \epsilon^{\frac{2}{3}-\delta} t + C \alpha^{\frac{1}{2}} \rho^{\frac{1}{2}} \epsilon t (1+t) , \quad (2.21)$$

holds uniformly in the infrared cutoff σ.

Proof. Our proof makes crucial use of the properties of the fiber ground state energy $E_\sigma(p)$ and of the corresponding dressed electron states $\Phi_\sigma^\rho(p)$, for $p \in \mathcal{S}$, given in Propositions 2.1 and 2.2 above. We define the Bogoliubov-transformed fiber Hamiltonians

$$K_\sigma^\rho(p) := W_{\nabla E_\sigma(p)}^\rho H_\sigma(p) (W_{\nabla E_\sigma(p)}^\rho)^* \quad (2.22)$$

for $p \in \mathcal{S}$, and the operator K_σ^ρ acting on $\mathcal{H}$,

$$K_\sigma^\rho := \int^\oplus I_p^{-1} K_\sigma^\rho(p) I_p \, dp, \quad (2.23)$$

and the perturbed operator $K_\sigma^V := K_\sigma^\rho + V_\epsilon$. Note that the dressed electron states $\Phi_\sigma^\rho(p)$, for $p \in \mathcal{S}$, are the ground states of $K_\sigma^\rho(p)$, i.e.,

$$K_\sigma^\rho(p) \, \Phi_\sigma^\rho(p) = E_\sigma(p) \, \Phi_\sigma^\rho(p) , \quad (2.24)$$

and the operator K_σ^ρ has the property that

$$K_\sigma^\rho \mathcal{J}_\sigma^\rho = \mathcal{J}_\sigma^\rho E_{\mathrm{eff},\sigma}(-i\nabla). \quad (2.25)$$

Next, we proceed to the proof of (2.21). We write the difference on the LHS of (2.21) as the integral of a derivative, and separate it into

$$\begin{aligned} e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon) - \mathcal{J}_\sigma^\rho(e^{-itH_{\mathrm{eff},\sigma}} u_0^\epsilon) &= -i \, e^{-itH_\sigma^V} \int_0^t ds \, e^{isH_\sigma^V} \, (H_\sigma^V \mathcal{J}_\sigma^\rho(u(s)) - \mathcal{J}_\sigma^\rho(H_{\mathrm{eff}} u(s)) \\ &=: \phi^1(t) + \phi^2(t) , \end{aligned} \quad (2.26)$$

by substituting $H_\sigma^V \to H_\sigma^V - K_\sigma^V + K_\sigma^V$ inside the integral and grouping terms suitably. The first term on the r.h.s. accounts for the radiation of infrared photons due to the motion of the dressed electron in the external potential, while the second term accounts for the influence of the external potential V_ϵ on the full QED dynamics (2.16), as compared to the effective Schrödinger evolution (2.20).

The first term on the r.h.s. of (2.26) has the form

$$\phi^1(t) := -i \, e^{-itH_\sigma^V} \int_0^t ds \, e^{isH_\sigma^V} \left(H_\sigma - K_\sigma^\rho \right) \mathcal{J}_\sigma^\rho(u_0^\epsilon) , \quad (2.27)$$

where we have used the cancelation of V in $H_\sigma^V - K_\sigma^V = H_\sigma - K_\sigma^\rho$. Using the fiber integral decomposition, we obtain

$$\|\phi^1(t)\|_{\mathcal{H}} \leq \sup_{p \in \mathcal{S}} \left\{ \frac{1}{|p|} \|(H_\sigma - K_\sigma^\rho)(p) \, \Phi_\sigma^\rho(p)\|_{\mathfrak{F}} \right\} \int_0^t \| \nabla u(s) \|_{L^2(\mathbb{R}^3)} \, ds . \quad (2.28)$$

In Appendix C we prove the following key result:

$$\sup_{p\in\mathcal{S}}\left\{\frac{1}{|p|}\,\|(H_\sigma-K_\sigma^\rho)(p)\,\Phi_\sigma^\rho(p)\|_{\mathfrak{F}}\right\}\;\leq\;C\alpha^{\frac12}\,\rho^{\frac12}\,,\tag{2.29}$$

uniformly in $\sigma\geq 0$. Furthermore, we have the estimate

$$\int_0^t \|\,\nabla u(s)\,\|_{L^2(\mathbb{R}^3)}\,ds\;\leq\;C\,\epsilon\,t\,(1+t)\,,\tag{2.30}$$

as shown below by using the condition $\|\nabla u_0^\epsilon\|_{L^2(\mathbb{R}^3)}\leq\epsilon$ on u_0^ϵ, and the fact that the potential V satisfies (1.14). We obtain

$$\|\phi^1(t)\|_{\mathcal{H}}\;\leq\;C\,t\,(1+t)\,\alpha^{\frac12}\,\rho^{\frac12}\,\epsilon,\tag{2.31}$$

which yields the second contribution to the r.h.s. of (2.21).

For the second term on the r.h.s. of (2.26), we have that

$$\begin{aligned}\phi^2(t)\;&:=\;-i\,e^{-itH_\sigma^V}\int_0^t ds\,e^{isH_\sigma^V}\,\big(\,K_\sigma^V\,\mathcal{J}_\sigma^\rho(u(s))-\mathcal{J}_\sigma^\rho(H_{\mathrm{eff}}u(s))\,\big)\\&=\;-i\,e^{-itH_\sigma^V}\int_0^t ds\,e^{isH_\sigma^V}\,\big(\,V_\epsilon\mathcal{J}_\sigma^\rho(u(s))-\mathcal{J}_\sigma^\rho(V_\epsilon u(s))\,\big)\,,\end{aligned}\tag{2.32}$$

using the fiber decomposition and the equation $K_\sigma^\rho(p)\,\Phi_\sigma^\rho(p)\;=\;E_\sigma(p)\,\Phi_\sigma^\rho(p)$. Let $\|\Phi\|_{C^\theta(\mathcal{S})}:=\sup_{p,q\in\mathcal{S}}\frac{\|\,\Phi(p)-\Phi(q)\,\|}{|p-q|^\theta}$. Below, we prove an estimate of the form

$$\|\phi^2(t)\|_{\mathcal{H}}\quad\leq\quad t\,C\,\|\widehat{|\nabla|^\theta V_\epsilon}\|_{L^1(\mathbb{R}^3)}\,(1\,+\,\|\Phi_\sigma^\rho\|_{C^\theta(\mathcal{S})})\,,\tag{2.33}$$

for $\theta<\frac23$. The key point here is that the θ-Hölder continuity of the fiber ground state $\Phi_\sigma^\rho(p)$ enables us to gain a θ derivative of the potential, yielding $\|\widehat{|\nabla|^\theta V_\epsilon}\|_{L^1(\mathbb{R}^3)}\leq C\epsilon^\theta$. To summarize, we have made use of the θ-Hölder continuity of $\Phi_\sigma^\rho(\,\cdot\,)$, which holds uniformly in σ, with $0<\sigma<\rho$, and we have used that

$$\|\widehat{|\nabla|^\theta V}\|_{L^1(\mathbb{R}^3)}\;\leq\;\gamma\,,\quad\text{where}\quad\gamma\;:=\;\|\widehat{V}(k)\|_{L^1}\;<\;\infty\,,\tag{2.34}$$

(see (1.14)).

Moreover, in Proposition B.4, we prove that $\|\Phi_\sigma^\rho\|_{C^\theta(\mathcal{S})}\;\leq\;C_\delta\,(1\,+\,\ln(\rho^{-1}))$. Collecting the estimates above, we arrive at

$$\|\phi^2(t)\|_{\mathcal{H}}\quad\leq\quad C_\delta\,t\,\epsilon^\theta\,(1+\ln(\rho^{-1})),\tag{2.35}$$

which yields the first term on the RHS of (2.21). □

Proof of (2.30). To verify (2.30), a simple calculation shows that

$$\nabla u(s)\quad=\quad e^{-isH_{\mathrm{eff},\sigma}}\,\nabla u_0^\epsilon\;-\,i\int_0^s dv\,e^{-ivH_{\mathrm{eff},\sigma}}\,\nabla V_\epsilon(x)\,e^{-i(s-v)H_{\mathrm{eff},\sigma}}\,u_0^\epsilon.\tag{2.36}$$

Using that $\|\nabla u_0^\epsilon\|_{L^2}\leq\epsilon$, and that

$$\|\nabla V_\epsilon\|_{L^\infty}\;\leq\;\|\widehat{\nabla V_\epsilon}\|_{L^1}\;\leq\;\gamma\,\epsilon\,,\tag{2.37}$$

where γ is defined in (2.34), we conclude that

$$\|\,\nabla u(s)\,\|_{L^2}\quad\leq\quad C\,\epsilon\,(1\,+\,s)\,,\tag{2.38}$$

and thus,

$$\int_0^t ds \, \| \nabla u(s) \|_{L^2} \, \leq \, C \, \epsilon \, t \, (1+t) \, . \tag{2.39}$$

This proves (2.30). □

Proof of (2.33). We define

$$\psi_s \quad := \quad V_\epsilon \mathcal{J}_\sigma^\rho(u(s)) - \mathcal{J}_\sigma^\rho(V_\epsilon u(s)). \tag{2.40}$$

Moreover, we define the generalized Fourier transform in the electron position, x, by

$$\widehat{\psi}(p) \; = \; \frac{1}{(2\pi)^{\frac{3}{2}}} \int dx \, e^{-i(p-P_f)x} \psi(x) \, . \tag{2.41}$$

By definitions (2.26) and (2.40) and the unitarity of the generalized Fourier transform proven in Appendix A, we have that

$$\|\phi^2(t)\|_{\mathcal{H}} \leq \int_0^t ds \, \|\psi_s\|_{L^2_x \otimes \mathfrak{F}} = \int_0^t ds \, \|\widehat{\psi_s}\|_{L^2_p \otimes \mathfrak{F}}. \tag{2.42}$$

Using the definition of $\mathcal{J}_\sigma^\rho$ and computing the Fourier transform, we find that

$$\hat{\psi}_s(p) \quad = \quad \int_{\mathbb{R}^3} dq \, \widehat{V}_\epsilon(p-q) \, \widehat{u}(s,q) \left(\chi_{\mathcal{S}_\mu}(q) \Phi_\sigma^\rho(q) \; - \; \chi_{\mathcal{S}_\mu}(p) \Phi_\sigma^\rho(p) \right). \tag{2.43}$$

It is important to note that, for any function $f \in L^2(\mathbb{R}^3)$ with $\mathrm{supp}(f) \subset \mathcal{S}_\mu$,

$$\mathrm{supp}(\widehat{V}_\epsilon * f) \, \subset \, \mathcal{S}, \tag{2.44}$$

for $\epsilon \leq \mu/3$, since we are assuming $\mathrm{supp}(\widehat{V}) \subset \{k||k| \leq 1\}$, so that $\mathrm{supp}(\widehat{V}_\epsilon) \subset \{k||k| \leq \epsilon\}$. Since the term in the integrand given by $(\widehat{u}(s)\chi_{\mathcal{S}_\mu}\Phi_\sigma^\rho)(q)$ is supported in $q \in \mathcal{S}_\mu$, so that, by (2.44), its convolution with $\widehat{V}_\epsilon$ has support in $\mathcal{S}$, we find

$$\hat{\psi}_s(p) \quad = \quad \mathbf{1}_{\mathcal{S}}(p) \int_{\mathbb{R}^3} dq \, \widehat{V}_\epsilon(p-q) \, \widehat{u}(s,q) \, (\chi_{\mathcal{S}_\mu}(q) \, \Phi_\sigma^\rho(q) - \chi_{\mathcal{S}_\mu}(p) \Phi_\sigma^\rho(p)) \, , \tag{2.45}$$

for $\epsilon \leq \mu/3$, where $\mathbf{1}_{\mathcal{S}}$ is the characteristic function of the set $\mathcal{S}$. Inserting $|p-q|^\theta |p-q|^{-\theta} = \mathbf{1}$ into (2.45), using the definition of $|\nabla|^\theta$ by its Fourier transform and using that, since $\chi_{\mathcal{S}_\mu}$ is a smooth function,

$$\sup_{p,q \in \mathcal{S}} |p-q|^{-\theta} \| \left(\chi_{\mathcal{S}_\mu}(q) \Phi_\sigma^\rho(q) \; - \; \chi_{\mathcal{S}_\mu}(p) \Phi_\sigma^\rho(p) \right) \|_{\mathfrak{F}} \leq C(1 \, + \, \|\Phi_\sigma^\rho\|_{C^\theta(\mathcal{S})}) \, , \tag{2.46}$$

we obtain the bound $\|\hat{\psi}_s\|_{L^2_x \otimes \mathfrak{F}} \leq C(1 + \|\Phi_\sigma^\rho\|_{C^\theta(\mathcal{S})}) \| \, |\mathbf{1}_{\mathcal{S}} \widehat{u}(s)| * |\widehat{|\nabla|^\theta V_\epsilon}| \, \|_{L^2(\mathcal{S})}$. Next, using Young's inequality, $\|f * g\|_{L^r} \leq \|f\|_{L^1} \|g\|_{L^r}$, we find that

$$\|\hat{\psi}_s\|_{L^2_x \otimes \mathfrak{F}} \leq C(1 \, + \, \|\Phi_\sigma^\rho\|_{C^\theta(\mathcal{S})}) \, \|\widehat{|\nabla|^\theta V_\epsilon}\|_{L^1(\mathbb{R}^3)} \sup_{s \in [0,t]} \|\mathbf{1}_{\mathcal{S}} \widehat{u}(s)\|_{L^2(\mathbb{R}^3)}. \tag{2.47}$$

Finally, observing that

$$\|\mathbf{1}_{\mathcal{S}} \widehat{u}(s)\|_{L^2(\mathbb{R}^3)} \, \leq \, \|\widehat{u}(s)\|_{L^2(\mathbb{R}^3)} = \|u(s)\|_{L^2(\mathbb{R}^3)} \, = \, \|u_0^\epsilon\|_{L^2(\mathbb{R}^3)} \, = \, 1 \, , \tag{2.48}$$

by unitarity of $e^{-itH_{\mathrm{eff},\sigma}}$, and using (2.42), we arrive at (2.33). □

3. The limit $\sigma \searrow 0$

In this section we remove the infrared cut-off from the evolution.

Proposition 3.1. *Under the conditions of Theorem 2.3, the strong limits*

$$s - \lim_{\sigma \searrow 0} e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon) \, = \, e^{-itH^V} \mathcal{J}_0^\rho(u_0^\epsilon) \tag{3.1}$$

and

$$s - \lim_{\sigma \searrow 0} \mathcal{J}_\sigma^\rho(e^{-itH_{\text{eff},\sigma}} u_0^\epsilon) \, = \, \mathcal{J}_0^\rho(e^{-itH_{\text{eff}}} u_0^\epsilon) \tag{3.2}$$

exist, for arbitrary $|t| < \infty$.

Proof. We write

$$e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon) - e^{-itH^V} \mathcal{J}_0^\rho(u_0^\epsilon) \, = \, (e^{-itH_\sigma^V} - e^{-itH^V}) \mathcal{J}_0^\rho(u_0^\epsilon) + e^{-itH_\sigma^V} (\mathcal{J}_\sigma^\rho - \mathcal{J}_0^\rho)(u_0^\epsilon). \tag{3.3}$$

Clearly,

$$\left\| e^{-itH_\sigma^V} (\mathcal{J}_\sigma^\rho - \mathcal{J}_0^\rho)(u_0^\epsilon) \right\| \, = \, \| (\mathcal{J}_\sigma^\rho - \mathcal{J}_0^\rho)(u_0^\epsilon) \| \, \leq \, \| u_0^\epsilon \|_{L^2} \sup_{p \in \mathcal{S}_\mu} \| \Phi_\sigma^\rho(p) - \Phi^\rho(p) \|_{\mathcal{F}} \, .$$

Thus,

$$\lim_{\sigma \searrow 0} \left\| e^{-itH_\sigma^V} (\mathcal{J}_\sigma^\rho - \mathcal{J}_0^\rho)(u_0^\epsilon) \right\| \, = \, 0 \, ,$$

follows from Proposition B.1.

Next, we discuss the first term on the right side of (3.3). In order to prove that it converges to 0, as $\sigma \searrow 0$, it suffices to show that H_σ^V converges to H^V in the norm resolvent sense; (see [24, Theorem VIII.21]), i.e.,

$$\lim_{\sigma \searrow 0} \left\| (H_\sigma^V + i)^{-1} - (H^V + i)^{-1} \right\| \, = \, 0.$$

From the second resolvent equation and the fact that $\| (H_\sigma^V + i)^{-1} \| \, \leq \, 1$, it follows that

$$\left\| (H_\sigma^V + i)^{-1} - (H^V + i)^{-1} \right\| \, = \, \left\| (H^V + i)^{-1} \, Q_\sigma \, (H_\sigma^V + i)^{-1} \right\| \, , \tag{3.4}$$

where

$$Q_\sigma \, := H^V - H_\sigma^V \, = \, \alpha^{\frac{1}{2}} A_{<\sigma}(x) \cdot v_\sigma + \frac{\alpha}{2} (A_{<\sigma}(x))^2 \, ,$$

and

$$v_\sigma \, := \, -i \nabla_x + \alpha^{\frac{1}{2}} A_\sigma(x)$$

is the velocity operator. Here $A_\sigma(x)$ is defined in (2.3), and

$$A_{<\sigma}(x) \, := \, \sum_\lambda \int_{|k| \leq \sigma} \frac{dk}{|k|^{1/2}} \left\{ \epsilon_\lambda(k) \, e^{-ikx} \otimes a_\lambda(k) \, + \, h.c. \right\} . \tag{3.5}$$

In order to estimate the norm of $Q_\sigma(H^V + i)^{-1}$, we use the following well-known lemma.

Lemma 3.2. *Let* $f, g \in \mathrm{L}^2(\mathbb{R}^3 \times \{+,-\}; \mathcal{B}(\mathcal{H}_{\mathrm{el}}))$ *be operator-valued functions such that* $\|(1+|k|^{-1})^{1/2} f\|, \|(1+|k|^{-1})^{1/2} g\| < \infty$. *Then*

$$\|a^{\#}(f)(H_f+1)^{-\frac{1}{2}}\| \leq \|(1+|k|^{-1})^{\frac{1}{2}} f\|, \tag{3.6}$$

$$\|a^{\#}(f)a^{\#}(g)(H_f+1)^{-1}\| \leq \|(1+|k|^{-1})^{\frac{1}{2}} f\| \|(1+|k|^{-1})^{\frac{1}{2}} g\|, \tag{3.7}$$

where $a^{\#}$ *stands for* a *or* a^*.

In particular, using the Kato-Rellich theorem, one easily shows that, for α small enough, $D(H^V) = D(-\Delta_x \otimes I + I \otimes H_f) \subset D(H_f)$. Thus, we have that

$$\big\|(H_f+1)(H^V+i)^{-1}\big\| \leq C,$$

which when combined with Lemma 3.2 yields

$$\Big\|\frac{\alpha}{2}(A_{<\sigma}(x))^2(H^V+i)^{-1}\Big\| \leq C\,\alpha\,\sigma. \tag{3.8}$$

Likewise one verifies that

$$\Big\|\alpha^{\frac{1}{2}} A_{<\sigma}(x)\cdot v_\sigma(H^V+i)^{-1}\Big\| \leq C\,\alpha^{\frac{1}{2}}\,\sigma^{\frac{1}{2}}, \tag{3.9}$$

since $0 \leq v_\sigma^2 \leq H^V + \|V\|_{L^\infty}$ is bounded relative to H^V. Estimates (3.8) and (3.9) yield

$$\big\|Q_\sigma(H^V+i)^{-1}\big\| \leq C\,\alpha^{\frac{1}{2}}\,\sigma^{\frac{1}{2}}.$$

By (3.4), we have shown that H^V_σ converges to H^V, as $\sigma \searrow 0$, in the norm resolvent sense. □

4. Proof of Theorem 1.1

In this section, we prove the bound in Theorem 1.1, which compares the full dynamics to the effective dynamics for the system without infrared cutoff. We have that

$$\begin{aligned}
&\|\, e^{-itH^V}\, \mathcal{J}_0^\rho(\, u_0^\epsilon\,) - \mathcal{J}_0^\rho(\, e^{-itH_{\mathrm{eff}}}\, u_0^\epsilon\,)\,\|_{\mathcal{H}} \\
&\leq \ \|\, e^{-itH^V_\sigma}\, \mathcal{J}_\sigma^\rho(\, u_0^\epsilon\,) - \mathcal{J}_\sigma^\rho(\, e^{-itH_{\mathrm{eff},\sigma}}\, u_0^\epsilon\,)\,\|_{\mathcal{H}} \\
&\quad + \|\, e^{-itH^V_\sigma}\, \mathcal{J}_\sigma^\rho(\, u_0^\epsilon\,) - e^{-itH^V}\, \mathcal{J}_0^\rho(\, u_0^\epsilon\,)\,\|_{\mathcal{H}} \\
&\quad + \|\, \mathcal{J}_\sigma^\rho(\, e^{-itH_{\mathrm{eff},\sigma}}\, u_0^\epsilon\,) - \mathcal{J}_0^\rho(\, e^{-itH_{\mathrm{eff}}}\, u_0^\epsilon\,)\,\|_{\mathcal{H}}\,,
\end{aligned} \tag{4.1}$$

for any t and $0 < \sigma < \rho \leq 1$. It follows from Theorem 2.3 that the first term on the r.s. of the inequality sign is bounded by $C_\delta\,(1+\ln(\rho^{-1}))\,\epsilon^{\frac{2}{3}-\delta}\,t + C\,\alpha^{\frac{1}{2}}\,\rho^{\frac{1}{2}}\,\epsilon\,t\,(1+t)$, uniformly in $\sigma > 0$.

From Proposition 3.1, it follows that the second and third term on the r.s. converge to zero, as $\sigma \searrow 0$.

We thus conclude that

$$\|\, e^{-itH^V}\, \mathcal{J}_0^\rho(\, u_0^\epsilon\,) - \mathcal{J}_0^\rho(\, e^{-itH_{\mathrm{eff}}}\, u_0^\epsilon\,)\,\|_{\mathcal{H}} \leq C_\delta\,(1+\ln(\rho^{-1}))\,\epsilon^{\frac{2}{3}-\delta}\,t + C\,\alpha^{\frac{1}{2}}\,\rho^{\frac{1}{2}}\,\epsilon\,t\,(1+t)\,, \tag{4.2}$$

by taking σ to zero. This concludes the proof of Theorem 1.1. □

Appendix A. Generalized Fourier transform

For $\psi \in \mathcal{H}$, we define the Fourier transform in the electron position, x, by

$$\widehat{\psi}(p) \;=\; \frac{1}{(2\pi)^{\frac{3}{2}}} \int dx\, e^{-i(p-P_f)x} \psi(x)\,. \tag{A.1}$$

Our claim is that

$$\phi^{\vee}(x) \;=\; \frac{1}{(2\pi)^{\frac{3}{2}}} \int dp\, e^{i(p-P_f)x} \phi(p) \tag{A.2}$$

corresponds to the inverse Fourier transform, in the following sense.

Lemma A.1. *The above linear operations define unitary maps $\mathcal{H} \to \mathcal{H}$, and are mutual inverses.*

Proof. By density, we may assume that ψ is smooth in x and of rapid decay, i.e., belongs to the space

$$\mathcal{S}_{\mathfrak{F}} \;:=\; \Big\{ \psi \in \mathcal{H} \,\Big|\, \sup_x \big\| x^\alpha \partial_x^\beta \psi \big\|_{\mathfrak{F}} \;<\; \infty \quad , \quad \forall \alpha, \beta \in \mathbb{N}_0^3 \Big\}\,. \tag{A.3}$$

Then,

$$\begin{aligned}
(\widehat{\psi})^{\vee}(x) \;&=\; \frac{1}{(2\pi)^{\frac{3}{2}}} \int dp\, e^{-i(p-P_f)x} \frac{1}{(2\pi)^{\frac{3}{2}}} \int dx'\, e^{i(p-P_f)x'}\, \psi(x') \\
&=\; \frac{1}{(2\pi)^3} \int dx' \int dp\, e^{-ip(x-x')} e^{iP_f(x-x')}\, \psi(x') \\
&=\; \int dx'\, \delta(x-x')\, e^{iP_f(x-x')}\, \psi(x') \\
&=\; \psi(x)\,.
\end{aligned} \tag{A.4}$$

Likewise, for $\phi \in \mathcal{S}_{\mathfrak{F}}$,

$$\begin{aligned}
(\phi^{\vee})\widehat{\ }(p) \;&=\; \frac{1}{(2\pi)^{\frac{3}{2}}} \int dx\, e^{i(p-P_f)x} \frac{1}{(2\pi)^{\frac{3}{2}}} \int dq\, e^{-i(q-P_f)x}\, \phi(q) \\
&=\; \frac{1}{(2\pi)^3} \int dq \int dx\, e^{i(p-q)x}\, \phi(q) \\
&=\; \int dq\, \delta(p-q)\, \phi(q) \\
&=\; \phi(p)\,.
\end{aligned} \tag{A.5}$$

From the density of $\mathcal{S}_{\mathfrak{F}}$ in $\mathcal{H}$, we infer that (A.1) and (A.2) define linear maps $\mathcal{H} \to \mathcal{H}$, and are mutual inverses. Moreover, the identity

$$\begin{aligned}
\int dp\, \big\| \widehat{\psi}(p) \big\|_{\mathfrak{F}}^2 \;&=\; \int dp \Big\langle \frac{1}{(2\pi)^{\frac{3}{2}}} \int dx e^{-i(p-P_f)x}\, \psi(x)\,,\; \frac{1}{(2\pi)^{\frac{3}{2}}} \int dx'\, e^{-i(p-P_f)x'}\, \psi(x') \Big\rangle_{\mathfrak{F}} \\
&=\; (2\pi)^{-3} \int dx \int dx'\, \underbrace{\Big(\int dp\, e^{ip(x-x')} \Big)}_{=(2\pi)^3 \delta(x-x')} \Big\langle e^{iP_f x} \psi(x)\,,\; e^{iP_f x'}\, \psi(x') \Big\rangle_{\mathfrak{F}} \\
&=\; \int dx\, \big\| \psi(x) \big\|_{\mathfrak{F}}^2
\end{aligned} \tag{A.6}$$

proves unitarity. □

As an application, we observe that

$$\begin{aligned} \widehat{P_{tot}\psi}(p) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int dx\, e^{i(p-P_f)x}(-i\nabla_x+P_f)\psi(x) \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int dx\,\big(\,(-i\nabla_x+P_f)e^{i(p-P_f)x}\,\big)\,\psi(x) \\ &= p\,\widehat{\psi}(p)\,. \end{aligned} \tag{A.7}$$

To each fiber space $\mathcal{H}_p$ there corresponds an isomorphism $I_p : \mathcal{H}_p \to \mathfrak{F}^b$ where $\mathfrak{F}^b$ is the Fock space corresponding to creation- and annihilation operators $b^*_\lambda(k)$ (given by $e^{-ikx}a^*_\lambda(k)$), and $b_\lambda(k)$ (given by $e^{ikx}a_\lambda(k)$) commuting with P_{tot}. The vacuum $\Omega_f \in \mathfrak{F}^b$ is given by $I_p(e^{ipx})$. To define I_p explicitly, we consider an improper state in $\mathcal{H}_p$ with a definite total momentum p describing the electron at position x together with n photons. Its wave function has the form

$$e^{i(p-k_1-\cdots-k_n)x}\,\phi^{(n)}(k_1,\lambda_1;\ldots;k_n,\lambda_n)\,. \tag{A.8}$$

Then the equation

$$\begin{aligned} & I_p\Big(\,e^{i(p-k_1-\cdots-k_n)x}\,\phi^{(n)}(k_1,\lambda_1;\ldots;k_n,\lambda_n)\,\Big) \qquad (A.9) \\ &= \sum_{\lambda_1,\ldots,\lambda_n}\int dk_1\cdots dk_n\,\phi^{(n)}(k_1,\lambda_1;\ldots;k_n,\lambda_n)\,b^*_{\lambda_1}(k_1)\,\cdots\,b^*_{\lambda_n}(k_n)\,\Omega_f \end{aligned}$$

defines I_p.

Next, we discuss the Hamiltonian (2.2). It is easy to see that H_σ is translation-invariant, so that $[H_\sigma, P_{tot}] = 0$. Accordingly,

$$(H_\sigma\psi)\widehat{\ }(p) \,=\, H_\sigma(p)\widehat{\psi}(p), \tag{A.10}$$

where $H_\sigma(p)$ is the fiber Hamiltonian corresponding to total momentum p. Applying the isomorphism $I_p : \mathcal{H}_p \to \mathfrak{F}^b$, $H_\sigma(p)$ is represented on $\mathfrak{F}^b$ by

$$H^b_\sigma(p) \,:=\, I_p H_\sigma(p) I_p^{-1} \,=\, \frac{1}{2}(\,p\,-\,P^b_f\,+\,\sqrt{\alpha}A^b_\sigma\,)^2\,+\,H^b_f \tag{A.11}$$

where

$$A^b_\sigma \,=\, \sum_\lambda \int_{\sigma\le|k|\le 1} dk\,\frac{dk}{|k|^{1/2}}\,\{\,\epsilon_\lambda(k)\,b_\lambda(k)\,+\,h.c.\,\} \tag{A.12}$$

and

$$P^b_f \,=\, \sum_\lambda \int\, k\,b^*_\lambda(k)\,b_\lambda(k)\,dk \tag{A.13}$$

$$H^b_f \,=\, \sum_\lambda \int\, |k|\,b^*_\lambda(k)\,b_\lambda(k)\,dk\,. \tag{A.14}$$

We note that

$$H_\sigma \,=\, \int^\oplus I_p^{-1} H^b_\sigma(p) I_p\,dp \tag{A.15}$$

is the direct integral decomposition of H_σ over the spectrum of P_{tot}.

Remark A.2. *Throughout this paper, we have usually dropped the superscripts in $H^b_\sigma(p)$ and $\mathfrak{F}^b$, etc., while keeping the notation $b^*_\lambda(k)$, $b_\lambda(k)$ for the creation- and annihilation operators on $\mathfrak{F}^b$.*

Appendix B. Hölder continuity of the ground state

We recall that $\Phi^\rho_\sigma(p)$ denotes a normalized ground state of the Bogoliubov transformed fiber Hamiltonian $K^\rho_\sigma(p) = W^\rho_{\nabla E_\sigma(p)}\, H_\sigma(p)\, (W^\rho_{\nabla E_\sigma(p)})^*$, with infrared cutoff $\sigma > 0$ (see (2.22)). Our aim in this appendix is to prove that, for a suitable choice of the vectors $\Phi^\rho_\sigma(p)$, the map $p \mapsto \Phi^\rho_\sigma(p)$ is θ-Hölder continuous, for $\theta < 2/3$.

For $\rho = 1$, we set

$$\Phi_\sigma(p) := \Phi^1_\sigma(p), \qquad K_\sigma(p) := K^1_\sigma(p). \tag{B.1}$$

We remark that

$$K^\rho_\sigma(p) \;=\; \big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^*\, K_\sigma(p)\, W^{\rho,1}_{\nabla E_\sigma(p)}\,, \qquad \Phi^\rho_\sigma(p) \;=\; \big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^*\, \Phi_\sigma(p)\,, \tag{B.2}$$

where

$$W^{\rho,1}_{\nabla E_\sigma(p)} \;:=\; \exp\Big[\alpha^{\frac12} \sum_\lambda \int_{\rho \le |k| \le 1} dk\, \frac{\nabla E_\sigma(p)\cdot \epsilon_\lambda(k)\, b_\lambda(k) - h.c.}{|k|^{1/2}(|k| - \nabla E_\sigma(p)\cdot k)}\Big]\,.$$

Letting

$$\mathfrak{F}_\sigma \;:=\; \bigoplus_{n\ge 0} \mathrm{Sym}\big(\, L^2(\{k \in \mathbb{R}^3, |k| \ge \sigma\} \,\times\, \{+,-\}\,)\,\big)^{\otimes n} \tag{B.3}$$

denote the Fock space of photons of energies $\geq \sigma$, and identifying $\mathfrak{F}_\sigma$ with a subspace of $\mathfrak{F}$, we observe that $K_\sigma(p)$ leaves $\mathfrak{F}_\sigma$ invariant. Let $\tilde K_\sigma(p)$ denote the restriction of $K_\sigma(p)$ to $\mathfrak{F}_\sigma$. An important property, proven in [3, 10, 13], is that there is an energy gap of size $\eta\sigma$, $\eta > 0$, in the spectrum of $\tilde K_\sigma(p)$ above the ground state energy $E_\sigma(p)$. Moreover, one can choose

$$\Phi_\sigma(p) \;=\; \tilde\Phi_\sigma(p) \otimes \Omega_f\,, \tag{B.4}$$

in the representation $\mathfrak{F} \simeq \mathfrak{F}_\sigma \otimes \mathfrak{F}_{<\sigma}$, where

$$\mathfrak{F}_{<\sigma} \;:=\; \bigoplus_{n\ge 0} \mathrm{Sym}\big(\, L^2(\{k \in \mathbb{R}^3, |k| \le \sigma\} \,\times\, \{+,-\}\,)\,\big)^{\otimes n}\,. \tag{B.5}$$

Now, by [10, 13], $\tilde\Phi_\sigma(p)$ can be chosen in the following way:

$$\tilde\Phi_\sigma(p) = \frac{\tilde\Pi_\sigma(p)\Omega_f}{\|\tilde\Pi_\sigma(p)\Omega_f\|}, \tag{B.6}$$

where Ω_f denotes the vacuum in Fock space and $\tilde\Pi_\sigma(p)$ is the rank-one projection onto the eigenspace associated with $E_\sigma(p) = \inf \mathrm{spec}(\tilde K_\sigma(p))$. We recall from [10, 13] that

$$\|\tilde\Pi_\sigma(p)\Omega_f\| \;\ge\; \frac{1}{3}, \tag{B.7}$$

for arbitrary $\sigma > 0$ and $|p| \leq 1/3$ provided that α is chosen sufficiently small.

Let N denote the number operator,

$$N \;=\; \sum_{\lambda} \int dk \, b^*_\lambda(k)\, b_\lambda(k)\,. \tag{B.8}$$

The following proposition has been proven in [8, 10, 13].

Proposition B.1. *For $\alpha \ll 1$ and $|p| \leq 1/3$, there exists a vector $\Phi(p)$ in the Fock space such that $\Phi_\sigma(p) \to \Phi(p)$, strongly, as $\sigma \to 0$. The following bound holds,*

$$\|N^{\frac{1}{2}} \Phi_\sigma(p)\| \;\leq\; C < \infty\,, \tag{B.9}$$

uniformly in $\sigma \geq 0$. Moreover, For all $\delta > 0$, there exists $\alpha_\delta > 0$ and $C_\delta < \infty$ such that, for all $0 \leq \alpha \leq \alpha_\delta$, $0 \leq \sigma' < \sigma \leq 1$ and $|p| \leq 1/3$,

$$\|\Phi_\sigma(p) - \Phi_{\sigma'}(p)\| \;\leq\; C_\delta\, \alpha^{\frac{1}{4}}\, \sigma^{1-\delta}, \tag{B.10}$$

$$|\nabla E_\sigma(p) - \nabla E_{\sigma'}(p)| \;\leq\; C_\delta\, \alpha^{\frac{1}{4}}\, \sigma^{1-\delta}. \tag{B.11}$$

As a consequence, we show the following corollary.

Corollary B.2. *Let $0 < \rho < 1$. For all $\delta > 0$, there exists $\alpha_\delta > 0$ such that, for all $0 \leq \alpha \leq \alpha_\delta$ and $|p| \leq 1/3$, there exists a vector $\Phi^\rho(p)$ in the Fock space such that $\Phi^\rho_\sigma(p) \to \Phi^\rho(p)$, strongly, as $\sigma \to 0$. Moreover, there exists a constant $C_\delta < \infty$ such that, for all $0 \leq \alpha \leq \alpha_\delta$, $0 \leq \sigma' < \sigma \leq 1$ and $|p| \leq 1/3$,*

$$\|\Phi^\rho_\sigma(p) - \Phi^\rho_{\sigma'}(p)\| \;\leq\; C_\delta\, \alpha^{\frac{1}{4}}\, \sigma^{1-\delta}\, (1 + \ln(\rho^{-1}))\,. \tag{B.12}$$

Proof. Using (B.2), we split

$$\Phi^\rho_\sigma(p) - \Phi^\rho_{\sigma'}(p) \;=\; \Big(\big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^* - \big(W^{\rho,1}_{\nabla E_{\sigma'}(p)}\big)^* \Big) \Phi_\sigma(p) \;+\; \big(W^{\rho,1}_{\nabla E_{\sigma'}(p)}\big)^* \big(\Phi_\sigma(p) - \Phi_{\sigma'}(p)\big). \tag{B.13}$$

By Proposition B.1 and unitarity of $W^{\rho,1}_{\nabla E_\sigma(p)}$, the second term is estimated as

$$\Big\| \big(W^{\rho,1}_{\nabla E_{\sigma'}(p)}\big)^* \big(\Phi_\sigma(p) - \Phi_{\sigma'}(p)\big) \Big\| \leq C_\delta\, \alpha^{\frac{1}{4}}\, \sigma^{1-\delta}\,. \tag{B.14}$$

The first term in the right side of (B.13) is estimated as

$$\begin{aligned} \Big\| \Big(\big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^* - \big(W^{\rho,1}_{\nabla E_{\sigma'}(p)}\big)^* \Big) \Phi_\sigma(p) \Big\| &= \Big\| \Big(\mathbf{1} - W^{\rho,1}_{\nabla E_\sigma(p)} \big(W^{\rho,1}_{\nabla E_{\sigma'}(p)}\big)^* \Big) \Phi_\sigma(p) \Big\| \\ &\leq \big\| B(\rho) \Phi_\sigma(p) \big\|\,, \end{aligned} \tag{B.15}$$

by unitarity of $W^{\rho,1}_{\nabla E_\sigma(p)}$ and the spectral theorem, where

$$B(\rho) := \alpha^{\frac{1}{2}} \sum_\lambda \int_{\rho \leq |k| \leq 1} dk \Big(\frac{\nabla E_\sigma(p) \cdot \epsilon_\lambda(k)\, b_\lambda(k) - h.c.}{|k|^{1/2}(|k| - \nabla E_\sigma(p) \cdot k)} - \frac{\nabla E_{\sigma'}(p) \cdot \epsilon_\lambda(k)\, b_\lambda(k) - h.c.}{|k|^{1/2}(|k| - \nabla E_{\sigma'}(p) \cdot k)} \Big)\,. \tag{B.16}$$

To estimate $\|B(\rho)\Phi_\sigma(p)\|$, we use the well known fact that, for any $f \in \mathrm{L}^2(\mathbb{R}^3 \times \{+,-\})$,

$$\|a^{\#}(f)(N+1)^{-\frac{1}{2}}\| \;\leq\; \sqrt{2} \|f\|_{L^2}\,. \tag{B.17}$$

Clearly,

$$\frac{\nabla E_\sigma(p)\cdot\epsilon_\lambda(k)}{|k|^{1/2}(|k|-\nabla E_\sigma(p)\cdot k)}-\frac{\nabla E_{\sigma'}(p)\cdot\epsilon_\lambda(k)}{|k|^{1/2}(|k|-\nabla E_{\sigma'}(p)\cdot k)}$$
$$=\frac{(\nabla E_\sigma(p)-\nabla E_{\sigma'}(p))\cdot\epsilon_\lambda(k)}{|k|^{1/2}(|k|-\nabla E_\sigma(p)\cdot k)}+\frac{\nabla E_{\sigma'}(p)\cdot\epsilon_\lambda(k)}{|k|^{1/2}(|k|-\nabla E_\sigma(p)\cdot k)}\frac{(\nabla E_\sigma(p)-\nabla E_{\sigma'}(p))\cdot k}{(|k|-\nabla E_{\sigma'}(p)\cdot k)}\,. \tag{B.18}$$

Hence, by (B.11) and the facts that $|\nabla E_\sigma(p)|, |\nabla E_{\sigma'}(p)|\le 1/2$ for α small enough (see Proposition 2.1 (2)), we obtain

$$\Big|\frac{\nabla E_\sigma(p)\cdot\epsilon_\lambda(k)}{|k|^{1/2}(|k|-\nabla E_\sigma(p)\cdot k)}-\frac{\nabla E_{\sigma'}(p)\cdot\epsilon_\lambda(k)}{|k|^{1/2}(|k|-\nabla E_{\sigma'}(p)\cdot k)}\Big|\;\le\;\frac{C_\delta\,\alpha^{\frac14}\,\sigma^{1-\delta}}{|k|^{\frac32}}\,. \tag{B.19}$$

Thus, (B.16) and (B.17) yield that

$$\begin{aligned}\big\|B(\rho)\Phi_\sigma(p)\big\| \;&\le\; C_\delta\,\alpha^{\frac34}\,\sigma^{1-\delta}\Big\|\frac{\mathbf{1}_{\rho\le|k|\le1}(|k|)}{|k|^{\frac32}}\Big\|_{L^2_k}\big\|(N+1)^{\frac12}\Phi_\sigma(p)\big\|\\ &\le\; C_\delta\,\alpha^{\frac34}\,\sigma^{1-\delta}\,\ln(\rho^{-1})\,.\end{aligned} \tag{B.20}$$

where we used (B.9) in the last inequality. Together with (B.13) – (B.15), this concludes the proof of the corollary. □

The following result follows from [10, 13] (it is also a consequence of (2.6) in Proposition 2.1 (2)).

Proposition B.3. *There exist $\alpha_c>0$ and $C>0$ such that, for all $0\le\alpha\le\alpha_c$ and p,p' satisfying $|p|\le1/3$, $|p'|\le1/3$,*

$$\big|\nabla E_\sigma(p)-\nabla E_\sigma(p')\big|\;\le\;C\,|p-p'|, \tag{B.21}$$

uniformly in $\sigma>0$.

We now prove the following proposition.

Proposition B.4. *Let $0<\rho<1$. For all $\delta>0$, there exist $\alpha_\delta>0$ and $C_\delta<\infty$ such that, for all $0\le\alpha\le\alpha_\delta$, $\sigma>0$ and $p,k\in\mathbb{R}^3$ satisfying $|p|\le1/3$, $|p+k|\le1/3$,*

$$\|\Phi^\rho_\sigma(p+k)-\Phi^\rho_\sigma(p)\|\;\le\;C_\delta\,(1+\ln(\rho^{-1}))\,|k|^{\frac23-\delta}. \tag{B.22}$$

We recall the following easy lemma (see e.g. [3]).

Lemma B.5. *Let $\mathcal{H}$ be a Hilbert space and let Π_1 and Π_2 be two rank-one projections in $\mathcal{H}$. Let $\Phi_1\in\operatorname{Ran}\Pi_1$, $\|\Phi_1\|=1$ and $\Phi_2\in\operatorname{Ran}\Pi_2$, $\|\Phi_2\|=1$. We have that*

$$\|\Pi_1-\Pi_2\|=\big|\langle\Phi_1,(\Pi_1-\Pi_2)\Phi_1\rangle\big|^{\frac12}=\big|\langle\Phi_2,(\Pi_2-\Pi_1)\Phi_2\rangle\big|^{\frac12}. \tag{B.23}$$

Proof of Proposition B.4

Step 1. We first prove that, for all $0<\sigma<\rho\le1$,

$$\|\Phi^\rho_\sigma(p+k)-\Phi^\rho_\sigma(p)\|\;\le\;C\,|k|\,\sigma^{-\frac12}\,. \tag{B.24}$$

We decompose

$$\begin{aligned}\Phi^{\rho}_{\sigma}(p+k)-\Phi^{\rho}_{\sigma}(p) \;&=\; \big(W^{\rho,1}_{\nabla E_\sigma(p+k)}\big)^*\,\Phi_\sigma(p+k)\;-\;\big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^*\,\Phi_\sigma(p)\\ &=\;\Big(\big(W^{\rho,1}_{\nabla E_\sigma(p+k)}\big)^*-\big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^*\Big)\,\Phi_\sigma(p)\\ &\quad+\;\big(W^{\rho,1}_{\nabla E_\sigma(p+k)}\big)^*\,\big(\Phi_\sigma(p+k)-\Phi_\sigma(p)\big)\,. \end{aligned} \tag{B.25}$$

To estimate the first term in the right side of (B.25), we proceed as in the proof of Corollary B.2. Namely, we have that

$$\begin{aligned}\Big\|\Big(\big(W^{\rho,1}_{\nabla E_\sigma(p+k)}\big)^*-\big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^*\Big)\Phi_\sigma(p)\Big\| &=\Big\|\Big(\mathbf{1}-W^{\rho,1}_{\nabla E_\sigma(p+k)}\big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^*\Big)\Phi_\sigma(p)\Big\|\\ &\leq\;\big\|C(\rho)\Phi_\sigma(p)\big\|\,, \end{aligned}\tag{B.26}$$

by the spectral theorem, where

$$C(\rho):=\alpha^{\frac12}\sum_\lambda\int_{\rho\leq|\tilde k|\leq 1}d\tilde k\,\Big(\frac{\nabla E_\sigma(p+k)\cdot\epsilon_\lambda(\tilde k)\,b_\lambda(\tilde k)-h.c.}{|\tilde k|^{1/2}(|\tilde k|-\nabla E_\sigma(p+k)\cdot\tilde k)}-\frac{\nabla E_\sigma(p)\cdot\epsilon_\lambda(\tilde k)\,b_\lambda(\tilde k)-h.c.}{|\tilde k|^{1/2}(|\tilde k|-\nabla E_\sigma(p)\cdot\tilde k)}\Big)\,.$$

Using Proposition B.3, one verifies that

$$\Big|\frac{\nabla E_\sigma(p+k)\cdot\epsilon_\lambda(\tilde k)}{|\tilde k|^{1/2}(|\tilde k|-\nabla E_\sigma(p+k)\cdot\tilde k)}-\frac{\nabla E_\sigma(p)\cdot\epsilon_\lambda(\tilde k)}{|\tilde k|^{1/2}(|\tilde k|-\nabla E_\sigma(p)\cdot\tilde k)}\Big|\;\leq\;\frac{C\,|k|}{|\tilde k|^{\frac32}}\,. \tag{B.27}$$

Hence (B.17) implies that

$$\begin{aligned}\big\|C(\rho)\Phi_\sigma(p)\big\|\;&\leq\;C\,|k|\,\Big\|\frac{\mathbf{1}_{\rho\leq|\tilde k|\leq1}(\tilde k)}{|\tilde k|^{\frac32}}\Big\|_{L^2_{\tilde k}}\big\|(N+1)^{\frac12}\Phi_\sigma(p)\big\|\\ &\leq\;C\,|k|\,\ln(\rho^{-1})\,, \end{aligned}\tag{B.28}$$

where we used (B.9) in the last inequality. Equations (B.26) and (B.28) yield

$$\begin{aligned}\Big\|\Big(\big(W^{\rho,1}_{\nabla E_\sigma(p+k)}\big)^*-\big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^*\Big)\Phi_\sigma(p)\Big\| &=\Big\|\Big(\mathbf{1}-W^{\rho,1}_{\nabla E_\sigma(p+k)}\big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^*\Big)\Phi_\sigma(p)\Big\|\\ &\leq C\,|k|\,\sigma^{-\frac12}\,\sigma^{\frac12}\,\ln(\rho^{-1})\\ &\leq C\,|k|\,\sigma^{-\frac12}\,, \end{aligned}\tag{B.29}$$

since $0<\sigma<\rho\leq1$.

It remains to estimate the second term in the right side of (B.25). By unitarity of $W^{\rho,1}_{\nabla E_\sigma(p+k)}$, it suffices to estimate $\|\Phi_\sigma(p+k)-\Phi_\sigma(p)\|$, and, by (B.4),

$$\|\Phi_\sigma(p+k)-\Phi_\sigma(p)\|=\|\tilde\Phi_\sigma(p+k)-\tilde\Phi_\sigma(p)\|\,. \tag{B.30}$$

It follows from (B.7) and Lemma B.5 that

$$\begin{aligned}\|\tilde\Phi_\sigma(p+k)-\tilde\Phi_\sigma(p)\|\;&\leq\;\Big(\frac{1}{\|\tilde\Pi_\sigma(p)\Omega_f\|}+\frac{1}{\|\tilde\Pi_\sigma(p+k)\Omega_f\|}\Big)\big\|\tilde\Pi_\sigma(p)\Omega_f-\tilde\Pi_\sigma(p+k)\Omega_f\big\|\\ &\leq\;6\big\|\tilde\Pi_\sigma(p)-\tilde\Pi_\sigma(p+k)\big\|\\ &=\;6\big|\big\langle\tilde\Phi_\sigma(p),\bar\Pi_\sigma(p+k)\tilde\Phi_\sigma(p)\big\rangle\big|^{\frac12}, \end{aligned}\tag{B.31}$$

where $\bar{\Pi}_\sigma(p+k) := I - \tilde{\Pi}_\sigma(p+k)$. Using that there is an energy gap of size $\eta\sigma$ above $E_\sigma(p+k)$ in the spectrum of the operator $\tilde{K}_\sigma(p+k)$, we can estimate

$$\begin{aligned} &\big|\big\langle \tilde{\Phi}_\sigma(p), \bar{\Pi}_\sigma(p+k)\tilde{\Phi}_\sigma(p)\big\rangle\big|^{\frac{1}{2}} \\ &\leq \big\|(\tilde{K}_\sigma(p+k) - E_\sigma(p+k))^{-1}\bar{\Pi}_\sigma(p+k)\big\|^{\frac{1}{2}} \big\|(\tilde{K}_\sigma(p+k) - E_\sigma(p+k))^{\frac{1}{2}}\tilde{\Phi}_\sigma(p)\big\| \\ &\leq \eta^{-\frac{1}{2}}\,\sigma^{-\frac{1}{2}}\,\big\|(\tilde{K}_\sigma(p+k) - E_\sigma(p+k))^{\frac{1}{2}}\tilde{\Phi}_\sigma(p)\big\|. \end{aligned} \tag{B.32}$$

Let

$$\nabla_p K_\sigma(p) := W_{\nabla E_\sigma(p)} \nabla_p H_\sigma(p) W^*_{\nabla E_\sigma(p)}\,, \tag{B.33}$$

where

$$\nabla_p H_\sigma(p) = p - P_f + \alpha^{\frac{1}{2}} A_\sigma\,. \tag{B.34}$$

Using the Feynman-Hellman formula,

$$\langle \tilde{\Phi}_\sigma(p), \nabla_p K_\sigma(p)\tilde{\Phi}_\sigma(p)\rangle = \nabla E_\sigma(p)\,, \tag{B.35}$$

together with the mean-value theorem and Proposition B.3, we have that (see also [7, Lemma 3.6])

$$\begin{aligned} &\big\|(\tilde{K}_\sigma(p+k) - E_\sigma(p+k))^{\frac{1}{2}}\tilde{\Phi}_\sigma(p)\big\|^2 \\ &= \big\langle \tilde{\Phi}_\sigma(p), (\tilde{K}_\sigma(p+k) - E_\sigma(p+k))\tilde{\Phi}_\sigma(p)\big\rangle \\ &= \big\langle \tilde{\Phi}_\sigma(p), (\tilde{K}_\sigma(p) + k\cdot(\nabla_p K_\sigma(p)) + k^2/2 - E_\sigma(p+k))\tilde{\Phi}_\sigma(p)\big\rangle \\ &= E_\sigma(p) - E_\sigma(p+k) + k\cdot(\nabla_p E_\sigma(p)) + k^2/2 \\ &\leq C\,k^2. \end{aligned} \tag{B.36}$$

Hence,

$$\big\|\big(\tilde{K}_\sigma(p+k) - E_\sigma(p+k)\big)^{\frac{1}{2}}\tilde{\Phi}_\sigma(p)\big\| \leq C\,|k|. \tag{B.37}$$

Combining (B.31), (B.32) and (B.37), we obtain that

$$\|\tilde{\Phi}_\sigma(p+k) - \tilde{\Phi}_\sigma(p)\| \leq C\,|k|\,\sigma^{-\frac{1}{2}}\,, \tag{B.38}$$

and hence (B.24) follows.

Step 2. We now prove that $\|\Phi^\rho_\sigma(p+k) - \Phi^\rho_\sigma(p)\| \leq C_\delta\,|k|^{2/3-\delta}$ (with $C_\delta < \infty$ for $\delta > 0$).

Suppose first that $\sigma \geq |k|^{2/3}$. Then by Step 1, we have that

$$\|\Phi^\rho_\sigma(p+k) - \Phi^\rho_\sigma(p)\| \leq C\,|k|\,|k|^{-\frac{1}{3}} = C\,|k|^{\frac{2}{3}}. \tag{B.39}$$

Newt, we assume that $\sigma \leq |k|^{2/3}$. We write

$$\begin{aligned} \|\Phi^\rho_\sigma(p+k) - \Phi^\rho_\sigma(p)\| \leq\ & \|\Phi^\rho_\sigma(p+k) - \Phi^\rho(p+k)\| + \|\Phi^\rho(p+k) - \Phi^\rho_{|k|^{2/3}}(p+k)\| \\ &+ \|\Phi^\rho_\sigma(p) - \Phi^\rho(p)\| + \|\Phi^\rho(p) - \Phi^\rho_{|k|^{2/3}}(p)\| \\ &+ \|\Phi^\rho_{|k|^{2/3}}(p+k) - \Phi^\rho_{|k|^{2/3}}(p)\|. \end{aligned} \tag{B.40}$$

By Corollary B.2, the first two lines are bounded by

$$\begin{aligned}&\|\Phi^\rho_\sigma(p+k)-\Phi^\rho(p+k)\|+\|\Phi^\rho(p+k)-\Phi^\rho_{|k|^{2/3}}(p+k)\|\\&+\|\Phi^\rho_\sigma(p)-\Phi^\rho(p)\|+\|\Phi^\rho(p)-\Phi^\rho_{|k|^{2/3}}(p)\|\\&\leq\ C_\delta\,\alpha^{\frac14}\,(1+\ln(\rho^{-1}))\,|k|^{\frac23(1-\delta)}\,,\end{aligned}\tag{B.41}$$

whereas by Step 1, the last term is bounded by $C\,|k|^{2/3}$. Setting $\delta'=2\delta/3$ and changing notations concludes the proof of the proposition. □

Appendix C. Proof of Estimate (2.29)

In this Appendix, we prove (2.29). It asserts that

$$\|(K^\rho_\sigma(p)\ -\ H_\sigma(p))\Phi^\rho_\sigma(p)\|_{\mathfrak F}\ \leq\ C\,\alpha^{\frac12}\,\rho^{\frac12}\,|p|\,,\tag{C.1}$$

for all $p\in\mathcal S$, for a constant independent of α and σ, ρ such that $0<\sigma<\rho\leq 1$.

To begin with, we note that

$$\|(K^\rho_\sigma(p)\ -\ H_\sigma(p))\Phi^\rho_\sigma(p)\|_{\mathfrak F}\ \ =\ \ \|(K^\rho_\sigma(p)\ -\ H_\sigma(p))\Psi_\sigma(p)\|_{\mathfrak F}\,,\tag{C.2}$$

which follows from

$$\begin{aligned}(H_\sigma-K^\rho_\sigma)(p)\Phi_\sigma(p)\ &=\ \big(H_\sigma(p)\ -\ W^\rho_{\nabla E_\sigma(p)}H_\sigma(p)(W^\rho_{\nabla E_\sigma(p)})^*\big)\,\Phi^\rho_\sigma(p)\\&=\ W^\rho_{\nabla E_\sigma(p)}(\,K^\rho_\sigma(p)\ -\ H_\sigma(p)\,)\Psi_\sigma(p)\,,\end{aligned}\tag{C.3}$$

and unitarity of $W^\rho_{\nabla E_\sigma(p)}$. Here we recall that $\Psi^\rho_\sigma(p)=(W^\rho_{\nabla E_\sigma(p)})^*\Phi^\rho_\sigma(p)$.

Next, we let

$$v^\sharp_\lambda(k)\ :=\ \alpha^{\frac12}\,\mathbf 1_{\sigma\leq|k|\leq\rho}(|k|)\,\frac{\nabla E_\sigma(p)\cdot\epsilon^\sharp_\lambda(k)}{|k|^{1/2}(|k|-\nabla E_\sigma(p)\cdot k)}\,,\tag{C.4}$$

(scalar-valued) and

$$w^\sharp_\lambda(k)\ :=\ \alpha^{\frac12}\,\mathbf 1_{\sigma\leq|k|\leq1}(|k|)\,\frac{\epsilon^\sharp_\lambda(k)}{|k|^{1/2}}\tag{C.5}$$

(vector-valued). We note that

$$|v_\lambda(k)|\ \leq\ C\,\alpha^{\frac12}\,|p|\,\frac{\mathbf 1_{\sigma\leq|k|\leq\rho}(|k|)}{|k|^{\frac32}}\tag{C.6}$$

and

$$|w_\lambda(k)|\ \leq C\,\alpha^{\frac12}\,\frac{\mathbf 1_{\sigma\leq|k|\leq1}(|k|)}{|k|^{\frac12}}\tag{C.7}$$

where we have used that $|\nabla E_\sigma(p)|\ \leq\ C\,|p|$, uniformly in the infrared cutoff $0\leq\sigma\leq1$.

Using that

$$W^\rho_{\nabla E_\sigma(p)}\,a^\sharp_\lambda(k)\,(W^\rho_{\nabla E_\sigma(p)})^*\ =\ a^\sharp_\lambda(k)\ +\ v^\sharp_\lambda(k)\,,\tag{C.8}$$

a straightforward calculation yields

$$\begin{aligned} & K_\sigma^\rho(p) \,-\, H_\sigma(p) \\ &= \; W^\rho_{\nabla E_\sigma(p)}\, H_\sigma(p)\, (W^\rho_{\nabla E_\sigma(p)})^* \,-\, H_\sigma(p) \\ &= \; (\nabla_p H_\sigma(p))\cdot V(p) \,+\, V(p)\cdot(\nabla_p H_\sigma(p)) \,+\, V^2(p) \,+\, Y(p)\,, \end{aligned} \tag{C.9}$$

where $\nabla_p H_\sigma(p)$ is defined in (B.34), and

$$V(p) \;:=\; \sum_\lambda \Big[a_\lambda(k v_\lambda) \,+\, a_\lambda^*(k v_\lambda) \,+\, 2Re(w_\lambda, v_\lambda) \,+\, (v_\lambda, k v_\lambda)\Big]\,, \tag{C.10}$$

(vector-valued operator) and

$$Y(p) \;:=\; \sum_\lambda \Big[a_\lambda(|k| v_\lambda) \,+\, a_\lambda^*(|k| v_\lambda) \,+\, (v_\lambda, |k| v_\lambda)\Big]\,, \tag{C.11}$$

(scalar-valued operator). We observe that both $V(p)$ and $Y(p)$ are proportional to $|\nabla E_\sigma(p)|$ since all terms are of first or higher order in v_λ (which is proportional to $|\nabla E_\sigma(p)| \leq C|p|$).

Let us first consider

$$\begin{aligned} \|Y(p)\,\Psi_\sigma(p)\|_{\mathfrak{F}} \;&\leq\; \Big\| \sum_\lambda a_\lambda(|k| v_\lambda)\Psi_\sigma(p)\Big\|_{\mathfrak{F}} \,+\, \Big\| \sum_\lambda a_\lambda^*(|k| v_\lambda)\Psi_\sigma(p)\Big\|_{\mathfrak{F}} \\ &\qquad + \sum_\lambda (v_\lambda, |k| v_\lambda) \|\Psi_\sigma(p)\|_{\mathfrak{F}} \\ &\leq\; 2\, \| |k|^{\frac12} v_\lambda\|_{L^2(\mathbb{R}^3\times\{+,-\})} \|(H_f+1)^{\frac12}\Psi_\sigma(p)\|_{\mathfrak{F}} \,+\, \sum_\lambda (v_\lambda, |k| v_\lambda), \end{aligned} \tag{C.12}$$

where we used Lemma 3.2 in the second inequality. Since $H_f \leq H_\sigma(p)$, we have that

$$\begin{aligned} \|(H_f+1)^{\frac12}\Psi_\sigma(p)\|_{\mathfrak{F}} \;&\leq\; \|(H_\sigma(p)+1)^{\frac12}\Psi_\sigma(p)\|_{\mathfrak{F}} \\ &=\; (E_\sigma(p)+1)^{\frac12} \;=\; \Big(\frac{p^2}{2} + \mathcal{O}(\alpha) + 1\Big)^{\frac12} \;\leq\; C, \end{aligned} \tag{C.13}$$

where we used the Rayleigh-Ritz principle in the last equality. Since

$$\| |k|^{\frac12} v_\lambda\|_{L^2(\mathbb{R}^3\times\{+,-\})} \;\leq\; C\,\alpha^{\frac12}\,\rho^{\frac12}\,|p|\,, \tag{C.14}$$

(see (C.6)) we conclude that

$$\|Y(p)\,\Psi_\sigma(p)\|_{\mathfrak{F}} \;\;\leq\;\; C\,\alpha^{\frac12}\,\rho^{\frac12}\,|p|\,. \tag{C.15}$$

To bound $\|V^2(p)\,\Psi_\sigma(p)\|$ and $\|V(p)\cdot(\nabla_p H_\sigma(p))\Psi_\sigma(p)\|_{\mathfrak{F}}$, we use the following lemma.

Lemma C.1. *For all $p \in \mathcal{S}$, $0 \leq \sigma \leq 1$ and α small enough,*

$$\|H(p)\Psi_\sigma(p)\| \;\;\leq\;\; C, \tag{C.16}$$

uniformly with respect to p, σ and α.

The proof of Lemma C.1 is given below.

Let us then consider

$$\|V^2(p)\,\Psi_\sigma(p)\| \;\;\leq\;\; \|V^2(p)(H_f+1)^{-1}\|\;\|(H_f+1)(H(p)+1)^{-1}\|\;\|(H(p)+1)\Psi_\sigma(p)\|.$$

The first term is bounded as above, using (3.7) in Lemma 3.2, which yields

$$\|V^2(p)(H_f+1)^{-1}\| \le C\,\alpha\,\rho\,p^2\,. \tag{C.17}$$

The second term is bounded by using the fact that

$$D(H(p)) = D(P_f^2 + H_f) \subset D(H_f), \tag{C.18}$$

and the last term is uniformly bounded by Lemma C.1. Combining these estimates we obtain

$$\|V^2(p)\,\Psi_\sigma(p)\| \;\le\; C\,\alpha\,\rho\,p^2\,. \tag{C.19}$$

Finally, to bound $\|V(p)\cdot(\nabla_p H_\sigma(p))\Psi_\sigma(p)\|_{\mathfrak{F}}$, we recall that

$$\nabla_p H_\sigma(p) \,=\, p \,-\, P_f \,+\, \alpha^{\frac12} A_\sigma\,, \tag{C.20}$$

where

$$A_\sigma \,=\, \sum_\lambda \big(a_\lambda(w_\lambda) + a_\lambda^*(w_\lambda)\big)\,. \tag{C.21}$$

We decompose

$$\begin{aligned}\|V(p)\cdot(\nabla_p H_\sigma(p))\Psi_\sigma(p)\|_{\mathfrak{F}} \;\le\;& \|V(p)\cdot p\Psi_\sigma(p)\|_{\mathfrak{F}} + \alpha^{\frac12}\|V(p)\cdot A_\sigma\Psi_\sigma(p)\|_{\mathfrak{F}} \\ &+\|V(p)\cdot P_f\Psi_\sigma(p)\|_{\mathfrak{F}}\,. \end{aligned} \tag{C.22}$$

The first two terms are estimated in the same way as in (C.15) and (C.19), which gives

$$\begin{aligned}\|V(p)\cdot p\Psi_\sigma(p)\|_{\mathfrak{F}} \;&\le\; C\,\alpha^{\frac12}\,\rho^{\frac12}\,p^2 \\ \alpha^{\frac12}\|V(p)\cdot A_\sigma\Psi_\sigma(p)\|_{\mathfrak{F}} \;&\le\; C\,\alpha\,\rho^{\frac12}\,|p|. \end{aligned} \tag{C.23}$$

It remains to estimate $\|V(p)\cdot P_f\Psi_\sigma(p)\|_{\mathfrak{F}}$. Let

$$\phi_{jl} := -i\sum_\lambda \big[a_\lambda(ik_jk_lv_\lambda) + a_\lambda^*(ik_jk_lv_\lambda)\big]\,. \tag{C.24}$$

Using that $[P_{f,j}, V(p)_l] = \phi_{jl}$, we find that

$$\begin{aligned}\|V(p)\cdot P_f\Psi_\sigma(p)\|_{\mathfrak{F}}^2 \;=\;& \sum_{j,l}\langle\Psi_\sigma(p), P_{f,j}V(p)_jV(p)_lP_{f,l}\Psi_\sigma(p)\rangle \\ =\;& \sum_{j,l}\langle\Psi_\sigma(p), \phi_{jj}V(p)_lP_{f,l}\Psi_\sigma(p)\rangle + \sum_{j,l}\langle\Psi_\sigma(p), V(p)_jP_{f,j}V(p)_lP_{f,l}\Psi_\sigma(p)\rangle \\ =\;& \sum_{j,l}\langle\Psi_\sigma(p), \phi_{jj}V(p)_lP_{f,l}\Psi_\sigma(p)\rangle + \sum_{j,l}\langle\Psi_\sigma(p), V(p)_j\phi_{jl}P_{f,l}\Psi_\sigma(p)\rangle \\ &+\sum_{j,l}\langle\Psi_\sigma(p), V(p)_jV(p)_lP_{f,j}P_{f,l}\Psi_\sigma(p)\rangle\,. \end{aligned} \tag{C.25}$$

Proceeding in the same way as in (C.19), we obtain that

$$\begin{aligned}\|V(p)_l\phi_{jj}\Psi_\sigma(p)\| \;&\le\; C\,\alpha\,\rho^2\,p^2\,, \\ \|\phi_{jl}V(p)_l\Psi_\sigma(p)\| \;&\le\; C\,\alpha\,\rho^2\,p^2\,, \\ \|V(p)_lV(p)_j\Psi_\sigma(p)\| \;&\le\; C\,\alpha\,\rho\,p^2\,. \end{aligned}$$

Furthermore,

$$\begin{aligned} \|P_{f,l}\Psi_\sigma(p)\| &\leq \|H_f\Psi_\sigma(p)\| \\ &\leq \|H_f(H(p)+1)^{-1}\|\|(H(p)+1)\Psi_\sigma(p)\| \leq C\,, \end{aligned} \tag{C.26}$$

by Lemma C.1 and the fact that $D(H(p)) \subset D(H_f)$, and, likewise,

$$\begin{aligned} \|P_{f,j}P_{f,l}\Psi_\sigma(p)\| &\leq \|P_f^2\Psi_\sigma(p)\| \\ &\leq \|P_f^2(H(p)+1)^{-1}\|\|(H(p)+1)\Psi_\sigma(p)\| \leq C\,, \end{aligned} \tag{C.27}$$

since $D(H(p)) \subset D(P_f^2)$. Combining (C.25) with the Cauchy-Schwarz inequality and the previous estimates, we obtain

$$\|V(p)\cdot(\nabla_p H_\sigma(p))\Psi_\sigma(p)\|_{\mathfrak{F}} \leq C\,\alpha^{\frac{1}{2}}\,\rho^{\frac{1}{2}}\,|p|. \tag{C.28}$$

The term $\|(\nabla_p H_\sigma(p))\cdot V(p)\Psi_\sigma(p)\|_{\mathfrak{F}}$ can be estimated exactly in the same way, which, together with (C.15) and (C.19), concludes the proof of Lemma 4.2.

We conclude with proving Lemma C.1.

Proof of Lemma C.1. Since $H_\sigma(p)\Psi_\sigma(p) = E_\sigma(p)\Psi_\sigma(p)$ and $\|\Psi_\sigma(p)\| = 1$, we can write

$$\begin{aligned} \|H(p)\Psi_\sigma(p)\| &\leq E_\sigma(p) + \|Q_\sigma(p)\Psi_\sigma(p)\| \\ &\leq \frac{p^2}{2} + \mathcal{O}(\alpha) + \|Q_\sigma(p)\Psi_\sigma(p)\|\,, \end{aligned} \tag{C.29}$$

where

$$Q_\sigma(p) := H(p) - H_\sigma(p) = \alpha^{\frac{1}{2}}A_{<\sigma}\cdot(p - P_f + \alpha^{\frac{1}{2}}A_\sigma) + \frac{\alpha}{2}A_\sigma^2\,.$$

Here A_σ is defined in (A.12), and

$$A_{<\sigma} := \sum_\lambda \int_{|k|\leq\sigma} dk\,\frac{dk}{|k|^{1/2}}\,\{\,\epsilon_\lambda(k)\,b_\lambda(k)\;+\;h.c.\,\}\,. \tag{C.30}$$

From Lemma 3.2, it follows, using that $D(H(p)) \subset D(H_f)$, that

$$\begin{aligned} \|\alpha^{\frac{1}{2}}(A_{<\sigma}\cdot p)\Psi_\sigma(p)\| &\leq C\,\alpha^{\frac{1}{2}}\,\sigma^{\frac{1}{2}}\,|p|\,\|(H(p)+1)\Psi_\sigma(p)\|\,, \\ \|\alpha(A_{<\sigma}\cdot A_\sigma)\Psi_\sigma(p)\| &\leq C\,\alpha\,\sigma^{\frac{1}{2}}\,\|(H(p)+1)\Psi_\sigma(p)\|\,, \\ \|\alpha(A_\sigma)^2\Psi_\sigma(p)\| &\leq C\,\alpha\,\|(H(p)+1)\Psi_\sigma(p)\|\,. \end{aligned} \tag{C.31}$$

Moreover, since $A_{<\sigma}$ and P_f commute, we can estimate

$$\begin{aligned} \|\alpha^{\frac{1}{2}}(A_{<\sigma}\cdot P_f)\Psi_\sigma(p)\|^2 &\leq \alpha\,\|A_{<\sigma}^2\Psi_\sigma(p)\|\|P_f^2\Psi_\sigma(p)\| \\ &\leq C\,\alpha\,\sigma\,\|(H(p)+1)\Psi_\sigma(p)\|^2\,, \end{aligned} \tag{C.32}$$

where in the last inequality we used again Lemma 3.2 together with $D(H(p)) \subset D(H_f) \cap D(P_f^2)$. From (C.29)–(C.32), we obtain that

$$\|H(p)\Psi_\sigma(p)\| \leq C\,+\,C\,\alpha^{\frac{1}{2}}\,\|H(p)\Psi_\sigma(p)\|\,, \tag{C.33}$$

for any $p \in \mathcal{S}$. Assuming that α is sufficiently small, the result follows. □

Acknowledgements. J.Fa., I.M.S., and T.C. are grateful to J.Fr. for hospitality at ETH Zürich. T.C. thanks I.M.S. for hospitality at the University of Toronto. The research of I.M.S. has been supported by NSERC under Grant NA 7901. The work of T.C. was supported by the NSF through the grants DMS-0940145 and DMS-1009448.

References

[1] V. Bach, T. Chen, J. Fröhlich, I. M. Sigal, *Smooth Feshbach map and oprator-theoretic renormalization group methods*, J. Funct. Anal., **203** (1), 44–92, (2003).
[2] V. Bach, T. Chen, J. Fröhlich, I. M. Sigal, *The renormalized electron mass in non-relativistic QED*. J. Funct. Anal., **243** (2), 426–535, (2007).
[3] V. Bach, J. Fröhlich and A. Pizzo, *Infrared-finite algorithms in QED: the groundstate of an atom interacting with the quantized radiation field.*, Comm. Math. Phys., **264**, 145–165, (2006).
[4] V. Bach, J. Fröhlich, I.M. Sigal, *Quantum electrodynamics of confined non-relativistic particles*, Adv. Math., **137** (2), 299–395, (1998).
[5] V. Bach, J. Fröhlich, I. M. Sigal, *Renormalization group analysis of spectral problems in quantum field theory*, Adv. Math. **137**, 205–298, (1998).
[6] T. Chen, *Infrared renormalization in non-relativistic QED and scaling criticality*, J. Funct. Anal., **254** (10), 2555–2647, (2008).
[7] T. Chen, J. Faupin, J. Fröhlich, I.M. Sigal, *Local decay in non-relativistic QED*, Comm. Math. Phys. **309**, 543–583, (2012).
[8] T. Chen, J. Fröhlich, *Coherent infrared representations in non-relativistic QED*, *Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday, Proc. Symp. Pure Math.*, AMS, (2007).
[9] T. Chen, J. Fröhlich, A. Pizzo, *Infraparticle scattering states in non-relativistic QED - I. The Bloch-Nordsieck paradigm*, Commun. Math. Phys., **294** (3), 761–825, (2010).
[10] T. Chen, J. Fröhlich, A. Pizzo, *Infraparticle scattering states in non-relativistic QED - II. Mass shell properties*, J. Math. Phys., **50** (1), 012103, (2009).
[11] J. Fröhlich, *On the infrared problem in a model of scalar electrons and massless, scalar bosons*, Ann. Inst. Henri Poincaré, Section Physique Théorique, **19** (1), 1–103, (1973).
[12] J. Fröhlich, *Existence of dressed one electron states in a class of persistent models*, Fortschritte der Physik, **22**, 159–198, (1974).
[13] J. Fröhlich, A. Pizzo, *The renormalized electron mass in nonrelativistic QED*, Commun. Math. Phys., **294**, 439–470, (2010).
[14] M. Griesemer, E. H. Lieb, M. Loss, *Ground states in non-relativistic quantum electrodynamics*, Invent. Math. **145**, no. 3, 557–595, (2001).
[15] C. Hainzl, R. Seiringer, *Mass renormalization and energy level shift in non-relativistic QED*, Adv. Theor. Math. Phys. **6**, no. 5, 847–871, (2003).
[16] F. Hiroshima, K. R. Ito, *Mass Renormalization in Non-relativistic Quantum Electrodynamics with Spin 1/2*, Preprint http://arxiv.org/abs/math-ph/0412026.
[17] F. Hiroshima, H. Spohn, *Mass renormalization in nonrelativistic quantum electrodynamics*, J. Math. Phys., **46**, no. 4, (2005).
[18] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag New-York, (1966).
[19] E. H. Lieb, M. Loss, *Self-energy of electrons in non-perturbative QED*, Conférence Moshé Flato 1999, Vol. I (Dijon), 327–344, Math. Phys. Stud., 21.
[20] E. H. Lieb, M. Loss, *A bound on binding energies and mass renormalization in models of quantum electrodynamics*, J. Statist. Phys., **108**, 1057–1069, (2002).
[21] E. H. Lieb, M. Loss, *The thermodynamic limit for matter interacting with Coulomb forces and with the quantized electromagnetic field. I. The lower bound*, Comm. Math. Phys., **258**, no. 3, 675–695, (2005).
[22] A. Pizzo, *One-particle (improper) states in Nelson's massless model*, Ann. Henri Poincaré, **4** (3), 439–486, (2003).
[23] A. Pizzo, *Scattering of an infraparticle: The one particle sector in Nelson's massless model*, Ann. Henri Poincaré, **6** (3), 553–606, (2005).
[24] M. Reed and B. Simon, *Methods of modern mathematical physics IV. Analysis of operators*, Academic press, New-York-London, (1978).

[25] H. Spohn, *Dynamics of charged particles and their radiation field*, Cambridge University Press, Cambridge, (2004).

[26] H. Spohn, S. Teufel, *Semiclassical motion of dressed electrons*, Rev. Math. Phys., **14** (1), 1–28, (2002).

(V. Bach) Institut fuer Analysis und Algebra Carl-Friedrich-Gauss-Fakultaet, Technische Universitaet Braunschweig, 38106 Braunschweig, Germany
E-mail address: v.bach@tu-bs.de

(T. Chen) Department of Mathematics, University of Texas at Austin, Austin TX 78712, USA
E-mail address: tc@math.utexas.edu

(J. Faupin) Institut de Mathématiques de Bordeaux, UMR-CNRS 5251, Université de Bordeaux 1, 351 cours de la libération, 33405 Talence Cedex, France
E-mail address: jeremy.faupin@math.u-bordeaux1.fr

(J. Fröhlich) Institut für Theoretische Physik, ETH Hönggerberg, CH-8093 Zürich, Switzerland
E-mail address: juerg@phys.ethz.ch

(I.M. Sigal) Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4, Canada
E-mail address: im.sigal@utoronto.ca