Entropic Chaoticity for the Steady State of a Current Carrying System.

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Abstract

The steady state for a system of N particle under the influence of an external field and a Gaussian thermostat and colliding with random "virtual" scatterers can be obtained explicitly in the limit of small field. We show the sequence of steady state distribution, as N varies, forms a chaotic sequence in the sense that the k particle marginal, in the limit of large N, is the k-fold tensor product of the 1 particle marginal. We also show that the chaoticity properties holds in the stronger form of entropic chaoticity.

1 introduction

Ever since its introduction by Kac [1] in 1956, the notion of a chaotic sequences has become an important concept in studying many body systems. Chaotic sequences and propagation of chaos are the principal tools for passing from many body problems to effective equations. The aim of this article is to give yet another example of the interplay of chaotic sequences and effective equations.

In [BDLR] the authors consider a system consisting of N particles moving in a 2 dimensional billiard and colliding with convex scatterers that form a dispersing billiard. The particle are subject to an external electric field E and a Gaussian thermostat that keeps the kinetic energy fixed. The equation of motion between collisions are thus:

$$\begin{cases} \dot{q}_i = v_i & i = 1, \dots, N \\ \dot{v}_i = \mathbf{F}_i = E - \dfrac{E \cdot j}{U} v_i + F_i \end{cases} \tag{1}$$

where

$$j(\mathbf{V}) = \sum_{i=1}^{N} v_i, \qquad U(\mathbf{V}) = \sum_{i=1}^{N} |v_i|^2 \tag{2}$$

and F_i is the force exerted on the ith particle by collisions with the fixed scatterers. Very little is known about billiards with more than one particle. In particular there is no existence theorem for the SRB measure of this system.

The authors introduced a stochastic version of the above model in which the deterministic collisions are replaced by Poisson distributed collision processes. More precisely, in the time interval

between t and $t+dt$, the i-th particle as a probability $|v_i|^\alpha dt$ of suffering a collision. When a collision happens, the velocity of the particle is randomly updated, i.e. if the particle's velocity direction before collision is $\omega = v/|v|$, after collision it will be distributed as $K(\omega' \cdot \omega)d\omega'$. The details of the collision kernel K are largely irrelevant. For what follows it will be enough to assume that $K(x) > 0$ for x in a open set $\mathcal{U} \in [-1, 1]$. We note that this stochastic process make sense for any dimension d.

The master equation for this process is given by

$$\begin{aligned}\frac{\partial W(\mathbf{Q},\mathbf{V},t;E)}{\partial t} = & - \sum_{i=1}^{N} v_i \frac{\partial W(\mathbf{Q},\mathbf{V},t;E)}{\partial q_i} - \sum_{i=1}^{N} \nabla_{v_i}\left[\Big(E - \frac{E\cdot j(\mathbf{V})}{U(\mathbf{V})}v_i\Big)W(\mathbf{Q},\mathbf{V},t;E)\right] \\ & + \sum_{i=1}^{N} |v_i|^\alpha \int_{\mathcal{S}^{d-1}(1)} K(\omega_i\cdot\omega_i')\big[W(\mathbf{Q},\mathbf{V}_i',t;E) - W(\mathbf{Q},\mathbf{V},t;E)\big]\, d\sigma^{d-1}(\omega')\end{aligned} \tag{3}$$

where $\mathcal{S}^m(R)$ is the m-dimensional sphere with radius R, $d\sigma^m(\cdot)$ is the uniform surfaces measure on $\mathcal{S}^m(R)$ and $j = J/U$ as in (2). Moreover

$$\mathbf{Q} = (q_1,\dots,q_N) \quad , \quad \mathbf{V} = (v_1,\dots,v_i,\dots,v_N) \quad \text{and} \quad \mathbf{V}_i' = (v_1,\dots,v_i',\dots,v_N), \tag{4}$$

and $v_i' = |v_i|\omega_i'$ if $v_i = |v_i|\omega_i$. Note that the variable $\mathbf{Q}$ is not part of the dynamics, i.e., if the initial condition $W(\mathbf{Q},\mathbf{V},0)$ is independent of $\mathbf{Q}$ so will be $W(\mathbf{Q},\mathbf{V},t)$. Moreover, if $W(\mathbf{Q},\mathbf{V},0)$ is concentrated on the surface of given energy U_0, that is if

$$W(\mathbf{V},0) = \delta(U(\mathbf{V}) - U_0)F(\mathbf{V},0)$$

then so will be the solution of .(3):

$$W(\mathbf{V},t;E) = \delta(U(\mathbf{V}) - U_0)F(\mathbf{V},t;E).$$

Finally if $F(\mathbf{V},0)$ is a symmetric function so is $F(\mathbf{V},t;E)$. Thus we will, henceforth, only consider symmetric spatially homogeneous solutions concentrated on the surface of energy $U_0 = N$.

Recall that the k-particle marginal $f_N^{(k)}(v_1,\dots,v_k)$ of a distribution $F_N(\mathbf{V})$ is defined by the equation

$$\int_{\mathcal{S}^{dN-1}(\sqrt{N})} \varphi(v_1,\dots,v_k)F_N(\mathbf{V})d\sigma^{dN-1}(\mathbf{V}) = \int_{\mathbb{R}^{dk}} \varphi(v_1,\dots,v_k)f_N^{(k)}(v_1,\dots,v_k)dv_1\cdots dv_k \ ,$$

where $\varphi(v_1,\dots,v_k)$ is any bounded continuous function on $\mathbb{R}^{dk}$. Simple computations show that

$$f_N^{(k)}(\mathbf{V}_k) = \sqrt{\frac{N}{N-|\mathbf{V}_k|}}\int_{\mathcal{S}^{d(N-k)-1}\left(\sqrt{N-|\mathbf{V}_k|^2}\right)} F_N(\mathbf{V}_k,\mathbf{V}^k)d\sigma^{d(N-k)-1}(\mathbf{V}^k) \ , \tag{5}$$

where $\mathbf{V}_k = (v_1,\dots,v_k)$ and $\mathbf{V}^k = (v_{k+1},\dots,v_N)$. A sequence of densities $\{F_N\}_{N=1}^\infty$ form a **chaotic sequence with marginal** f if for any bounded continuous function φ

$$\lim_{N\to\infty}\int_{\mathcal{S}^{dN-1}(\sqrt{N})} \varphi(\mathbf{V}_k)F_N(\mathbf{V})d\sigma^{dN-1}(\mathbf{V}) = \int_{\mathbb{R}^{dk}} \varphi(\mathbf{V}_k)\prod_{j=1}^{k} f(v_j)dv_1\cdots dv_k \tag{6}$$

It was shown in [2] that for finite time t the master equations (3) propagates chaos, i.e., the solution of the master equation (3) forms a chaotic sequence if the initial condition does. More precisely, for any bounded continuous function on $\mathbb{R}^{dk}$, $\varphi(v_1, \ldots, v_k)$, we have

$$\lim_{N\to\infty} \int_{\mathcal{S}^{dN-1}} \varphi(\mathbf{V}_k) F(\mathbf{V}, t; E) d\sigma(\mathbf{V}) = \int_{\mathbb{R}^{dk}} \varphi(\mathbf{V}_k) \prod_{j=1}^{k} f(v_j, t; E) dv_1 \cdots dv_k$$

where

$$f(v_1, t; E) = \lim_{N\to\infty} f_N^{(1)}(v_1, t; E) \ .$$

It was shown in [2] that $f(v_1, t; E)$ satisfies the Boltzmann equation

$$\frac{f(v,t;E)}{dt} + \frac{\partial}{\partial v}\Big[\big(E - (E\cdot\hat{j}(t,E))v\big)f\Big] = |v|^\alpha \int_{\mathcal{S}^{d-1}(1)} K(\omega\cdot\omega')\big[f(v',t;E) - f(v,t;E)\big]\, d\sigma^{d-1}(\omega') \tag{7}$$

where $\hat{j}(t, E)$ is given by the self consistent condition

$$\hat{j}(t,E) = \frac{1}{u}\int v f(v,t;E) dv$$

with

$$u = \int |v|^2 f(v,t;E) dv.$$

The initial condition is given by $f(v) = \lim_{N\to\infty} f_N^{(1)}(v, 0)$. It is easy to see that u is independent of time and since we have chosen $U_0 = N$, $u = 1$.

Concerning steady states, the situation is far from clear. In [3] and [4] it was shown that a steady state $F_{\rm ss}(\mathbf{V}; E)$ exists for the Kac master equation (3) provided that $E \neq 0$. If $E = 0$ any density $F(\mathbf{V})$ that depends only on the magnitude of the velocities furnishes a stationary state. It is, however, an open question whether $F_{\rm ss}(\mathbf{V}; E)$ tends to a limiting distribution as $E \to 0$. Interestingly, assuming the *existence* of the limiting distribution, it can be computed exactly and it is given by

$$F_{\rm ss}(\mathbf{V}; 0) = \frac{1}{\widetilde{Z}_N}\delta(U(\mathbf{V}) - N)F(\mathbf{V}) \tag{8}$$

with

$$F_N(\mathbf{V}) = \frac{1}{\left(\sum_{i=1}^N |v_i|^{2+\alpha}\right)^{\frac{dN-1}{2+\alpha}}} \tag{9}$$

where $\widetilde{Z}_N$ is the normalization constant

$$\widetilde{Z}_N = \int_{\mathcal{S}^{dN-1}(\sqrt{N})} F_N(\mathbf{V}) d\sigma^{Nd-1}(\mathbf{V}) \ . \tag{10}$$

For details, the reader should consult [3] and [4]. Thus, the electric field 'selects' the right steady state as the electric field tends to zero.

A similar problem exists on the level of the Boltzmann equation. Again it is possible to show that the steady state $f_{\rm ss}(v; E)$ for the Boltzmann equation (7) exists and is unique if $E \neq 0$. This

clearly implies the existence of a steady state current $\hat{j}_{\rm ss}(E)$. In [5] it was shown that, assuming that $f(v) = \lim_{E\to 0} f_{\rm ss}(v,E)$ exists and that $\hat{j}_{\rm ss}(E) = O(E)$, one has

$$f(v) = \frac{\mu^{\frac{d}{2}}}{c} e^{-(\sqrt{\mu}|v|)^{2+\alpha}} \tag{11}$$

where c and μ are uniquely determined by normalization and the condition $u = 1$. One easily get

$$\mu = \frac{\Gamma\left(\frac{d+2}{2+\alpha}\right)}{\Gamma\left(\frac{d}{2+\alpha}\right)} \qquad\qquad c = \frac{|\mathcal{S}^{d-1}(1)|\Gamma\left(\frac{d}{2+\alpha}\right)}{(2+\alpha)} \tag{12}$$

which for $\alpha = 1$ and $d = 2$ gives

$$\mu = \frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \approx 0.65948 \qquad\qquad c = \frac{2\pi}{3}\Gamma\left(\frac{2}{3}\right) \approx 2.83605$$

For details the reader may consult [5] where the existence of the small field limit of the steady state distribution is proved for $d = 1$.

It is now natural to ask whether the distribution (8) is chaotic with marginal f given by (11). For the reasons mentioned above, this cannot be deduced from the previous results on propagation of chaos since these results do not hold uniform in time. A more serious impediment is the fact that the small field limits of the steady states are not known to exist. As explained before, the limit as $E \to 0$ selects a steady state for the Kac master equation as well as for its Boltzmann version. It is far from clear that the selection mechanism is such as to preserve chaoticity. In this note we prove that, nevertheless, the distribution (8) is chaotic with marginal (11).

THEOREM 1.1. *Let $f_N(v)$ be the one particle marginal of $F_N(\mathbf{V})$ i.e., defined by*

$$f_N^{(1)}(v_1) = \frac{\sqrt{N}}{\sqrt{N-|v_1|^2}} \frac{1}{\widetilde{Z}_N} \int_{\mathcal{S}^{dN-1-1}(\sqrt{N-|v_1|^2})} F_N(\mathbf{V}) d\sigma(\mathbf{V}^1) \tag{13}$$

and set

$$f(v) = \frac{\mu^{\frac{d}{2}}}{c} e^{-(\sqrt{\mu}|v|)^{2+\alpha}} \tag{14}$$

with the constants given by (12)*. Then for any bounded continuous function $\varphi(v)$*

$$\lim_{N\to\infty} \int_{\mathbb{R}^d} \varphi(v) f_N^{(1)}(v) dv = \int_{\mathbb{R}^d} \varphi(v) f(v) dv \tag{15}$$

and for every k, the k particle marginal $f_N^{(k)}(v_1,\dots,v_k)$ of $F(\mathbf{V})$ satisfies

$$\lim_{N\to\infty} \int_{\mathbb{R}^{kd}} \varphi(v_1,\dots,v_k) f_N^{(k)}(v_1,\dots,v_k) dv_1\cdots dv_k = \int_{\mathbb{R}^{kd}} \varphi(v_1,\dots,v_k) \prod_{i=1}^{k} f(v_k) dv_1\cdots dv_k \tag{16}$$

where, again, φ is any bounded continuous function on $\mathbb{R}^{kd}$. Thus $F_N(\mathbf{V})$ from a chaotic sequence with marginal f.

2 Proof of Theorem 1.1.

The following elementary lemma sets the stage for the proof. It will be expressed in terms of the probability distribution

$$g(w) := \frac{e^{-|w|^{2+\alpha}}}{\int_{\mathbb{R}^d} e^{-|w|^{2+\alpha}} dw} \ .$$

LEMMA 2.1. *The following formulas hold for $F_N(\mathbf{V})$:*

$$F_N(\mathbf{V}) = \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})} \frac{1}{Z_N} \int_0^\infty t^{dN-1} \prod_{j=1}^N g(v_j t) \frac{dt}{t} \ , \tag{17}$$

$$Z_N = \frac{(2+\alpha)}{\Gamma\left(\frac{dN-1}{2+\alpha}\right)} \int_{\mathbb{R}^{dN}} \frac{\prod_{i=1}^N g(w_i)}{|\mathbf{W}|} \, d\mathbf{W} \tag{18}$$

and

$$\begin{aligned} f_N^{(k)}(\mathbf{V}_k) : &= \sqrt{\frac{N}{N-|\mathbf{V}_k|}} \int_{\mathcal{S}^{d(N-k)-1}(\sqrt{N-|\mathbf{V}_k|^2})} F_N(\mathbf{V}_k, \mathbf{V}^k) d\sigma^{d(N-k)-1}(\mathbf{V}^k) \\ &= \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha}) Z_N} \sqrt{\frac{N}{(N-|\mathbf{V}_k|)^{dk+1}}} \times \\ &\qquad \int_{\mathbb{R}^{d(N-k)}} \prod_{j=1}^k g\left(\frac{v_j |\mathbf{W}^k|}{\sqrt{N-|\mathbf{V}_k|^2}} \right) |\mathbf{W}^k|^{dk-1} \prod_{j=k+1}^N g(w_j) d\mathbf{W}^k \ . \end{aligned} \tag{19}$$

Proof. Formula (17) follows from (9) and

$$A^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty s^\gamma e^{-As} \frac{ds}{s} \ , \tag{20}$$

valid for all $A > 0$ and $\gamma > 0$ by setting

$$A = \left(\sum_{i=1}^N |v_i|^{2+\alpha} \right) \ , \ \gamma = dN - 1$$

and substituting $s = t^{2+\alpha}$. The normalization constant $\widetilde{Z}_N$ given in (10) is then given as

$$\frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})} \int_{\mathcal{S}^{Nd-1}(\sqrt{N})} \int_0^\infty t^{dN-1} \prod_{j=1}^N e^{-(|v_j|t)^{2+\alpha}} \frac{dt}{t} d\sigma^{Nd-1}(\mathbf{V})$$

which, using Fubini's theorem and changing variables $v_j = \sqrt{N} w_j$ equals

$$\frac{(2+\alpha) N^{\frac{Nd-1}{2}}}{\Gamma(\frac{dN-1}{2+\alpha})} \int_0^\infty \int_{\mathcal{S}^{Nd-1}(1)} \prod_{j=1}^N e^{-(|w_j|t\sqrt{N})^{2+\alpha}} d\sigma^{Nd-1}(\mathbf{W}) t^{dN-1} \frac{dt}{t} \ .$$

One more variable change $r = \sqrt{N} t$ yields

$$\frac{(2+\alpha)}{\Gamma(\frac{dN-1}{2+\alpha})} \int_0^\infty \int_{\mathcal{S}^{Nd-1}(1)} \prod_{j=1}^N e^{-(|w_j|r)^{2+\alpha}} d\sigma^{Nd-1}(\mathbf{W}) r^{dN-1} \frac{dr}{r} \ .$$

Hence

$$Z_N = \frac{(2+\alpha)}{\Gamma(\frac{dN-1}{2+\alpha})} \int_0^\infty \int_{\mathcal{S}^{Nd-1}(1)} \prod_{j=1}^N g(w_j r) d\sigma^{Nd-1}(\mathbf{W}) r^{dN-1} \frac{dr}{r} \ .$$

This integral is (18) written in terms of polar coordinates. To see (19) we start with (5) and find

$$f_N^{(k)}(\mathbf{V}_k) = \sqrt{\frac{N}{N-|\mathbf{V}_k|}} \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})} \frac{1}{Z_N} \int_{\mathcal{S}^{d(N-k)-1}(\sqrt{N-|\mathbf{V}_k|^2})} \int_0^\infty t^{dN-1} \prod_{j=1}^N g(v_j t) \frac{dt}{t} d\sigma^{d(N-k)-1}(\mathbf{V}^k) \ .$$

Once more, using Fubini's theorem and changing variables $v_j = \sqrt{N-|\mathbf{V}_k|^2} w_j, j = k+1, \ldots, N$ yields

$$\sqrt{\frac{N}{N-|\mathbf{V}_k|^2}} \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})} \frac{1}{Z_N} (N-|\mathbf{V}_k|^2)^{\frac{d(N-k)-1}{2}} \times$$
$$\int_0^\infty t^{dN-1} \prod_{j=1}^k g(v_j t) \int_{\mathcal{S}^{d(N-k)-1}(1)} \prod_{j=k+1}^N g\left(\sqrt{N-|\mathbf{V}_k|^2} w_j t\right) d\sigma^{d(N-k)-1}(\mathbf{W}^k) \frac{dt}{t} \ .$$

Changing variables $r = \sqrt{N-|\mathbf{V}_k|^2} t$ yields the expression

$$\sqrt{\frac{N}{N-|\mathbf{V}_k|^2}} \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})} \frac{1}{Z_N} \frac{1}{(N-|\mathbf{V}_k|^2)^{\frac{dk}{2}}} \times$$
$$\int_0^\infty r^{d(N-k)-1} \prod_{j=1}^k g\left(\frac{v_j r}{\sqrt{N-|\mathbf{V}_k|^2}}\right) r^{dk-1} \int_{\mathcal{S}^{d(N-k)-1}(1)} \prod_{j=k+1}^N g\left(w_j r\right) d\sigma^{d(N-k)-1}(\mathbf{W}^k) dr \ .$$

which equals

$$\frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha}) Z_N} \sqrt{\frac{N}{(N-|\mathbf{V}_k|^2)^{dk+1}}} \int_{\mathbb{R}^{d(N-k)}} \prod_{j=1}^k g\left(\frac{v_j |\mathbf{W}^k|}{\sqrt{N-|\mathbf{V}_k|^2}}\right) |\mathbf{W}^k|^{dk-1} \prod_{j=k+1}^N g\left(w_j\right) d\mathbf{W}^k \ .$$

□

LEMMA 2.2. *We have the following limit*

$$\lim_{N\to\infty} \int_{\mathbb{R}^{Nd}} \frac{\sqrt{N}}{|\mathbf{W}|} \prod_{j=1}^N g(w_j) d\mathbf{W} = \frac{1}{\sqrt{\mu}}$$

where

$$\mu := \int_{\mathbb{R}^d} |w|^2 g(w) dw \ ,$$

so that

$$\lim_{N\to\infty} \sqrt{N} \Gamma\left(\frac{dN-1}{2+\alpha}\right) Z_N = \frac{(2+\alpha)}{\sqrt{\mu}} \ .$$

Proof. The proof follows with a slight modification from the law of large numbers. We denote

$$\mathbb{P}(A) := \int_A \prod_{j=1}^N g(w_j) d\mathbf{W} \ , \quad \text{and } \sigma^2 := \int_{\mathbb{R}^d} (|w|^2 - \mu)^2 g(w) dw \ .$$

If

$$A_\varepsilon = \left\{ \mathbf{W} \in \mathbb{R}^{Nd} : \left| \frac{|\mathbf{W}|^2}{N} - \mu \right| > \varepsilon \right\}$$

then it is a standard estimate that

$$\mathbb{P}(A_\varepsilon) \leq \frac{\sigma^2}{\varepsilon^2 N} \ . \tag{21}$$

From this it follows readily that

$$\frac{1}{\sqrt{\mu+\varepsilon}} \mathbb{P}(A_\varepsilon^c) \leq \int_{A_\varepsilon^c} \frac{\sqrt{N}}{|\mathbf{W}|} \Pi_{j=1}^N g(w_j) d\mathbf{W} \leq \frac{1}{\sqrt{\mu-\varepsilon}} \mathbb{P}(A_\varepsilon^c) \ . \tag{22}$$

The set A_ε can be written as $A_\varepsilon = A_\varepsilon^< \cup A_\varepsilon^>$ where

$$A_\varepsilon^< = \{ \mathbf{W} \in \mathbb{R}^{Nd} : \frac{|\mathbf{W}|^2}{N} < \mu - \varepsilon \} \text{ and } A_\varepsilon^> = \{ \mathbf{W} \in \mathbb{R}^{Nd} : \frac{|\mathbf{W}|^2}{N} > \mu + \varepsilon \} \ .$$

Clearly

$$\int_{A_\varepsilon^>} \frac{\sqrt{N}}{|\mathbf{W}|} \prod_{j=1}^N g(w_j) d\mathbf{W} \leq \frac{1}{\sqrt{\mu+\varepsilon}} \mathbb{P}(A_\varepsilon^>) \ ,$$

however, to estimate the corresponding integral in the set $A_\varepsilon^<$ is a little bit trickier, on account of the singularity of $\frac{1}{|\mathbf{W}|}$. Fix some number $a < \sqrt{\frac{\mu-\varepsilon}{d}}$ independent of N so that for N sufficiently large

$$\int_{C^d} \frac{g(w)}{|w|^{\frac{1}{N}}} dw < \frac{1}{2}$$

where $C^d = [-a, a]^d$. Note that the cube $C^{Nd} = [-a, a]^{Nd} \subset A_\varepsilon^<$. By the arithmetic-geometric mean inequality

$$\sqrt{\frac{N}{|\mathbf{W}|^2}} \leq \prod_{j=1}^N |w_j|^{-1/N} \tag{23}$$

so that

$$\int_{C^{Nd}} \frac{\sqrt{N}}{|\mathbf{W}|} \prod_{j=1}^N g(w_j) d\mathbf{W} \leq \left(\int_{C^d} \frac{g(w)}{|w|^{\frac{1}{N}}} dw \right)^N \leq \frac{1}{2^N} \ .$$

It remains to estimate the integral

$$\int_{A_\varepsilon^< \setminus C^{Nd}} \frac{\sqrt{N}}{|\mathbf{W}|} \Pi_{j=1}^N g(w_j) d\mathbf{W} \ .$$

Here we note that the ball centered at the origin of radius a is a subset of C^{Nd} and hence

$$\sqrt{\frac{N}{|\mathbf{W}|^2}} \geq \frac{\sqrt{N}}{a}$$

and this leads to the estimate

$$\int_{A_\varepsilon^< \setminus C^{Nd}} \frac{\sqrt{N}}{|\mathbf{W}|} \prod_{j=1}^N g(w_j) d\mathbf{W} \leq \frac{\sqrt{N}}{a} \mathbb{P}(A_\varepsilon^< \setminus C^{Nd}) \ .$$

Collecting these bounds yields

$$\int_{A_\varepsilon} \frac{\sqrt{N}}{|\mathbf{W}|} \prod_{j=1}^N g(w_j) d\mathbf{W} \leq \frac{\sqrt{N}}{a} \mathbb{P}(A_\varepsilon^< \setminus C^{Nd}) + \frac{1}{2^N} + \frac{1}{\sqrt{\mu+\varepsilon}} \mathbb{P}(A_\varepsilon^>) \tag{24}$$

and finally

$$\begin{aligned} \frac{1}{\sqrt{\mu+\varepsilon}} - \frac{1}{\sqrt{\mu+\varepsilon}} \mathbb{P}(A_\varepsilon) &\leq \int_{\mathbb{R}^{Nd}} \frac{\sqrt{N}}{|\mathbf{W}|} \prod_{j=1}^N g(w_j) d\mathbf{W} \\ &\leq \frac{1}{\sqrt{\mu-\varepsilon}} + \frac{\sqrt{N}}{a} \mathbb{P}(A_\varepsilon^< \setminus C^{Nd}) + \frac{1}{2^N} + \frac{1}{\sqrt{\mu+\varepsilon}} \mathbb{P}(A_\varepsilon^>) \end{aligned} \tag{25}$$

and using the estimate (21) we find that

$$\frac{1}{\sqrt{\mu+\varepsilon}}(1 - \frac{\sigma^2}{\varepsilon^2 N}) \leq \int_{\mathbb{R}^{Nd}} \frac{\sqrt{N}}{|\mathbf{W}|} \Pi_{j=1}^N g(w_j) d\mathbf{W} \leq \frac{1}{\sqrt{\mu-\varepsilon}}(1 + \frac{\sigma^2}{\varepsilon^2 N}) + \frac{\sqrt{N}}{a} \frac{\sigma^2}{\varepsilon^2 N} + \frac{1}{2^N}$$

By choosing $\varepsilon = N^{-1/8}$ the lemma follows. □

We now turn our attention to $f_N^{(k)}$. We first need the following elementary Lemma.

LEMMA 2.3. *Let* $\mathbf{Y} = (y_1, \ldots, y_N)$ *be defined by*

$$\begin{cases} y_i = \frac{|\mathbf{V}^k| v_i}{\sqrt{N - |\mathbf{V}_k|^2}} & 1 \leq i \leq k \\ y_i = v_i & k+1 \leq i \leq N \ . \end{cases}$$

Then

$$v_i = y_i \sqrt{\frac{N}{|\mathbf{Y}|^2}} \qquad 0 \leq i \leq k \ . \tag{26}$$

Moreover the Jacobian determinant is given by

$$\left| \frac{\partial(v_1, \ldots, v_k)}{\partial(y_1, \ldots, y_k)} \right| = \frac{|\mathbf{Y}^k|^2}{N} \left(\frac{N}{|\mathbf{Y}|^2} \right)^{\frac{dk}{2}+1} = \frac{|\mathbf{V}^k|^2}{N} \left(\frac{N^2 - |\mathbf{V}_k|^2}{|\mathbf{V}^k|^2} \right)^{\frac{dk}{2}+1}$$

Proof. We have

$$|\mathbf{Y}_k|^2 = \frac{|\mathbf{V}^k|^2 |\mathbf{V}_k|^2}{N - |\mathbf{V}_k|^2}$$

so that

$$N - |\mathbf{V}_k|^2 = \frac{N |\mathbf{Y}^k|^2}{|\mathbf{Y}|^2}$$

from which (26) follows immediately. To conclude the proof of the Lemma it is enough to observe that

$$\frac{\partial v_i}{\partial y_j} = \sqrt{\frac{N}{|\mathbf{Y}|^2}} \left(\delta_{i,j} - \frac{y_i y_j}{|\mathbf{Y}|^2} \right)$$

□

Let now $\varphi(\mathbf{V}_k)$ be a continuous function on $\mathbb{R}^{dk}$ such that

$$\sup_{\mathbf{V}_k\in\mathbb{R}^{dk}} \varphi(\mathbf{V}_k) < K \ .$$

Using Lemma 2.3 and 17 we get

$$\int d\mathbf{V}_k \varphi(\mathbf{V}_k) f_N^{(k)}(\mathbf{V}_k) =$$

$$\frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})Z_N}\int_{\mathbb{R}^{dk}} d\mathbf{Y}_k \prod_{i=1}^k g(y_i) \int_{\mathbb{R}^{d(N-k)}} d\mathbf{Y}^k \frac{\prod_{j=k+1}^N g(y_j)}{|\mathbf{Y}|}\varphi\left(y_1\frac{\sqrt{N}}{|\mathbf{Y}|},\ldots,y_k\frac{\sqrt{N}}{|\mathbf{Y}|^2}\right) :=$$

$$\int_{\mathbb{R}^{dk}} dy_1\cdots dy_k \prod_{i=1}^k g(y_i) H_N(y_1,\ldots,y_k) \tag{27}$$

Since

$$\frac{|\mathbf{Y}^k|^2}{|\mathbf{Y}^k|^2+|\mathbf{Y}_k|^2} < 1$$

we get

$$|H_N(y_1,\ldots,y_k)| \le K\frac{\Gamma\left(\frac{d(N-k)-1}{2+\alpha}\right)Z_{N-k}}{\Gamma\left(\frac{dN-1}{2+\alpha}\right)Z_N} \le K'$$

for a suitable constant K'. For the last inequality we have used Lemma 2.2. The proof can then be completed by applying the Dominated Convergence Theorem if we can show that

$$\lim_{N\to\infty} H_N(y_1,\ldots,y_N) = \varphi\left(\frac{y_1}{\mu},\ldots,\frac{y_N}{\mu}\right)$$

pointwise in $\mathbb{R}^{dk}$. The proof of this fact follows closely that of Lemma 2.2. From (24) it follows that

$$\int_{A_\varepsilon} d\mathbf{Y}^k \prod_{j=k+1}^N g(y_j)\frac{\sqrt{N}}{|\mathbf{Y}^k|}\sqrt{\frac{|\mathbf{Y}^k|^2}{|\mathbf{Y}^k|^2+|\mathbf{Y}_k|^2}}\left|\varphi\left(y_1\frac{\sqrt{N}}{|\mathbf{Y}|},\ldots,y_k\frac{\sqrt{N}}{|\mathbf{Y}|}\right)-\varphi\left(\frac{y_1}{\mu},\ldots,\frac{y_N}{\mu}\right)\right| \le$$

$$2K\left(\frac{\sqrt{N}}{a}\mathbb{P}(A^\varepsilon\setminus C^{Nd}) + \frac{1}{2^N} + \frac{1}{\sqrt{\mu+\varepsilon}}\mathbb{P}(A_\varepsilon^>)\right)$$

while

$$\int_{\mathbb{R}^{Nd}\setminus A_\varepsilon} d\mathbf{Y}^k \prod_{j=k+1}^N g(y_j)\frac{\sqrt{N}}{|\mathbf{Y}^k|}\sqrt{\frac{|\mathbf{Y}^k|^2}{|\mathbf{Y}^k|^2+|\mathbf{Y}_k|^2}}\left|\varphi\left(y_1\frac{\sqrt{N}}{|\mathbf{Y}|},\ldots,y_k\frac{\sqrt{N}}{|\mathbf{Y}|}\right)-\varphi\left(\frac{y_1}{\mu},\ldots,\frac{y_N}{\mu}\right)\right| \le$$

$$\frac{1}{\sqrt{\mu-\varepsilon}}\sup_{\lambda\in I_N}|\varphi(\lambda y_i,\ldots,\lambda y_k)-\varphi(y_i,\ldots,y_k)|$$

where I_N is the interval

$$I_N = \left[\sqrt{\frac{N}{N(\mu+\varepsilon)+|\mathbf{Y}_k|^2}}, \sqrt{\frac{N}{N(\mu-\varepsilon)+|\mathbf{Y}_k|^2}}\right] \ .$$

Collecting these estimates we get

$$\begin{aligned}\left|H_N(y_i,\dots,y_k) \;-\; \varphi\left(\frac{y_1}{\mu},\dots,\frac{y_N}{\mu}\right)\right| \leq \\ \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})\sqrt{N}Z_N}\left[\frac{1}{\sqrt{\mu-\varepsilon}}\sup_{\lambda\in I_N}|\varphi(\lambda y_i,\dots,\lambda y_k)-\varphi\left(\frac{y_i}{\mu},\dots,\frac{y_k}{\mu}\right)|+\right. \\ \left. 2K\left(\frac{\sqrt{N}}{a}\mathbb{P}(A^\varepsilon\setminus C^{Nd})+\frac{1}{2^N}+\frac{1}{\sqrt{\mu+\varepsilon}}\mathbb{P}(A_\varepsilon^>)\right)\right] \qquad (28)\end{aligned}$$

from which Theorem 1.1 follows by choosing $\varepsilon = N^{-\frac{1}{8}}$.

3 Extension and Remarks.

It is easy to extend the results of the previous section in a couple of interesting directions. We first observe that one can give a stronger definition of chaoticity by requiring that given a sequence of functions $F_N(\mathbf{V})$ on $\mathcal{S}^{dN-1}(\sqrt{N})$, the entropy per particle of this sequence converge to the entropy of the 1 particle marginal. More precisely if

$$S_N = \int_{\mathcal{S}^{dN-1}(\sqrt{N})} F_N(\mathbf{V})\log F_N(\mathbf{V})d\sigma(\mathbf{V}$$

is the entropy of the N particles system than

$$\lim_{N\to\infty}\frac{S_N}{N} = \int_{\mathbb{R}^d} f(v)\log f(v)dv$$

where, as before

$$f(v) = \lim_{N\to\infty} f_N^{(1)}(v).$$

If this is true we say that the sequence F_N is **entropically chaotic**, see [6].

COROLLARY 3.1. *The sequence $F_N(\mathbf{V})$ given by (9) is entropically chaotic and*

$$\lim_{N\to\infty} N^{-1}\int_{\mathcal{S}^{dN-1}(\sqrt{N})} F_N(\mathbf{V})\log F_N(\mathbf{V})d\sigma(\mathbf{V}) = \log\left(\frac{\mu^{\frac{d}{2}}}{c}\right) - \frac{d}{2+\alpha} = \int_{\mathbb{R}^d} f(v)\log f(v)dv \qquad (29)$$

Proof. We will just report here the minor modification to the proof of Lemma 2.2 needed to prove the corollary. We observe that

$$x\log x = \lim_{\delta\to 0}\frac{x^{1+\delta}-x}{\delta}$$

Applying this to (20) we get

$$A^{-\gamma}\log A^{-\gamma} = \frac{1}{\Gamma(\gamma)}\int_0^\infty s^\gamma \log s^\gamma e^{-As}ds - \gamma\psi(\gamma)A^{-\gamma}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. Following the proof of Lemma 2.1, we find

$$\begin{aligned}\frac{S_N}{N} \;=\; -\frac{\log\tilde{Z}_N}{N} - \frac{dN-1}{(2+\alpha)N}\psi\left(\frac{dN-1}{2+\alpha}\right)+ \\ \frac{dN-1}{N}\frac{(2+\alpha)}{\Gamma\left(\frac{dN-1}{2+\alpha}\right)Z_N}\int_{\mathbb{R}^{dN}}\log\left(\frac{|\mathbf{W}|}{\sqrt{N}}\right)\frac{\prod_{i=1}^N g(w_i)}{|\mathbf{W}|}\,d\mathbf{W} \qquad (30)\end{aligned}$$

where $\tilde{Z}_N = c^N Z_N$. Using Stirling formula we get that

$$\lim_{N\to\infty}\left(\frac{\log \tilde{Z}_N}{N} - \frac{dN-1}{(2+\alpha)N}\psi\left(\frac{dN-1}{2+\alpha}\right)\right) = -\log c - \frac{d}{2+\alpha}$$

Finally we need to compute the integral in the last term of (30). It is very easy to adapt the proof of Lemma 2.2 to this expression. Indeed we have that

$$\log\left(\frac{|\mathbf{W}|}{\sqrt{N}}\right)\frac{\sqrt{N}}{|\mathbf{W}|} < \max\left(e, \frac{\log\sqrt{\mu+\epsilon}}{\sqrt{\mu+\epsilon}}\right)$$

for $\mathbf{W} \in A_\varepsilon^>$. On the other hand, the function $|(\log x)/x|$ is increasing for $x < 1$ so that from (23) we get

$$\left|\log\left(\frac{|\mathbf{W}|}{\sqrt{N}}\right)\frac{\sqrt{N}}{|\mathbf{W}|}\right| \le \frac{\sum_{i=1}^N |\log|w_i||}{N}\prod\frac{1}{|w_i|^{\frac{1}{N}}} \qquad \text{for } \frac{|\mathbf{W}|}{\sqrt{N}} < 1$$

One can now proceed like in Lemma 2.2 with the only difference that a must be chosen such that

$$\int_{C^d}\frac{|\log|w_i||}{|w|^{\frac{1}{N}}}g(w)dw < \frac{1}{2}$$

We finally obtain that

$$\lim_{N\to\infty}\int_{\mathbb{R}^{dN}}\log\left(\frac{|\mathbf{W}|}{\sqrt{N}}\right)\frac{\sqrt{N}}{|\mathbf{W}|}\prod_{i=1}^N g(w_i)\,d\mathbf{W} = \frac{\log\sqrt{\mu}}{\sqrt{\mu}}$$

Collecting the above computation we get the first equality in (29). The second equality is immediate.

□

A last interesting extension regards the first order correction in E. In [4], under the assumption that the limit $|E| \to 0$ exists, it was shown that

$$F_{\rm ss}(\mathbf{V}, E) = \frac{1}{\widetilde{Z}_N}\left(F_N(\mathbf{V}) + \sum_{i=1}^N E\cdot c(\omega_i)|v_i|H_N(\mathbf{V}) + o(|E|)\right)$$

where $v_i = |v_i|\omega_i$ and

$$H_N(\mathbf{V}) = \frac{dN-1}{\left(\sum_{i=1}^N |v_i|^{2+\alpha}\right)^{\frac{dN-1}{2+\alpha}+1}} = \frac{1}{|v_1|^{2+\alpha}}v_1\cdot\nabla_{v_1}F_N(\mathbf{V}),$$

$c(\omega)$ is the unique solution of

$$[(\mathrm{Id} - \mathcal{K})c](\omega) = \omega$$

where $\mathcal{K}$ is the convolution operator generated by K, that is

$$(\mathcal{K}c)(\omega) = \int_{\mathcal{S}^{d-1}(1)} K(\omega\cdot\omega')c(\omega')d\sigma^{d-1}(\omega').$$

Since $-c(-\omega)$ is also a solution if $c(\omega)$ is, we have, by uniqueness, that $c(\omega) = -c(-\omega)$. As a consequence

$$\int_{\mathcal{S}^{d-1}(1)} c(\omega') d\sigma^{d-1}(\omega') = 0.$$

Calling $h_N^{(k)}$ the marginal of H_N, we easily get that

$$h_N^{(k)}(v_1, \dots, v_k) = \frac{1}{|v_1|^{2+\alpha}} v_1 \cdot \nabla_{v_1} f_N^{(k)}(v_1, \dots, v_k)$$

It is easy to see, from (27), that we can take the limit for $N \to \infty$ on both side and obtain

$$\lim_{N\to\infty} h_N^{(k)}(v_1, \dots, v_k) = h(v_1) \prod_{i=2}^{k} f(v_k)$$

where

$$h(v) = \frac{1}{|v_1|^{2+\alpha}} v \cdot \nabla_v f(v) = (2+\alpha)\mu^{\frac{2+\alpha}{2}} f(v)$$

Combining the above results we get that the k particle marginal of $F_{\rm ss}$ is

$$\lim_{N\to\infty} f_{\rm ss}^{(k)}(v_1, \dots, v_k; E) = \left(1 + (2+\alpha)\mu^{\frac{2+\alpha}{2}} \sum_{i=1}^{k} E \cdot c(\omega_i)|v_i|\right) \prod_{i=1}^{k} f(v_k) \tag{31}$$

This is consistent with the results on the Boltzmann equation (7). To solve the steady state equation of (7) one as to make an assumption on the form of $\hat{j}_{\rm ss}(E)$ for small $|E$. It is natural to assume that

$$\hat{j}_{\rm ss}(E) = \tau \underline{\kappa} E + o(|E|) \tag{32}$$

where $\underline{\kappa}$ is the conductivity tensor for the system with 1 particle and energy 1, that is

$$\underline{\kappa} = \frac{1}{|\mathcal{S}^{d-1}(1)|} \int_{\mathcal{S}^{d-1}(1)} c(\omega) \otimes \omega \, d\sigma^{d-1}(\omega).$$

Under this assumption one finds that

$$f_{\rm ss}(v, E) = \left(1 + (2+\alpha)\nu^{\frac{2+\alpha}{2}} E \cdot c(\omega)|v|\right) \tilde{f}(v) + o(|E|) \tag{33}$$

where

$$\tilde{f}(v) = \frac{\nu^{\frac{d}{2}}}{b} e^{-(\sqrt{\nu}|v|)^{2+\alpha}}$$

with ν and b uniquely determined by normalization and (32). One easily see that the average energy of this solution is

$$u = \int_{\mathbb{R}^d} |v^2| \tilde{f}(v) \, dv = \left(\frac{\nu}{\mu}\right)^{\frac{2+\alpha}{2}}$$

so that, requiring $u = 1$ we get back the large N limit of the one particle marginal of $F_{\rm ss}$. Clearly the first order in E of the k-fold tensor product of (33) give us back (31). We finally notice that the above results tell us that the current per particle at small field for a large system is $(2+\alpha)\mu^{\frac{2+\alpha}{2}}$ times the current of the one particle system, if the energy per particle is 1.

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