New expressions for the wave operators of Schrödinger operators in $\mathbb{R}^3$

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Abstract

We prove new and explicit formulas for the wave operators of Schrödinger operators in $\mathbb{R}^3$. These formulas put into light the very special role played by the generator of dilations and validate the topological approach of Levinson's theorem introduced in a previous publication. Our results hold for general (not spherically symmetric) potentials decaying fast enough at infinity, without any assumption on the absence of eigenvalue or resonance at 0-energy.

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1 Introduction and main theorem

The purpose of this work is to establish explicit and completely new expressions for the wave operators of Schrödinger operators in $\mathbb{R}^3$, and as a by-product to validate the use of the topological approach of Levinson's theorem.

The set-up is the standard one. We consider in the Hilbert space $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^3)$ the free Schrödinger operator $H_0 := -\Delta$ and the perturbed Schrödinger operator $H := -\Delta + V$, with V a measurable bounded real function on $\mathbb{R}^3$ decaying fast enough at infinity. In such a situation, it is well-known that the wave operators

$$W_\pm := \text{s-}\lim_{t \to \pm\infty} \mathrm{e}^{itH}\,\mathrm{e}^{-itH_0} \tag{1.1}$$

exist and are asymptotically complete [2, 21, 23], and as a consequence that the scattering operator $S := W_+^* W_-$ is a unitary operator in $\mathcal{H}$. Moreover, it is also well-known that one can write time-independent expressions for $W_\pm$ by using the stationary formulation of scattering theory (see [3, 19, 25]).

Among the many features of the wave operators, their mapping properties between weighted Hilbert spaces, weighted Sobolev spaces and L^p-spaces have attracted a lot of attention (see for instance the seminal papers [14, 27, 28, 30] and the preprint [5] which contains an interesting historical overview and many references). Also, recent technics developed for the study of the wave operators have been used to obtain dispersive estimates for Schrödinger operators [6, 7, 8, 29]. Our point here, which can be inscribed in this line of general works on wave operators, is to show that the time-independent expressions for $W_\pm$ can be made completely explicit, up to a compact term. Namely, if $\mathscr{B}(\mathcal{H})$ (resp. $\mathscr{K}(\mathcal{H})$) denotes the set of bounded (resp. compact) operators in $\mathcal{H}$, and if A stands for the generator of dilations in $\mathbb{R}^3$, then we have the following result:

Theorem 1.1. *Let V satisfy $|V(x)| \leq \text{Const.} (1 + |x|)^{-\sigma}$ with $\sigma > 7$ for almost every $x \in \mathbb{R}^3$. Then, one has in $\mathscr{B}(\mathcal{H})$ the equalities*

$$W_- = 1 + R(A)(S - 1) + K \qquad and \qquad W_+ = 1 + \big(1 - R(A)\big)\big(S^* - 1\big) + K', \qquad (1.2)$$

with $R(A) := \frac{1}{2}\big(1 + \tanh(\pi A) - i \cosh(\pi A)^{-1}\big)$ and $K, K' \in \mathscr{K}(\mathcal{H})$.

We stress that the absence of eigenvalue or resonance at 0-energy is not assumed. On the other hand, if such an implicit hypothesis is made, then the same result holds under a weaker assumption on the decay of V at infinity. We also note that no spherical symmetry is imposed on V.

Our motivation for proving Theorem 1.1 was the observation made in [16] (and applied to various situations in [4, 10, 17, 20, 24]) that Levinson's theorem can be reinterpreted as an index theorem, with a proof based on an explicit expression for the wave operators. The main idea is to show that the wave operators belong to a certain C^*-algebra. Once such an affiliation property is settled, the machinery of non-commutative topology leads naturally to an index theorem. In its original form, this index theorem corresponds to Levinson's theorem; that is, the equality between the number of bound states of the operator H and an expression (trace) involving the scattering operator S. For more complex scattering systems, other topological equalities involving higher degree traces can also be derived (see [15] for more explanations).

For the scattering theory of Schrödinger operators in $\mathbb{R}^3$, the outcomes of this topological approach have been detailed in [18]: It has been shown how Levinson's theorem can be interpreted as an index theorem, and how one can derive from it various formulas for the number of bound states of H in terms of the scattering operator and a second operator related to the 0-energy. However, a technical argument was missing, and an implicit assumption had to be made accordingly. Theorem 1.1 makes this implicit assumption no longer necessary, and thus allows one to apply all the results of [18] (see Remark 2.8 for some more comments).

Let us now present a more detailed description of our results. As mentioned above, our goal was to obtain an explicit formula for the wave operators, as required by the C^*-algebras framework. However, neither the time dependant formula (1.1), nor the stationary approach as presented for instance in [25], provided us with a sufficiently precise answer. This motivated us to show in Theorem 2.6 of Section 2 that the difference $W_- - 1$ is unitarily equivalent to a product of three explicit bounded operators. The result is exact and no compact operator as in the statement of Theorem 1.1 has to be added. In addition, each of the three operators is either an operator of multiplication by an operator-valued function, or a simple function of the generator of dilation in $\mathsf{L}^2(\mathbb{R}_+)$. For these reasons, we expect that the formula of Theorem 2.6 might have various applications, as for example for the mapping properties of W_-. Finally, the commutation of two of the three operators reveals the presence of the scattering operator up to a compact term, as stated in Theorem 1.1. One deduces from this new expression for W_- the corresponding expression for W_+.

As a conclusion, we emphasize once more that the present work validates the use of the topological approach of Levinson's theorem, as presented in [18]. It also implicitly shows that this C^*-algebraic approach of scattering theory leads to new questions and new results, as exemplified by the explicit formula presented in Theorem 1.1.

Notations: $\mathbb{N} := \{0, 1, 2, \ldots\}$ is the set of natural numbers, $\mathbb{R}_+ := (0, \infty)$, and $\mathscr{S}$ is the Schwartz space on $\mathbb{R}^3$. The sets $\mathcal{H}_t^s$ are the weighted Sobolev spaces over $\mathbb{R}^3$ with index $s \in \mathbb{R}$ associated to derivatives and index $t \in \mathbb{R}$ associated to decay at infinity [1, Sec. 4.1] (with the convention that $\mathcal{H}^s := \mathcal{H}_0^s$ and $\mathcal{H}_t := \mathcal{H}_t^0$). The three-dimensional Fourier transform $\mathscr{F}$ is a topological isomorphism of $\mathcal{H}_t^s$ onto $\mathcal{H}_s^t$ for any $s, t \in \mathbb{R}$. Given two Banach spaces $\mathcal{G}_1$ and $\mathcal{G}_2$, $\mathscr{B}(\mathcal{G}_1, \mathcal{G}_2)$ (resp. $\mathscr{K}(\mathcal{G}_1, \mathcal{G}_2)$) stands for the set of bounded (resp. compact) operators from $\mathcal{G}_1$ to $\mathcal{G}_2$. Finally, $\otimes$ (resp. $\odot$) stands for the closed (resp. algebraic) tensor product of Hilbert spaces or of operators.

2 New expressions for the wave operators

We start by introducing the Hilbert spaces we use throughout the paper; namely, $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^3)$, $\mathfrak{h} := \mathsf{L}^2(\mathbb{S}^2)$ and $\mathscr{H} := \mathsf{L}^2(\mathbb{R}_+; \mathfrak{h})$ with respective scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ indexed accordingly. The Hilbert space

$\mathscr{H}$ hosts the spectral representation of the operator $H_0 = -\Delta$ with domain $\mathcal{D}(H_0) = \mathcal{H}^2$, *i.e.*, there exists a unitary operator $\mathscr{F}_0 : \mathcal{H} \to \mathscr{H}$ satisfying

$$(\mathscr{F}_0 H_0 f)(\lambda) = \lambda (\mathscr{F}_0 f)(\lambda) \equiv (L \mathscr{F}_0 f)(\lambda), \quad f \in \mathcal{D}(H_0), \text{ a.e. } \lambda \in \mathbb{R}_+,$$

with L the maximal multiplication operator in $\mathscr{H}$ by the variable in $\mathbb{R}_+$. The explicit formula for $\mathscr{F}_0$ is

$$\big((\mathscr{F}_0 f)(\lambda)\big)(\omega) = \big(\tfrac{\lambda}{4}\big)^{1/4} (\mathscr{F}f)(\sqrt{\lambda}\omega) = \big(\tfrac{\lambda}{4}\big)^{1/4} \big(\gamma(\sqrt{\lambda})\mathscr{F}f\big)(\omega), \quad f \in \mathscr{S}, \ \lambda \in \mathbb{R}_+, \ \omega \in \mathbb{S}^2, \quad (2.1)$$

with $\gamma(\lambda) : \mathscr{S} \to \mathfrak{h}$ the trace operator given by $\big(\gamma(\lambda)f\big)(\omega) := f(\lambda\omega)$.

The potential $V \in \mathsf{L}^\infty(\mathbb{R}^3; \mathbb{R})$ of the perturbed Hamiltonian $H := H_0 + V$ satisfies for some $\sigma > 0$ the condition

$$|V(x)| \leq \text{Const.}\, \langle x \rangle^{-\sigma}, \quad \text{a.e. } x \in \mathbb{R}^3, \quad (2.2)$$

with $\langle x \rangle := \sqrt{1 + x^2}$. Since V is bounded, H is self-adjoint with domain $\mathcal{D}(H) = \mathcal{D}(H_0)$. Also, it is well-known [21, Thm. 12.1] that the wave operators defined by (1.1) exist and are asymptotically complete if $\sigma > 1$. In stationary scattering theory one defines the wave operators in terms of suitable limits of the resolvents of H_0 and H on the real axis. We shall mainly use this second approach, noting that for this model both definitions for the wave operators do coincide (see [25, Sec. 5.3]).

Now, we recall from [25, Eq. 2.7.5] that for suitable $f, g \in \mathcal{H}$ the stationary expressions for the wave operators are given by

$$\langle W_\pm f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}} \mathrm{d}\lambda \lim_{\varepsilon \searrow 0} \frac{\varepsilon}{\pi} \big\langle R_0(\lambda \pm i\varepsilon)f, R(\lambda \pm i\varepsilon)g \big\rangle_{\mathcal{H}},$$

where $R_0(z) := (H_0 - z)^{-1}$ and $R(z) := (H - z)^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, are the resolvents of the operators H_0 and H. We also recall from [25, Sec. 1.4] that the limit $\lim_{\varepsilon \searrow 0} \big\langle \delta_\varepsilon(H_0 - \lambda)f, g \big\rangle_{\mathcal{H}}$ with $\delta_\varepsilon(H_0 - \lambda) := \frac{\varepsilon}{\pi} R_0(\lambda \mp i\varepsilon) R_0(\lambda \pm i\varepsilon)$ exists for a.e. $\lambda \in \mathbb{R}$ and that

$$\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}} \mathrm{d}\lambda \lim_{\varepsilon \searrow 0} \big\langle \delta_\varepsilon(H_0 - \lambda)f, g \big\rangle_{\mathcal{H}}.$$

Thus, taking into account the second resolvent equation, one infers that

$$\big\langle (W_\pm - 1)f, g \big\rangle_{\mathcal{H}} = -\int_{\mathbb{R}} \mathrm{d}\lambda \lim_{\varepsilon \searrow 0} \big\langle \delta_\varepsilon(H_0 - \lambda)f, \big(1 + V R_0(\lambda \pm i\varepsilon)\big)^{-1} V R_0(\lambda \pm i\varepsilon)g \big\rangle_{\mathcal{H}}.$$

We now derive new expressions for the wave operators in the spectral representation of H_0; that is, for the operators $\mathscr{F}_0(W_\pm - 1)\mathscr{F}_0^*$. So, let φ, ψ be suitable elements of $\mathscr{H}$ (precise conditions will be specified in Theorem 2.6 below), then one obtains that

$$\big\langle \mathscr{F}_0(W_\pm - 1)\mathscr{F}_0^*\varphi, \psi \big\rangle_{\mathscr{H}}$$
$$= -\int_{\mathbb{R}} \mathrm{d}\lambda \lim_{\varepsilon \searrow 0} \big\langle V\big(1 + R_0(\lambda \mp i\varepsilon)V\big)^{-1} \mathscr{F}_0^* \delta_\varepsilon(L - \lambda)\varphi, \mathscr{F}_0^*(L - \lambda \mp i\varepsilon)^{-1}\psi \big\rangle_{\mathcal{H}}$$
$$= -\int_{\mathbb{R}} \mathrm{d}\lambda \lim_{\varepsilon \searrow 0} \int_0^\infty \mathrm{d}\mu \big\langle \big\{ \mathscr{F}_0 V\big(1 + R_0(\lambda \mp i\varepsilon)V\big)^{-1} \mathscr{F}_0^* \delta_\varepsilon(L - \lambda)\varphi \big\}(\mu), (\mu - \lambda \mp i\varepsilon)^{-1}\psi(\mu) \big\rangle_{\mathfrak{h}}.$$

Using the short hand notation $T(z) := V\big(1 + R_0(z)V\big)^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, one thus gets the equality

$$\big\langle \mathscr{F}_0(W_\pm - 1)\mathscr{F}_0^*\varphi, \psi \big\rangle_{\mathscr{H}}$$
$$= -\int_{\mathbb{R}} \mathrm{d}\lambda \lim_{\varepsilon \searrow 0} \int_0^\infty \mathrm{d}\mu \big\langle \big\{ \mathscr{F}_0 T(\lambda \mp i\varepsilon) \mathscr{F}_0^* \delta_\varepsilon(L - \lambda)\varphi \big\}(\mu), (\mu - \lambda \mp i\varepsilon)^{-1}\psi(\mu) \big\rangle_{\mathfrak{h}}. \quad (2.3)$$

The next step is to exchange the integral over μ and the limit $\varepsilon \searrow 0$ in the previous expression. To do it properly, we need a series of preparatory lemmas. First of all, we recall that for $\lambda > 0$ the trace operator $\gamma(\lambda)$

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extends to an element of $\mathscr{B}(\mathcal{H}_t^s, \mathfrak{h})$ for each $s > 1/2$ and $t \in \mathbb{R}$ and that the map $\mathbb{R}_+ \ni \lambda \mapsto \gamma(\lambda) \in \mathscr{B}(\mathcal{H}_t^s, \mathfrak{h})$ is continuous [12, Sec. 3]. As a consequence, the operator $\mathscr{F}_0(\lambda) : \mathscr{S} \to \mathfrak{h}$ given by $\mathscr{F}_0(\lambda)f := (\mathscr{F}_0 f)(\lambda)$ extends to an element of $\mathscr{B}(\mathcal{H}_t^s, \mathfrak{h})$ for each $s \in \mathbb{R}$ and $t > 1/2$, and the map $\mathbb{R}_+ \ni \lambda \mapsto \mathscr{F}_0(\lambda) \in \mathscr{B}(\mathcal{H}_t^s, \mathfrak{h})$ is continuous.

We shall now strengthen these standard results.

Lemma 2.1. *Let $s \geq 0$ and $t > 3/2$. Then, the functions*

$$(0, \infty) \ni \lambda \mapsto \lambda^{\pm 1/4} \mathscr{F}_0(\lambda) \in \mathscr{B}(\mathcal{H}_t^s, \mathfrak{h})$$

are continuous and bounded.

Proof. The continuity of the functions $(0, \infty) \ni \lambda \mapsto \lambda^{\pm 1/4} \mathscr{F}_0(\lambda) \in \mathscr{B}(\mathcal{H}_t^s, \mathfrak{h})$ follows from what has been said before. For the boundedness, it is sufficient to show that the map $\lambda \mapsto \lambda^{-1/4} \|\mathscr{F}_0(\lambda)\|_{\mathscr{B}(\mathcal{H}_t^s, \mathfrak{h})}$ is bounded in a neighbourhood of 0, and that the map $\lambda \mapsto \lambda^{1/4} \|\mathscr{F}_0(\lambda)\|_{\mathscr{B}(\mathcal{H}_t^s, \mathfrak{h})}$ is bounded in a neighbourhood of $+\infty$. The first bound follows from the asymptotic development for small $\lambda > 0$ of the operator $\gamma(\sqrt{\lambda}) \mathscr{F} \in \mathscr{B}(\mathcal{H}_t^s, \mathfrak{h})$ (see [13, Sec. 5]) and the second bound follows from [26, Thm. 1.1.4] which implies that the map $\lambda \mapsto \lambda^{1/4} \|\mathscr{F}_0(\lambda)\|_{\mathscr{B}(\mathcal{H}_t^s, \mathfrak{h})}$ is bounded on $\mathbb{R}_+$. Note that only the case $s = 0$ is presented in [26, Thm. 1.1.4], but the extension to the case $s \geq 0$ is trivial since $\mathcal{H}_t^s \subset \mathcal{H}_t^0$ for any $s > 0$. $\square$

One immediately infers from Lemma 2.1 that the function $\mathbb{R}_+ \ni \lambda \mapsto \|\mathscr{F}_0(\lambda)\|_{\mathscr{B}(\mathcal{H}_t^s, \mathfrak{h})} \in \mathbb{R}$ is continuous and bounded for any $s \geq 0$ and $t > 3/2$. Also, one can strengthen the statement of Lemma 2.1 in the case of the minus sign:

Lemma 2.2. *Let $s > -1$ and $t > 3/2$. Then, $\mathscr{F}_0(\lambda) \in \mathscr{K}(\mathcal{H}_t^s, \mathfrak{h})$ for each $\lambda \in \mathbb{R}_+$, and the function $\mathbb{R}_+ \ni \lambda \mapsto \lambda^{-1/4} \mathscr{F}_0(\lambda) \in \mathscr{K}(\mathcal{H}_t^s, \mathfrak{h})$ is continuous, admits a limit as $\lambda \searrow 0$ and vanishes as $\lambda \to \infty$.*

Proof. The inclusion $\mathscr{F}_0(\lambda) \in \mathscr{K}(\mathcal{H}_t^s, \mathfrak{h})$ follows from the compact embedding $\mathcal{H}_t^s \subset \mathcal{H}_{t'}^{s'}$ for any $s' < s$ and $t' < t$ (see for instance [1, Prop. 4.1.5]).

For the continuity and the existence of the limit as $\lambda \searrow 0$ one can use the same argument as the one used in the proof of Lemma 2.1. For the limit as $\lambda \to \infty$, we define the regularizing operator $\langle P \rangle^{-s} := (1 - \Delta)^{-s/2}$ and then observe that $\lambda^{-1/4} \mathscr{F}_0(\lambda) \langle P \rangle^{-s} = \lambda^{-1/4} (1 + \lambda)^{-s/2} \mathscr{F}_0(\lambda)$ for each $\lambda \in \mathbb{R}_+$ (see (2.1)). It follows that $\lim_{\lambda \to \infty} \|\lambda^{-1/4} \mathscr{F}_0(\lambda)\|_{\mathscr{B}(\mathcal{H}_t^s, \mathfrak{h})} = 0$ if and only if $\lim_{\lambda \to \infty} \|\lambda^{-1/4} (1 + \lambda)^{-s/2} \mathscr{F}_0(\lambda)\|_{\mathscr{B}(\mathcal{H}_t, \mathfrak{h})} = 0$. So, the claim follows from Lemma 2.1 (with the positive sign) as long as $-1/4 - s/2 < 1/4$, which is equivalent to the condition $s > -1$. $\square$

From now on, we use the notation $C_c(\mathbb{R}_+; \mathcal{G})$ for the set of compactly supported and continuous functions from $\mathbb{R}_+$ to some Hilbert space $\mathcal{G}$. With this notation and what precedes, we note that the multiplication operator $M : C_c(\mathbb{R}_+; \mathcal{H}_t^s) \to \mathscr{H}$ given by

$$(M\xi)(\lambda) := \lambda^{-1/4} \mathscr{F}_0(\lambda) \xi(\lambda), \quad \xi \in C_c(\mathbb{R}_+; \mathcal{H}_t^s), \ \lambda \in \mathbb{R}_+, \tag{2.4}$$

extends for $s \geq 0$ and $t > 3/2$ to an element of $\mathscr{B}\big(\mathsf{L}^2(\mathbb{R}_+; \mathcal{H}_t^s), \mathscr{H}\big)$.

The next step is to deal with the limit $\varepsilon \searrow 0$ of the operator $\delta_\varepsilon(L - \lambda)$ in Equation (2.3). For that purpose, we shall use the continuous extension of the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ to a duality $\langle \cdot, \cdot \rangle_{\mathcal{H}_t^s, \mathcal{H}_{-t}^{-s}}$ between $\mathcal{H}_t^s$ and $\mathcal{H}_{-t}^{-s}$.

Lemma 2.3. *Take $s \geq 0$, $t > 3/2$, $\lambda \in \mathbb{R}_+$ and $\varphi \in C_c(\mathbb{R}_+; \mathfrak{h})$. Then, we have*

$$\lim_{\varepsilon \searrow 0} \big\| \mathscr{F}_0^* \delta_\varepsilon(L - \lambda)\varphi - \mathscr{F}_0(\lambda)^* \varphi(\lambda) \big\|_{\mathcal{H}_{-t}^{-s}} = 0.$$

Proof. By definition of the norm of $\mathcal{H}_{-t}^{-s}$, one has

$$\left\|\mathscr{F}_0^* \delta_\varepsilon(L-\lambda)\varphi - \mathscr{F}_0(\lambda)^*\varphi(\lambda)\right\|_{\mathcal{H}_{-t}^{-s}}$$

$$= \sup_{f\in\mathscr{S},\,\|f\|_{\mathcal{H}_t^s}=1} \left|\left\langle f, \mathscr{F}_0^*\delta_\varepsilon(L-\lambda)\varphi - \mathscr{F}_0(\lambda)^*\varphi(\lambda)\right\rangle_{\mathcal{H}_t^s,\mathcal{H}_{-t}^{-s}}\right|$$

$$= \sup_{f\in\mathscr{S},\,\|f\|_{\mathcal{H}_t^s}=1} \left|\frac{1}{\pi}\int_0^\infty \mathrm{d}\mu \left\langle \mathscr{F}_0(\mu)f, \frac{\varepsilon}{(\mu-\lambda)^2+\varepsilon^2}\varphi(\mu)\right\rangle_{\mathfrak{h}} - \left\langle \mathscr{F}_0(\lambda)f, \varphi(\lambda)\right\rangle_{\mathfrak{h}}\right|$$

$$\leq \sup_{f\in\mathscr{S},\,\|f\|_{\mathcal{H}_t^s}=1} \left|\frac{1}{\pi}\int_0^\infty \mathrm{d}\mu \left\langle \big(\mathscr{F}_0(\mu)-\mathscr{F}_0(\lambda)\big)f, \frac{\varepsilon}{(\mu-\lambda)^2+\varepsilon^2}\varphi(\mu)\right\rangle_{\mathfrak{h}}\right| \tag{2.5}$$

$$+ \sup_{f\in\mathscr{S},\,\|f\|_{\mathcal{H}_t^s}=1} \left|\frac{1}{\pi}\int_0^\infty \mathrm{d}\mu \left\langle \mathscr{F}_0(\lambda)f, \frac{\varepsilon}{(\mu-\lambda)^2+\varepsilon^2}\big(\varphi(\mu)-\varphi(\lambda)\big)\right\rangle_{\mathfrak{h}}\right| \tag{2.6}$$

$$+ \sup_{f\in\mathscr{S},\,\|f\|_{\mathcal{H}_t^s}=1} \left|\frac{1}{\pi}\int_0^\infty \mathrm{d}\mu \left\langle \mathscr{F}_0(\lambda)f, \frac{\varepsilon}{(\mu-\lambda)^2+\varepsilon^2}\varphi(\lambda)\right\rangle_{\mathfrak{h}} - \left\langle \mathscr{F}_0(\lambda)f, \varphi(\lambda)\right\rangle_{\mathfrak{h}}\right| \tag{2.7}$$

Clearly, the term (2.7) converges to 0 as $\varepsilon \searrow 0$, as expected. Furthermore, the term (2.5) converges to 0 as $\varepsilon \searrow 0$ because of the continuity and the boundedness of the function $\lambda \mapsto \|\mathscr{F}_0(\lambda)\|_{\mathscr{B}(\mathcal{H}_t^s,\mathfrak{h})}$ (mentioned just after Lemma 2.1) together with the boundedness of the map $\lambda \mapsto \|\varphi(\lambda)\|_{\mathfrak{h}}$. Finally, the term (2.6) also converges to 0 as $\varepsilon \searrow 0$ because of the continuity and the boundedness of the function $\lambda \mapsto \varphi(\lambda) \in \mathfrak{h}$ together with the boundedness of the function $\lambda \mapsto \|\mathscr{F}_0(\lambda)\|_{\mathscr{B}(\mathcal{H}_t^s,\mathfrak{h})}$. $\qquad\square$

The next necessary result concerns the limits $T(\lambda \pm i0) := \lim_{\varepsilon\searrow 0} T(\lambda \pm i\varepsilon)$, $\lambda \in \mathbb{R}_+$. Fortunately, it is already known (see for example [13, Lemma 9.1]) that if $\sigma > 1$ in (2.2) then the limit $\big(1 + R_0(\lambda+i0)V\big)^{-1} := \lim_{\varepsilon\searrow 0}\big(1 + R_0(\lambda+i\varepsilon)V\big)^{-1}$ exists in $\mathscr{B}(\mathcal{H}_{-t}, \mathcal{H}_{-t})$ for any $t \in (1/2, \sigma-1/2)$, and that the map $\mathbb{R}_+ \ni \lambda \mapsto \big(1 + R_0(\lambda+i0)V\big)^{-1} \in \mathscr{B}(\mathcal{H}_{-t}, \mathcal{H}_{-t})$ is continuous. Corresponding results for $T(\lambda+i\varepsilon)$ follow immediately. Note that only the limits from the upper half-plane have been computed in [13], even though similar results for $T(\lambda - i0)$ could have been derived. Due to this lack of information in the literature and for the simplicity of the exposition, we consider from now on only the wave operator W_-.

Lemma 2.4. *Take $\sigma > 5$ in (2.2) and let $t \in (5/2, \sigma - 5/2)$. Then, the function*

$$\mathbb{R}_+ \ni \lambda \mapsto \lambda^{1/4} T(\lambda+i0)\mathscr{F}_0(\lambda)^* \in \mathscr{B}(\mathfrak{h}, \mathcal{H}_{\sigma-t})$$

is continuous and bounded, and the multiplication operator $B : C_c\big(\mathbb{R}_+; \mathfrak{h}\big) \to \mathsf{L}^2(\mathbb{R}_+; \mathcal{H}_{\sigma-t})$ given by

$$(B\varphi)(\lambda) := \lambda^{1/4} T(\lambda+i0)\mathscr{F}_0(\lambda)^*\varphi(\lambda) \in \mathcal{H}_{\sigma-t}, \quad \varphi \in C_c\big(\mathbb{R}_+;\mathfrak{h}\big),\ \lambda \in \mathbb{R}_+, \tag{2.8}$$

extends to an element of $\mathscr{B}\big(\mathscr{H}, \mathsf{L}^2(\mathbb{R}_+; \mathcal{H}_{\sigma-t})\big)$.

Proof. The continuity of the function $\lambda \mapsto \lambda^{1/4} T(\lambda+i0)\mathscr{F}_0(\lambda)^* \in \mathscr{B}(\mathfrak{h}, \mathcal{H}_{\sigma-t})$ follows from what has been said before. For the boundedness, it is sufficient to show that the function

$$\mathbb{R}_+ \ni \lambda \mapsto \lambda^{1/4}\big\|T(\lambda+i0)\mathscr{F}_0(\lambda)^*\big\|_{\mathscr{B}(\mathfrak{h},\mathcal{H}_{\sigma-t})} \tag{2.9}$$

is bounded in a neighbourhood of 0 and in a neighbourhood of $+\infty$.

For $\lambda > 1$, we know from [13, Lemma 9.1] that the function $\lambda \mapsto \|T(\lambda+i0)\|_{\mathscr{B}(\mathcal{H}_{-t},\mathcal{H}_{\sigma-t})}$ is bounded. We also know from Lemma 2.1 that the function $\mathbb{R}_+ \ni \lambda \mapsto \lambda^{1/4}\|\mathscr{F}_0(\lambda)^*\|_{\mathscr{B}(\mathfrak{h},\mathcal{H}_{-t})}$ is bounded. Thus, the function (2.9) stays bounded in a neighbourhood of $+\infty$.

For λ in a neighbourhood of 0, we use asymptotic developments for $T(\lambda+i0)$ and $\mathscr{F}_0(\lambda)^*$. The development for $\mathscr{F}_0(\lambda)^*$ (to be found in [13, Sec. 5]) can be written as follows. For each $s \in \mathbb{R}$, there exist $\gamma_0^*, \gamma_1^* \in \mathscr{B}(\mathfrak{h}, \mathcal{H}_{-t}^s)$ such that

$$\mathscr{F}_0(\lambda)^* = \big(\tfrac{\lambda}{4}\big)^{1/4}\big(\gamma_0^* - i\lambda^{1/2}\gamma_1^* + o(\lambda^{1/2})\big) \quad \text{in} \quad \mathscr{B}(\mathfrak{h}, \mathcal{H}_{-t}^s) \text{ as } \lambda \searrow 0.$$

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The development for $T(\lambda + i0)$ as $\lambda \searrow 0$ has been computed in [13, Lemmas 4.1 to 4.5]. It varies drastically depending on the presence of 0-energy eigenvalue and/or 0-energy resonance. We reproduce here the most singular behavior possible (*cf.* [13, Lemma 4.5]):

$$T(\lambda + i0) = \lambda^{-1} V P_0 V - i\lambda^{-1/2} C + O(1) \quad \text{in} \quad \mathscr{B}(\mathcal{H}^1_{-t}; \mathcal{H}_{\sigma-t}) \text{ as } \lambda \searrow 0,$$

with P_0 the orthogonal projection onto $\ker(H)$ and $C \in \mathscr{B}(\mathcal{H}^1_{-t}; \mathcal{H}_{\sigma-t})$. Now, using these expressions for $\mathscr{F}_0(\lambda)^*$ and $T(\lambda + i0)$, one can write $\lambda^{1/4} T(\lambda + i0) \mathscr{F}_0(\lambda)^*$ as a sum of terms bounded in $\mathscr{B}(\mathfrak{h}, \mathcal{H}_{\sigma-t})$ as $\lambda \searrow 0$ plus a term $\frac{1}{\sqrt{2}} \lambda^{-1/2} V P_0 V \gamma_0^*$ which is apparently unbounded. However, we know from the proof of [13, Thm. 5.3] that $P_0 V \gamma_0^* = 0$. Thus, all the terms in the asymptotic development of $\lambda^{1/4} T(\lambda + i0) \mathscr{F}_0(\lambda)^*$ are effectively bounded in $\mathscr{B}(\mathfrak{h}, \mathcal{H}_{\sigma-t})$ as $\lambda \searrow 0$, and thus the claim about boundedness is proved. The claim on the operator B is then a simple consequence of what precedes. $\qquad \square$

Remark 2.5. *If one assumes that H has no 0-energy eigenvalue and/or no 0-energy resonance, then one can prove Lemma 2.4 under a weaker assumption on the decay of V at infinity. However, even if the absence of 0-energy eigenvalue and 0-energy resonance is generic, we do not want to make such an implicit assumption in the sequel. The condition on V is thus imposed adequately.*

Before deriving our main result, we recall the action of the dilation group $\{U_\tau^+\}_{\tau \in \mathbb{R}}$ in $\mathsf{L}^2(\mathbb{R}_+)$, namely,

$$(U_\tau^+ f)(\lambda) := \mathrm{e}^{\tau/2} f(\mathrm{e}^\tau \lambda), \quad f \in C_\mathrm{c}(\mathbb{R}_+),\ \lambda \in \mathbb{R}_+,\ \tau \in \mathbb{R},$$

and denote its self-adjoint generator by A_+. We also introduce the function $\vartheta \in C(\mathbb{R}) \cap \mathsf{L}^\infty(\mathbb{R})$ given by

$$\vartheta(\nu) := \frac{1}{2}\big(1 - \tanh(2\pi\nu) - i \cosh(2\pi\nu)^{-1}\big), \quad \nu \in \mathbb{R}. \tag{2.10}$$

Finally, we recall that the Hilbert spaces $\mathsf{L}^2(\mathbb{R}_+; \mathcal{H}_t^s)$ and $\mathscr{H}$ can be naturally identified with the Hilbert spaces $\mathsf{L}^2(\mathbb{R}_+) \otimes \mathcal{H}_t^s$ and $\mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{h}$.

Theorem 2.6. *Take $\sigma > 7$ in (2.2) and let $t \in (7/2, \sigma - 7/2)$. Then, one has in $\mathscr{B}(\mathscr{H})$ the equality*

$$\mathscr{F}_0(W_- - 1)\mathscr{F}_0^* = -2\pi i\, M\,\big\{\vartheta(A_+) \otimes 1_{\mathcal{H}_{\sigma-t}}\big\}B, \tag{2.11}$$

with M and B defined in (2.4) and (2.8).

The proof below consists in two parts. First, we show that the expression (2.3) is well-defined for φ and ψ in dense subsets of $\mathscr{H}$ (and thus equal to $\big\langle \mathscr{F}_0(W_\pm - 1) \mathscr{F}_0^* \varphi, \psi \big\rangle_{\mathscr{H}}$ due to the computations presented at the beginning of the section). Second, we show that the expression (2.3) is equal to $\big\langle -2\pi i\, M\,\big\{\vartheta(A_+) \otimes 1_{\mathcal{H}_{\sigma-t}}\big\}B\varphi, \psi\big\rangle_{\mathscr{H}}$.

Proof. Take $\varphi \in C_\mathrm{c}(\mathbb{R}_+; \mathfrak{h})$ and $\psi \in C_\mathrm{c}^\infty(\mathbb{R}_+) \odot C(\mathbb{S}^2)$, and set $s := \sigma - t > 7/2$. Then, we have for each $\varepsilon > 0$ and $\lambda \in \mathbb{R}_+$ the inclusions

$$g_\varepsilon(\lambda) := \lambda^{1/4} T(\lambda + i\varepsilon) \mathscr{F}_0^* \delta_\varepsilon(L - \lambda)\varphi \in \mathcal{H}_s \qquad \text{and} \qquad f(\lambda) := \lambda^{-1/4} \mathscr{F}_0(\lambda)^* \psi(\lambda) \in \mathcal{H}_{-s}.$$

It follows that the expression (2.3) is equal to

$$-\int_{\mathbb{R}} \mathrm{d}\lambda \lim_{\varepsilon \searrow 0} \int_0^\infty \mathrm{d}\mu \big\langle T(\lambda + i\varepsilon) \mathscr{F}_0^* \delta_\varepsilon(L - \lambda)\varphi, (\mu - \lambda + i\varepsilon)^{-1} \mathscr{F}_0(\mu)^* \psi(\mu)\big\rangle_{\mathcal{H}_s, \mathcal{H}_{-s}}$$

$$= -\int_{\mathbb{R}_+} \mathrm{d}\lambda \lim_{\varepsilon \searrow 0} \int_0^\infty \mathrm{d}\mu \bigg\langle g_\varepsilon(\lambda), \frac{\lambda^{-1/4}\mu^{1/4}}{\mu - \lambda + i\varepsilon}\, f(\mu)\bigg\rangle_{\mathcal{H}_s, \mathcal{H}_{-s}}.$$

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Now, using the formula $(\mu - \lambda + i\varepsilon)^{-1} = -i\int_0^\infty \mathrm{d}z\, \mathrm{e}^{i(\mu-\lambda)z}\,\mathrm{e}^{-\varepsilon z}$ and then applying Fubini's theorem, one obtains that

$$
\lim_{\varepsilon \searrow 0}\int_0^\infty \mathrm{d}\mu\, \left\langle g_\varepsilon(\lambda), \frac{\lambda^{-1/4}\mu^{1/4}}{\mu - \lambda + i\varepsilon}\, f(\mu)\right\rangle_{\mathcal{H}_s,\mathcal{H}_{-s}}
$$
$$
= -i\lim_{\varepsilon \searrow 0}\int_0^\infty \mathrm{d}z\, \mathrm{e}^{-\varepsilon z}\left\langle g_\varepsilon(\lambda), \int_0^\infty \mathrm{d}\mu\, \mathrm{e}^{i(\mu-\lambda)z}\,\lambda^{-1/4}\mu^{1/4}f(\mu)\right\rangle_{\mathcal{H}_s,\mathcal{H}_{-s}}
$$
$$
= -i\lim_{\varepsilon \searrow 0}\int_0^\infty \mathrm{d}z\, \mathrm{e}^{-\varepsilon z}\left\langle g_\varepsilon(\lambda), \int_{-\lambda}^\infty \mathrm{d}\nu\, \mathrm{e}^{i\nu z}\left(\frac{\nu+\lambda}{\lambda}\right)^{1/4}f(\nu+\lambda)\right\rangle_{\mathcal{H}_s,\mathcal{H}_{-s}}. \tag{2.12}
$$

Furthermore, the integrant in (2.12) can be bounded independently of $\varepsilon \in (0,1)$. Indeed, one has

$$
\left|\mathrm{e}^{-\varepsilon z}\left\langle g_\varepsilon(\lambda), \int_{-\lambda}^\infty \mathrm{d}\nu\, \mathrm{e}^{i\nu z}\left(\frac{\nu+\lambda}{\lambda}\right)^{1/4}f(\nu+\lambda)\right\rangle_{\mathcal{H}_s,\mathcal{H}_{-s}}\right|
$$
$$
\leq \left\|g_\varepsilon(\lambda)\right\|_{\mathcal{H}_s}\left\|\int_{-\lambda}^\infty \mathrm{d}\nu\, \mathrm{e}^{i\nu z}\left(\frac{\nu+\lambda}{\lambda}\right)^{1/4}f(\nu+\lambda)\right\|_{\mathcal{H}_{-s}}, \tag{2.13}
$$

and we know from Lemma 2.3 and the paragraph following it that $g_\varepsilon(\lambda)$ converges to $g_0(\lambda) := \lambda^{1/4}\, T(\lambda + i0)\mathscr{F}_0^*(\lambda)\varphi(\lambda)$ in $\mathcal{H}_s$ as $\varepsilon \searrow 0$. Therefore, the family $\|g_\varepsilon(\lambda)\|_{\mathcal{H}_s}$ (and thus the r.h.s. of (2.13)) is bounded by a constant independent of $\varepsilon \in (0,1)$.

In order to exchange the integral over z and the limit $\varepsilon \searrow 0$ in (2.12), it remains to show that the second factor in (2.13) belongs to $\mathsf{L}^1(\mathbb{R}_+, \mathrm{d}z)$. For that purpose, we denote by h_λ the trivial extension of the function $(-\lambda, \infty) \ni \nu \mapsto \left(\frac{\nu+\lambda}{\lambda}\right)^{1/4}f(\nu+\lambda) \in \mathcal{H}_{-s}$ to all of $\mathbb{R}$, and then note that the second factor in (2.13) can be rewritten as $(2\pi)^{1/2}\|(\mathscr{F}_1^* h_\lambda)(z)\|_{\mathcal{H}_{-s}}$, with $\mathscr{F}_1$ the one-dimensional Fourier transform. To estimate this factor, observe that if P_1 denotes the self-adjoint operator $-i\nabla$ on $\mathbb{R}$, then

$$
\left\|(\mathscr{F}_1^* h_\lambda)(z)\right\|_{\mathcal{H}_{-s}} = \langle z\rangle^{-2}\left\|(\mathscr{F}_1^*\langle P_1\rangle^2 h_\lambda)(z)\right\|_{\mathcal{H}_{-s}}, \quad z \in \mathbb{R}_+.
$$

Consequently, one would have that $\|(\mathscr{F}_1^* h_\lambda)(z)\|_{\mathcal{H}_{-s}} \in \mathsf{L}^1(\mathbb{R}_+, \mathrm{d}z)$ if the norm $\left\|(\mathscr{F}_1^*\langle P_1\rangle^2 h_\lambda)(z)\right\|_{\mathcal{H}_{-s}}$ were bounded independently of z. Now, if $\psi = \eta \otimes \xi$ with $\eta \in C_\mathrm{c}^\infty(\mathbb{R}_+)$ and $\xi \in C(\mathbb{S}^2)$, then one has for any $x \in \mathbb{R}^3$

$$
\big(f(\nu+\lambda)\big)(x) = \frac{1}{4\pi^{3/2}}\,\eta(\nu+\lambda)\int_{\mathbb{S}^2}\mathrm{d}\omega\, \mathrm{e}^{i\sqrt{\nu+\lambda}\,\omega\cdot x}\,\xi(\omega).
$$

Therefore, one has

$$
\big(h_\lambda(\nu)\big)(x) = \begin{cases} \frac{1}{4\pi^{3/2}}\left(\frac{\nu+\lambda}{\lambda}\right)^{1/4}\eta(\nu+\lambda)\int_{\mathbb{S}^2}\mathrm{d}\omega\, \mathrm{e}^{i\sqrt{\nu+\lambda}\,\omega\cdot x}\,\xi(\omega) & \nu > -\lambda \\ 0 & \nu \leq -\lambda, \end{cases} \tag{2.14}
$$

which in turns implies that

$$
\left|\big\{(\mathscr{F}_1^*\langle P_1\rangle^2 h_\lambda)(z)\big\}(x)\right| \leq \mathrm{Const.}\,\langle x\rangle^2,
$$

with a constant independent of $x \in \mathbb{R}^3$ and $z \in \mathbb{R}_+$. Since the r.h.s. belongs to $\mathcal{H}_{-s}$ for $s > 7/2$, one concludes that $\left\|(\mathscr{F}_1^*\langle P_1\rangle^2 h_\lambda)(z)\right\|_{\mathcal{H}_{-s}}$ is bounded independently of z for each $\psi = \eta \otimes \xi$, and thus for each $\psi \in C_\mathrm{c}^\infty(\mathbb{R}_+)\odot C(\mathbb{S}^2)$ by linearity. As a consequence, one can apply Lebesgue dominated convergence theorem and obtain that (2.12) is equal to

$$
-i\left\langle g_0(\lambda), \int_0^\infty \mathrm{d}z\int_{\mathbb{R}}\mathrm{d}\nu\, \mathrm{e}^{i\nu z}\, h_\lambda(\nu)\right\rangle_{\mathcal{H}_s,\mathcal{H}_{-s}}.
$$

With this equality, one has concluded the first part of the proof; that is, one has justified the equality between the expression (2.3) and $\left\langle \mathscr{F}_0(W_\pm - 1)\mathscr{F}_0^*\varphi, \psi\right\rangle_{\mathscr{H}}$ on the dense sets of vectors introduced at the beginning of the proof.

The next task is to show that $\langle \mathscr{F}_0(W_\pm - 1)\mathscr{F}_0^*\varphi, \psi\rangle_{\mathscr{H}}$ is equal to $\langle -2\pi i\, M\{\vartheta(A_+) \otimes 1_{\mathcal{H}_{-s}}\}B\varphi, \psi\rangle_{\mathscr{H}}$. For that purpose, we write χ_+ for the characteristic function for $\mathbb{R}_+$. Since h_λ has compact support, we obtain the following equalities in the sense of distributions (with values in $\mathcal{H}_{-s}$):

$$
\int_0^\infty \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}\nu\, \mathrm{e}^{i\nu z}\, h_\lambda(\nu) = \sqrt{2\pi}\int_{\mathbb{R}} \mathrm{d}\nu\, \big(\mathscr{F}_1^*\chi_+\big)(\nu)\, h_\lambda(\nu)
$$

$$
= \sqrt{2\pi}\int_{-\lambda}^\infty \mathrm{d}\nu\, \big(\mathscr{F}_1^*\chi_+\big)(\nu)\left(\frac{\nu+\lambda}{\lambda}\right)^{1/4} f(\nu+\lambda)
$$

$$
= \sqrt{2\pi}\int_{\mathbb{R}} \mathrm{d}\mu\, \big(\mathscr{F}_1^*\chi_+\big)\big(\lambda(\mathrm{e}^\mu - 1)\big)\,\lambda\, \mathrm{e}^{5\mu/4} f(\mathrm{e}^\mu\, \lambda) \qquad (\mathrm{e}^\mu\, \lambda := \nu + \lambda)
$$

$$
= \sqrt{2\pi}\int_{\mathbb{R}} \mathrm{d}\mu\, \big(\mathscr{F}_1^*\chi_+\big)\big(\lambda(\mathrm{e}^\mu - 1)\big)\,\lambda\, \mathrm{e}^{3\mu/4}\big\{\big(U_\mu^+ \otimes 1_{\mathcal{H}_{-s}}\big)f\big\}(\lambda).
$$

Then, by using the fact that $\mathscr{F}_1^*\chi_+ = \sqrt{\frac{\pi}{2}}\,\delta_0 + \frac{i}{\sqrt{2\pi}}\,\mathrm{Pv}\frac{1}{(\cdot)}$ with δ_0 the Dirac delta distribution and Pv the principal value, one gets that

$$
\int_0^\infty \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}\nu\, \mathrm{e}^{i\nu z}\, h_\lambda(\nu) = \int_{\mathbb{R}} \mathrm{d}\mu\left(\pi\,\delta_0(\mathrm{e}^\mu - 1) + i\,\mathrm{Pv}\frac{\mathrm{e}^{3\mu/4}}{\mathrm{e}^\mu - 1}\right)\big\{\big(U_\mu^+ \otimes 1_{\mathcal{H}_{-s}}\big)f\big\}(\lambda).
$$

So, by considering the identity

$$
\frac{\mathrm{e}^{3\mu/4}}{\mathrm{e}^\mu - 1} = \frac{1}{4}\left(\frac{1}{\sinh(\mu/4)} + \frac{1}{\cosh(\mu/4)}\right)
$$

and the equality [11, Table 20.1]

$$
\big(\mathscr{F}_1\bar{\vartheta}\big)(\nu) := \sqrt{\frac{\pi}{2}}\,\delta_0\big(\mathrm{e}^\nu - 1\big) + \frac{i}{4\sqrt{2\pi}}\,\mathrm{Pv}\left(\frac{1}{\sinh(\nu/4)} + \frac{1}{\cosh(\nu/4)}\right),
$$

with ϑ defined in (2.10), one infers that

$$
\langle \mathscr{F}_0(W_- - 1)\mathscr{F}_0^*\varphi, \psi\rangle_{\mathscr{H}}
$$
$$
= i\int_{\mathbb{R}_+} \mathrm{d}\lambda\left\langle g_0(\lambda), \int_{\mathbb{R}} \mathrm{d}\mu\left\{\pi\,\delta_0\big(\mathrm{e}^\mu - 1\big)\right.\right.
$$
$$
\left.\left. + \frac{i}{4}\,\mathrm{Pv}\left(\frac{1}{\sinh(\mu/4)} + \frac{1}{\cosh(\mu/4)}\right)\right\}\big\{\big(U_\mu^+ \otimes 1_{\mathcal{H}_{-s}}\big)f\big\}(\lambda)\right\rangle_{\mathcal{H}_s, \mathcal{H}_{-s}}
$$
$$
= i\sqrt{2\pi}\int_{\mathbb{R}_+} \mathrm{d}\lambda\left\langle g_0(\lambda), \int_{\mathbb{R}} \mathrm{d}\mu\,\big(\mathscr{F}_1\bar{\vartheta}\big)(\mu)\big\{\big(U_\mu^+ \otimes 1_{\mathcal{H}_{-s}}\big)f\big\}(\lambda)\right\rangle_{\mathcal{H}_s, \mathcal{H}_{-s}}.
$$

Finally, by recalling that $\{\vartheta(A_+) \otimes 1_{\mathcal{H}_{-s}}\}f = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}\mu\,\big(\mathscr{F}_1\bar{\vartheta}\big)(\mu)\big(U_\mu^+ \otimes 1_{\mathcal{H}_{-s}}\big)f$, that $g_0(\lambda) = (B\varphi)(\lambda)$ and that $f = M^*\psi$, one obtains

$$
\langle \mathscr{F}_0(W_- - 1)\mathscr{F}_0^*\varphi, \psi\rangle_{\mathscr{H}} = 2\pi i\int_{\mathbb{R}_+} \mathrm{d}\lambda\left\langle(B\varphi)(\lambda), \big\{\big(\vartheta(A_+)^* \otimes 1_{\mathcal{H}_{-s}}\big)M^*\psi\big\}(\lambda)\right\rangle_{\mathcal{H}_s, \mathcal{H}_{-s}}
$$
$$
= \big\langle -2\pi i\, M\{\vartheta(A_+) \otimes 1_{\mathcal{H}_{-s}}\}B\varphi, \psi\big\rangle_{\mathscr{H}}.
$$

This concludes the proof, since the sets of vectors $\varphi \in C_c(\mathbb{R}_+; \mathfrak{h})$ and $\psi \in C_c^\infty(\mathbb{R}_+) \odot C(\mathbb{S}^2)$ are dense in $\mathscr{H}$. $\qquad\square$

We now derive a technical lemma which will be essential for the proof of Theorem 1.1.

Lemma 2.7. *Take $s > -1$ and $t > 3/2$. Then, the difference*

$$\{\vartheta(A_+) \otimes 1_{\mathfrak{h}}\}M - M\{\vartheta(A_+) \otimes 1_{\mathcal{H}_t^s}\}$$

belongs to $\mathscr{K}\big(\mathsf{L}^2(\mathbb{R}_+; \mathcal{H}_t^s), \mathscr{H}\big).$

Proof. (i) The unitary operator $\mathscr{G} : \mathsf{L}^2(\mathbb{R}) \to \mathsf{L}^2(\mathbb{R}_+)$ given by

$$(\mathscr{G}f)(\lambda) := \lambda^{-1/2} f\big(\ln(\lambda)\big), \quad f \in C_c^\infty(\mathbb{R}), \ \lambda \in \mathbb{R}_+,$$

satisfies $(\mathscr{G}^* U_\tau^+ \mathscr{G} f)(x) = f(x + \tau)$ and $(\mathscr{G}^* \mathrm{e}^{i\tau \ln(\mathcal{L})} \mathscr{G} f)(x) = \mathrm{e}^{i\tau x} f(x)$ for each $x, \tau \in \mathbb{R}$, with $\mathcal{L}$ the maximal multiplication operator in $\mathsf{L}^2(\mathbb{R}_+)$ by the variable in $\mathbb{R}_+$. It follows that $\mathscr{G}^* A_+ \mathscr{G} = P_1$ on $\mathcal{D}(P_1)$ and that $\mathscr{G}^* \ln(\mathcal{L}) \mathscr{G} = X_1$ on $\mathcal{D}(X_1)$, with P_1 and X_1 the self-adjoint operators of momentum and position in $\mathsf{L}^2(\mathbb{R})$.

Now, take f_1, f_2 two complex-valued continuous functions on $\mathbb{R}$ having limits at $\pm\infty$; that is, $f_1, f_2 \in C([-\infty, \infty])$. Then, a standard result of Cordes implies the inclusion $[f_1(P_1), f_2(X_1)] \in \mathscr{K}\big(\mathsf{L}^2(\mathbb{R})\big)$ (see for instance [1, Thm. 4.1.10]). Conjugating this inclusion with the unitary operator $\mathscr{G}$, one thus infers that $\big[f_1(A_+), f_3(\mathcal{L})\big] \in \mathscr{K}\big(\mathsf{L}^2(\mathbb{R}_+)\big)$ with $f_3 := f_2 \circ \ln \in C([0, \infty])$.

(ii) We know from Lemma 2.2 and Definition (2.4) that

$$(M\xi)(\lambda) := m(\lambda)\xi(\lambda), \quad \xi \in C_c(\mathbb{R}_+; \mathcal{H}_t^s), \ \lambda \in \mathbb{R}_+,$$

with $m \in C\big([0, \infty]; \mathscr{K}(\mathcal{H}_t^s, \mathfrak{h})\big)$. We also know that the algebraic tensor product $C([0, \infty]) \odot \mathscr{K}(\mathcal{H}_t^s, \mathfrak{h})$ is dense in $C\big([0, \infty]; \mathscr{K}(\mathcal{H}_t^s, \mathfrak{h})\big)$, when $C\big([0, \infty]; \mathscr{K}(\mathcal{H}_t^s, \mathfrak{h})\big)$ is equipped with the uniform topology (see [22, Thm. 1.15]). So, for each $\varepsilon > 0$ there exist $n \in \mathbb{N}^*$, $a_j \in C([0, \infty])$ and $b_j \in \mathscr{K}(\mathcal{H}_t^s, \mathfrak{h})$ such that such that $\big\| M - \sum_{j=1}^n a_j(\mathcal{L}) \otimes b_j \big\|_{\mathscr{B}(\mathsf{L}^2(\mathbb{R}_+; \mathcal{H}_t^s), \mathscr{H})} < \varepsilon$. Therefore, in order to prove the claim, it is sufficient to show that the operator

$$\{\vartheta(A_+) \otimes 1_{\mathfrak{h}}\}\left\{\sum_{j=1}^n a_j(\mathcal{L}) \otimes b_j\right\} - \left\{\sum_{j=1}^n a_j(\mathcal{L}) \otimes b_j\right\}\{\vartheta(A_+) \otimes 1_{\mathcal{H}_t^s}\} = \sum_{j=1}^n \big[\vartheta(A_+), a_j(\mathcal{L})\big] \otimes b_j \quad (2.15)$$

is compact. But, we know that $b_j \in \mathscr{K}(\mathcal{H}_t^s, \mathfrak{h})$ and that $\big[\vartheta(A_+), a_j(\mathcal{L})\big] \in \mathscr{K}\big(\mathsf{L}^2(\mathbb{R}_+)\big)$ due to point (i). So, it immediately follows that the operator (2.15) is compact, since finite sums and tensor products of compact operators are compact operators (see [9, Thm. 2]). $\square$

Before giving the proof of Theorem 1.1, we recall the action of the dilation group $\{U_\tau\}_{\tau \in \mathbb{R}}$ in $\mathcal{H}$, namely,

$$(U_\tau f)(x) := \mathrm{e}^{3\tau/2} f(\mathrm{e}^\tau x), \quad f \in C_c(\mathbb{R}^3), \ x \in \mathbb{R}^3, \ \tau \in \mathbb{R},$$

and denote its self-adjoint generator by A. The image $\mathscr{F}_0 R(A) \mathscr{F}_0^*$ of $R(A) := \frac{1}{2}\big(1 + \tanh(\pi A) - i \cosh(\pi A)^{-1}\big)$ in $\mathscr{B}(\mathscr{H})$ can be easily computed. Indeed, one has the decomposition $\mathscr{F}_0 = \mathscr{U}\mathscr{F}$, with $\mathscr{U} : \mathcal{H} \to \mathscr{H}$ given by $\big((\mathscr{U}f)(\lambda)\big)(\omega) := \big(\frac{\lambda}{4}\big)^{1/4} f(\sqrt{\lambda}\,\omega)$ for each $f \in \mathscr{S}$, $\lambda \in \mathbb{R}_+$ and $\omega \in \mathbb{S}^2$. Furthermore, one has the identities $\mathscr{F} A \mathscr{F}^* = -A$ on $\mathcal{D}(A)$ and $\mathscr{U} A \mathscr{U}^* = 2A_+ \otimes 1_{\mathfrak{h}}$ on $\mathcal{D}(A_+ \otimes 1_{\mathfrak{h}})$. Therefore, one obtains that

$$\mathscr{F}_0 R(A) \mathscr{F}_0^* = \vartheta(A_+) \otimes 1_{\mathfrak{h}}.$$

Proof of Theorem 1.1. Set $s = 0$ and $t \in (7/2, \sigma - 7/2)$. Then, we deduce from Theorem 2.6, Lemma 2.7 and the above paragraph that

$$\begin{aligned}
W_- - 1 &= -2\pi i\, \mathscr{F}_0^* M\{\vartheta(A_+) \otimes 1_{\mathcal{H}_{\sigma-t}}\} B \mathscr{F}_0 \\
&= -2\pi i\, \mathscr{F}_0^* \{\vartheta(A_+) \otimes 1_{\mathfrak{h}}\} M B \mathscr{F}_0 + K \\
&= R(A) \mathscr{F}_0^* (-2\pi i M B) \mathscr{F}_0 + K,
\end{aligned}$$

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with $K \in \mathscr{K}(\mathcal{H})$. Comparing $-2\pi i M B$ with the usual expression for the scattering matrix $S(\lambda)$ (see for example [13, Eq. (5.1)]), one observes that $-2\pi i M B = \int_{\mathbb{R}_+}^{\oplus} \mathrm{d}\lambda \left(S(\lambda) - 1 \right)$. Since $\mathscr{F}_0$ defines the spectral representation of H_0, one obtains that

$$W_- - 1 = R(A)(S - 1) + K. \tag{2.16}$$

The formula for $W_+ - 1$ follows then from (2.16) and the relation $W_+ = W_- S^*$. $\qquad\square$

Remark 2.8. *Formulas* (1.2) *were already obtained in [18] under an implicit assumption. The only difference is that the operator $R(A)$ is replaced in [18] by an operator $\varphi(A)$ slightly more complicated. The resulting formulas for the wave operators differ by a compact term, but compact operators do not play any role in the algebraic construction (both expressions for the wave operators belong to the C^*-algebra constructed in [18, Sec. 4] and thus coincide after taking the quotient by the ideal of compact operators). Consequently, the topological approach of Levinson's theorem presented in [18] also applies here, with the implicit assumption no longer necessary.*

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