A notion of nonlocal curvature

Nicola Abatangelo[1,2] and Enrico Valdinoci[2]

(1) – Laboratoire Amiénois de Mathématique Fondamentale et Appliquée
UMR CNRS 7352, Université de Picardie Jules Verne
33 Rue St Leu, 80039 Amiens (France).

(2) – Dipartimento di Matematica Federigo Enriques
Università degli Studi di Milano,
Via Saldini 50, I-20133 Milano (Italy).

Abstract

We consider a nonlocal (or fractional) curvature and we investigate similarities and differences with respect to the classical case. In particular, we show that the nonlocal mean curvature may be seen as an average of suitable nonlocal directional curvatures and there is a natural asymptotic convergence to the classical case.

Nevertheless, differently from the classical cases, minimal and maximal nonlocal directional curvatures are not in general attained at perpendicular directions and in fact one can arbitrarily prescribe the set of extremal directions for nonlocal directional curvatures.

Also the classical directional curvatures naturally enjoy some linear properties that are lost in the nonlocal framework. In this sense, nonlocal directional curvatures are somewhat intrinsically nonlinear.

1 Introduction

This note is aimed at the study of a nonlocal notion of curvature of a surface. This interest is motivated by the fractional Laplacian context in which several "classical" problems have been recently rephrased by attracting the attention of a large numbers of researchers. In particular, in the differential geometry and geometric analysis framework, several new results have been recently obtained for the *diffusion by mean curvature* and the closely related problem of *minimal surfaces* in a nonlocal setting (see, e.g., [Imb09, CRS10, CS10, CV11, AdPM11, CG, SV, DFPV, BFV, FMM, CV] and [Val] for a recent review).

As we shall see, the concept of nonlocal mean curvature may be naturally associated to a suitable average of appropriate nonlocal directional curvatures, which asymptotically approach their classical counterpart. On the other hand, the nonlocal curvatures seem to have a more messy and fanciful behavior than the classical ones. In particular, differently from the classical cases, minimal and maximal nonlocal directional curvatures are not in general attained at orthogonal directions and the set of extremal directions for nonlocal directional curvatures may be prescribed somehow arbitrarily (the precise statements of these results are contained in Paragraph 2.3).

The paper is organized as follows: after a brief setting of the notation in Paragraph 1.1, the introductory Paragraph 1.2 recalls some basic definitions and facts on classical curvatures of smooth surfaces (of course, this part may be easily skipped by the expert reader but we included it in order to make a clear comparison between the classical setting and the nonlocal one); in Section 2 we introduce our definition of *nonlocal directional curvature* and give some ideas of the context in which it arises; finally, we state some theorems which compare similarities and differences between the local and the nonlocal setting. The remaining sections are devoted to proofs and explicit computations.

Though the motivation of this paper arises in the framework of nonlocal minimal surfaces and integro-differential operators of fractional type, which are subjects that involve a very advanced technology, this paper is completely self-contained and no prior knowledge on the topic is required to follow the proofs. Also, we put an effort in keeping all the arguments as elementary as possible and accessible to a wide audience.

1.1 Notation

In the following we will always use:

- n to denote the dimension of the Euclidean space $\mathbb{R}^n$, with $n \geqslant 3$, whose points are sometimes written in the form $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

- $\mathcal{C}E$ to denote the complementary set of $E \subseteq \mathbb{R}^n$, i.e. $\mathcal{C}E := \mathbb{R}^n \setminus E$,

- $\mathcal{H}^{n-2}$ to denote the $(n-2)$-dimensional Hausdorff measure,

- S^{n-2} to denote the $(n-2)$-dimensional unit sphere in $\mathbb{R}^{n-1}$, namely

$$S^{n-2} := \{e \in \mathbb{R}^{n-1} : |e| = 1\};$$

  with a slight abuse of notation, we will also identify S^{n-2} and the set

$$\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| = 1, x_n = 0\} \subseteq \mathbb{R}^n$$

  (notice that the latter set is simply an $(n-2)$-dimensional sphere lying in an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ and this justifies our notation),

- ω_{n-2} to denote the $(n-2)$-dimensional Hausdorff measure of the $(n-2)$-dimensional sphere, that is $\omega_{n-2} := \mathcal{H}^{n-2}(S^{n-2})$,

- χ_E, where $E \subseteq \mathbb{R}^n$, for the characteristic function of E, i.e.

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \in \mathcal{C}E, \end{cases}$$

- $\widetilde{\chi}_E$ for the difference $\chi_E - \chi_{\mathcal{C}E}$, namely

$$\widetilde{\chi}_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ -1 & \text{if } x \in \mathcal{C}E, \end{cases}$$

- $\langle Ax, x\rangle$, when A is an $n \times n$ real symmetric matrix and $x \in \mathbb{R}^n$, to denote the quadratic form on $\mathbb{R}^n$ represented by A and evaluated at x, i.e. if $A = \{A_{ij}\}_{i,j=1,\dots,n}$,

$$\langle Ax, x\rangle := \sum_{i,j=1}^{n} A_{ij} x_i x_j,$$

- $\fint_\Omega$, when $\Omega \subseteq \mathbb{R}^n$ has finite measure, for the integral operator $\dfrac{1}{|\Omega|}\displaystyle\int_\Omega$.

Also, we will sometimes write multiple integrals by putting in evidence the integration variables according to the expression

$$\int_X dx \int_Y dy \int_Z dz\ f(x,y,z) \qquad \text{in place of} \qquad \int_X \left[\int_Y \left[\int_Z f(x,y,z)\, dz\right]\, dy\right]\, dx.$$

Moreover, we will reserve the name s for a fractional parameter that, in our scaling, is taken in $(0, 1/2)$.

1.2 Summary on classical curvatures

In order to make a clear comparison between some classical facts and the nonlocal framework, we recall here a few basic results. Namely, some well-known facts on the classical concept of curvature show a nice and deep interplay between geometry, analysis and algebra that may risk to be not evident from the beginning. In particular, the mean curvature, which is a geometrical object, may be described in normal coordinates by the Laplacian, which comes from analysis, and also may be seen as the trace of a linear map, and here an algebraic notion shows up. The interplay between these disciplines has some striking consequences: let us recall two of them.

First of all, we recall that, given a C^2 surface S, a point $p \in S$ and a vector e in the tangent space of S at p, one may define the classical notion of *directional curvature of S at p in direction e* by the curvature at p of the curve C lying in the intersection between S and the two-dimensional plane spanned by e and the normal vector of S at p (see Figure 1). We denote by K_e the directional curvature in direction e.

It is well-known that this directional curvature may be easily computed in normal coordinates. Namely, suppose we are given a set $E \subseteq \mathbb{R}^n$ such that $0 \in \partial E$, and suppose that $S = \partial E$ is described as a graph in normal coordinates, meaning that, in an open ball $B_r \subseteq \mathbb{R}^n$, ∂E coincides with the graph of a C^2 function $\varphi : B_r \cap \mathbb{R}^{n-1} \to \mathbb{R}$ with $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. Then the directional curvature in direction e is given by

$$K_e = \langle D^2\varphi(0)\, e, e\rangle = D^2_e\varphi(0), \quad e \in \mathbb{R}^{n-1}, |e| = 1$$

where $D^2\varphi(0)$ is the Hessian matrix of φ evaluated at 0.

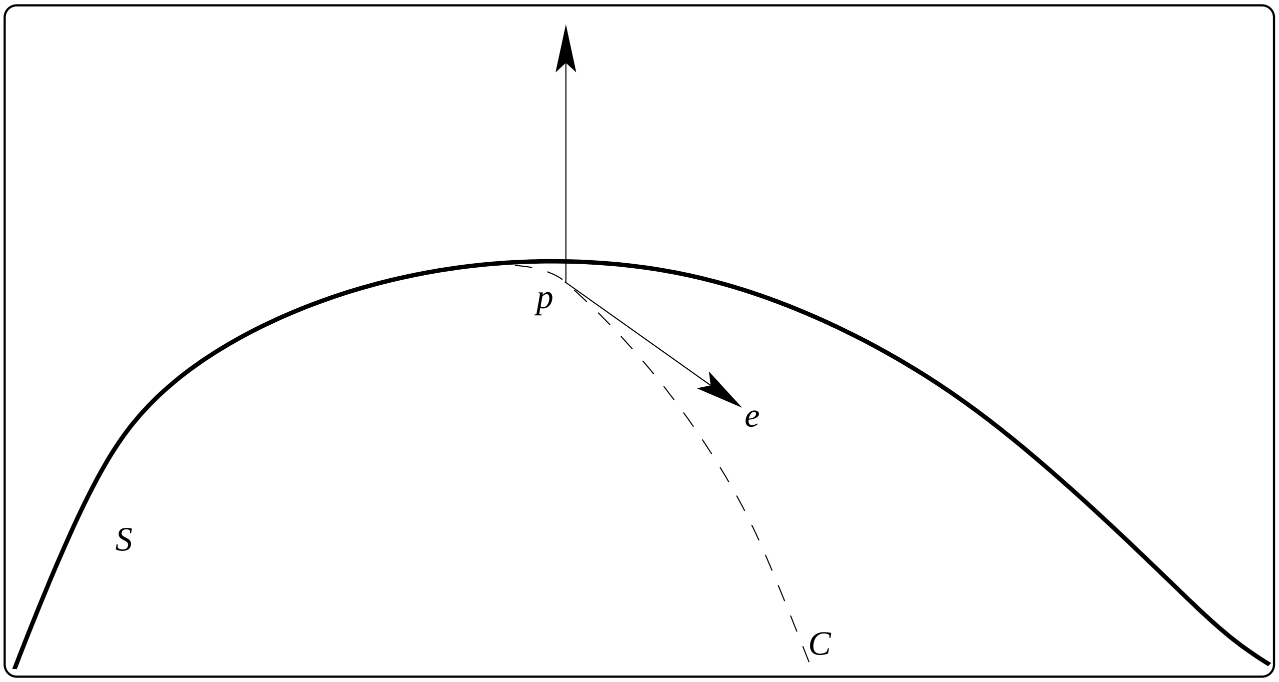

Figure 1: The notion of classical directional curvature of the surface S at the point p

Since $D^2\varphi(0)$ is a real symmetric matrix, it will admit $n-1$ real eigenvalues $\lambda_1 \leqslant \ldots \leqslant \lambda_{n-1}$ called *principal curvatures*. Moreover, associated to this eigenvalues, there is an orthonormal basis of eigenvectors $v_1, \ldots, v_{n-1}$ called *principal directions*.

The arithmetic mean of the principal curvatures is called *mean curvature* and we denote it by H, namely

$$H := \frac{\lambda_1 + \cdots + \lambda_{n-1}}{n-1}.$$

The above mentioned algebraic formulation implies that the principal directions $v_1, \ldots, v_{n-1}$ can be always chosen orthogonally, which is a somehow surprising geometric outcome that allows to easily compute any directional curvature once the principal curvatures are known:

Theorem 1. *In the above hypotheses, every directional curvature can be calculated using principal curvatures; given a vector* $e = \alpha_1 v_1 + \ldots + \alpha_{n-1} v_{n-1}$, *with* $\alpha_1^2 + \ldots + \alpha_{n-1}^2 = 1$, *then*

$$K_e = \langle D^2\varphi(0)\, e, e\rangle = \lambda_1 \alpha_1^2 + \ldots + \lambda_{n-1}\alpha_{n-1}^2.$$

Remark 2. We point out that Theorem 1 implies also that all directional curvatures are bounded below by λ_1 and above by λ_{n-1} and that λ_1 and λ_{n-1} are attained along orthogonal directions. In particular, when $n = 3$, the two principal curvatures are the minimum and the maximum of the directional curvature K_e for $e \in S^1$.

Remark 3. In a sense, Theorem 1 shows a sort of linear phenomenon that drives the classical directional curvatures. As we will see in the forthcoming Remark 9, this linear feature is lost by the nonlocal directional curvatures, that are somewhat intrinsically nonlinear in nature.

Furthermore, the spherical average of directional curvatures may be reconstructed by the arithmetic mean of the principal curvatures, that is the normalized integral of K_e over $e \in S^{n-2}$ coincides with the normalized trace of the Hessian matrix, thus reducing the (difficult, in general) computation of an integral on the sphere to a (simple, in general) sum of finitely many terms (that are the eigenvalues of the Hessian matrix) and this clearly provides an important computational simplification:

Theorem 4. *In the above hypotheses, there are two different but equivalent ways to compute the mean curvature, since*

$$\fint_{S^{n-2}} K_e \, d\mathcal{H}^{n-2}(e) = \fint_{S^{n-2}} \langle D^2\varphi(0)\, e, e\rangle \, d\mathcal{H}^{n-2}(e) = \frac{\lambda_1 + \ldots + \lambda_{n-1}}{n-1} = H.$$

Proof. By symmetry

$$\int_{S^{n-2}} \alpha_i^2 \, d\mathcal{H}^{n-2}(\alpha) = \int_{S^{n-2}} \alpha_1^2 \, d\mathcal{H}^{n-2}(\alpha) \tag{1}$$

for any $i \in \{1, \ldots, n-1\}$. By summing up in (1) we obtain

$$\omega_{n-2} = \int_{S^{n-2}} 1 \, d\mathcal{H}^{n-2}(\alpha) = \int_{S^{n-2}} \sum_{i=1}^{n-1} \alpha_i^2 \, d\mathcal{H}^{n-2}(\alpha) = (n-1) \int_{S^{n-2}} \alpha_1^2 \, d\mathcal{H}^{n-2}(\alpha).$$

Using again (1) we deduce from the identity above that

$$\omega_{n-2} = (n-1) \int_{S^{n-2}} \alpha_i^2 \, d\mathcal{H}^{n-2}(\alpha)$$

for any $i \in \{1, \ldots, n-1\}$. As a consequence

$$\begin{aligned}
\frac{1}{\omega_{n-2}} \int_{S^{n-2}} \langle D^2\varphi(0)\, e, e\rangle \, d\mathcal{H}^{n-2}(e) &= \frac{1}{\omega_{n-2}} \int_{S^{n-2}} \left(\lambda_1 \alpha_1^2 + \ldots + \lambda_{n-1}\alpha_{n-1}^2\right) \, d\mathcal{H}^{n-2}(\alpha) \\
&= \frac{1}{\omega_{n-2}} \sum_{i=1}^{n-1} \lambda_i \int_{S^{n-2}} \alpha_i^2 \, d\mathcal{H}^{n-2}(\alpha) \\
&= \frac{\lambda_1 + \ldots + \lambda_{n-1}}{\omega_{n-2}} \cdot \frac{\omega_{n-2}}{n-1}
\end{aligned}$$

as desired. □

Associated with these concepts, there is also a theory of *motion by mean curvature*. Let us think of a bounded set $\Omega \subseteq \mathbb{R}^n$ whose shape changes in time according to local features of its boundary, i.e. each point x_0 of the boundary moves along the normal direction to $\partial\Omega$ at x_0 and with a speed given by the mean curvature of $\partial\Omega$ at x_0. In [MBO92], with the aid of [Ish95] and [Eva93], it is possible to find the following approximation of this motion. Let evolve the function $\widetilde{\chi}_\Omega$ according to the heat equation

$$\begin{cases} \partial_t u(x,t) = \Delta u(x,t) \\ u(x,0) = \widetilde{\chi}_\Omega(x) \end{cases} \tag{2}$$

Then the set $\Omega_\varepsilon = \{u(x,\varepsilon) > 0\}$, for small $\varepsilon > 0$, has a boundary close to the evolution of $\partial\Omega$ by mean curvature. Before passing to the nonlocal case, we would like to bring to the reader's attention two facts:

1. we underline how the evolution of a point $x_0 \in \partial\Omega$ depends only on the shape of Ω in a neighborhood of x_0,

2. we recall that if a set E has minimal perimeter in a region U, then it has zero mean curvature at each point of $\partial E \cap U$, see [Giu84], and we can say that this is the Euler-Lagrange equation associated to the minimization of the perimeter of a set; therefore a set with minimal perimeter will be a stationary solution to the motion by mean curvature.

2 Nonlocal directional curvatures

From now on we take $s \in (0, 1/2)$ and think of a set $E \subseteq \mathbb{R}^n$, with C^2 boundary ∂E.

2.1 Nonlocal definitions

We introduce here the nonlocal objects that will play the role of directional and mean curvatures (for details, heuristics and justifications of our definitions see Paragraph 2.2).

Definition 5. The **nonlocal mean curvature** of ∂E at the point $p \in \partial E$ is

$$H_s := \frac{1}{\omega_{n-2}} \int_{\mathbb{R}^n} \frac{\widetilde{\chi}_E(x)}{|x-p|^{n+2s}}\, dx. \tag{3}$$

Denote now by ν be a normal unit vector for ∂E at p. Let also e be any unit vector in the tangent space of ∂E at p and[1] let $\pi(e)$ the two-dimensional open half-plane

$$\pi(e) := \{y \in \mathbb{R}^n : y = \rho e + h\nu,\ \rho > 0,\ h \in \mathbb{R}\}.$$

We endow $\pi(e)$ with the induced two-dimensional Lebesgue measure, that is we define the integration over $\pi(e)$ by the formula

$$\int_{\pi(e)} g(y)\, dy := \int_0^{+\infty} d\rho \int_{\mathbb{R}} dh\, g(\rho e + h\nu). \tag{4}$$

[1] Notice that $\pi(e)$ is simply the portion of the two-dimensional plane spanned by e and ν given by the vectors with positive scalar product with respect to e. We point out that a change of the orientation of ν does not change $\pi(e)$ which is therefore uniquely defined. Needless to say, such two-dimensional plane plays an important role even in the classical setting, see Figure 1.

Definition 6. We define the **nonlocal directional curvature** of ∂E at the point $p \in \partial E$ in direction e the quantity

$$K_{s,e} := \int_{\pi(e)} \frac{|y' - p'|^{n-2}\, \widetilde{\chi}_E(y)}{|y-p|^{n+2s}}\, dy. \tag{5}$$

Without loss of generality, we may and do consider now a normal frame of coordinates in which p coincides with the origin 0 of $\mathbb{R}^n$, and the tangent space of S at 0 is the horizontal hyperplane $\{x_n = 0\}$. In this way we may take also

$$\nu = (0, \dots, 0, 1). \tag{6}$$

With this choice, (3) and (5) become

$$H_s = \frac{1}{\omega_{n-2}} \int_{\mathbb{R}^n} \frac{\widetilde{\chi}_E(x)}{|x|^{n+2s}}\, dx \tag{7}$$

and

$$K_{s,e} = \int_{\pi(e)} \frac{|y'|^{n-2}\, \widetilde{\chi}_E(y)}{|y|^{n+2s}}\, dy. \tag{8}$$

As a matter of fact, since the function $\widetilde{\chi}_E(x)/|x|^{n+2s}$ is not in the space $L^1(\mathbb{R}^n)$, the integral in (7) has to be taken in the principal value sense, that is

$$\lim_{\varepsilon \searrow 0} \int_{\mathcal{C}B_\varepsilon} \frac{\widetilde{\chi}_E(x)}{|x|^{n+2s}}\, dx. \tag{9}$$

Similarly, the integral in (8) may be taken in the principal value sense as

$$\lim_{\varepsilon \searrow 0} \int_{\pi(e)\setminus B_\varepsilon} \frac{|y'|^{n-2} \widetilde{\chi}_E(y)}{|y|^{n+2s}}\, dy. \tag{10}$$

Next observation points out that these definitions are well-posed, thanks to the smoothness of ∂E:

Lemma 7. *The limits in* (9) *and* (10) *exist and are finite.*

We postpone the proof of Lemma 7 to Section 8. Though the definition of the nonlocal direction curvature may look rather mysterious at a first glance, it finds a concrete justification thanks to the following result:

Theorem 8. *In the above setting*

$$H_s = \fint_{S^{n-2}} K_{s,e}\, d\mathcal{H}^{n-2}(e).$$

Namely, Theorem 8 states that the nonlocal mean curvature is the average of the nonlocal directional curvatures, thus providing a nonlocal counterpart of Theorem 4. See Section 3 for the proof of Theorem 8.

In the particular case in which the set E is characterized as the subgraph of a function $f \in C^2(\mathbb{R}^{n-1})$ (that, due to our normalization setting, satisfies $f(0) = 0$ and $\nabla f(0) = 0$), namely if $E = \{x_n < f(x')\}$, then formula (8) may be written directly in terms of f, according to the expression

$$K_{s,e} = 2 \int_0^{+\infty} d\rho\, \rho^{n-2} \int_0^{f(\rho e)} \frac{dh}{(\rho^2 + h^2)^{s+n/2}}. \tag{11}$$

The proof of (11) is deferred to Section 4.

Remark 9. We point out that if f is a radial function, i.e. $f(\rho e) = \phi(\rho)$ for some $\phi : [0, +\infty) \to \mathbb{R}$, equation (11) becomes

$$\begin{aligned} K_{s,e} &= 2\int_0^{+\infty} d\rho \, \frac{1}{\rho^{2+2s}} \int_0^{\phi(\rho)} \frac{dh}{(1+\rho^{-2}h^2)^{s+n/2}} \quad = \quad 2\int_0^{+\infty} d\rho \, \frac{1}{\rho^{1+2s}} \int_0^{\frac{\phi(\rho)}{\rho}} \frac{d\tau}{(1+\tau^2)^{s+n/2}} \\ &= \int_0^{+\infty} \frac{1}{\rho^{1+2s}} \cdot F\left(\frac{\phi(\rho)}{\rho}\right) \, d\rho, \end{aligned}$$

where

$$F(t) := \int_0^t \left(1+\tau^2\right)^{-s-n/2} d\tau.$$

We observe that the function F is nonlinear, thus the nonlocal directional curvature depends on the graph of the set in a nonlinear fashion. Comparing with Theorem 1, we notice that this phenomenon is in sharp contrast with the classical case.

Remark 10. In our setting $K_{s,-e}$ is, in general, not equal to $K_{s,e}$, differently from the classical case in which $K_{-e} = K_e$. For a notion of fractional directional curvature that is even on S^{n-2} one may consider $\tilde{K}_{s,e} := (K_{s,e} + K_{s,-e})/2$. Of course the results presented in this paper hold for $\tilde{K}_{s,e}$ too (with obvious minor modifications).

2.2 The context in which nonlocal curvatures naturally come forth

We now give some further motivation for the study of curvatures of nonlocal type. A few years ago the notion of *s-minimal set* has been introduced, see [CRS10]. Roughly speaking, one can think of the problem of minimizing functionals which have a strong nonlocal flavor, meaning that these functionals take into account power-like interactions between distant objects. In particular, one is interested in the functional

$$\mathcal{J}(A, B) = \frac{1}{\omega_{n-1}} \int_A \int_B \frac{dx\, dy}{|x-y|^{n+2s}}$$

for every measurable sets $A, B \subseteq \mathbb{R}^n$, and in the minimization of the functional

$$\mathrm{Per}_s(E, U) := \mathcal{J}(E \cap U, U \setminus E) + \mathcal{J}(E \cap U, \mathcal{C}E \cap \mathcal{C}U) + \mathcal{J}(E \setminus U, U \setminus E),$$

called the *s-perimeter of E in U*, where $E, U \subseteq \mathbb{R}^n$ are measurable sets and U is bounded.

A set $E_\star \subseteq \mathbb{R}^n$ that minimizes $\mathrm{Per}_s(E, U)$ among all the measurable sets $E \subseteq \mathbb{R}^n$ such that $E \setminus U = E_\star \setminus U$ is called s-minimal. In this framework, U can be viewed as an ambient space, meaning the space in which one is free to modify the set E, while the shape of E is fixed outside U and $E \setminus U$ plays the role of a boundary datum.

As the reader may have noticed, the notation $\mathrm{Per}_s(E, U)$ and the name "s-perimeter" strongly remind the notation $\mathrm{Per}(E, U)$ for the perimeter of a set E in U, see [Giu84] (and indeed s-minimal sets are the natural nonlocal generalizations of sets with minimal perimeter). For instance, it is proved in [CV11, AdPM11] that, as $s \nearrow \frac{1}{2}$, the s-perimeter reduces to the classical perimeter, namely

$$\lim_{s \nearrow \frac{1}{2}} (1-2s)\,\mathrm{Per}_s(E, B_r) = \mathrm{Per}(E, B_r) \quad \text{for a.e. } r > 0. \tag{12}$$

While, in the classical setting, sets with minimal perimeter satisfy the zero mean curvature equation, it is proved in [CRS10] that if $E_\star$ is an s-minimal set and $p \in \partial E$, then

$$\int_{\mathbb{R}^n} \frac{\widetilde{\chi}_{E_\star}(x)}{|x-p|^{n+2s}}\, dx = 0. \tag{13}$$

Of course this equation makes sense if $\partial E_\star$ is smooth enough near p, so in general [CRS10] has to deal with (13) in a suitable weak (and in fact viscosity) sense. In this setting, one can say that (13) is the Euler-Lagrange equation of the functional Per_s and so, by analogy with the classical case, it is natural to consider the left hand side of (13) as a nonlocal mean curvature.

This justifies Definition 5. Furthermore, in [Imb09, CS10] a nonlocal approximation scheme of motion by mean curvature has been developed. This scheme differs from the classical one recalled in Paragraph 1.2 since it substitutes the standard heat equation in (2) with its nonlocal counterpart

$$\begin{cases} \partial_t u(x,t) = -(-\Delta)^s u(x,t) \\ u(x,0) = \widetilde{\chi}_\Omega(x). \end{cases}$$

Here $(-\Delta)^s$ is the *fractional Laplacian* operator, see e.g. [DNPV12] for a gentle introduction to this kind of operators. With this modification it has been proved that the counterpart of the normal velocity at a point $x_0 \in \partial\Omega$ is given by the quantity

$$\int_{\mathbb{R}^n} \frac{\widetilde{\chi}_\Omega(x)}{|x-x_0|^{n+2s}}\, dx. \tag{14}$$

2.3 Some comparisons between classical and nonlocal directional curvatures

Now we turn to the study of the objects that we have introduced in the last paragraph, by stating some properties. Our goal is threefold: first we study the directions in which maximal curvatures are attained, then we are interested in asymptotics for $s \nearrow \frac{1}{2}$, finally we present an example dealing with the relation between the nonlocal mean curvature and the average of extremal nonlocal directional curvatures.

First of all, we establish that the counterparts of Theorem 1 and Remark 2 do not hold in the nonlocal framework. Indeed, the direction that maximizes the nonlocal directional curvature is not, in general, orthogonal to the one that minimizes it. Even more, one can prescribe arbitrarily the set of directions that maximize and minimize the nonlocal directional curvature, according to the following result:

Theorem 11 (Directions of extremal nonlocal curvatures)**.** *For any two disjoint, nonvoid, closed subsets*

$$\Sigma_-, \Sigma_+ \subseteq S^{n-2},$$

there exists a set $E \subseteq \mathbb{R}^n$ such that ∂E is C^2, $0 \in \partial E$ and

$$K_{s,e_-} < K_{s,e} < K_{s,e_+}, \qquad \textit{for any } e_- \in \Sigma_-,\ e \in S^{n-2} \setminus (\Sigma_+ \cup \Sigma_-),\ e_+ \in \Sigma_+,$$

i.e. the minimum and maximum of the nonlocal directional curvatures are attained at any point of Σ_- and Σ_+ respectively.

We remark that, in the statement above, it is not necessary to assume any smoothness on the boundary of the sets Σ_- and Σ_+ in S^{n-2}

Next result points out that the definition of nonlocal directional curvature is consistent with the classical concept of directional curvature and reduces to it in the limit:

Theorem 12 (Asymptotics to $\frac{1}{2}$)**.** *For any $e \in S^{n-2}$*

$$\lim_{s \nearrow \frac{1}{2}} (1-2s) K_{s,e} = K_e \tag{15}$$

and

$$\lim_{s \nearrow \frac{1}{2}} (1-2s) H_s = H,$$

where K_e (resp., H) is the directional curvature of E in direction e (resp., the mean curvature of E) at 0.

Notice that Theorem 12 may be seen as an extension of the asymptotics in (12) for the directional and mean curvatures.

A further remark is that, differently from the local case, in the nonlocal one it is not possible to calculate the mean curvature simply by taking the arithmetic mean of the principal curvatures (this in dimension $n = 3$ reduces to the half of the sum between the maximal and the minimal directional curvatures). This phenomenon is a consequence of Theorem 12 and it may also be detected by an explicit example:

Example 13. Let $E = \{(x, y, z) \in \mathbb{R}^3 : z \leqslant 8x^2y^2\}$. Let H_s be the nonlocal mean curvature at $0 \in \partial E$. Let also $K_{s,e}$ be the nonlocal principal curvature at 0 in direction e,

$$\lambda_- := \min_{e \in S^1} K_{s,e} \quad \text{and} \quad \lambda_+ := \max_{e \in S^1} K_{s,e}.$$

Then λ_- is attained at $(0, 1)$, λ_+ is attained at $(\sqrt{2}/2, \sqrt{2}/2)$, and $H_s \neq (\lambda_- + \lambda_+)/2$.

Of course, Example 13 is in sharp contrast with the classical case, recall Remark 2. For the proof of the claims related to Example 13 see Section 7.

3 Proof of Theorem 8

Given a function G, we apply (4) to the function $g(y) := |y'|^{n-2} G(y)$. For this, we recall the normal coordinates convention in (6) and, with a slight abuse of notation we identify the vector $e = (e_1, \dots, e_{n-1}, 0) \in \mathbb{R}^n$ with $(e_1, \dots, e_{n-1}) \in \mathbb{R}^{n-1}$, so that we write

$$\pi(e) \ni y = \rho e + h\nu = (\rho e, h). \tag{16}$$

Then (4) reads

$$\int_{\pi(e)} dy\ |y'|^{n-2} G(y) = \int_0^{+\infty} d\rho \int_{\mathbb{R}} dh\ \rho^{n-2} G(\rho e, h).$$

We integrate this identity over $e \in S^{n-2}$: by recognizing the polar coordinates in $\mathbb{R}^{n-1}$ we obtain

$$\begin{aligned} &\int_{S^{n-2}} d\mathcal{H}^{n-2}(e) \int_{\pi(e)} dy\ |y'|^{n-2} G(y) = \int_{\mathbb{R}} dh \int_{S^{n-2}} d\mathcal{H}^{n-2}(e) \int_0^{+\infty} d\rho\ \rho^{n-2} G(\rho e, h) \\ &= \int_{\mathbb{R}} dh \int_{\mathbb{R}^{n-1}} dx'\ G(x', h) = \int_{\mathbb{R}^n} dx\ G(x). \end{aligned}$$

We apply this formula to $G(y) := \widetilde{\chi}_E(y)/|y|^{n+2s}$ and we recall (7) and (8), so to conclude that

$$\int_{S^{n-2}} d\mathcal{H}^{n-2}(e)\ K_{s,e} = \int_{S^{n-2}} d\mathcal{H}^{n-2}(e) \int_{\pi(e)} dy\ \frac{|y'|^{n-2}\widetilde{\chi}_E(y)}{|y|^{n+2s}} = \int_{\mathbb{R}^n} dx\ \frac{\widetilde{\chi}_E(x)}{|x|^{n+2s}} = \omega_{n-2}\, H_s,$$

establishing Theorem 8.

4 Proof of (11)

We exploit again the notation in (16) and (4) applied to $g(y) := |y'|^{n-2}\widetilde{\chi}_E(y)/|y|^{n+2s}$, to see that

$$\int_{\pi(e)} dy\ \frac{|y'|^{n-2}\widetilde{\chi}_E(y)}{|y|^{n+2s}} = \int_0^{+\infty} d\rho \int_{\mathbb{R}} dh\ \frac{\rho^{n-2}\widetilde{\chi}_E(\rho e, h)}{(\rho^2 + h^2)^{s+n/2}}. \tag{17}$$

Now we observe that $\widetilde{\chi}_E(\rho e, h) = 1$ if $h < -|f(\rho e)|$ and $\widetilde{\chi}_E(\rho e, h) = -1$ if $h > |f(\rho e)|$, being E the subgraph of f. Therefore, for any fixed $e \in S^{n-2}$ the map

$$h \longmapsto \frac{\rho^{n-2}\widetilde{\chi}_E(\rho e, h)}{(\rho^2 + h^2)^{s+n/2}}$$

is odd for $h \in \mathbb{R} \setminus [-|f(\rho e)|, |f(\rho e)|]$ and therefore

$$\int_{\mathbb{R}\setminus[-|f(\rho e)|,|f(\rho e)|]} dh\ \frac{\rho^{n-2}\widetilde{\chi}_E(\rho e, h)}{(\rho^2 + h^2)^{s+n/2}}\ =\ 0. \tag{18}$$

The subgraph property also gives that

$$\begin{aligned} &\int_{-|f(\rho e)|}^{|f(\rho e)|} dh\ \frac{\rho^{n-2}\widetilde{\chi}_E(\rho e, h)}{(\rho^2 + h^2)^{s+n/2}} = \begin{cases} 2\displaystyle\int_0^{|f(\rho e)|} dh\ \frac{\rho^{n-2}}{(\rho^2 + h^2)^{s+n/2}} & \text{if } f(\rho e) \geqslant 0, \\[2ex] -2\displaystyle\int_{-|f(\rho e)|}^{0} dh\ \frac{\rho^{n-2}}{(\rho^2 + h^2)^{s+n/2}} & \text{if } f(\rho e) \leqslant 0 \end{cases} \\ &= 2\int_0^{f(\rho e)} dh\ \frac{\rho^{n-2}}{(\rho^2 + h^2)^{s+n/2}}. \end{aligned}$$

This, (17) and (18) give that

$$\int_{\pi(e)} dy\, \frac{|y'|^{n-2}\widetilde{\chi}_E(y)}{|y|^{n+2s}} = 2\int_0^{+\infty} d\rho \int_0^{f(\rho e)} dh\, \frac{\rho^{n-2}}{(\rho^2+h^2)^{s+n/2}},$$

and so (11) follows now from (8).

5 Proof of Theorem 11

In $\mathbb{R}^n$ define the set $E = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n \leqslant f(x')\}$. We will construct f in such a way to make it a regular function, say at least C^2. For this, we fix two closed and disjoint sets Σ_- and Σ_+ in S^{n-2}, and we take $a \in C^\infty(S^{n-2}, [0,1])$ in such a way that

$$\begin{array}{ll} a(e) = 0 & \text{for any } e \in \Sigma_-, \\ a(e) \in (0,1) & \text{for any } e \in S^{n-2} \setminus (\Sigma_- \cup \Sigma_+), \\ a(e) = 1 & \text{for any } e \in \Sigma_+. \end{array} \tag{19}$$

The existence of such a is warranted by a strong version of the smooth Urysohn Lemma (notice that $a \in C^\infty(S^{n-2})$ in spite of the fact that no regularity assumption has been taken on Σ_- and Σ_+ and that a takes values 0 and 1 only in $\Sigma_- \cup \Sigma_+$). We provide the details of the construction of a for the facility of the reader. For this we observe that Σ_- is a closed set in $\mathbb{R}^{n-1}$. So, by Theorem 1.1.4 in [KP99], there exists $f_- \in C^\infty(\mathbb{R}^{n-1})$ such that $f_-(p) = 0$ for any $p \in \Sigma_-$ and $f_-(p) \neq 0$ for any $p \in \mathbb{R}^{n-1} \setminus \Sigma_-$. Then the function $g_-(p) := \big(f_-(p)\big)^2$ satisfies that $g_-(p) = 0$ for any $p \in \Sigma_-$ and $g_-(p) > 0$ for any $p \in \mathbb{R}^{n-1} \setminus \Sigma_-$. Similarly, there exists $g_+ \in C^\infty(\mathbb{R}^{n-1})$ such that $g_+(p) = 0$ for any $p \in \Sigma_+$ and $g_+(p) > 0$ for any $p \in \mathbb{R}^{n-1} \setminus \Sigma_+$. Then the function

$$\mathbb{R}^{n-1} \ni p \longmapsto a(p) := \frac{g_-(p)}{g_+(p) + g_-(p)}$$

satisfies (19) as desired.

Now we take an even function $\phi \in C_0^\infty(\mathbb{R}, [0,1])$ with $\phi(\rho) > 0$ if $\rho \in (1,2)$,

$$\lim_{\rho \to 0^+} \phi''(\rho) = \lim_{\rho \to 0^+} \phi'(\rho) = \lim_{\rho \to 0^+} \phi(\rho) = 0.$$

Then we define f using the polar coordinates of $\mathbb{R}^{n-1}$, namely we set

$$f(x') := a(e)\phi(\rho), \qquad \text{where } \rho = |x'| \text{ and } e = \frac{x'}{|x'|}.$$

By construction f is C^2 in the whole of $\mathbb{R}^{n-1}$ (in particular, in a neighborhood of 0). Also

$$0 = a(e_-)\phi(\rho) \leqslant \phi(\rho)a(e) \leqslant \phi(\rho)a(e_+) \text{ for all } \rho > 0, e \in S^{n-2} \tag{20}$$

whenever we choose $e_- \in \Sigma_-$, $e_+ \in \Sigma_+$ and $e \in S^{n-2} \setminus (\Sigma_- \cup \Sigma_+)$, and strict inequalities occur whenever $\rho \in (1,2)$. Therefore, by (11) and (20),

$$K_{s,e_-} = 0 = 2\int_0^{+\infty} d\rho\ \rho^{n-2} \int_0^{\phi(\rho)a(e_-)} \frac{dh}{(\rho^2+h^2)^{s+n/2}} \leqslant 2\int_0^{+\infty} d\rho\ \rho^{n-2} \int_0^{\phi(\rho)a(e)} \frac{dh}{(\rho^2+h^2)^{s+n/2}} = K_{s,e}$$

and

$$K_{s,e} = 2\int_0^{+\infty} d\rho\ \rho^{n-2} \int_0^{\phi(\rho)a(e)} \frac{dh}{(\rho^2+h^2)^{s+n/2}} \leqslant 2\int_0^{+\infty} d\rho\ \rho^{n-2} \int_0^{\phi(\rho)a(e_+)} \frac{dh}{(\rho^2+h^2)^{s+n/2}} = K_{s,e_+},$$

hence $K_{s,e}$ attains its minimum at any point of Σ_- and its maximum at any point of Σ_+ (and only there).

6 Proof of Theorem 12

For simplicity, we consider here the case in which E is a subgraph, namely that there exists $f \in C^2(\mathbb{R}^{n-1})$, such that $f(0) = 0$, $\nabla f(0) = 0$ and $E = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n \leqslant f(x')\} \subseteq \mathbb{R}^n$. Such assumption can be easily dropped a posteriori just working in local coordinates and observing that the contribution to $K_{s,e}$ coming from far is bounded uniformly[2] when $s \nearrow 1/2$ and so it does not contribute to the limit in (15).

So, to prove Theorem 12, we take equation (11) and split the integral in $\eta > 0$

$$K_{s,e} = 2\int_0^{\eta} d\rho\ \rho^{n-2} \int_0^{f(\rho e)} \frac{dh}{(\rho^2+h^2)^{s+n/2}} + 2\int_\eta^{+\infty} d\rho\ \rho^{n-2} \int_0^{f(\rho e)} \frac{dh}{(\rho^2+h^2)^{s+n/2}}. \tag{21}$$

Let us start from the second addendum:

$$\begin{aligned} \left| \int_\eta^{+\infty} d\rho\ \rho^{n-2} \int_0^{f(\rho e)} \frac{dh}{(\rho^2+h^2)^{s+n/2}} \right| &\leqslant \int_\eta^{+\infty} d\rho \int_0^{+\infty} dh\, \frac{\rho^{n-2}}{(\rho^2+h^2)^{s+n/2}} \\ &\leqslant \int_\eta^{+\infty} d\rho \int_0^{+\infty} dh\, \frac{(\rho^2+h^2)^{(n-2)/2}}{(\rho^2+h^2)^{s+n/2}} \leqslant \int_{\mathbb{R}^2\setminus B_\eta} |x|^{-2-2s}\, dx \\ &= 2\pi \int_\eta^{+\infty} r^{-1-2s}\, dr = \frac{\pi}{s}\, \eta^{-2s}. \end{aligned} \tag{22}$$

[2] In further detail, recalling (4),

$$\begin{aligned} \left| \int_{\pi(e)\setminus B_1} \frac{|y'|^{n-2}\, \widetilde{\chi}_E(y)}{|y|^{n+2s}}\, dy \right| &\leqslant \int_{\pi(e)} \frac{|y|^{n-2}\, \chi_{\mathcal{C}B_1}(y)}{|y|^{n+2s}}\, dy \\ &\leqslant \int_0^{+\infty} d\rho \int_{-\infty}^{+\infty} dh\, \chi_{\mathbb{R}\setminus(-1,1)}(\rho^2+h^2)\ (\rho^2+h^2)^{-1-s} \leqslant \int_{\mathbb{R}^2\setminus B_1} dx |x|^{-2-2s} \\ &= 2\pi \int_1^{+\infty} dr\, r^{-1-2s} = \frac{\pi}{s} \end{aligned}$$

that is uniformly bounded as $s \nearrow 1/2$. This means that we can suppose that ∂E is a graph in, say, B_1 and replace it outside B_1 without affecting the statement of Theorem 12.

Now we look at the first addendum in (21): for this we write, for $\eta \in (0,1)$ sufficiently small,

$$D_e^2 f(0)\frac{\rho^2}{2} - \varepsilon(\rho) \;\leqslant\; f(\rho e) \;\leqslant\; D_e^2 f(0)\frac{\rho^2}{2} + \varepsilon(\rho), \qquad \rho \in (0,\eta) \tag{23}$$

where $\varepsilon : (0,\eta) \to (0,+\infty)$ is defined as

$$\varepsilon(\rho) := \sup_{|\xi'|\leqslant\rho} |D_e^2 f(\xi') - D_e^2 f(0)|\,\frac{\rho^2}{2}.$$

Let also

$$E(\eta) := \sup_{0<\rho\leqslant\eta} \frac{\varepsilon(\rho)}{\rho^2} = \frac{1}{2}\sup_{|\xi'|\leqslant\eta} |D_e^2 f(\xi') - D_e^2 f(0)|.$$

and observe that

$$E(\eta) \searrow 0 \;\text{ as } \eta \searrow 0. \tag{24}$$

Then, if we denote by $D := \frac{1}{2}D_e^2 f(0)$, we have that

$$\begin{aligned}
\left|\int_0^\eta d\rho\, \rho^{n-2}\int_0^{f(\rho e)} \frac{dh}{(\rho^2+h^2)^{s+n/2}} - \frac{D}{1-2s}\right| \;\leqslant\; & \left|\int_0^1 d\rho\, \rho^{n-2}\int_0^{D\rho^2} \frac{dh}{(\rho^2+h^2)^{s+n/2}} - \frac{D}{1-2s}\right| \\
& + \left|\int_0^\eta d\rho\, \rho^{n-2}\int_0^{f(\rho e)} \frac{dh}{(\rho^2+h^2)^{s+n/2}} - \int_0^\eta d\rho\, \rho^{n-2}\int_0^{D\rho^2} \frac{dh}{(\rho^2+h^2)^{s+n/2}}\right| \\
& + \left|\int_\eta^1 d\rho\, \rho^{n-2}\int_0^{D\rho^2} \frac{dh}{(\rho^2+h^2)^{s+n/2}}\right|.
\end{aligned} \tag{25}$$

The latter term is uniformly bounded as $s \nearrow 1/2$: indeed

$$\begin{aligned}
\left|\int_\eta^1 d\rho\, \rho^{n-2}\int_0^{D\rho^2} \frac{dh}{(\rho^2+h^2)^{s+n/2}}\right| &\leqslant \int_\eta^1 d\rho\, \rho^{n-2}\cdot\frac{|D|\rho^2}{\rho^{n+2s}} \;=\; \int_\eta^1 d\rho\,\frac{|D|}{\rho^{2s}} \\
&= |D|\,\frac{1-\eta^{1-2s}}{1-2s} \;\xrightarrow[s\nearrow\frac{1}{2}]{}\; -|D|\,\ln\eta,
\end{aligned} \tag{26}$$

and therefore, for s close to $1/2$,

$$\left|\int_\eta^1 d\rho\, \rho^{n-2}\int_0^{D\rho^2} \frac{dh}{(\rho^2+h^2)^{s+n/2}}\right| \;\leqslant\; -\widetilde{D}\,\ln\eta, \tag{27}$$

where $\widetilde{D}$ is a positive constant depending only on D.

The central term in (25) may be estimated using (23): indeed

$$\begin{aligned}
&\left| \int_0^\eta d\rho\, \rho^{n-2} \int_0^{f(\rho e)} \frac{dh}{(\rho^2+h^2)^{s+n/2}} - \int_0^\eta d\rho\, \rho^{n-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2+h^2)^{s+n/2}} \right| \\
\leqslant\quad & \int_0^\eta d\rho\, \rho^{n-2} \left| \int_{D\rho^2}^{f(\rho e)} \frac{dh}{(\rho^2+h^2)^{s+n/2}} \right| \quad\leqslant\quad \int_0^\eta \rho^{n-2} \frac{|f(\rho e) - D\rho^2|}{\rho^{n+2s}}\, d\rho \\
\leqslant\quad & \int_0^\eta \frac{\varepsilon(\rho)}{\rho^{2+2s}}\, d\rho \quad\leqslant\quad E(\eta) \int_0^\eta \rho^{-2s}\, d\rho \quad=\quad E(\eta)\cdot \frac{\eta^{1-2s}}{1-2s}.
\end{aligned} \tag{28}$$

It remains now to estimate the first term on the right hand side of (25). For this we apply the change of variable $t = h/\rho$ and we see that

$$\begin{aligned}
&\left| \rho^{n-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2+h^2)^{s+n/2}} - D\rho^{-2s} \right| = |D|\rho^{-2s} \left| \frac{1}{D\rho^2} \int_0^{D\rho^2} \frac{dh}{\left(1+\frac{h^2}{\rho^2}\right)^{s+n/2}} - 1 \right| \\
&= |D|\rho^{-2s} \left| \frac{1}{D\rho} \int_0^{D\rho} \frac{dt}{(1+t^2)^{s+n/2}} - 1 \right|.
\end{aligned}$$

Now, since the map $t \longmapsto 1/\left(1+t^2\right)^{s+n/2}$ is decreasing, we have that

$$\frac{1}{(1+D^2\rho^2)^{s+n/2}} \;\leqslant\; ⨍_0^{|D|\rho} \frac{dt}{(1+t^2)^{s+n/2}} \;\leqslant\; 1.$$

Using this and a Taylor expansion, we obtain

$$\begin{aligned}
&\left| \rho^{n-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2+h^2)^{s+n/2}} - D\rho^{-2s} \right| = |D|\rho^{-2s} \left(1 - \frac{1}{|D|\rho} \int_0^{|D|\rho} \frac{dt}{(1+t^2)^{s+n/2}} \right) \\
&\leqslant |D|\rho^{-2s} \left(1 - \frac{1}{(1+D^2\rho^2)^{s+n/2}} \right) \leqslant |D|\rho^{-2s} \cdot (\beta D^2\rho^2) = \beta|D|^3\rho^{2-2s}
\end{aligned}$$

for any $\rho \in (0,1)$, where β is a suitable positive constant only depending on n. By integrating this estimate over the interval $(0,1)$ we conclude that

$$\left| \int_0^1 d\rho\, \rho^{n-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2+h^2)^{s+n/2}} - \frac{D}{1-2s} \right| \;=\; \left| \int_0^1 d\rho \left[\rho^{n-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2+h^2)^{s+n/2}} - D\rho^{-2s} \right] \right| \;\leqslant\; \widetilde{\beta}, \tag{29}$$

for a suitable $\widetilde{\beta}$ possibly depending on D and n, but independent of s.

Resuming all this calculations in one formula and putting together the information in equations (21), (22), (25), (27), (28) and (29), we obtain that

$$\left| K_{s,e} - \frac{2\,D}{1-2s} \right| \;\leqslant\; \frac{2\pi}{s}\eta^{-2s} - \widetilde{D}\,\ln\eta + 2E(\eta)\cdot \frac{\eta^{1-2s}}{1-2s} + 2\widetilde{\beta}, \tag{30}$$

hence

$$\limsup_{s\nearrow\frac{1}{2}}(1-2s)\left|K_{s,e}-\frac{2\,D}{1-2s}\right| \leqslant E(\eta).$$

But, by (24), $E(\eta)$ is arbitrarily small. Hence we can conclude

$$\lim_{s\nearrow\frac{1}{2}}(1-2s)\left|K_{s,e}-\frac{2\,D}{1-2s}\right| = 0$$

which also implies

$$\lim_{s\nearrow\frac{1}{2}}(1-2s)K_{s,e} = 2\,D = D_e^2 f(0),$$

that is the desired claim. Moreover, since estimate (30) is uniform on S^{n-2}, we have also the convergence

$$(1-2s)H_s = \frac{1-2s}{\omega_{n-2}}\int_{S^{n-2}} K_{s,e}\,d\mathcal{H}^{n-2}(e) \xrightarrow[s\nearrow\frac{1}{2}]{} \frac{1}{\omega_{n-2}}\int_{S^{n-2}} D_e^2 f(0)\,d\mathcal{H}^{n-2}(e) = H.$$

This completes the proof of Theorem 12.

7 Proof of the claims in Example 13

In $\mathbb{R}^3$ define

$$E = \{(x,y,z)\in\mathbb{R}^3 : z \leqslant f(x,y) = 8x^2y^2\}$$

and denote by (ρ,θ,z) the cylindrical coordinates, i.e.

$$\rho = \sqrt{x^2+y^2}, \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}.$$

In this way, we set $e := (\cos\theta,\sin\theta)\in S^1$ and we have that $(x,y)=\rho e$. Notice that

$$1-\cos 4\theta = 1-\cos^2 2\theta+\sin^2 2\theta = 2\sin^2 2\theta = 8\sin^2\theta\cos^2\theta\cdot\frac{\rho^4}{\rho^4} = \frac{8x^2y^2}{(x^2+y^2)^2}$$

therefore the function f, written in terms of (ρ,θ), becomes

$$f(x,y) = f(\rho e) = \widetilde{f}(\rho,\theta) = (1-\cos 4\theta)\rho^4.$$

With a slight abuse of notation, we denote by $K_{s,\theta}$ the nonlocal directional curvature of E at 0 in direction $e=(\cos\theta,\sin\theta)$ (i.e. $K_{s,\theta}$ is short for $K_{s,(\cos\theta,\sin\theta)}$). We recall that, in this case, we can calculate nonlocal curvatures exploiting identity (11), i.e.

$$K_{s,\theta} = 2\int_0^{+\infty} d\rho \int_0^{\widetilde{f}(\rho,\theta)} dz\,\frac{\rho}{(\rho^2+z^2)^{s+3/2}}. \tag{31}$$

In the above domain of integration it holds that

$$0\leqslant z\leqslant \widetilde{f}(\rho,\theta) = (1-\cos 4\theta)\rho^4$$

and so

$$\rho \geqslant \sqrt[4]{\frac{z}{1-\cos 4\theta}}.$$

Therefore, integrating first in the ρ variable, we deduce from (31) that

$$\begin{aligned} K_{s,\theta} &= \int_0^{+\infty} dz \int_{\sqrt[4]{\frac{z}{1-\cos 4\theta}}}^{+\infty} d\rho \, \frac{2\rho}{(\rho^2+z^2)^{s+3/2}} = \int_0^{+\infty} dz \, \frac{\left(\rho^2+z^2\right)^{-s-1/2}}{-s-\frac{1}{2}} \Bigg|_{\rho=\sqrt[4]{\frac{z}{1-\cos 4\theta}}}^{\rho=+\infty} \\ &= \frac{2}{2s+1} \int_0^{+\infty} \frac{dz}{\left(z^2+\sqrt{\frac{z}{1-\cos 4\theta}}\right)^{s+1/2}} = \frac{2}{2s+1} \int_0^{+\infty} \frac{(1-\cos 4\theta)^{s/2+1/4}}{\left(z^2\sqrt{1-\cos 4\theta}+\sqrt{z}\right)^{s+1/2}} \, dz. \end{aligned} \tag{32}$$

We concentrate now on maximal and minimal nonlocal directional curvatures. Since $K_{s,\theta}$ is a nonnegative quantity (because $\widetilde{f}(\rho,\theta)$ is a nonnegative function) and since $K_{s,0} = K_{s,\pi/2} = 0$, then $K_{s,\theta}$ attains its minimum in 0 and $\pi/2$. Also, $\widetilde{f}(\rho,\pi/4) \geqslant \widetilde{f}(\rho,\theta)$ for every positive ρ and $\theta \in [0,\pi)$, thus $K_{s,\theta}$ attains its maximum for $\theta = \pi/4$. On the one hand we have that the arithmetic mean of the maximal and minimal nonlocal principal curvatures is given by

$$\frac{K_{s,0}+K_{s,\pi/4}}{2} = \frac{1}{2} K_{s,\pi/4} = \frac{2^{s/2+1/4}}{2s+1} \int_0^{+\infty} \frac{dz}{\left(\sqrt{2}z^2+\sqrt{z}\right)^{s+1/2}}, \tag{33}$$

thanks to (32).

On the other hand, the nonlocal mean curvature is

$$H_s = \frac{1}{2\pi} \int_0^{2\pi} K_{s,\theta} \, d\theta$$

and we are going to estimate this quantity, in order to show that it is not equal to the arithmetic mean of the principal nonlocal curvatures (i.e. to the quantity in (33)). Note that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} K_{s,\theta} \, d\theta &= \frac{1}{(2s+1)\pi} \int_0^{2\pi} d\theta \int_0^{+\infty} dz \, \frac{(1-\cos 4\theta)^{s/2+1/4}}{\left(z^2\sqrt{1-\cos 4\theta}+\sqrt{z}\right)^{s+1/2}} \\ &\geqslant \frac{1}{(2s+1)\pi} \int_0^{2\pi} (1-\cos 4\theta)^{s/2+1/4} d\theta \cdot \int_0^{+\infty} \frac{dz}{\left(\sqrt{2}z^2+\sqrt{z}\right)^{s+1/2}} \\ &= \frac{1}{\pi \, 2^{s/2+1/4}} \int_0^{2\pi} (1-\cos 4\theta)^{s/2+1/4} d\theta \cdot \underbrace{\frac{2^{s/2+1/4}}{2s+1} \int_0^{+\infty} \frac{dz}{\left(\sqrt{2}z^2+\sqrt{z}\right)^{s+1/2}}}_{=K_{s,\pi/4}/2} \\ &= \frac{1}{\pi \, 2^{s/2+1/4}} \int_0^{2\pi} (1-\cos 4\theta)^{s/2+1/4} d\theta \cdot \frac{1}{2} K_{s,\pi/4}. \end{aligned} \tag{34}$$

On the other hand, with the substitution $\varphi := 4\theta$ and recalling that $s + 1/2 \in (0, 1)$, we see that

$$\begin{aligned}\frac{1}{\pi\, 2^{s/2+1/4}} \int_0^{2\pi} (1-\cos 4\theta)^{s/2+1/4} d\theta \quad &= \quad \frac{1}{4\pi}\int_0^{8\pi} \left(\sqrt{\frac{1-\cos\varphi}{2}}\right)^{s+1/2} d\varphi \quad = \quad \frac{1}{\pi}\int_0^{2\pi} \left(\sin\frac{\varphi}{2}\right)^{s+1/2} d\varphi \\ &\geqslant \quad \frac{1}{\pi}\int_0^{2\pi} \sin\frac{\varphi}{2}\, d\varphi \quad = \quad -\frac{2}{\pi}\cos\frac{\varphi}{2}\Big|_0^{2\pi} \quad = \quad \frac{4}{\pi} \quad > \quad 1.\end{aligned}$$

By combining this with (33) and (34) we conclude that

$$H_s \ \gneqq \ \frac{1}{2} K_{s,\pi/4} = \frac{K_{s,0} + K_{s,\pi/4}}{2}.$$

This establishes the claims in Example 13.

8 Proof of Lemma 7

We prove (10) since the one of (9) is alike. Let

$$\sigma_\varepsilon := \int_{\pi(e)\setminus B_\varepsilon} \frac{|y'|^{n-2}\widetilde{\chi}_E(y)}{|y|^{n+2s}}\, dy.$$

Notice that, for any $\varepsilon > \varepsilon' > 0$,

$$|\sigma_{\varepsilon'} - \sigma_\varepsilon| = \left| \int_{\pi(e)\cap(B_\varepsilon\setminus B_{\varepsilon'})} \frac{|y'|^{n-2}\widetilde{\chi}_E(y)}{|y|^{n+2s}}\, dy \right| \tag{35}$$

Since ∂E is C^2, in normal coordinates we may suppose that, for small $\varepsilon > 0$,

$$E \cap B_\varepsilon \subseteq \{x = (x', x_n) \in \mathbb{R}^n \ : \ x_n \leqslant M|x'|^2\} \ \text{ and } \ (\mathcal{C}E) \cap B_\varepsilon \subseteq \{x \in \mathbb{R}^n \ : \ x_n \geqslant -M|x'|^2\},$$

for a suitable $M > 0$. This provides a cancellation of the contributions outside the set $E_{\varepsilon,\varepsilon'} := \{x \in B_\varepsilon \setminus B_{\varepsilon'} : x_n \leqslant M|x'|^2\}$, namely

$$\begin{aligned}\left| \int_{\pi(e)\cap(B_\varepsilon\setminus B_{\varepsilon'})} \frac{|y'|^{n-2}\widetilde{\chi}_E(y)}{|y|^{n+2s}}\, dy \right| &= \left| \int_{\pi(e)\cap E_{\varepsilon,\varepsilon'}} \frac{|y'|^{n-2}\widetilde{\chi}_E(y)}{|y|^{n+2s}}\, dy \right| \\ &\leqslant \int_{\pi(e)\cap E_{\varepsilon,\varepsilon'}} \frac{|y'|^{n-2}}{|y|^{n+2s}}\, dy \leqslant \int_{\pi(e)} \frac{\chi_{E_{\varepsilon,\varepsilon'}}(y)}{|y'|^{2+2s}}\, dy,\end{aligned}$$

since $|y| \geqslant |y'|$. That is, by (4),

$$\left| \int_{\pi(e)\cap(B_\varepsilon\setminus B'_\varepsilon)} \frac{|y'|^{n-2}\widetilde{\chi}_E(y)}{|y|^{n+2s}}\, dy \right| \leqslant \int_{\varepsilon'}^{\varepsilon} d\rho \int_{-M\rho^2}^{M\rho^2} dh\ \rho^{-2-2s} \leqslant 2M \int_{\varepsilon'}^{\varepsilon} d\rho\ \rho^{-2s} = \frac{2M(\varepsilon - \varepsilon')}{1-2s}$$

which is infinitesimal with ε (recall that $s \in (0, 1/2)$ by assumption). This and (35) imply that for any $\varepsilon_n \searrow 0$, σ_{ε_n} is a Cauchy sequence, as desired.

Acknowledgements

EV is supported by the ERC Grant 277749 Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities (EPSILON). It is a pleasure to thank Serena Dipierro for extensive and valuable discussions.

References

[AdPM11] L. Ambrosio, G. De Philippis and L. Martinazzi. Γ-convergence of nonlocal perimeter functionals. *Manuscripta Math.*, 134(3-4):377–403, 2011.

[BFV] B. Barrios, A. Figalli and E. Valdinoci. Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5).*

[CG] M.C. Caputo and N. Guillen. Regularity for non-local almost minimal boundaries and applications. *http://arxiv.org/abs/1003.2470.*

[CRS10] L. Caffarelli, J.-M. Roquejoffre and O. Savin. Nonlocal minimal surfaces. *Comm. Pure Appl. Math.*, 63(9):1111–1144, 2010.

[CS10] L. Caffarelli and P. Souganidis. Convergence of nonlocal threshold dynamics approximations to front propagation. *Arch. Ration. Mech. Anal.*, 195(1):1–23, 2010.

[CV] L. Caffarelli and E. Valdinoci. Regularity properties of nonlocal minimal surfaces via limiting arguments. *http://arxiv.org/abs/1105.1158.*

[CV11] L. Caffarelli and E. Valdinoci. Uniform estimates and limiting arguments for nonlocal minimal surfaces. *Calc. Var. Partial Differential Equations*, 41(1-2):203–240, 2011.

[DFPV] S. Dipierro, A. Figalli, G. Palatucci and E. Valdinoci. Asymptotics of the s-perimeter as $s \searrow 0$. *http://arxiv.org/abs/1204.0750.*

[DNPV12] E. Di Nezza, G. Palatucci and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[Eva93] L.C. Evans. Convergence of an algorithm for mean curvature motion. *Indiana Univ. Math. J.*, 42(2):533–557, 1993.

[FMM] N. Fusco, V. Millot and M. Morini. A quantitative isoperimetric inequality for fractional perimeters. *http://it.similarsites.com/goto/cvgmt.sns.it.*

[Giu84] E. Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[Imb09] C. Imbert. Level set approach for fractional mean curvature flows. *Interfaces Free Bound.*, 11(1):153–176, 2009.

[Ish95] H. Ishii. A generalization of the Bence, Merriman and Osher algorithm for motion by mean curvature. In *Curvature flows and related topics (Levico, 1994)*, volume 5 of *GAKUTO Internat. Ser. Math. Sci. Appl.*, pages 111–127. Gakkōtosho, Tokyo, 1995.

[KP99] S.G. Krantz and H.R. Parks. *The geometry of domains in space*, in *Birkhäuser Advanced Texts: Basler Lehrbücher*, Birkhäuser Boston, Boston, MA, 1999.

[MBO92] B. Merriman, J.K. Bence and S. Osher. *Diffusion Generated Motion by Mean Curvature.* CAM report. Department of Mathematics, University of California, Los Angeles, 1992.

[SV] O. Savin and E. Valdinoci. Regularity of nonlocal minimal cones in dimension 2. *Calc. Var. Partial Differential Equations.*

[Val] E. Valdinoci. A fractional framework for perimeters and phase transitions. *http://arxiv.org/abs/1210.5612.*