A BIFURCATION RESULT FOR NON-LOCAL FRACTIONAL EQUATIONS

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Abstract. In the present paper we consider problems modeled by the following non-local fractional equation

$$\begin{cases} (-\Delta)^s u - \lambda u = \mu f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

where $s \in (0,1)$ is fixed, $(-\Delta)^s$ is the fractional Laplace operator, λ and μ are real parameters, Ω is an open bounded subset of $\mathbb{R}^n$, $n > 2s$, with Lipschitz boundary and f is a function satisfying suitable regularity and growth conditions. A critical point result for differentiable functionals is exploited, in order to prove that the problem admits at least one non-trivial and non-negative (non-positive) solution, provided the parameters λ and μ lie in a suitable range.

The existence result obtained in the present paper may be seen as a bifurcation theorem, which extends some results, well known in the classical Laplace setting, to the non-local fractional framework.

Contents

1. Introduction

Fractional and non-local operators appear in concrete applications in many fields such as, among the others, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves. This is one of the reason why, recently, non-local fractional problems are widely studied in the literature.

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In [4, 7, 8, 11, 19, 20, 21, 23, 24, 25, 26, 27, 30] (see also the references therein) the authors studied non-local fractional Laplacian equations with superlinear and subcritical or critical nonlinearities, while in [12, 13] the asymptotically linear case was exploited.

There are a lot interesting problems in the standard framework of the Laplacian (and, more generally, of uniformly elliptic operators), widely studied in the literature. A natural question is whether or not the existence results got in this classical context can be extended to the non-local framework of the fractional Laplacian type operators.

In particular in this paper we are interested in equations depending on parameters. In many mathematical problems deriving from applications the presence of one (or more) parameter is a relevant feature, and the study of how solutions depend on parameters is an important topic. Most of the results in this direction were obtained through bifurcation theory (for an extensive treatment of this matters we refer to [1, 2] and their bibliography). However, some interesting results can be obtained also by means of variational techniques (for this see [3] and the references therein), or as a combination of the two methods.

Aim of this paper is to study the existence of solutions for the following general non-local equation depending on two real parameters

$$\begin{cases} \mathcal{L}_K u + \lambda u + \mu f(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.1}$$

using variational techniques.

Here Ω is an open bounded subset of $\mathbb{R}^n$, $n > 2s$ ($s \in (0,1)$), with smooth boundary $\partial\Omega$, while $\mathcal{L}_K$ is the integrodifferential operator defined as follows

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \Big(u(x+y) + u(x-y) - 2u(x)\Big) K(y)\, dy\,, \quad x \in \mathbb{R}^n\,, \tag{1.2}$$

with the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ such that

$$mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\{|x|^2, 1\}\,; \tag{1.3}$$

$$\text{there exists } \theta > 0 \text{ such that } K(x) \geqslant \theta |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,; \tag{1.4}$$

$$K(x) = K(-x) \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,. \tag{1.5}$$

A model for K is given by the singular kernel $K(x) = |x|^{-(n+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$, which, up to normalization factors, may be defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy\,, \quad x \in \mathbb{R}^n\,. \tag{1.6}$$

The homogeneous Dirichlet datum in (1.1) is given in $\mathbb{R}^n \setminus \Omega$ and not simply on the boundary $\partial\Omega$, as it happens in the classical case of the Laplacian, consistently with the non-local nature of the operator $\mathcal{L}_K$.

Along this paper the nonlinearity in (1.1) is a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ verifying the following growth condition:

$$\begin{gathered} \text{there exist } a_1, a_2 \geqslant 0 \text{ and } q \in (1, 2^*), 2^* := 2n/(n-2s)\,, \text{ such that} \\ |f(x,t)| \leqslant a_1 + a_2 |t|^{q-1} \text{ a.e. } x \in \Omega,\, t \in \mathbb{R}\,. \end{gathered} \tag{1.7}$$

Assumption (1.7) says that f has a subcritical growth. Here the exponent 2^* is the fractional critical Sobolev exponent. Notice that when $s = 1$ the above exponent reduces to the classical critical Sobolev exponent $2_* := 2n/(n-2)$.

The aim of this paper is to prove the existence of a non-trivial weak solution for problem (1.1). With this respect, we would like to note that the trivial function $u \equiv 0$ in $\mathbb{R}^n$ is a solution of problem (1.1) if and only if $f(\cdot, 0) = 0$. Hence, in order to get our goal, in the case when $f(\cdot, 0) = 0$ we need some extra assumptions on f. Precisely, in this setting we assume the following condition, which is a sort of subquadratical growth assumption at zero:

$$
\begin{gathered}
\text{there exist a non-empty open set } D \subseteq \Omega \text{ and a set } B \subseteq D \\
\text{of positive Lebesgue measure such that} \\
\limsup_{t\to 0^+} \frac{\operatorname{essinf}_{x\in B} F(x,t)}{t^2} = +\infty \quad \text{and} \quad \liminf_{t\to 0^+} \frac{\operatorname{essinf}_{x\in D} F(x,t)}{t^2} > -\infty\,.
\end{gathered} \tag{1.8}
$$

Here F is the primitive of the nonlinearity f with respect to the second variable, i.e.

$$
F(x,t) := \int_0^t f(x,\tau)d\tau\,, \tag{1.9}
$$

for a.e. $x \in \Omega$ and any $t \in \mathbb{R}$.

As a model for f we can take the function $f(x,t) := a(x)|t|^{r-2}t + b(x)|t|^{q-2}t + c(x)$, with $1 < r < 2 \leqslant q < 2^*$ and $a, b, c \in L^\infty(\Omega)$. If $c \equiv 0$ a.e. in Ω, we assume also that $\operatorname{essinf}_{x\in\Omega} a(x) > 0$.

Equation (1.1) in the model case $\mathcal{L}_K = -(-\Delta)^s$ becomes

$$
\begin{cases} (-\Delta)^s u - \lambda u = \mu f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.10}
$$

which is the counterpart of the following Laplace equation

$$
\begin{cases} -\Delta u - \lambda u = \mu f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega\,. \end{cases} \tag{1.11}
$$

Problem (1.11) was widely studied in the literature. For instance, we refer to [15, 16], where nonlinear differential equations driven by general uniformly elliptic operators (and not only by the Laplacian) are considered under certain assumptions about the nonlinear term f.

In this paper we will prove the existence of non-trivial weak solutions of problem (1.1) using variational and topological methods. By a weak solutions of (1.1) we mean a solution of the following problem

$$
\begin{cases} \displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx \\ \qquad\qquad\qquad\qquad\displaystyle = \mu\int_\Omega f(x,u(x))\varphi(x)dx \quad \forall\, \varphi \in X_0 \\ u \in X_0\,. \end{cases} \tag{1.12}
$$

Problem (1.12) represents the weak formulation of (1.1). Note that, in order to write such a weak formulation, we need to assume (1.5).

The space X_0 in which we set problem (1.12) is a functional space, inspired by, but not equivalent to, the usual fractional Sobolev space. This new space was introduced in [22] (see also [23]). The choice of this space is motivated by the fact that it allows us to correctly encode the Dirichlet boundary datum in the weak formulation. We will recall its definition in Section 2, in order to make the present paper self-contained.

1.1. **Main theorems.** In this paper we will show the existence of a non-trivial non-negative (non-positive) weak solutions for problem (1.1), namely non-trivial non-negative (non-positive) solution of (1.12), using variational and topological methods.

To this purpose we will consider the Euler–Lagrange functional $\mathcal{J}_{K,\lambda,\mu}: X_0 \to \mathbb{R}$ associated with problem (1.12), given by

$$\begin{aligned}\mathcal{J}_{K,\lambda,\mu}(u) := \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}|u(x)-u(y)|^2K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx \\ - \mu\int_\Omega F(x,u(x))dx\,,\end{aligned} \tag{1.13}$$

where F is the function defined in (1.9).

It is well known that $\mathcal{J}_{K,\lambda,\mu}$ is Frechét differentiable in X_0 and that its critical points are solutions of problem (1.12). Then, in order to get our goal, we will look for non-trivial critical points of this functional in the space X_0. A similar variational approach has been used in [9], where the authors studied the existence of a non-trivial solution for singular elliptic equations through the Caffarelli-Kohn-Nirenberg inequality.

The main result of the present paper can be stated as follows:

Theorem 1. *Let $s \in (0,1)$, $n > 2s$, Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and let λ_1 be the first eigenvalue of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data, where $K : \mathbb{R}^n\setminus\{0\} \to (0,+\infty)$ is a function satisfying conditions* (1.3)–(1.5)*. Further, let $f : \Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function verifying* (1.7)*. In addition, if $f(x,0) = 0$ for a.e. $x \in \Omega$, assume also* (1.8)*.*

Then, for any $\lambda < \lambda_1$ there exists $\mu_\lambda > 0$, depending on λ, such that, for any $\mu \in (0,\mu_\lambda)$ problem (1.1) *admits at least one non-trivial weak solution $u_\mu \in X_0$. Also $\mu_\lambda = +\infty$, provided $q \in (1,2)$.*

Moreover,

$$\lim_{\mu\to 0^+}\|u_\mu\|_{X_0} = 0$$

and the function

$$\mu \mapsto \mathcal{J}_{K,\lambda,\mu}(u_\mu)$$

is negative and strictly decreasing in $(0,\mu_\lambda)$.

Actually, using a truncation argument, we can prove that problem (1.1) admits a non-trivial non-negative (non-positive) weak solution, provided $f(\cdot,0) = 0$ (see Corollary 3 and Subsection 4.1 for more details). In general, when $f(\cdot,0) \neq 0$, problem (1.1) admits changing-sign solutions, as it happens if we look at the classical case of the Laplacian.

As an application of Theorem 1, we can consider the following model problem

$$\begin{cases} (-\Delta)^s u - \lambda u = \mu\big(a(x)|u|^{r-2}u + b(x)|u|^{q-2}u + c(x)\big) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n\setminus\Omega\,. \end{cases} \tag{1.14}$$

In this framework, Theorem 1 (here we take into account also the result stated in Corollary 3) reduces to the following result:

Theorem 2. *Let $s \in (0,1)$, $n > 2s$, Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and let $\lambda_{1,s}$ be the first eigenvalue of $(-\Delta)^s$ with homogeneous Dirichlet boundary data. Furthermore, assume that $1 < r < 2 \leqslant q < 2^*$ and $a,b,c : \Omega \to \mathbb{R}$ are functions such that $a,b,c \in L^\infty(\Omega)$. In addition, if $c \equiv 0$ a.e. in Ω, assume that* $\operatorname{essinf}_{x\in\Omega} a(x) > 0$*.*

Then, for any $\lambda < \lambda_{1,s}$ there exists $\mu_\lambda > 0$ such that for any $\mu \in (0,\mu_\lambda)$ problem (1.14) *admits at least one non-trivial weak solution $u_\mu \in H^s(\mathbb{R}^n)$ such that $u_\mu = 0$ a.e. in $\mathbb{R}^n\setminus\Omega$ and*

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u_\mu(x)-u_\mu(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \to 0$$

as $\mu \to 0^+$. Also $\mu_\lambda = +\infty$, provided $b \equiv 0$ a.e. in Ω.

Furthermore, if $c \equiv 0$ a.e. in Ω, the solution u_μ is non-negative in $\mathbb{R}^n$.

The proof of Theorem 1 is based on variational and topological techniques. Precisely, in the sequel we will perform the variational principle of Ricceri contained in [17] (see also [5, Theorem 2.1; part a)]). For several related topics and a careful analysis of the abstract framework we refer to the recent monograph [14].

Thanks to assumption (1.7), we will prove the existence of a weak solution u_μ for problem (1.1), provided $\lambda < \lambda_1$ and μ is sufficiently small. If $f(\cdot, 0) \neq 0$, the trivial function does not solve equation (1.1) and so $u_\mu \not\equiv 0$ in the space X_0. Otherwise, if $f(\cdot, 0) = 0$, the proof of the fact that u_μ is not the trivial function is more difficult and it relies on the subquadratical growth assumption (1.8), which will be crucial for our argument.

Theorem 1 may be seen as a bifurcation result for problem (1.1). For more details on this we refer to Subsection 4.2.

Moreover, Theorem 1 and Theorem 2 extend the existence results, well known in the classical context of (1.11), to the non-local setting. In particular, Theorem 1 represents the non-local counterpart of [18, Theorem 1].

The paper is organized as follows. In Section 2 we recall the definition of the functional space we work in and we give some notations. In Section 3 we prove Theorem 1, while Section 4 is devoted to some comments on the results of the paper. In particular, in Corollary 3 we prove that, under the condition that $f(\cdot, 0) = 0$, the solution given by Theorem 1 has constant sign, i.e. Theorem 1 provides non-negative (non-positive) solutions. Finally, in Section 5 we give an application of Theorem 1, studying in particular a non-local equation driven by the fractional Laplacian. Here we provide the proof of Theorem 2.

2. Some preliminaries

This section is devoted to the notations used along the paper. We also give some preliminary results which will be useful in the sequel.

2.1. The functional space X_0.

In this subsection we briefly recall the definition of the functional space X_0, firstly introduced in [22], and we give some notations. The reader familiar with this topic may skip this section and go directly to the next one.

The functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x,y) \mapsto (g(x) - g(y))\sqrt{K(x-y)} \text{ is in } L^2\big((\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy\big),$$

(here $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$). Also, we denote by X_0 the following linear subspace of X

$$X_0 := \big\{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\big\}.$$

We remark that X and X_0 are non-empty, since $C_0^2(\Omega) \subseteq X_0$ by [22, Lemma 11]. Moreover, the space X is endowed with the norm defined as

$$\|g\|_X := \|g\|_{L^2(\Omega)} + \Big(\int_Q |g(x) - g(y)|^2 K(x-y)dx\,dy\Big)^{1/2}, \tag{2.1}$$

where $Q = (\mathbb{R}^n \times \mathbb{R}^n) \setminus \mathcal{O}$ and $\mathcal{O} = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^n \times \mathbb{R}^n$. It is easily seen that $\|\cdot\|_X$ is a norm on X (see, for instance, [23] for a proof).

By [23, Lemmas 6 and 7] in the sequel we can take the function

$$X_0 \ni v \mapsto \|v\|_{X_0} = \Big(\int_Q |v(x) - v(y)|^2 K(x-y)\,dx\,dy\Big)^{1/2} \tag{2.2}$$

as norm on X_0. Also $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space (for this see [23, Lemmas 7]), with scalar product

$$\langle u, v\rangle_{X_0} := \int_Q \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\, K(x-y)\,dx\,dy. \tag{2.3}$$

Note that in (2.2) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^n \times \mathbb{R}^n$, since $v \in X_0$ (and so $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$).

In what follows, we denote by λ_1 the first eigenvalue of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data, namely the first eigenvalue of the problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases}$$

For the existence and the basic properties of this eigenvalue we refer to [24, Proposition 9 and Appendix A], where a spectral theory for general integrodifferential non-local operators was developed. Further properties can be also found in [19, 26, 28].

When $\lambda < \lambda_1$ we can take as a norm on X_0 the function

$$X_0 \ni v \mapsto \|v\|_{X_0,\lambda} = \Big(\int_Q |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda \int_\Omega |v(x)|^2\,dx\Big)^{1/2}\,, \tag{2.4}$$

since for any $v \in X_0$ it holds true (for this see [24, Lemma 10])

$$m_\lambda \|v\|_{X_0} \leqslant \|v\|_{X_0,\lambda} \leqslant M_\lambda \|v\|_{X_0}\,, \tag{2.5}$$

where

$$m_\lambda := \min\Big\{\sqrt{\frac{\lambda_1-\lambda}{\lambda_1}},\,1\Big\} \quad \text{and} \quad M_\lambda := \max\Big\{\sqrt{\frac{\lambda_1-\lambda}{\lambda_1}},\,1\Big\}\,.$$

Note that (2.5) is a consequence of the variational characterization of λ_1 given in [24, Proposition 9] and of the choice of λ. Of course, also in (2.4) the integral on Q can be replaced by the integral in all $\mathbb{R}^n \times \mathbb{R}^n$.

While for a general kernel K satisfying conditions (1.3)–(1.5) we have that $X_0 \subset H^s(\mathbb{R}^n)$, in the model case $K(x) = |x|^{-(n+2s)}$ the space X_0 consists of all the functions of the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ which vanish a.e. outside Ω (see [25, Lemma 7]).

Here $H^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \Big(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}. \tag{2.6}$$

Before concluding this subsection, we recall the embedding properties of X_0 into the usual Lebesgue spaces (see [23, Lemma 8]). The embedding $j : X_0 \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [1, 2^*]$, while it is compact whenever $\nu \in [1, 2^*)$. Hence, for any $\nu \in [1, 2^*]$ there exists a positive constant c_ν such that

$$\|v\|_{L^\nu(\mathbb{R}^n)} \leqslant c_\nu \|v\|_{X_0} \leqslant c_\nu m_\lambda^{-1} \|v\|_{X_0,\lambda} \quad \text{for any } v \in X_0\,. \tag{2.7}$$

In our setting we used also the fact that the norms defined in (2.2) and (2.4) are equivalent, as stated in (2.5).

For further details on the fractional Sobolev spaces we refer to [10] and to the references therein, while for other details on X and X_0 we refer to [22], where these functional spaces were introduced, and also to [19, 20, 23, 24, 25, 26, 27], where various properties of these spaces were proved.

2.2. **A critical points result for differentiable functionals.** In order to prove our main result, stated in Theorem 1, in the following we will perform the variational principle of Ricceri (see the quoted paper [17]) in the form given in [5, Theorem 2.1; part *a*)]. For the sake of clarity, we recall it here below:

Theorem. ([5, Theorem 2.1; part *a*)]) *Let Y be a reflexive real Banach space, let $\Phi, \Psi : Y \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive in Y and Ψ is sequentially weakly upper semicontinuous in Y. Let J_μ be the functional defined as $J_\mu := \Phi - \mu\Psi$, $\mu \in \mathbb{R}$, and for any*

$r > \inf_Y \Phi$ *let* φ *be the function defined as*

$$\varphi(r) := \inf_{u\in\Phi^{-1}\big((-\infty,r)\big)} \frac{\displaystyle\sup_{v\in\Phi^{-1}\big((-\infty,r)\big)} \Psi(v)-\Psi(u)}{r-\Phi(u)}\,.$$

Then, for any $r > \inf_Y \Phi$ *and any* $\mu \in (0, 1/\varphi(r))$[1], *the restriction of the functional* J_μ *to* $\Phi^{-1}\big((-\infty,r)\big)$ *admits a global minimum, which is a critical point (precisely a local minimum) of* J_μ *in* Y.

3. The existence of a non-trivial weak solution

This section is devoted to the proof of the main results of the present paper, that is Theorem 1.

Our approach will be variational and will consist in looking for critical points of the functional $\mathcal{J}_{K,\lambda,\mu}$ naturally associated with problem (1.12) (see formula (1.13)).

First of all, we would like to note that, in general, $\mathcal{J}_{K,\lambda,\mu}$ can be unbounded from below in X_0. Indeed, for instance, in the case when $f(x,t) := 1+|t|^{q-2}t$ with $q\in(2,2^*)$, for any fixed $u\in X_0\setminus\{0\}$ we get

$$\begin{aligned}\mathcal{J}_{K,\lambda,\mu}(tu) &= \frac{t^2}{2}\|u\|^2_{X_0,\lambda} - \mu\int_\Omega F(x,tu(x))\,dx\\ &= \frac{t^2}{2}\|u\|^2_{X_0,\lambda} - \mu t\|u\|_{L^1(\Omega)} - \frac{\mu t^q}{q}\|u\|^q_{L^q(\Omega)} \to -\infty\end{aligned}$$

as $t\to+\infty$.

Hence, in order to find critical points of $\mathcal{J}_{K,\lambda,\mu}$ we can not argue, in general, by direct minimization. For this reason, along the present paper, we will perform a critical point theorem. Precisely, we will apply [5, Theorem 2.1; part *a*)] (recalled in Subsection 2.2) to the Euler–Lagrange functional $\mathcal{J}_{K,\lambda,\mu}$ associated with problem (1.12) .

We would like to note that the subcritical assumption (1.7) will be crucial in order to prove the existence of a weak solution u_μ for problem (1.1), provided μ is sufficiently small. While the subquadratical growth condition (1.8) will be useful, in the case when $f(\cdot,0)=0$, in order to show that u_μ is not the trivial function. Notice that, if $f(\cdot,0)\neq 0$, the trivial function does not solve equation (1.1) and so, obviously, $u_\mu\not\equiv 0$.

Finally, we would like to note that, obviously, condition (1.8) is satisfied if the following stronger assumption holds true:

$$\begin{aligned}&\text{there exists a non-empty open set } B\subseteq\Omega\\ &\text{of positive Lebesgue measure such that}\\ &\lim_{t\to 0^+}\frac{\operatorname{essinf}_{x\in B}F(x,t)}{t^2} = +\infty\,,\end{aligned} \tag{3.1}$$

where F, as usual, is the function defined in (1.9).

3.1. **Proof of Theorem 1.** The idea of the proof consists in applying [5, Theorem 2.1; part *a*)]) (recalled in Subsection 2.2) to the functional $\mathcal{J}_{K,\lambda,\mu}$.

To this purpose, we write the functional $\mathcal{J}_{K,\lambda,\mu}$ as follows:

$$\mathcal{J}_{K,\lambda,\mu}(u) = \Phi_{K,\lambda}(u) - \mu\Psi(u), \quad u\in X_0,$$

with

$$\Phi_{K,\lambda}(u) := \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}|u(x)-u(y)|^2K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx\,,$$

---

[1] Note that, by definition, $\varphi(r)\geqslant 0$ for any $r>\inf_Y\Phi$. Here and in the following, if $\varphi(r)=0$, by $1/\varphi(r)$ we mean $+\infty$, i.e. we set $1/\varphi(r)=+\infty$.

as well as

$$\Psi(u) := \int_\Omega F(x, u(x))\, dx\,.$$

First of all, note that X_0 is a Hilbert space (see [23, Lemmas 7]) and the functionals $\Phi_{K,\lambda}$ and Ψ are Frechét differentiable in X_0.

Also, note that the map

$$u \mapsto \|u\|^2_{X_0,\lambda}$$

is lower semicontinuous in the weak topology of X_0, so that the functional $\Phi_{K,\lambda}$ is lower semicontinuous in the weak topology of X_0.

Moreover, the application

$$u \mapsto \int_\Omega F(x, u(x))dx$$

is continuous in the weak topology of X_0. Indeed, if $\{u_j\}_{j\in\mathbb{N}}$ is a sequence in X_0 such that $u_j \rightharpoonup u$ weakly in X_0, then, by (2.7) and [6, Theorem IV.9], up to a subsequence, u_j converges to u strongly in $L^\nu(\Omega)$ and a.e. in Ω as $j \to +\infty$, and it is dominated by some function $\kappa_\nu \in L^\nu(\Omega)$, i.e.

$$|u_j(x)| \leqslant \kappa_\nu(x) \quad \text{a.e. } x \in \Omega \ \text{ for any } j \in \mathbb{N} \tag{3.2}$$

for any $\nu \in [1, 2^*)$.

Then, by the continuity of F and (1.7) it follows that

$$F(x, u_j(x)) \to F(x, u(x)) \ \text{ a.e. } x \in \Omega$$

as $j \to \infty$ and

$$|F(x, u_j(x))| \leqslant a_1|u_j(x)| + \frac{a_2}{q}|u_j(x)|^q \leqslant a_1\kappa_1(x) + \frac{a_2}{q}\,(\kappa_q(x))^q \in L^1(\Omega)$$

a.e. $x \in \Omega$ and for any $j \in \mathbb{N}$. Hence, by applying the Lebesgue Dominated Convergence Theorem in $L^1(\Omega)$, we have that

$$\int_\Omega F(x, u_j(x))\, dx \to \int_\Omega F(x, u(x))\, dx$$

as $j \to \infty$, that is the map

$$u \mapsto \int_\Omega F(x, u(x))dx$$

is continuous from X_0 with the weak topology to $\mathbb{R}$. Thus, the functional Ψ is continuous from X_0 with the weak topology to $\mathbb{R}$.

Hence, we have shown that the functionals $\Phi_{K,\lambda}$ and Ψ have the regularity required by [5, Theorem 2.1; part *a*)] (see Subsection 2.2).

Now, let $\lambda < \lambda_1$. By (2.4) for any $u \in X_0$

$$\Phi_{K,\lambda}(u) = \frac{1}{2}\|u\|^2_{X_0,\lambda}\,, \tag{3.3}$$

so that the functional $\Phi_{K,\lambda}$ is coercive in X_0 and $\inf\limits_{u\in X_0} \Phi_{K,\lambda}(u) = 0$.

Now, let $r > 0$ and let $\varphi_{K,\lambda}$ be the function defined as follows

$$\varphi_{K,\lambda}(r) := \inf_{u\in\Phi^{-1}_{K,\lambda}\big((-\infty,r)\big)} \frac{\displaystyle\sup_{v\in\Phi^{-1}_{K,\lambda}\big((-\infty,r)\big)} \Psi(v) - \Psi(u)}{r - \Phi_{K,\lambda}(u)}\,. \tag{3.4}$$

It is easy to see that $\varphi_{K,\lambda}(r) \geqslant 0$ for any $r > 0$.

Then, by [5, Theorem 2.1; part a)],

$$\begin{gathered}\text{for any } r>0 \text{ and any } \mu\in \Big(0, 1/\varphi_{K,\lambda}(r)\Big) \text{ the restriction}\\ \text{of } \mathcal{J}_{K,\lambda,\mu} \text{ to } \Phi_{K,\lambda}^{-1}\big((-\infty,r)\big) \text{ admits a global minimum } u_{\mu,r},\\ \text{which is a critical point (namely a local minimum) of } \mathcal{J}_{K,\lambda,\mu} \text{ in } X_0\,.\end{gathered} \tag{3.5}$$

Remember that, when $\varphi_{K,\lambda}(r)=0$, by $1/\varphi_{K,\lambda}(r)$ we mean $+\infty$.

Let μ_λ be defined as follows

$$\mu_\lambda := \sup_{r>0} \frac{1}{\varphi_{K,\lambda}(r)}\,.$$

Note that $\mu_\lambda>0$, since $\varphi_{K,\lambda}(r)\geqslant 0$ for any $r>0$.

Now, let us fix $\bar\mu\in(0,\mu_\lambda)$. First of all, thanks to the definition of μ_λ, it is easy to see that

$$\text{there exists } \bar r_{\bar\mu}>0 \text{ such that } \bar\mu\leqslant 1/\varphi_{K,\lambda}(\bar r_{\bar\mu})\,. \tag{3.6}$$

Then, by (3.5) applied with $r=\bar r_{\bar\mu}$, we have that for any μ such that

$$0<\mu<\bar\mu\leqslant 1/\varphi_{K,\lambda}(\bar r_{\bar\mu})\,,$$

the function

$$u_\mu := u_{\mu,\bar r_{\bar\mu}}$$

is a global minimum of the functional $\mathcal{J}_{K,\lambda,\mu}$ restricted to $\Phi_{K,\lambda}^{-1}\big((-\infty,\bar r_{\bar\mu})\big)$, i.e.

$$\mathcal{J}_{K,\lambda,\mu}(u_\mu)\leqslant \mathcal{J}_{K,\lambda,\mu}(u), \quad \text{for any } u\in X_0 \text{ such that } \Phi_{K,\lambda}(u)<\bar r_{\bar\mu} \tag{3.7}$$

and

$$\Phi_{K,\lambda}(u_\mu)<\bar r_{\bar\mu}\,, \tag{3.8}$$

and also u_μ is a critical point of $\mathcal{J}_{K,\lambda,\mu}$ in X_0 and so it is a weak solution of problem (1.1).

In this way we have shown that for any $\lambda<\lambda_1$ and any $\mu\in(0,\mu_\lambda)$, problem (1.1) admits a weak solution $u_\mu\in X_0$.

Now, we have to prove that $\mu_\lambda=+\infty$, provided $q\in(1,2)$. To this purpose, note that by (1.7), one has

$$F(x,t)\leqslant a_1|t|+\frac{a_2}{q}|t|^q\,, \tag{3.9}$$

for a.e. $x\in\Omega$ and any $t\in\mathbb{R}$. Thus, for any $u\in X_0$ we get

$$\Psi(u)=\int_\Omega F(x,u(x))dx\leqslant a_1\|u\|_{L^1(\Omega)}+\frac{a_2}{q}\|u\|^q_{L^q(\Omega)}. \tag{3.10}$$

Also, by (3.3), for any $u\in X_0$ such that $\Phi_{K,\lambda}(u)<r$, with $r>0$, we easily get that

$$\|u\|_{X_0,\lambda}<\sqrt{2r}\,. \tag{3.11}$$

Hence, by (2.7), (3.10) and (3.11) we obtain that for any $u\in X_0$ such that $\Phi_{K,\lambda}(u)<r$

$$\begin{aligned}\Psi(u)&\leqslant a_1\|u\|_{L^1(\Omega)}+\frac{a_2}{q}\|u\|^q_{L^q(\Omega)}\\ &\leqslant \frac{a_1c_1}{m_\lambda}\|u\|_{X_0,\lambda}+\frac{a_2c_q^q}{qm_\lambda^q}\|u\|^q_{X_0,\lambda}\\ &< \frac{a_1c_1}{m_\lambda}(2r)^{1/2}+\frac{a_2c_q^q}{qm_\lambda^q}(2r)^{q/2}\,,\end{aligned}$$

so that

$$\sup_{u\in\Phi_{K,\lambda}^{-1}\big((-\infty,r)\big)}\Psi(u)\leqslant \frac{\sqrt{2}a_1c_1}{m_\lambda}\,r^{1/2}+\frac{2^{q/2}a_2c_q^q}{qm_\lambda^q}\,r^{q/2}$$

for any $r>0$.

Then, denoting by χ the following function

$$\chi(r) := \frac{\displaystyle\sup_{u\in\Phi_{K,\lambda}^{-1}\big((-\infty,r)\big)} \Psi(u)}{r}\,, \quad r>0\,,$$

we have

$$\chi(r) \leqslant \frac{\sqrt{2}a_1c_1}{m_\lambda}\, r^{-1/2} + \frac{2^{q/2}a_2c_q^q}{qm_\lambda^q}\, r^{q/2-1}, \tag{3.12}$$

for every $r>0$.

Now, observe that, by definition of $\varphi_{K,\lambda}$ and of χ and by (3.12) for any $r>0$ we have

$$\begin{aligned}\varphi_{K,\lambda}(r) &\leqslant \frac{\displaystyle\sup_{u\in\Phi_{K,\lambda}^{-1}\big((-\infty,r)\big)} \Psi(u)}{r}\\ &= \chi(r)\\ &\leqslant \frac{\sqrt{2}a_1c_1}{m_\lambda}\, r^{-1/2} + \frac{2^{q/2}a_2c_q^q}{qm_\lambda^q}\, r^{q/2-1}\,,\end{aligned}$$

just because $\Phi_{K,\lambda}(0)=\Psi(0)=0$. Namely,

$$\frac{1}{\varphi_{K,\lambda}(r)} \geqslant \frac{qm_\lambda^q}{\sqrt{2}a_1c_1qm_\lambda^{q-1}r^{-1/2} + 2^{q/2}a_2c_q^q\, r^{q/2-1}}\,,$$

so that

$$\mu_\lambda = \sup_{r>0}\frac{1}{\varphi_{K,\lambda}(r)} \geqslant \sup_{r>0}\frac{qm_\lambda^q}{\sqrt{2}a_1c_1qm_\lambda^{q-1}r^{-1/2} + 2^{q/2}a_2c_q^q\, r^{q/2-1}} = +\infty\,, \tag{3.13}$$

provided $q\in(1,2)$. Hence, $\mu_\lambda=+\infty$ if $q\in(1,2)$ and the assertion of Theorem 1 is proved.

Now, we have to show that for any $\mu\in(0,\mu_\lambda)$ the solution u_μ found here above is not the trivial function. If $f(\cdot,0)\neq 0$, then it easily follows that $u_\mu\not\equiv 0$ in X_0, since the trivial function does not solve problem (1.1).

Let us consider the case when $f(\cdot,0)=0$ and let us fix $\bar\mu\in(0,\mu_\lambda)$ and $\mu\in(0,\bar\mu)$. Finally, let u_μ be as in (3.7) and (3.8). In this setting, in order to prove that $u_\mu\not\equiv 0$ in X_0, first we claim that there exists a sequence $\big\{w_j\big\}_{j\in\mathbb{N}}$ in X_0 such that

$$\limsup_{j\to+\infty}\frac{\Psi(w_j)}{\Phi_{K,\lambda}(w_j)} = +\infty\,. \tag{3.14}$$

By the assumption on the limsup in (1.8) there exists a sequence $\{\xi_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}^+$ such that $\xi_j\to 0^+$ as $j\to+\infty$ and

$$\lim_{j\to+\infty}\frac{\mathrm{essinf}_{x\in B}\,F(x,\xi_j)}{\xi_j^2} = +\infty, \tag{3.15}$$

namely, we have that for any $M>0$ and j sufficiently large

$$\mathrm{essinf}_{x\in B}\,F(x,\xi_j) > M\xi_j^2\,. \tag{3.16}$$

Now, let C be a set of positive Lebesgue measure such that $C\subset B$. Also, let $v\in X_0$ be a function such that

i) $v(x)\in[0,1]$ for every $x\in\mathbb{R}^n$;
ii) $v(x)=1$ for every $x\in C$;
iii) $v(x)=0$ for every $x\in\Omega\setminus D$.

Of course C exists since B has positive Lebesgue measure, while the function v exists thanks to the fact that $C_0^2(\Omega) \subseteq X_0$ (see [22, Lemma 11]).

Finally, let $w_j := \xi_j v$ for any $j \in \mathbb{N}$. It is easily seen that $w_j \in X_0$ for any $j \in \mathbb{N}$ (actually, $w_j \in C_0^2(\Omega)$ if v does). Furthermore, taking into account the properties of v stated in $i)$–$iii)$, the fact that $C \subset B \subseteq D \subseteq \Omega$ and $F(\cdot, 0) = 0$, and (3.16) we have

$$
\begin{aligned}
\frac{\Psi(w_j)}{\Phi(w_j)} &= \frac{\displaystyle\int_\Omega F(x, w_j(x))\,dx}{\Phi_{K,\lambda}(w_j)} \\
&= \frac{\displaystyle\int_C F(x, w_j(x))\,dx + \int_{D\setminus C} F(x, w_j(x))\,dx}{\Phi_{K,\lambda}(w_j)} \\
&= \frac{\displaystyle\int_C F(x, \xi_j)\,dx + \int_{D\setminus C} F(x, \xi_j v(x))\,dx}{\Phi_{K,\lambda}(w_j)} \\
&\geqslant \frac{\displaystyle 2M\,\mathrm{meas}(C)\xi_j^2 + 2\int_{D\setminus C} F(x, \xi_j v(x))\,dx}{\xi_j^2 \|v\|^2_{X_0,\lambda}}\,,
\end{aligned}
\tag{3.17}
$$

for j sufficiently large, thanks to (3.3).

Now we have to distinguish two different cases, i.e. the case when the liminf in (1.8) is $+\infty$ (and so the liminf is actually a limit) and the one in which the liminf in (1.8) is finite.

Case 1: $\displaystyle\lim_{t\to 0^+} \frac{\mathrm{essinf}_{x\in D} F(x,t)}{t^2} = +\infty\,.$

Then, there exists $\rho_M > 0$ such that for any t with $0 < t < \rho_M$

$$
\mathrm{essinf}_{x\in D} F(x,t) \geqslant M t^2\,. \tag{3.18}
$$

Since $\xi_j \to 0^+$ and $0 \leqslant v \leqslant 1$ in Ω, then $w_j(x) = \xi_j v(x) \to 0^+$ as $j \to +\infty$ uniformly in $x \in \Omega$. Hence, $0 \leqslant w_j(x) < \rho_M$ for j sufficiently large and for any $x \in \Omega$. Hence, as a consequence of (3.17) and (3.18) (used here with $t = w_j(x)$, j large), we deduce that

$$
\begin{aligned}
\frac{\Psi(w_j)}{\Phi(w_j)} &\geqslant \frac{\displaystyle 2M\,\mathrm{meas}(C)\xi_j^2 + 2\int_{D\setminus C} F(x, \xi_j v(x))\,dx}{\xi_j^2 \|v\|^2_{X_0,\lambda}} \\
&\geqslant \frac{\displaystyle 2M\,\mathrm{meas}(C)\xi_j^2 + 2M\xi_j^2\int_{D\setminus C} v^2(x)\,dx}{\xi_j^2 \|v\|^2_{X_0,\lambda}} \\
&= \frac{\displaystyle 2M\,\mathrm{meas}(C) + 2M\int_{D\setminus C} v^2(x)\,dx}{\|v\|^2_{X_0,\lambda}}\,,
\end{aligned}
$$

for j sufficiently large. The arbitrariness of M gives (3.14) and so the claim is proved.

Case 2: $\displaystyle\liminf_{t\to 0^+} \frac{\mathrm{essinf}_{x\in D} F(x,t)}{t^2} = \ell \in \mathbb{R}\,.$

Then, for any $\varepsilon > 0$ there exists $\rho_\varepsilon > 0$ such that for any t with $0 < t < \rho_\varepsilon$

$$
\mathrm{essinf}_{x\in D} F(x,t) \geqslant (\ell - \varepsilon) t^2\,. \tag{3.19}
$$

Arguing as above, we can suppose that $0 \leqslant w_j(x) = \xi_j v(x) < \rho_\varepsilon$ for j large enough and any $x \in \Omega$. Thus, by (3.17) and (3.19) (used with $t = \xi_j v(x)$ with j large) we get

$$\begin{aligned}\frac{\Psi(w_j)}{\Phi(w_j)} &\geqslant \frac{2M\,\mathrm{meas}(C)\xi_j^2 + 2\displaystyle\int_{D\setminus C} F(x, \xi_j v(x))\,dx}{\xi_j^2 \|v\|^2_{X_0,\lambda}} \\ &\geqslant \frac{2M\,\mathrm{meas}(C)\xi_j^2 + 2(\ell-\varepsilon)\xi_j^2\displaystyle\int_{D\setminus C} v^2(x)\,dx}{\xi_j^2 \|v\|^2_{X_0,\lambda}} \\ &= \frac{2M\,\mathrm{meas}(C) + 2(\ell-\varepsilon)\displaystyle\int_{D\setminus C} v^2(x)\,dx}{\|v\|^2_{X_0,\lambda}}\,,\end{aligned} \tag{3.20}$$

provided j is sufficiently large.

Choosing $M > 0$ large enough, say

$$M\,\mathrm{meas}(C) > \max\left\{0, -2\ell\int_{D\setminus C} v^2(x)\,dx\right\},$$

and $\varepsilon > 0$ small enough so that

$$\varepsilon\int_{D\setminus C} v^2(x)\,dx < \frac{M\,\mathrm{meas}(C)}{2} + \ell\int_{D\setminus C} v^2(x)\,dx\,,$$

by (3.20) we get

$$\begin{aligned}\frac{\Psi(w_j)}{\Phi(w_j)} &\geqslant \frac{2M\,\mathrm{meas}(C) + 2(\ell-\varepsilon)\displaystyle\int_{D\setminus C} v^2(x))\,dx}{\|v\|^2_{X_0,\lambda}} \\ &\geqslant \frac{2}{\|v\|^2_{X_0,\lambda}}\left(M\,\mathrm{meas}(C) + \ell\int_{D\setminus C} v^2(x))\,dx - M\,\mathrm{meas}(C)/2 - \ell\int_{D\setminus C} v^2(x)\,dx\right) \\ &= \frac{M\,\mathrm{meas}(C)}{\|v\|^2_{X_0,\lambda}}\end{aligned}$$

for j large enough. Also in this case the arbitrariness of M gives assertion (3.14).

Now, note that

$$\|w_j\|_{X_0,\lambda} = |\xi_j|\,\|v\|_{X_0,\lambda} \to 0,$$

as $j \to +\infty$, so that for j large enough

$$\|w_j\|_{X_0,\lambda} < \sqrt{2\bar r_{\bar\mu}}\,,$$

where $\bar r_{\bar\mu}$ is given in (3.6). As a consequence of this and taking into account (3.3)

$$w_j \in \Phi_{K,\lambda}\big((-\infty, \bar r_{\bar\mu})\big)\,, \tag{3.21}$$

provided j is large enough. Also, by (3.14) and the fact that $\mu > 0$

$$\mathcal{J}_{K,\lambda,\mu}(w_j) = \Phi_{K,\lambda}(w_j) - \mu\Psi(w_j) < 0, \tag{3.22}$$

for j sufficiently large.

Since u_μ is a global minimum of the restriction of $\mathcal{J}_{K,\lambda,\mu}$ to $\Phi^{-1}_{K,\lambda}\big((-\infty, \bar r_\mu)\big)$ (see (3.7)), by (3.21) and (3.22) we conclude that

$$\mathcal{J}_{K,\lambda,\mu}(u_\mu) \leqslant \mathcal{J}_{K,\lambda,\mu}(w_j) < 0 = \mathcal{J}_{K,\lambda,\mu}(0)\,, \tag{3.23}$$

so that $u_\mu \not\equiv 0$ in X_0. Thus, u_μ is a non-trivial weak solution of problem (1.1). The arbitrariness of μ and $\bar\mu$ gives that $u_\mu \not\equiv 0$ for any $\mu \in (0, \mu_\lambda)$.

Moreover, from (3.23) we get that the map

$$(0, \mu_\lambda) \ni \mu \mapsto \mathcal{J}_{K,\lambda,\mu}(u_\mu) \ \text{ is negative.} \tag{3.24}$$

Now, we claim that $\lim\limits_{\mu\to 0^+}\|u_\mu\|_{X_0}=0$. By (2.5) it is enough to show that

$$\|u_\mu\|_{X_0,\,\lambda}\to 0, \tag{3.25}$$

as $\mu\to 0^+$.

For this, let again $\bar\mu\in(0,\mu_\lambda)$ and $\mu\in(0,\bar\mu)$. Bearing in mind (3.3) and the fact that $\Phi_{K,\,\lambda}(u_\mu)<\bar r_{\bar\mu}$, for any $\mu\in(0,\bar\mu)$ (see (3.8)), one has that

$$\Phi_{K,\,\lambda}(u_\mu)=\frac{1}{2}\|u_\mu\|^2_{X_0,\,\lambda}<\bar r_{\bar\mu}\,,$$

that is

$$\|u_\mu\|_{X_0,\,\lambda}<\sqrt{2\bar r_{\bar\mu}}\,.$$

As a consequence of this and by using the growth condition (1.7) together with the property (2.7), it follows that

$$\begin{aligned}\left|\int_\Omega f(x,u_\mu(x))u_\mu(x)dx\right|&\leqslant a_1\|u_\mu\|_{L^1(\Omega)}+a_2\|u_\mu\|^q_{L^q(\Omega)}\\ &\leqslant \frac{a_1c_1}{m_\lambda}\|u_\mu\|_{X_0,\,\lambda}+\frac{a_2c_q^q}{m_\lambda^q}\|u_\mu\|^q_{X_0,\,\lambda}\\ &< \frac{a_1c_1}{m_\lambda}\,(2\bar r_{\bar\mu})^{1/2}+\frac{a_2c_q^q}{m_\lambda^q}\,(2\bar r_{\bar\mu})^{q/2}=:M_{\bar r_{\bar\mu}}\,.\end{aligned} \tag{3.26}$$

Since u_μ is a critical point of $\mathcal{J}_{K,\,\lambda,\,\mu}$, then $\langle\mathcal{J}'_{K,\,\lambda,\,\mu}(u_\mu),\varphi\rangle=0$, for any $\varphi\in X_0$ and every $\mu\in(0,\bar\mu)$. In particular $\langle\mathcal{J}_{K,\,\lambda,\,\mu}(u_\mu),u_\mu\rangle=0$, that is

$$\langle\Phi'_{K,\,\lambda}(u_\mu),u_\mu\rangle=\mu\int_\Omega f(x,u_\mu(x))u_\mu(x)dx \tag{3.27}$$

for every $\mu\in(0,\bar\mu)$.

Then, from (3.26) and (3.27), it follows that

$$0\leqslant\|u_\mu\|^2_{X_0,\,\lambda}=\langle\Phi'_{K,\,\lambda}(u_\mu),u_\mu\rangle=\mu\int_\Omega f(x,u_\mu(x))u_\mu(x)\,dx<\mu\,M_{\bar r_{\bar\mu}}$$

for any $\mu\in(0,\bar\mu)$. Letting $\mu\to 0^+$, we get $\lim\limits_{\mu\to 0^+}\|u_\mu\|_{X_0,\,\lambda}=0$, as claimed.

Finally, we have to show that the map

$$\mu\mapsto\mathcal{J}_{K,\,\lambda,\,\mu}(u_\mu)\ \text{ is strictly decreasing in }\ (0,\mu_\lambda)\,.$$

For this we observe that for any $u\in X_0$

$$\mathcal{J}_{K,\,\lambda,\,\mu}(u)=\mu\left(\frac{\Phi_{K,\,\lambda}(u)}{\mu}-\Psi(u)\right)\,. \tag{3.28}$$

Now, let us fix $0<\mu_1<\mu_2<\bar\mu<\mu_\lambda$ and let u_{μ_i} be the global minimum of the functional $\mathcal{J}_{K,\,\lambda,\,\mu_i}$ restricted to $\Phi_{K,\,\lambda}\big((-\infty,\bar r_{\bar\mu})\big)$ for $i=1,2$ (for this see (3.7)). Also, let

$$m_{\mu_i}:=\left(\frac{\Phi_{K,\,\lambda}(u_{\mu_i})}{\mu_i}-\Psi(u_{\mu_i})\right)=\inf_{v\in\Phi^{-1}_{K,\,\lambda}\big((-\infty,\bar r_{\bar\mu})\big)}\left(\frac{\Phi_{K,\,\lambda}(v)}{\mu_i}-\Psi(v)\right)\,,\quad i=1,2\,.$$

Clearly, (3.24), (3.28) and the positivity of μ imply that

$$m_{\mu_i}<0\quad\text{for}\ \ i=1,2\,. \tag{3.29}$$

Moreover,

$$m_{\mu_2}\leqslant m_{\mu_1}\,, \tag{3.30}$$

thanks to the fact that $0<\mu_1<\mu_2$ and $\Phi_{K,\,\lambda}\geqslant 0$ by (3.3). Then, by (3.28)–(3.30) and again by the fact that $0<\mu_1<\mu_2$, we get that

$$\mathcal{J}_{K,\,\lambda,\,\mu_2}(u_{\mu_2})=\mu_2 m_{\mu_2}\leqslant\mu_2 m_{\mu_1}<\mu_1 m_{\mu_1}=\mathcal{J}_{K,\,\lambda,\,\mu_1}(u_{\mu_1})\,,$$

so that the map $\mu \mapsto \mathcal{J}_{K,\lambda,\mu}(u_\mu)$ is strictly decreasing in $(0,\bar\mu)$. The arbitrariness of $\bar\mu < \mu_\lambda$ shows that $\mu \mapsto \mathcal{J}_{K,\lambda,\mu}(u_\mu)$ is strictly decreasing in $(0,\mu_\lambda)$. This concludes the proof of Theorem 1.

4. Some comments on the main result

In this section we discuss some properties of the weak solutions of problem (1.1) provided by Theorem 1. Also, we give some comments on the main result of the paper.

4.1. **Existence of constant-sign solutions.** First of all, we would like to note that, as in the classical case of the Laplacian, it is possible to prove that the solution of problem (1.1) given by Theorem 1 has constant sign, as stated in the following result:

Corollary 3. *Let all the assumptions of Theorem 1 be satisfied and assume $f(\cdot,0)=0$. Then, problem* (1.1) *admits a non-negative weak solution $u_+ \in X_0$ which is not identically zero.*

Proof. In order to prove the existence of a non-negative solution of problem (1.12) it is enough to introduce the function

$$F_+(x,t) := \int_0^t f_+(x,\tau)d\tau\,,$$

with

$$f_+(x,t) := \begin{cases} f(x,t) & \text{if } t \geqslant 0 \\ 0 & \text{if } t<0 \end{cases}$$

for a.e. $x\in\Omega$ and $t\in\mathbb{R}$.

First of all, note that both f_+ and F_+ are well defined a.e. $x\in\Omega$ and $t\in\mathbb{R}$. Furthermore, since $f(\cdot,0)=0$, then f_+ is a Carathéodory function in $\Omega\times\mathbb{R}$ and so $t\mapsto F_+(\cdot,t)$ is differentiable in $\mathbb{R}$. Moreover, it is easily seen that f_+ and F_+ satisfy conditions (1.7) and (1.8), respectively.

Let $\mathcal{J}^+_{K,\lambda,\mu} : X_0 \to \mathbb{R}$ be the functional defined as follows

$$\mathcal{J}^+_{K,\lambda,\mu}(u) := \Phi_{K,\lambda}(u) - \mu\Psi_+(u)\,,$$

with

$$\Psi_+(u) := \int_\Omega F_+(x,u(x))\,dx\,.$$

It is easy to see that the functional Ψ_+ is well defined, is Fréchet differentiable at any $u\in X_0$ (being F_+ differentiable in $\mathbb{R}$ and since (1.7) holds true for f_+) and has the regularity properties required by [5, Theorem 2.1; part a)] (see Subsection 2.2). For this it is enough to argue as in the proof of Theorem 1. Also, for any $\varphi\in X_0$

$$\begin{aligned}\langle(\mathcal{J}^+_{K,\lambda,\mu})'(u),\varphi\rangle = &\int_{\mathbb{R}^n\times\mathbb{R}^n}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy\\ &-\lambda\int_\Omega u(x)\varphi(x)\,dx - \mu\int_\Omega f_+(x,u(x))\varphi(x)\,dx\,.\end{aligned} \tag{4.1}$$

Hence, by [5, Theorem 2.1; part a)], there exists a critical point $u_+\in X_0$ of $\mathcal{J}^+_{K,\lambda,\mu}$.

Also $u_+\not\equiv 0$ in X_0. Indeed, since $f(\cdot,0)=0$, also $f_+(\cdot,0)=0$ and so, in order to prove that $u_+\not\equiv 0$, we can argue exactly as in the proof of Theorem 1, jus replacing f with f_+, F with F_+ and Ψ with Ψ_+ in formulas (3.14)–(3.23).

We claim that u_+ is non-negative in $\mathbb{R}^n$. For this we take $\varphi := (u_+)^-$ in (4.1), where v^- is the negative part of v, i.e. $v^- := \max\{-v,0\}$. We remark that, since $u_+\in X_0$, we have that $(u_+)^- \in X_0$, by [22, Lemma 12], and so the choice of such φ is admissible. In this way, since u_+ is a critical point of $\mathcal{J}^+_{k,\lambda,\mu}$, we get

$$
\begin{aligned}
0 &= \langle (\mathcal{J}^+_{K,\lambda,\mu})'(u_+), (u_+)^- \rangle \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_+(x)-u_+(y)\big)\big((u_+)^-(x)-(u_+)^-(y)\big)K(x-y)\,dx\,dy \\
&\qquad - \lambda \int_\Omega u_+(x)(u_+)^-(x)\,dx - \mu\int_\Omega f_+(x,u_+(x))(u_+)^-(x)\,dx \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_+(x)-u_+(y)\big)\big((u_+)^-(x)-(u_+)^-(y)\big)K(x-y)\,dx\,dy \\
&\qquad - \lambda \int_\Omega \big|(u_+)^-(x)\big|^2\,dx\,,
\end{aligned}
\tag{4.2}
$$

thanks to the definition of f_+ and of negative part.

Now, we claim that for any $w \in X_0$ the following relation holds true a.e. $x, y \in \mathbb{R}^n$

$$
\big(w(x)-w(y)\big)\big(w^-(x)-w^-(y)\big) \leqslant -\big|w^-(x)-w^-(y)\big|^2\,. \tag{4.3}
$$

Indeed, writing $w = w^+ - w^-$ and taking into account that

$$
w^+(x)w^-(x) = 0 \ \text{ and } \ w^+(x)w^-(y) \geqslant 0 \ \text{ a.e. } x, y \in \mathbb{R}^n\,,
$$

we get

$$
\begin{aligned}
(w(x)-w(y))(w^-(x)-w^-(y)) &= (w^+(x)-w^+(y))(w^-(x)-w^-(y)) - (w^-(x)-w^-(y))^2 \\
&= -w^+(x)w^-(y) - w^+(y)w^-(x) - (w^-(x)-w^-(y))^2 \\
&\leqslant -\big|w^-(x)-w^-(y)\big|^2
\end{aligned}
$$

a.e. $x, y \in \mathbb{R}^n$. Hence, the claim (4.3) is proved.

Thus, by (4.2) and (4.3) applied here with $w = u_+$ we get

$$
\begin{aligned}
0 &= \langle (\mathcal{J}^+_{K,\lambda,\mu})'(u_+), (u_+)^- \rangle \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_+(x)-u_+(y)\big)\big((u_+)^-(x)-(u_+)^-(y)\big)K(x-y)\,dx\,dy - \lambda\int_\Omega \big|(u_+)^-(x)\big|^2\,dx \\
&\leqslant -\int_{\mathbb{R}^n\times\mathbb{R}^n} \big|(u_+)^-(x)-(u_+)^-(y)\big|^2 K(x-y)\,dx\,dy - \lambda\int_\Omega \big|(u_+)^-(x)\big|^2\,dx \\
&= -\|(u_+)^-\|^2_{X_0} - \lambda\|(u_+)^-\|^2_{L^2(\Omega)} \\
&\leqslant -\kappa_\lambda\,\|(u_+)^-\|^2_{X_0}\,,
\end{aligned}
$$

recalling that the kernel K is positive and the variational characterization of λ_1. Here κ_λ is the positive constant given by $\kappa_\lambda = \min\big\{1, 1+\lambda/\lambda_1\big\}$. Hence, $\|(u_+)^-\|_{X_0} = 0$, so that $(u_+)^- \equiv 0$ a.e. in $\mathbb{R}^n$, that is $u_+ \geqslant 0$ a.e. in $\mathbb{R}^n$. The assertion is proved. □

As a remark we would like to note that, if we replace condition (1.8) with the following one

$$
\begin{gathered}
\text{there exist a non-empty open set } D \subseteq \Omega \text{ and a set } B \subseteq D \\
\text{of positive Lebesgue measure such that} \\
\limsup_{t\to 0^-} \frac{\operatorname{essinf}_{x\in B} F(x,t)}{t^2} = +\infty \quad \text{and} \quad \liminf_{t\to 0^-} \frac{\operatorname{essinf}_{x\in D} F(x,t)}{t^2} > -\infty,
\end{gathered}
$$

then problem (1.1) admits a non-trivial weak solution. For this we can argue exactly as in the proof of Theorem 1.

Of course, in this case we can prove that this solution in non-positive in $\mathbb{R}^n$, provided $f(\cdot,0)=0$. To this purpose, it is enough to consider the functional

$$\mathcal{J}^-_{K,\lambda,\mu}(u):=\Phi_{K,\lambda}(u)-\mu\Psi_-(u)\,,\ \ u\in X_0$$

with

$$\Psi_-(u):=\int_\Omega F_-(x,u(x))\,dx\,,$$

and

$$F_-(x,t):=\int_0^t f_-(x,\tau)d\tau\,,\qquad f_-(x,t):=\begin{cases}0 & \text{if } t>0\\ f(x,t) & \text{if } t\leqslant 0\end{cases}$$

a.e. $x\in\Omega$ and $t\in\mathbb{R}$, and argue as in Corollary 3.

4.2. **Final remarks.** This subsection is devoted to some comments on the main results of the paper.

First of all, we would like to note that Theorem 1 is a bifurcation result, since $\mu=0$ is a bifurcation point for problem (1.1), in the sense that the pair $(0,0)$ belongs to the closure of the set

$$\big\{(u_\mu,\mu)\in X_0\times(0,+\infty):u_\mu\ \text{ is a non-trivial weak solution of (1.1)}\big\}$$

in $X_0\times\mathbb{R}$.

Indeed, by Theorem 1 we have that

$$\|u_\mu\|_{X_0,\lambda}\to 0\ \text{ as }\ \mu\to 0^+\,.$$

Hence, there exist two sequences $\{u_j\}_{j\in\mathbb{N}}$ in X_0 and $\{\mu_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}^+$ (here $u_j:=u_{\mu_j}$) such that

$$\mu_j\to 0^+\ \text{ and }\ \|u_j\|_{X_0,\lambda}\to 0,$$

as $j\to+\infty$.

Moreover, we would like to stress that for any $\mu_1,\mu_2\in(0,\mu_\lambda)$, with $\mu_1\neq\mu_2$, the solutions u_{μ_1} and u_{μ_2} given by Theorem 1 are different, thanks to the fact that the map

$$(0,\mu_\lambda)\ni\mu\mapsto\mathcal{J}_{K,\lambda,\mu}(u_\mu)$$

is strictly decreasing.

As a final remark, we give an estimate from below for the parameter μ_λ appearing in Theorem 1. Indeed, while when $q\in(1,2)$ Theorem 1 assures that $\mu_\lambda=+\infty$, the exact value of μ_λ is not known in the other cases, that is when $q\in[2,2^*)$.

Following the proof of Theorem 1 (see formula (3.13)) we have that

$$\begin{aligned}\mu_\lambda&:=\sup_{r>0}\frac{1}{\varphi_{K,\lambda}(r)}\\ &\geqslant\sup_{r>0}\frac{qm_\lambda^q}{\sqrt{2}a_1c_1qm_\lambda^{q-1}r^{-1/2}+2^{q/2}a_2c_q^q\,r^{q/2-1}}\\ &=\begin{cases}\dfrac{m_\lambda^2}{a_2c_2^2} & \text{if}\quad q=2\\[2ex] \dfrac{qm_\lambda^q}{\sqrt{2}a_1c_1qm_\lambda^{q-1}r_{\max}^{-1/2}+2^{q/2}a_2c_q^q\,r_{\max}^{q/2-1}} & \text{if}\quad q\in(2,2^*),\end{cases}\end{aligned}$$

where

$$r_{\max}:=\frac{m_\lambda^2}{2}\left(\frac{a_1c_1q}{a_2c_q^q(q-2)}\right)^{2/(q-1)}\,,$$

while a_1 and a_2 are as in (1.7), m_λ is given in (2.5) and c_1 and c_q are as in (2.7).

Hence, if the term f is sublinear at infinity (i.e. $q\in(1,2)$ in (1.7)), then Theorem 1 ensures that, for any $\lambda<\lambda_1$ and $\mu>0$, problem (1.1) admits at least one non-trivial weak solution, while for $q \in [2, 2^*)$ we get the existence of a non-trivial solution only if μ is small enough. Indeed, our approach allows us to treat problem (1.1), in the case when $q \in [2, 2^*)$, only if μ is sufficiently small, say $\mu < \mu_\lambda$. It should be interesting to investigate the existence of solution for (1.1) in the superlinear case, when μ is large (i.e. $\mu \geqslant \mu_\lambda$). As far as we know, also in the classical Laplacian setting this question is open.

Finally, notice that, when $q \in (1, 2)$ the existence of solutions for problem (1.1) can be obtained using classical direct methods (see, for instance [29, Chapter I]), but, as it happens with the arguments used along the present paper, we do not know a priori if the solution provided by these classical theorems is the trivial function or not (of course, we refer to the case $f(\cdot, 0) = 0$, otherwise $u \equiv 0$ does not solve (1.1)). Hence, if $f(\cdot, 0) = 0$, also when using classical methods, we need to assume extra conditions on f, in order to prove that the solution of the problem is not the trivial function.

5. An application to the fractional Laplacian case

This section is devoted to an application of Theorem 1 in the case of a non-local equation driven by the fractional Laplace operator $(-\Delta)^s$, when the nonlinearity f is a power-type function, as in the model. Namely, here we consider problem (1.14) and we prove Theorem 2.

5.1. **Proof of Theorem 2.** To our purpose, it is enough to apply Theorem 1 in the case when $\mathcal{L}_K = -(-\Delta)^s$ and

$$f(x,t) := a(x)|t|^{r-2}t + b(x)|t|^{q-2}t + c(x) \ \text{ in } \ \Omega \times \mathbb{R},$$

with $1 < r < 2 \leqslant q < 2^*$, while a, b and c are as in Theorem 2.

It is easy to verify that

$$|f(x,t)| \leqslant 3 \max\{\|a\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}\}(1 + |t|^{q-1})$$

a.e. $x \in \Omega$ and $t \in \mathbb{R}$, so that condition (1.7) holds true.

Furthermore, if $c \equiv 0$ in Ω (and so $f(\cdot, 0) = 0$), a direct computation shows that

$$\begin{aligned}
\lim_{t \to 0^+} \frac{\operatorname{essinf}_{x\in\Omega} F(x,t)}{t^2} &= \lim_{t \to 0^+} \frac{\operatorname{essinf}_{x\in\Omega} \Big(qa(x)|t|^r + rb(x)|t|^q\Big)}{qrt^2} \\
&\geqslant \lim_{t \to 0^+} \left(\frac{1}{r} \operatorname{essinf}_{x\in\Omega} a(x)|t|^{r-2} + \frac{1}{q} \operatorname{essinf}_{x\in\Omega} b(x)|t|^{q-2}\right) \\
&= +\infty,
\end{aligned}$$

thanks to the choice of r and q (i.e. $r < 2 \leqslant q$) and to the fact that $\operatorname{essinf}_{x\in\Omega} a(x) > 0$. Hence, assumption (3.1) with $B = \Omega$ (and so (1.8) with $B = D = \Omega$) is verified.

As a consequence of this, by Theorem 1 we get that for any $\lambda < \lambda_{1,s}$ there exists $\mu_\lambda > 0$ such that problem (1.14) admits a non-trivial weak solution u_μ in X_0 for any $\mu \in (0, \mu_\lambda)$ such that

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_\mu(x) - u_\mu(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \to 0$$

as $\mu \to 0^+$.

It is easily seen that $u_\lambda = +\infty$ if $b \equiv 0$ a.e. in Ω, since, in this case, f has a subliner growth.

Since in the model case in which $K(x) = |x|^{-(n+2s)}$ the space X_0 can be characterized as follows (see [25, Lemma 7-b)]

$$X_0 = \big\{v \in H^s(\mathbb{R}^n) : v = 0 \ \text{ a.e. in } \mathbb{R}^n \setminus \Omega\big\},$$

we get that $u_\mu \in H^s(\mathbb{R}^n)$ and $u_\mu = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

Finally, the non-negativity of the function u_μ comes from Corollary 3. This concludes the proof of Theorem 2.

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