ON AN INVERSE HYPERBOLIC PROBLEM

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Abstract. We consider the inverse problem of recovery of an initial condition of a hyperbolic PDE. This problem is also called sometimes "thermoacoustic tomography". In the past publications both stability estimates and convergent numerical methods for this problem were obtained only under some restrictive conditions imposed on the principal part of the elliptic operator. In this paper logarithmic stability estimates are obatined for an arbitrary variable principal part of that operator. Convergence of the Quasi-Reversibility Method to the exact solution is also established for this case. Both complete and incomplete data collection cases are considered.

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1. Introduction. We consider the inverse problem of recovery of one of initial conditions of the hyperbolic equation from boundary measurements. This problem is also called sometimes "thermoacoustic tomography". In thermoacoustic tomography (TAT) a short radio frequency pulse is sent in a biological tissue [2, 19]. Some energy is absorbed. It is well known that malignant legions absorb much more energy than healthy ones. Then the tissue expands and radiates a pressure wave satisfying equation

$$u_{tt} = c^2(x)\,\Delta u, x \in \mathbb{R}^3. \tag{1.1}$$

Let $u(x,0) = f(x)$ and $u_t(x,0) = 0$. The function $u(x,t)$ is measured by transducers at certain locations either at the boundary of the medium of interest or outside of this medium. The function $f(x)$ characterizes the absorption of the medium. Hence, if one would know the function $f(x)$, then one would know locations of malignant spots. The inverse problem consists in determining $f(x)$ using those measurements.

Both stability estimates and convergent numerical methods for the problem of determining of one of initial conditions of equation (1.1) are currently known only under some restrictive conditions imposed on the coefficient $c(x)$ (section 2). In addition, except of the case $c(x) \equiv 1$ in [42], those numerical methods are known only for complete data collection, i.e. when boundary measurements are given at the entire boundary of the domain of interest.

The goal of this publication is to obtain logarithmic stability estimates as well as convergent numerical methods for the inverse problem of determining an initial condition in a general hyperbolic PDE without any restrictions on its coefficients, except of some natural ones. Naturally, stability estimates imply uniqueness. Both complete and incomplete data collection cases are considered. We assume here that the data are given on the infinite time interval $t \in (0,\infty)$. It is shown in Remarks 3.1 (section 3) that the case $t \in (0,\infty)$ is well acceptable in applications.

First, we apply a well known analog of the Laplace transform to obtain a similar inverse problem for a parabolic PDE. In the complete data case the logarithmic stability estimate follows from the previous result of the author [30]. In the case when the data are given on a hyperplane, we significantly modify the proof of Theorem 1 of [31] when establishing our logarithmic stability estimate. More precisely, we prove this estimate for an integral inequality rather than for the parabolic PDE. We need this generalization to establish convergence rate of our numerical method. Results of both publications [30, 31] were obtained via Carleman estimates. In particular, a non-standard Carleman estimate was derived in [31], see Lemma 3.1 in section 3.

Let $\Omega \subset \{x_1 > 0\}$ be a bounded domain with the boundary $\partial\Omega \in C^5$. Let $T > 0$. Denote

$$Q_T = \Omega \times (0,T)\,, S_T = \partial\Omega \times (0,T)\,, P = \{x_1 = 0\}\,, P_T = P \times (0,T)\,.$$

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Below $C^{k+\alpha}, C^{2k+\alpha,k+\alpha/2}$ are Hölder spaces, where $k \geq 0$ is an integer and $\alpha \in (0,1)$. Consider the elliptic operator L of the second order with its principal part L_0,

$$Lu = \sum_{i,j=1}^{n} a_{i,j}(x) u_{x_i x_j} + \sum_{j=1}^{n} d_j(x) u_{x_j} + d_0(x) u, x \in \mathbb{R}^n, \tag{1.2}$$

$$L_0 u = \sum_{i,j=1}^{n} a_{i,j}(x) u_{x_i x_j}, \tag{1.3}$$

$$a_{i,j} \in C^{k+\alpha}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n), d_j, d_0 \in C^{k+\alpha}(\mathbb{R}^n), k \geq 2, \alpha \in (0,1), \tag{1.4}$$

$$\mu_1 |\eta|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x) \eta_i \eta_j \leq \mu_2 |\eta|^2, \forall x, \eta \in \mathbb{R}^n; \mu_1, \mu_2 = const. > 0. \tag{1.5}$$

Let the function $f(x)$ be such that

$$f \in C^{p+\alpha}(\mathbb{R}^n), p \geq 4, f(x) = 0, x \in \mathbb{R}^n \diagdown \Omega. \tag{1.6}$$

Consider the following Cauchy problem

$$u_{tt} = Lu, x \in \mathbb{R}^n, t \in (0, \infty), \tag{1.7}$$

$$u(x,0) = f(x), u_t(x,0) = 0. \tag{1.8}$$

Using the standard method of energy estimates [46], one can easily find verifiable sufficient conditions imposed on coefficient of the operator L and the initial condition f guaranteeing the validity of Assumptions 1,2. Assumption 2 is stronger than Assumption 1.

Assumption 1. We assume that integers $k \geq 2, p \geq 4$ in (1.4), (1.6) are such that there exists unique solution $u \in C^2(\mathbb{R}^n \times [0,T]), \forall T > 0$.

Assumption 2. We assume that integers $k \geq 2, p \geq 4$ in (1.4), (1.6) are such that there exists unique solution $u \in C^4(\mathbb{R}^n \times [0,T]), \forall T > 0$ satisfying

$$\|u\|_{C^4(\mathbb{R}^n \times [0,T])} \leq \delta e^{bT}, \tag{1.9}$$

where constants $\delta, b > 0$ and $b = b(L)$ depends on coefficients of the operator L.

Although the inequality (1.9) can be established via energy estimates as long as $u \in C^4(\mathbb{R}^n \times [0,T]), \forall T > 0$, we assume its validity for brevity. We consider the following two Inverse Problems:

Inverse Problem 1 (IP1). *Suppose that conditions (1.2)-(1.6) and Assumption 1 hold. Let* $u \in C^2(\mathbb{R}^n \times [0,T]), \forall T > 0$ *be the solution of the problem (1.7), (1.8). Assume that the function* $f(x)$ *is unknown. Determine this function, assuming that the following function* $\varphi_1(x,t)$ *is known*

$$u\mid_{S_\infty} = \varphi_1(x,t). \tag{1.10}$$

Inverse Problem 2 (IP2). *Suppose that conditions (1.2)-(1.6) are Assumption 1 hold. Let* $u \in C^2(\mathbb{R}^n \times [0,T]), \forall T > 0$ *be the solution of the problem (1.7), (1.8). Assume that the function* $f(x)$ *is unknown. Determine this function, assuming that the following function* $\varphi_2(x,t)$ *is known*

$$u\mid_{x \in P_\infty} = \varphi_2(x,t). \tag{1.11}$$

IP1 has complete data collection, since the function φ_1 is known at the entire boundary of the domain of interest Ω. On the other hand, IP2 represents a special case of incomplete data collection, since $\Omega \subset \{x_1 > 0\}$.

In section 2 we briefly discuss published results. In section 3 we prove logarithmic stability theorems. In section 4 we construct the convergent numerical method. For brevity, we are not considering the most general cases here, such as, e.g. minimal smoothness assumptions, etc.

2. Brief Overview of Published Results. TAT has attracted a significant interest in the past several years. We now give a brief overview of mathematical results focusing on those, which are the most relevant ones to this paper. We refer to [41] for a review paper. There are a number of publications in which explicit formulas for the reconstruction of the function $f(x)$ are given for the case of IP1 with $L = \Delta$, see, e.g. [17, 18, 19, 41]. In particular, the case of incomplete data was considered in [42]. Naturally these formulas imply certain stability estimates as well as numerical methods with good performances.

Another approach to IP1, IP2 is via analyzing the case when both Dirichlet and Neumann data are given at S_T for IP1 and at P_T for IP2. An elementary, well known and stable procedure of deriving the Neumann condition from the Dirichlet condition for both IP1 and IP2 is described in section 3 for the parabolic PDE, and a very similar procedure takes place in the hyperbolic case. Consider now IP1. Since both Dirichlet and Neumann data at S_T are stably delivered by this procedure, then the problem of determining the initial condition $f(x)$ can be reformulated as the Cauchy problem with the lateral Dirichlet and Neumann data at S_T. This problem consists in determining the function $u(x,t)$ inside of the time cylinder Q_T. If one would estimate the norm $\|u\|_{H^1(Q_T)}$ via norms of those Cauchy data, then the trace theorem would imply an estimate of the target function $f(x)$.

The Lipschitz stability estimate for that Cauchy problem plays a fundamental role in the control theory since it is used for proofs of exact controllability theorems. The first Lipschitz stability estimate for that problem was proved in 1986 in [52] for equation (1.1) with $c \equiv 1$ with the aim of applying to the control theory. However, the method of multipliers, which was proposed in [52], cannot handle neither variable lower order terms of the hyperbolic operator nor a variable coefficient $c(x)$. On the other hand, Carleman estimates are not sensitive to lower order terms of PDE operators and can also handle the case of a variable function $c(x)$.

For the first time, the Carleman estimate was applied to that Cauchy problem in [26]. As a result, the Lipschitz stability estimate was obtained for the Cauchy problem with the lateral data at S_T for the case of the hyperbolic equation (1.7) with the principal part $L_0 = \Delta$ of the operator L and variable lower order terms. Next, the result of [26] was extended in [25, 28] to the case of a hyperbolic inequality, which is more general than the hyperbolic PDE. Although in publications [25, 26, 28] $c \equiv 1$, it is clear from [25, 26] that the key idea consists in applying the Carleman estimate, while a specific form of the principal part of the hyperbolic operator is less important. This thought is reflected in the proof of Theorem 3.4.8 of the book [23]. Thus, the Lipschitz stability estimate for the variable coefficient $c(x)$ was obtained in section 2.4 of the book [38] as well as in [14]. Note that in [38] the case of a hyperbolic inequality was considered. The technique of [25] was used in the control theory in, e.g. [47, 48].

In the case of parabolic and elliptic operators, Carleman estimates are known for rather arbitrary variable principal parts [23, 38, 50]. On the other hand, in the hyperbolic case the Carleman estimate is known only when the principal part of the operator is the same as in (1.1) and also

$$\left(x - x_0, \nabla\left(c^{-2}(x)\right)\right) + \alpha \geq 0, x \in \overline{\Omega} \text{ for an } \alpha = const. > 0, \tag{2.1}$$

for a certain point x_0, where $(,)$ is the scalar product in $\mathbb{R}^n$ [9, 23, 37, 38]. Thus, above Lipschitz stability results hold only if (2.1) is in place. Clearly, (2.1) is valid for $c \equiv 1$. The second way of proving such results is via imposing the non-trapping condition on the function $c(x)$ [4, 55]. The third way is to impose on $c(x)$ a condition of the Riemannian geometry, which would guarantee the availability of the Carleman estimate, see, e.g. [56] for such a condition (in [56] a related problem of the recovery of the coefficient $c(x)$ in (1.1) is considered, see below about this topic). Since the latter two conditions cannot be analytically verified for a generic case, then they are equally restrictive with (2.1). A slight variation of (2.1) guarantees non-trapping, see formula (3.24) in [53]. Unlike [25, 28, 38], the techniques of [4, 55] were not applied to hyperbolic inequalities. Uniqueness theorems for TAT were obtained in [2, 19, 55] for equation (1.1). Stability estimates for an arbitrary positive $c(x)$ were not obtained in the past.

In addition, to the Lipschitz stability, the Quasi-Reversibility Method (QRM) for the above Cauchy problem with the lateral data was developed in [26] and numerically tested in [14, 27, 29]. We refer to [49] for the originating work on QRM. The convergence of the QRM solution to the exact solution was proven on the basis of above Lipschitz stability results. Numerical testing has consistently demonstrated a high degree of robustness. In particular, accurate results were obtained in [29] with up to 50% noise in the data. Later some other numerical methods were proposed in [2, 55]. Convergence of all numerical methods mentioned here was proven only for the complete data collection case of IP1 and under some restrictive conditions imposed on $c(x)$.

A problem, which is closely related to TAT, and, at the same time, is more complicated, because of its nonlinearity, is the Coefficient Inverse Problem (CIP) of recovering the unknown coefficient $c(x), x \in \Omega$ in (1.1) from a single boundary measurement. All currently known uniqueness and stability results for multidimensional CIPs with single measurement data are proven by the method, which was proposed for the first time in [11, 12, 34], also see, e.g. [9, 35, 36, 37, 38, 40] for some follow up publications. In particular, the case of the recovery of the coefficient $c(x)$ satisfying (2.1) was considered in [9, 36, 37, 38]. It is clear from the first publications [11, 12, 34, 35, 37] that this method basically says that as soon as a Carleman estimate is valid for a PDE operator, then uniqueness and at least Hölder stability can be proven for a corresponding CIP, see [35, 37] for an abstract scheme of this method. Therefore, this technique works for CIPs for hyperbolic, parabolic, elliptic and non-stationary Schrödinger equations, as well as for some other equations, see, e.g. [5, 13, 15, 20, 21, 22, 23, 51, 59] and references cited therein for an incomplete list of publications; a good survey can be found in [59]. A combination of ideas of [11, 12, 34] and [25, 26] led to Lipschitz stability estimates for CIPs for hyperbolic PDEs, see, e.g. [20, 21, 22, 39].

A well known restriction of the method of [11, 12, 34] is that it requires at least one of initial conditions to be non-vanishing in the entire domain of interest. However, the latter does not seem to be an over-restrictive condition from the applied standpoint. Indeed, the most interesting scenario is the case of the $\delta-$function in the initial condition with a single location of the point source. Replacement of the $\delta-$function with a narrow Gaussian makes that technique working, at least in a generic case. At the same time, the boundary data resulting from this replacement are very close to the original ones. Therefore, if the underlying numerical method for a corresponding CIP is stable, as it should be, then this replacement should not affect its performance.

Recently an approximately globally convergent numerical method was developed for the above CIP for equation (1.1) [6, 7, 8, 9, 10, 32, 43, 44]. Its main new element is that it does not need *a priori* known good first guess about the solution (such a guess is rarely known in applications). That numerical technique works within the framework of a certain approximate mathematical model, see [9, 10, 44] for the definition of the approximate global convergence property. Numerical testing of this method for both computationally simulated [6, 7, 9, 43] and the most challenging case of *blind* experimental data [9, 32, 44] has consistently demonstrated its accurate performance. These tests have verified that approximate mathematical model. In works [33, 54] this numerical method was extended to the case of a 2-d CIP for an elliptic equation with the point source running along a straight line. This technique, in turn was verified in [57] for the case of experimental data with applications to imaging of strokes in brains of small animals.

3. Logarithmic Stability.

3.1. Transformation. First, we consider the following well known Laplace-like transformation [38, 50], which transforms the hyperbolic Cauchy problem in a similar parabolic Cauchy problem,

$$\mathcal{L}g = \overline{g}(t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{\tau^2}{4t}\right) g(\tau)\, d\tau. \tag{3.1}$$

The transformation (3.1) is an analog of the Laplace transform, and it is one-to-one. It is valid for, e.g. all functions $g \in C[0,\infty)$ which satisfy $|g(t)| \leq A_g e^{k_g t}$, where A_g and k_g are positive constants depending on g. Using energy estimates, and either of Assumptions 1,2, one can prove that the function u together with its corresponding derivatives satisfies this condition. Obviously

$$\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{\tau^2}{4t}\right)\right] = \frac{\partial^2}{\partial \tau^2}\left[\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{\tau^2}{4t}\right)\right].$$

Hence, if $g \in C^2[0,\infty)$ and $g'(0)=0$, then $\mathcal{L}(g'') = \overline{g}'(t)$. Changing variables in (3.1) $\tau \Leftrightarrow z, \tau/2\sqrt{t} := z$, we obtain $\lim_{t\to 0^+} \overline{g}(t) = g(0)$. Hence, $\mathcal{L}u = v$, where the function $v(x,t)$ is the solution of the following parabolic Cauchy problem

$$v_t = Lv, x \in \mathbb{R}^n, \tag{3.2}$$

$$v(x,0) = f(x). \tag{3.3}$$

Below we work only with the function v. It is well known [45] that there exists unique solution

$v \in C^{4+\alpha,2+\alpha/2}(\mathbb{R}^n \times [0,T]), \forall T > 0$ of the problem (3.2), (3.3). We use the $C^{4+\alpha,2+\alpha/2}$ space instead of $C^{2+\alpha,1+\alpha/2}$ because we need this a little bit higher smoothness in section 3. Recall that we are not trying to use in this paper minimal smoothness assumptions, for brevity. Given either of Assumptions 1,2, we conclude that indeed this solution $v = \mathcal{L}u$.

Since we can work with any value of T in the parabolic case, we set everywhere below $T := 1$ for the sake of definiteness. Denote

$$\mathcal{L}\varphi_1 = \overline{\varphi}_1(x,t) = v\mid_{S_1}, \ \mathcal{L}\varphi_2 = \overline{\varphi}_2(x,t) = v\mid_{P_1}. \tag{3.4}$$

Then

$$\overline{\varphi}_1 \in C^{4+\alpha,2+\alpha/2}\left(\overline{S}_1\right), \overline{\varphi}_2 \in C^{4+\alpha,2+\alpha/2}\left(\overline{P}_1\right). \tag{3.5}$$

Let

$$\overline{\psi}_1(x,t) = \partial_\nu v\mid_{S_1}, \overline{\psi}_2(x,t) = \partial_{x_1} v\mid_{P_1}. \tag{3.6}$$

Then by Theorem 5.2 of Chapter IV of [45] there exist constants $C_\Omega, C_P > 0$ such that

$$\left\|\overline{\psi}_1\right\|_{C^{1+\alpha,\alpha/2}\left(\overline{S}_1\right)} \leq C_\Omega \left\|\overline{\varphi}_1\right\|_{C^{2+\alpha,1+\alpha/2}\left(\overline{S}_1\right)}, \tag{3.7}$$

$$\left\|\overline{\psi}_2\right\|_{C^{1+\alpha,\alpha/2}\left(\overline{P}_1\right)} \leq C_P \left\|\overline{\varphi}_2\right\|_{C^{2+\alpha,1+\alpha/2}\left(\overline{P}_1\right)}. \tag{3.8}$$

We intentionally use here weaker norms of functions $\overline{\varphi}_1, \overline{\varphi}_2$ than those in (3.5) because these norms are involved in our stability estimates. The constant $C_\Omega = C_\Omega(L,\Omega)$ depends only on coefficients of the operator L and the domain Ω. The constant $C_P = C_P(L,P)$ depends only on coefficients of the operator L and the hyperplane P.

We now describe an elementary and well known procedure (see, e.g. (35)-(37) in [19]) of finding the normal derivative of the function v either at S_1 (in the case of IP1) or at the plane P_1 (in the case of IP2). In the case of IP1 we solve the initial boundary value problem for equation (3.2) for $(x,t) \in (\mathbb{R}^n \diagdown \Omega) \times (0,1)$ with the zero initial condition in $\mathbb{R}^n \diagdown \Omega$ (because of (1.6)) and the Dirichlet boundary condition $v\mid_{S_1} = \overline{\varphi}_1$. Then we uniquely find the normal derivative $\partial_\nu v\mid_{S_1} = \overline{\psi}_1$. Similarly, in the case of IP2, we uniquely find the Neumann boundary condition $\partial_{x_1} v\mid_{P_1} = \overline{\psi}_2$. Estimates (3.7), (3.8) ensure the stability of this procedure.

Therefore, each problem IP1, IP2 is replaced with a similar problem for the parabolic PDE (3.2) with both Dirichlet and Neumann boundary data. These data are given at S_1 for IP1 and at P_1 for IP2. Uniqueness of the solution of each of these parabolic inverse problems follows from standard theorems about uniqueness of the continuation of solutions of parabolic PDEs with the data at the lateral surface [23, 50, 38] as well as from theorems of this section.

Remarks 3.1.

1. In fact, in both problems IP1 and IP2 one can estimate the norm $\|f\|_{L_2(\Omega)}$ via norms $\|\overline{\varphi}_1\|_{L_2(S_1)}$ and $\|\overline{\varphi}_2\|_{L_2(P_1)}$ respectively, which is stronger than estimates via Hölder norms (3.7), (3.8) in Theorems 3.1, 3.2, 3.4, 3.5. Indeed, using estimates of the fundamental solution of the parabolic equation in §§

11-13 of Chapter IV of the book [45] as well as formula (14.2) of the same Chapter of that book, one can prove that $v \in H^{2,1}(\mathbb{R}^n \times (0,1))$. Consider, for example IP2. To obtain the estimate in that weaker norm, one can solve the initial boundary value problem for equation (3.2) with the boundary condition $v \mid_{P_1} = \overline{\varphi}_1(x,t)$ (see (3.4)) and the initial condition $v(x,0) = 0$ for $x \in \{x_1 < 0\}$ (see (1.6)) in the time cylinder $\{x_1 < 0\} \times (0,1)$. Let $\sigma = const. > 0$ and $P^{\sigma} = \{x_1 = -\sigma\}$. Suppose for simplicity that $L = \Delta$ for $x \in \{x_1 < 0\}$. Then the well known analytic formula of the solution of this initial boundary value problem implies that norms $\|v\|_{H^1(P_1^{\sigma})}, \|v_{x_1}\|_{L_2(P_1^{\sigma})}$ can be estimated via the norm $\|\overline{\varphi}_1\|_{L_2(P_1)}$. Next, the norm $\|f\|_{L_2(\Omega)}$ can be estimated via norms $\|v\|_{H^1(P_1^{\sigma})}, \|v_{x_1}\|_{L_2(P_1^{\sigma})}$ (Theorem 3.3). Hence, $\|f\|_{L_2(\Omega)}$ can be estimated via $\|\overline{\varphi}_2\|_{L_2(P_1)}$. A similar argument is valid for IP1. However, we do not follow this root here for brevity.

2. Since the kernel of the transform $\mathcal{L}$ decays rapidly with $\tau \to \infty$, then the condition $t \in (0,\infty)$ in (1.10), (1.11) is not a serious restriction from the applied standpoint. In addition, if having the data in (1.10), (1.11) only on a finite time interval $t \in (0,T)$ and knowing an upper estimate of a norm of the function f in (1.8), one can estimate the error in the integral (3.1) when integrating over $\tau \in (T,\infty)$. Indeed, norms of the function $u(x,t)$ and its derivatives grow with $t \to \infty$ not faster than e^{at} with a certain $a = const. > 0$, compare with the kernel of the integral (3.1). This error will be small if either T is large or t is small in (3.1), (3.2). Next, this error can be incorporated in the stability estimates of theorems of this section. We do not follow this root here for brevity.

3. Another argument about $t \in (0,\infty)$ comes from the recent experience of the author of working with time resolved experimental data for wave processes [8, 9, 10, 32, 44]. The author has learned that almost all time resolved experimental data for wave processes in non-attenuating media are highly oscillatory due to some unknown processes in measurement devices, see graphs of those data in these references. Because of high oscillations, these data are not governed by a hyperbolic PDE even for the case of the free space, where the wave equation is supposed to work (see the graphs of the experimental data for reference medium in [8, 9, 32]). To make the data fit a description by a hyperbolic PDE, one might try to de-convolute them via solving a Volterra-like convolution integral equation of the first kind. The kernel of this equation reflects a certain property of the measurement device. However, the problem of solving this equation ill-posed. Besides, corresponding properties of measurement devices are usually both unknown and nonlinear. This makes the de-convolution problem quite problematic. Therefore, to make the inverse algorithm work, it was necessary to preprocess the experimental data first by a new data preprocessing procedure. This procedure uses only a small portion of the real data and immerses it in a specially processed computationally simulated data for the uniform medium. Since the case of the uniform medium can be solved analytically, then there is no problem to know the immersed data for all $t \in (0,\infty)$. Since accurate imaging results were obtained in [9, 32, 44] for the case of blind experimental data, then that data preprocessing procedure was unbiased.

3.2. Logarithmic stability estimates for Inverse Problem 1. To prove convergence of the QRM (Theorem 4.1), it is convenient to consider a more general parabolic inequality in the integral form instead of equation (3.2). Let $K > 0$ be a constant. Consider the function $w \in H^{2,1}(Q_1)$ satisfying the following inequality

$$\int\limits_{Q_1} (w_t - Lw)^2 \, dxdt \leq K^2, K = const. \geq 0 \tag{3.9}$$

Since IP1 and IP2 are linear problems, it is sufficient to establish stability estimates for the case when the input data are sufficiently small, since one can rescale the data next.

Theorem 3.1. *Let conditions (1.2)-(1.6) be fulfilled and let Assumption 1 holds. Let the function $w \in H^{4,2}(Q_T)$ satisfies the inequality (3.9). Denote*

$$\psi_0(x,t) = w \mid_{S_1}, \psi_1(x,t) = \partial_{\nu} w \mid_{S_1}, \; g(x) = w(x,0).$$

Denote

$$F = \|\psi_0\|_{H^1(S_1)} + \|\psi_1\|_{L_2(S_1)} + K. \tag{3.10}$$

Assume that an upper bound for $\|\nabla g\|_{L_2(\Omega)}$ is known,

$$\left(\sum_{i=1}^{n} \|g_{x_i}\|_{L_2(\Omega)}^2 \right)^{1/2} := \|\nabla g\|_{L_2(\Omega)} \leq C_2 = const. \tag{3.11}$$

Then there exist a constant $M = M(L, \Omega) > 0$ and a sufficiently small number $\delta_0 = \delta_0(L, \Omega, C_2) \in (0, 1)$, both dependent on coefficients of the operator L and the domain Ω, such that if $F \in (0, \delta_0)$, then the following logarithmic stability estimate is valid

$$\|g\|_{L_2(\Omega)} \leq \frac{MC_2}{\sqrt{\ln(F^{-1})}}. \tag{3.12}$$

In particular, in the case of IP1 we assume that

$$\|\nabla f\|_{L_2(\Omega)} \leq C_2 \tag{3.13}$$

and for $\|\overline{\varphi}_1\|_{C^{2+\alpha,1+\alpha/2}(\overline{S}_T)} \in (0, \delta_0)$ (3.12) becomes

$$\|f\|_{L_2(\Omega)} \leq \frac{MC_2}{\sqrt{\ln\left(\|\overline{\varphi}_1\|_{C^{2+\alpha,1+\alpha/2}(\overline{S}_1)}^{-1} \right)}}. \tag{3.14}$$

Remarks 3.2.

1. We assume that $w \in C^{2,1}\left(\overline{Q}_T\right)$ rather than $w \in H^{2,1}(Q_T)$ because of the boundary term u_t^2 which is involved in the Carleman estimate for the operator $\partial_t - L_0$, see Lemmata 3.1, 3.2. However, one can use $w \in H^{2,1}(Q_T)$ if assuming that $\psi_0 = 0$, see Corollary 3.1.

2. Below in this paper $M = M(L, \Omega) > 0$ denotes a generic positive constant depending on L, Ω. Estimates (3.12), (3.13) are the so-called "conditional stability estimates", which is often the case in ill-posed problems [9, 50]. For another example we refer to Hölder stability estimates for solutions of ill-posed problems for PDEs, see, e.g. [23, 38, 50]. The knowledge of the upper bound C_2 for the gradient in (3.11), (3.13) corresponds well with the Tikhonov concept of compact sets as sets of "admissible" solutions of ill-posed problems [3, 9, 16, 24, 50, 58]. Indeed, since by (1.6) $f\mid_{\partial\Omega} = 0$, then $\|f\|_{L_2(\Omega)} \leq C\|\nabla f\|_{L_2(\Omega)} \leq CC_2$, where the constant $C > 0$ depends only on the domain Ω. Thus, in this case the function f belongs to a compact set in $L_2(\Omega)$, and this set is determined by the constant C_2.

Proof of Theorem 3.1. First, we prove (3.12). Let $\beta \in (0, 1)$ be an arbitrary number. Then it follows from Theorem 2 of [30] that there exists a constant $\varepsilon = \varepsilon(L, \Omega, \beta) \in (0, 1)$ such that

$$\|g\|_{L_2(\Omega)} \leq \frac{MC_2}{\beta\sqrt{\ln\left[(\varepsilon F)^{-1}\right]}} + M\left(\frac{1}{\varepsilon}\right)^{\beta} F^{1-\beta}, \tag{3.15}$$

as long as $F \in (0, 1)$. We can fix β via, e.g. setting $\beta := 1/2$. It is clear therefore that there exists a sufficiently small number $\delta_0 = \delta_0(L, \Omega, C_2) \in (0, 1)$ such that if $F \in (0, \delta_0)$, then (3.15) implies (3.12).

We now prove (3.13). It follows from (3.2) that (3.9) holds for the function $w := v$ with $K = 0$. Since $v \in C^{4+\alpha,2+\alpha/2}(\mathbb{R}^n \times [0, T]), \forall T > 0$, then $v \in H^2(Q_1)$. Using (3.7), we obtain that (3.10) becomes $F \leq (C_\Omega + 1)\|\overline{\varphi}_1\|_{C^{2+\alpha,1+\alpha/2}(\overline{S}_1)}$. Hence, (3.13) follows from (3.12). $\square$

Theorem 3.2 follows immediately from Theorem 3.1.

Theorem 3.2. *Suppose that Assumption 2 holds. Consider IP1 and let $\|\nabla f\|_{L_2(\Omega)} \leq C_2$. Then there exist a sufficiently small number $\delta_0 = \delta_0(L, \Omega, C_2) \in (0, 1)$ such that if in (1.9) $\delta \in (0, \delta_0)$, then the following logarithmic stability estimate is valid*

$$\|f\|_{L_2(\Omega)} \leq \frac{M_b C_2}{\sqrt{\ln(\delta^{-1})}},$$

where the constant $M_b = M_b(L, \Omega, b) > 0$ depends only on coefficients of the operator L, the domain Ω and the number b.

3.3. Logarithmic stability estimate for Inverse Problem 2. Unlike the finite domain of the paper [30], where the integral inequality (3.9) was considered, the logarithmic stability estimate of the paper [31] in the infinite domain was obtained for the case of the pointwise inequality

$$|v_t - L_0 v| \leq A\left(|\nabla v| + |v|\right), A = const. > 0, \tag{3.16}$$

where L_0 is the principal part of the operator L, see (1.2), (1.3). However, to prove the convergence of the numerical method of section 4, we need to estimate the initial condition for the case of the integral inequality, like the one in (3.9). The Carleman estimate of [31] is not a standard one. Indeed, unlike the standard Carleman estimate for the parabolic operator [9, 38, 50], the integration domain of [31] is contained in the strip $\{|t-\varepsilon| < \tau\varepsilon, \tau \in (0,1)\}$, and that Carleman estimate does not break when $\varepsilon \to 0^+$.

There are two main differences between Theorem 3.3 (below) and Theorem 1 of [31]. First, we work now with the integral inequality instead of the pointwise inequality (3.16) of [31]. Second, it is assumed in [31] that the inequality (3.16) is valid in $\Psi \times (0,T)$, where $\Psi \subseteq \mathbb{R}^n$ is an unbounded domain, the domain of interest $\Phi \subset \Psi$, $P \cap \overline{\Psi} \neq \varnothing$, and it is also assumed that the Dirichlet boundary condition $v\mid_{\Psi \times (0,T)}$is given. Theorem 3.3 does not use the assumption about the knowledge of this Dirichlet boundary condition.

Denote $\overline{x} = (x_2, ..., x_n)$. Changing variables $(x', t') = \left(\sqrt{d}x, dt\right)$ with an appropriate constant $d > 0$ and keeping the same notations for new variables for brevity, we obtain that

$$\Omega \subset \left\{x_1 + |\overline{x}|^2 < \frac{1}{4}, x_1 > 0\right\}. \tag{3.17}$$

Let $\varepsilon \in (0,1)$ be a sufficiently small number. Consider the following functions $\psi(x,t), \varphi(x,t)$,

$$\psi(x,t) = x_1 + |\overline{x}|^2 + \frac{(t-\varepsilon)^2}{\varepsilon^2} + \frac{1}{8}, \tag{3.18}$$

$$\varphi(x,t) = \exp\left(\frac{\psi^{-\nu}}{\varepsilon}\right), \tag{3.19}$$

where $\nu > 1$ is a large parameter which will be defined later. The function $\varphi(x,t)$ is the Carleman Weight Function (CWF) in the Carleman estimate of Lemma 3.1. The main difference between $\varphi(x,t)$ in (3.19) and the standard CWF for the parabolic operator [9, 38, 50] is that the small parameter ε is involved in both $\psi(x,t)$ in (3.18) and, in a different format, in $\varphi(x,t)$. Denote

$$G_0 = \left\{(x,t) : \psi(x,t) < \frac{3}{4}, x_1 > 0\right\}, \tag{3.20}$$

$$G_\omega = \left\{(x,t) : \psi(x,t) < \frac{3}{4} - \omega, x_1 > 0\right\}, \forall \omega \in \left(0, \frac{1}{8}\right). \tag{3.21}$$

Using (3.17)-(3.21), we obtain

$$G_\omega \subset G_0, \varphi^2(x,t) \geq \exp\left[\frac{2}{\varepsilon}\left(\frac{3}{4} - \omega\right)^{-\nu}\right] \text{ in } G_\omega, \tag{3.22}$$

$$G_0 \subset \left\{|t-\varepsilon| < \varepsilon\sqrt{\frac{5}{8}}\right\} \subset \{t \in (0,1)\}, \tag{3.23}$$

$$\Omega \subset PG_0, \tag{3.24}$$

$$\partial G_0 = \partial_1 G_0 \cup \partial_2 G_0, \partial_1 G_0 = \{x_1 = 0\} \cap \overline{G}_0, \partial_2 G_0 = \left\{ \psi(x,t) = \frac{3}{4}, x_1 > 0, t > 0 \right\}. \tag{3.25}$$

In (3.24) PG_0 is the orthogonal projection of the domain G_0 on $\{t = 0\}$. Hence, (3.24) follows from (3.17). The following lemma is a modified formulation of Theorem 2 of [31].

Lemma 3.1. *Let coefficients of the operator L_0 in (1.3) satisfy conditions (1.4), (1.5). Then there exists a sufficiently large constant $\nu_0 = \nu_0(L_0, \Omega) > 1$ and a sufficiently small number $\varepsilon_0 = \varepsilon_0(L_0, \Omega) \in (0, 1)$ such that the following Carleman estimate holds for all $\nu \geq \nu_0, \varepsilon \in (0, \varepsilon_0)$ and all functions $u \in C^{2,1}(\overline{G}_0)$*

$$\frac{M\nu^3}{\varepsilon^3} \exp\left(\frac{2 \cdot 8^\nu}{\varepsilon}\right) \int\limits_{\partial_1 G_0} \left(u^2 + |\nabla u|^2 + u_t^2\right) d\overline{x}dt$$

$$+ \frac{M\nu^3}{\varepsilon^3} \left(\frac{3}{4}\right)^{-2\nu} \exp\left[\frac{2}{\varepsilon}\left(\frac{3}{4}\right)^{-\nu}\right] \int\limits_{\partial_2 G_0} \left(u^2 + |\nabla u|^2 + u_t^2\right) d\sigma$$

$$+ \int\limits_{G_0} (u_t - L_0 u)^2 \varphi^2(x,t)\, dxdt$$

$$\geq M \int\limits_{G_0} \left(\frac{\nu}{\varepsilon} |\nabla u|^2 + \frac{\nu^4}{\varepsilon^3} \psi^{-2\nu} u^2\right) \varphi^2(x,t)\, dxdt.$$

It follows from (3.23) that Lemma 3.1 provides the Carleman estimate in the narrow strip $\left\{ |t - \varepsilon| < \varepsilon\sqrt{5/8} \right\}$. We need this strip then to estimate the initial condition. At the same time, it is also important in numerical studies of the QRM to estimate its solution in a not narrow strip. This can be done via the standard Carleman estimate. Therefore, we introduce now notations, which are similar with (3.18)-(3.25), except that a narrow strip with respect to t is not used. Let

$$\theta(x,t) = x_1 + |\overline{x}|^2 + 5\left(t - \frac{1}{2}\right)^2 + \frac{1}{8}, \tag{3.26}$$

$$\xi(x,t) = \exp\left(\lambda \theta^{-\nu}\right), \tag{3.27}$$

where $\lambda > 1$ is a large parameter which is chosen later. Denote

$$D_0 = \left\{ (x,t) : \theta(x,t) < \frac{3}{4}, x_1 > 0 \right\}, \tag{3.28}$$

$$D_\omega = \left\{ (x,t) : \theta(x,t) < \frac{3}{4} - \omega, x_1 > 0 \right\}, \forall \omega \in \left(0, \frac{1}{8}\right). \tag{3.29}$$

Using (3.17), (3.26) and (3.28), we obtain

$$D_0 \subset \left\{ \left| t - \frac{1}{2} \right| < \frac{1}{2\sqrt{2}} \right\} \subset \{ t \in (0,1) \}. \tag{3.30}$$

Lemma 3.2 follows from the Carleman estimate for the parabolic operator of Lemma 3 of §1 of Chapter 4 of the book [50].

Lemma 3.2. *Let coefficients of the operator L_0 in (1.3) satisfy conditions (1.4), (1.5). Then there exist sufficiently large constants $\nu_0 = \nu_0(L_0,\Omega) > 1, \lambda_0 = \lambda_0(L_0,\Omega) > 1$ such that the following Carleman estimate holds for all $\nu \geq \nu_0, \lambda \geq \lambda_0$ and all functions $u \in C^{2,1}\left(\overline{D}_0\right)$*

$$M\lambda^3\nu^3 \exp\left(2\lambda \cdot 8^{\nu}\right) \int\limits_{\partial_1 D_0} \left(u^2 + |\nabla u|^2 + u_t^2\right) d\overline{x}dt$$

$$+M\lambda^3\nu^3 \left(\frac{3}{4}\right)^{-2\nu} \exp\left[2\lambda\left(\frac{3}{4}\right)^{-\nu}\right] \int\limits_{\partial_2 D_0} \left(u^2 + |\nabla u|^2 + u_t^2\right) d\sigma$$

$$+\int\limits_{D_0} \left(u_t - L_0 u\right)^2 \xi^2(x,t)\, dxdt$$

$$\geq M \int\limits_{D_0} \left(\lambda\nu |\nabla u|^2 + \lambda^3\nu^3\psi^{-2\nu}u^2\right) \xi^2(x,t)\, dxdt.$$

Theorem 3.3. *Let conditions (1.2)-(1.5), (3.17) be fulfilled and let Assumption 1 holds. Let the bounded domain $\Phi \subset \{x_1 > 0\}$ be such that*

$$\left\{x_1 + |\overline{x}|^2 < \frac{5}{8}, x_1 > 0\right\} \subseteq \Phi. \tag{3.31}$$

Denote $\partial_1\Phi_1 = \overline{\Phi}_1 \cap P_1$. Let the function $w \in C^{2,1}\left(\overline{\Phi}_1\right)$ satisfies the following integral inequality

$$\int\limits_{\Phi_1} \left(w_t - Lw\right)^2 dxdt \leq K^2, K = const. \geq 0. \tag{3.32}$$

Let

$$\psi_0(x,t) = w\mid_{\partial_1\Phi_1}, \psi_1(x,t) = \partial_{x_1} w\mid_{\partial_1\Phi_1}, g(x) = w(x,0), x \in \Omega.$$

Denote

$$F = \|\psi_0\|_{H^1(\partial_1\Phi_1)} + \|\psi_1\|_{L_2(\partial_1\Phi_1)} + K. \tag{3.33}$$

Assume that an upper bound for $\|w\|_{H^1(\Phi_1)}$ is known,

$$\|w\|_{H^1(\Phi_1)} \leq C_3 = const. \tag{3.34}$$

Then there exists a sufficiently small number $\delta_0 = \delta_0(L,\Omega,C_3) \in (0,1)$ such that if the number F is so small that $F \in (0,\delta_0)$, then the following logarithmic stability estimate is valid

$$\|g\|_{L_2(\Omega)} \leq \frac{MC_3}{\sqrt{\ln\left(F^{-1}\right)}}. \tag{3.35}$$

In addition, for every $\omega \in (0,1/8)$ there exists a number $\rho = \rho(L,\Omega,\omega) \in (0,1)$ such that for $F \in (0,\delta_0)$ the following Hölder stability estimate is valid

$$\|w\|_{L_2(D_{3\omega})} + \|\nabla w\|_{L_2(D_{3\omega})} \leq MC_3F^{\rho}. \tag{3.36}$$

Remark 3.3. We indicate here the dependence of the number $\delta_0 = \delta_0 (L, \Omega, C_3)$ from the domain Ω rather than from the domain Φ because Φ can be constructed depending on Ω. As to $w \in C^{2,1}\left(\overline{\Phi}_1\right)$ rather than $w \in H^{2,1}(\Phi_1)$, see the first Remark 3.2 and Corollary 3.1.

Proof. By (3.17), (3.20), (3.23) and (3.31) $\Omega \subset \Phi, G_0 \subset \Phi_1$. Choose a number $\omega \in (0, 1/8)$. Then by (3.17), (3.21) and (3.24)

$$\Omega \subset PG_{3\omega} \subset \Phi. \tag{3.37}$$

Let $\chi(x,t)$ be such a function that

$$\chi \in C^{2,1}\left(\overline{\Phi}_1\right), \chi(x,t) = \begin{cases} 1, (x,t) \in G_{2\omega}, \\ 0, (x,t) \in G_0 \diagdown G_\omega, \\ \text{between 0 and 1 otherwise.} \end{cases} \tag{3.38}$$

Let $\overline{w} = \chi w$. Hence,

$$\overline{w}(x,t) = 0 \text{ in } G_0 \diagdown G_\omega. \tag{3.39}$$

Then

$$\int\limits_{G_0} (\overline{w}_t - L\overline{w})^2 \varphi^2 dxdt \leq \int\limits_{G_0} (w_t - Lw)^2 \chi^2 \varphi^2 dxdt + M \int\limits_{G_\omega \diagdown G_{2\omega}} \left(|\nabla w|^2 + w^2\right) \varphi^2 dxdt.$$

Hence, (3.19) and (3.32) imply that

$$\int\limits_{G_0} (\overline{w}_t - L\overline{w})^2 \varphi^2 dxdt \leq K^2 \exp\left(\frac{2 \cdot 8^\nu}{\varepsilon}\right) + M \int\limits_{G_\omega \diagdown G_{2\omega}} \left(|\nabla w|^2 + w^2\right) \varphi^2 dxdt.$$

By (3.22) this inequality can be rewritten as

$$\int\limits_{G_0} (\overline{w}_t - L\overline{w})^2 \varphi^2 dxdt \leq K^2 \exp\left(\frac{2 \cdot 8^\nu}{\varepsilon}\right)$$

$$+M \exp\left[\frac{1}{\varepsilon}\left(\frac{3}{4} - 2\omega\right)^{-\nu}\right] \int\limits_{G_\omega \diagdown G_{2\omega}} \left(|\nabla w|^2 + w^2\right) dxdt.$$

Hence, (3.34) implies that

$$\int\limits_{G_0} (\overline{w}_t - L\overline{w})^2 \varphi^2 dxdt \leq K^2 \exp\left(\frac{2 \cdot 8^\nu}{\varepsilon}\right) + MC_3^2 \exp\left[\frac{2}{\varepsilon}\left(\frac{3}{4} - 2\omega\right)^{-\nu}\right]. \tag{3.40}$$

On the other hand, using Lemma 3.1, (3.22), (3.25), (3.38) and (3.39), we obtain

$$\int\limits_{G_0} (\overline{w}_t - L\overline{w})^2 \varphi^2 dxdt \geq \int\limits_{G_0} (\overline{w}_t - L_0\overline{w})^2 \varphi^2 dxdt - M \int\limits_{G_0} \left(|\nabla \overline{w}|^2 + \overline{w}^2\right) \varphi^2 dxdt$$

$$\geq M \int\limits_{G_0} \left(\frac{\nu}{\varepsilon} |\nabla \overline{w}|^2 + \frac{\nu^4}{\varepsilon^3} \psi^{-2\nu} \overline{w}^2\right) \varphi^2 (x,t) dxdt - M \int\limits_{G_0} \left(|\nabla \overline{w}|^2 + \overline{w}^2\right) \varphi^2 dxdt \tag{3.41}$$

$$-\frac{M\nu^3}{\varepsilon^3}\exp\left(\frac{2\cdot 8^\nu}{\varepsilon}\right)\int\limits_{\partial_1 G_0}\left[\psi_0^2+(\nabla\psi_0)^2+(\partial_t\psi_0)^2+\psi_1^2\right]d\overline{x}dt.$$

Fix $\nu:=\nu_0(L_0,\Omega)>1$. There exists sufficiently small number $\varepsilon_1=\varepsilon_1(L_0,\Omega)\in(0,1)$ such that for all $\varepsilon\in(0,\varepsilon_1)$

$$M\int\limits_{G_0}\left(\frac{\nu}{\varepsilon}|\nabla\overline{w}|^2+\frac{\nu^4}{\varepsilon^3}\psi^{-2\nu}\overline{w}^2\right)\varphi^2(x,t)\,dxdt-M\int\limits_{G_0}\left(|\nabla\overline{w}|^2+\overline{w}^2\right)\varphi^2dxdt$$

$$\geq\frac{M}{2}\int\limits_{G_0}\left(\frac{\nu}{\varepsilon}|\nabla\overline{w}|^2+\frac{\nu^4}{\varepsilon^3}\psi^{-2\nu}\overline{w}^2\right)\varphi^2(x,t)\,dxdt.$$

Hence, (3.40) and (3.41) imply that

$$\int\limits_{G_0}\left(\frac{1}{\varepsilon}|\nabla\overline{w}|^2+\frac{1}{\varepsilon^3}\overline{w}^2\right)\varphi^2(x,t)\,dxdt\leq\frac{M}{\varepsilon^3}\exp\left(\frac{2\cdot 8^\nu}{\varepsilon}\right)F^2 \tag{3.42}$$

$$+MC_3^2\exp\left[\frac{2}{\varepsilon}\left(\frac{3}{4}-2\omega\right)^{-\nu}\right].$$

On the other hand, since $G_{3\omega}\subset G_0$, then (3.22) and (3.38) imply that

$$\int\limits_{G_0}\left(\frac{\nu}{\varepsilon}|\nabla\overline{w}|^2+\frac{\nu^4}{\varepsilon^3}\psi^{-2\nu}\overline{w}^2\right)\varphi^2(x,t)\,dxdt\geq$$

$$\frac{1}{\varepsilon}\exp\left[\frac{2}{\varepsilon}\left(\frac{3}{4}-3\omega\right)^{-\nu}\right]\int\limits_{G_{3\omega}}\left(|\nabla w|^2+w^2\right)dxdt.$$

Hence, using (3.42), we obtain

$$\int\limits_{G_{3\omega}}\left(|\nabla w|^2+w^2\right)dxdt \tag{3.43}$$

$$\leq M\exp\left(\frac{8^{\nu+1}}{\varepsilon}\right)F^2+MC_3^2\exp\left\{-\frac{2}{\varepsilon}\left(\frac{3}{4}-3\omega\right)^{-\nu}\left[1-\left(\frac{1-4\omega}{1-8\omega/3}\right)\right]\right\},\forall\varepsilon\in(0,\varepsilon_1).$$

Since by (3.18) and (3.21)

$$\Theta^\varepsilon=\left\{x_1+|\overline{x}|^2<\frac{1}{4},x_1>0\right\}\times\left\{t:|t-\varepsilon|<\varepsilon\sqrt{\frac{3}{8}-3\omega}\right\}\subset G_{3\omega}. \tag{3.44}$$

Furthermore, by (3.17) $\Omega\subset P\Theta^\varepsilon$. Therefore, the mean value theorem, (3.43) and (3.44) imply that there exists such a point $t^*\in\left\{t:|t-\varepsilon|<\varepsilon\sqrt{3/8-3\omega}\right\}$ that

$$\int\limits_{\Omega}w^2(x,t^*)\,dx\leq M\exp\left(\frac{8^{\nu+1}}{\varepsilon}\right)F^2+MC_3^2\exp\left(-\frac{a}{\varepsilon}\right), \tag{3.45}$$

$$a=2\left(\frac{3}{4}-3\omega\right)^{-\nu}\left[1-\left(\frac{1-4\omega}{1-8\omega/3}\right)\right]>0.$$

We have

$$w\left(x,t^{*}\right)=g\left(x\right)+\int\limits_{0}^{t}w_{t}\left(x,\tau\right)d\tau. \tag{3.46}$$

Let $\varepsilon_{2}=\varepsilon\left(1+\sqrt{3/8-3\omega}\right)$. Using (3.34), (3.44) and (3.46), we obtain

$$\int\limits_{\Omega}w^{2}\left(x,t^{*}\right)dx\geq\left\|g\right\|_{L_{2}(\Omega)}^{2}-\varepsilon_{2}\int\limits_{0}^{\varepsilon_{2}}\int\limits_{\Omega}w_{t}^{2}\left(x,t\right)dt\geq\left\|g\right\|_{L_{2}(\Omega)}^{2}-\varepsilon_{2}\left\|w_{t}\right\|_{L_{2}(Q_{T})}^{2}$$

$$\geq\left\|g\right\|_{L_{2}(\Omega)}^{2}-MC_{3}^{2}\varepsilon.$$

Hence, (3.45) implies that

$$\left\|g\right\|_{L_{2}(\Omega)}^{2}\leq MC_{3}^{2}\varepsilon+M\exp\left(\frac{8^{\nu+1}}{\varepsilon}\right)F^{2}+MC_{3}^{2}\exp\left(-\frac{a}{\varepsilon}\right). \tag{3.47}$$

Choose $\varepsilon=\varepsilon\left(F,\omega\right)$ such that

$$\exp\left(\frac{8^{\nu+1}}{\varepsilon}\right)F=\exp\left(-\frac{a}{\varepsilon}\right). \tag{3.48}$$

Hence,

$$\varepsilon=\frac{8^{\nu+1}+a}{\ln\left(1/F\right)}:=\frac{\overline{a}}{\ln\left(1/F\right)}. \tag{3.49}$$

To ensure that in (3.49) $\varepsilon\in\left(0,\varepsilon_{1}\right)$, we should assume that F is so small that

$$F\in\left(0,\exp\left(-\frac{\overline{a}}{\varepsilon_{1}}\right)\right):=\left(0,\delta_{0}\right). \tag{3.50}$$

By (3.48) and 3.49)

$$MC_{3}^{2}\varepsilon+M\exp\left(\frac{8^{\nu+1}}{\varepsilon}\right)F^{2}+MC_{3}^{2}\exp\left(-\frac{a}{\varepsilon}\right)=\frac{MC_{3}^{2}}{\ln\left(1/F\right)}+MF^{a/\overline{a}}\left(1+C_{3}^{2}\right). \tag{3.51}$$

Hence, assuming that in (3.50) $\delta_{0}=\delta_{0}\left(L,\Omega,C_{3}\right)$ is sufficiently small and using (3.47) and (3.51), we obtain (3.35).

We now prove (3.36). By (3.26)-(3.30) $D_{0}\subset\Phi_{1}$.Using (3.26)-(3.29) and Lemma 3.2, we obtain similarly with (3.43)

$$\int\limits_{D_{3\omega}}\left(\left|\nabla w\right|^{2}+w^{2}\right)dxdt$$

$$\leq M\exp\left(8^{\nu+1}\lambda\right)F^{2}+MC_{3}^{2}\exp\left\{-2\lambda\left(\frac{3}{4}-3\omega\right)^{-\nu}\left[1-\left(\frac{1-4\omega}{1-8\omega/3}\right)\right]\right\},\forall\lambda\geq\lambda_{1},$$

where $\lambda_1 = \lambda_1(L_0, \Omega) > 1$ is a sufficiently large number. Hence, we obtain similarly with the above

$$\int\limits_{D_{3\omega}} \left(|\nabla w|^2 + w^2 \right) dxdt \leq MC_3^2 F^{a/\overline{a}} := MC_3^2 F^{2\rho}. \ \square$$

Corollary 3.1. *Theorem 3.3 remains true if the condition $w \in C^{2,1}\left(\overline{\Phi}_1\right)$ is replaced with the following two conditions: $w\mid_{\partial_1 \Phi_1} = 0$ and $w \in H^{2,1}(\Phi_1)$.*

Proof. Using density arguments, we conclude that Lemma 3.1 remains true if the function $u \in H^{2,1}(G_0)$, $u\mid_{\partial_1 G_0} = 0$ and $u(x,t) = 0$ in a neighborhood of the hypersurface $\partial_2 G_0$. Next, using the function $\chi(x,t)$ in (3.38), we obtain (3.35) similarly with the above. The proof of (3.36) is also similar. $\square$

Theorem 3.4. *Let conditions (1.2)-(1.6) be fulfilled and let Assumption 1 holds. Consider IP2 and let $\|\nabla f\|_{L_2(\Omega)} \leq C_2$. Then there exists a sufficiently small number $\delta_0 = \delta_0(L, \Omega, C_2) \in (0,1)$ such that if $\|\overline{\varphi}_2\|_{C^{2+\alpha,1+\alpha/2}\left(\overline{P}_1\right)} \in (0, \delta_0)$, then the following logarithmic stability estimate is valid*

$$\|f\|_{L_2(\Omega)} \leq \frac{MC_2}{\sqrt{\ln\left(\|\overline{\varphi}_2\|^{-1}_{C^{2+\alpha,1+\alpha/2}\left(\overline{P}_1\right)} \right)}}.$$

Proof. Let $v \in C^{4+\alpha,2+\alpha/2}(\mathbb{R}^n \times [0,1])$ be the solution of the Cauchy problem (3.2), (3.3). The standard technique for parabolic equations ensures that

$$\|v\|_{H^1(\mathbb{R}^n \times (0,1))} \leq M\left(\|f\|_{L_2(\Omega)} + \|\nabla f\|_{L_2(\Omega)} \right) \leq M\|f\|_{L_2(\Omega)} + MC_2,$$

which replaces (3.34). Hence, using (3.8), the fact that now in (3.32) $w = v, K = 0$ and repeating arguments of the proof of Theorem 3.3 we obtain that the following estimate holds instead of (3.47)

$$\|f\|^2_{L_2(\Omega)} \left[1 - M\varepsilon - M\exp\left(-\frac{a}{\varepsilon} \right) \right] \leq MC_2^2 \varepsilon + M\exp\left(\frac{8^{\nu+1}}{\varepsilon} \right) F^2 + MC_2^2 \exp\left(-\frac{a}{\varepsilon} \right),$$

where $F = \|\overline{\varphi}_2\|_{C^{2+\alpha,1+\alpha/2}\left(\overline{P}_1\right)}$. Since ε is sufficiently small, then

$$\|f\|^2_{L_2(\Omega)} \left[1 - M\varepsilon - M\exp\left(-\frac{a}{\varepsilon} \right) \right] \geq \frac{1}{2}\|f\|^2_{L_2(\Omega)}.$$

Hence,

$$\|f\|^2_{L_2(\Omega)} \leq MC_2^2 \varepsilon + M\exp\left(\frac{8^{\nu+1}}{\varepsilon} \right) F^2 + MC_2^2 \exp\left(-\frac{a}{\varepsilon} \right).$$

The rest of the proof is the same as the proof of Theorem 3.3 after (3.47). $\square$

The following theorem follows immediately from Theorem 3.4.

Theorem 3.5. *Consider IP2. Suppose that Assumption 2 holds. Let $\|\nabla f\|_{L_2(\Omega)} \leq C_2$. Then there exist a sufficiently small number $\delta_0 = \delta_0(L, \Omega, C_2) \in (0,1)$ such that if in (1.9) $\delta \in (0, \delta_0)$, then*

$$\|f\|_{L_2(\Omega)} \leq \frac{M_b C_2}{\sqrt{\ln\left(\delta^{-1}\right)}},$$

where the constant $M_b = M_b(L, \Omega, b) > 0$ depends only on coefficients of the operator L, the domain Ω and the number b.

4. The Quasi-Reversibility Method (QRM). We construct the QRM only for the more difficult case of IP2. We work now with the parabolic PDE (3.2). The case of IP1 is similar, and it was considered in [30]. It is clear from the material of this section that, compared with [30], the simplification here is that the norm $\|\nabla v_t\|^2_{L_2(Q_1)}$ in the regularization term should not be used.

Without a loss of generality, we assume that (3.17) holds. Let the bounded domain $\Phi \subset \{x_1 > 0\}$ be the same as in Theorem 3.3, i.e. (3.31) holds. It is most convenient to realize the QRM via finite differences [29, 43]. Besides, we want to have a convenient formula for the data extension from the hyperplane P inside of the domain Φ. Hence, we assume for the sake of definiteness, that Φ is a rectangular prism,

$$\Phi = \left\{ x : x_1 \in (0,1)\,, \overline{x} \in (-1,1)^{n-1} \right\}. \tag{4.1}$$

By (3.17), (3.18), (3.20), (3.23), (3.26)-(3.30) and (3.31)

$$G_0 \subset \Phi_1 = \Phi \times (0,1)\,, \Omega \subset \Phi, \tag{4.2}$$

$$D_0 \subset \Phi_1. \tag{4.3}$$

Let $Z = \left\{ x_1 = 0, \overline{x} \in (-1,1)^{n-1} \right\} = \partial\Phi \cap P$. Since it was described in section 3 how to obtain the Neumann boundary condition for both IP1 and IP2 and since (3.4)-(3.8) take place, we assume now that we have both Dirichlet and Neumann boundary conditions at $Z_1 = Z \times (0,1)\,,$

$$v\mid_{Z_1} = \overline{\varphi}_2\,(x,t)\,, \partial_{x_1} v\mid_{Z_1} = \overline{\psi}_2\,(x,t)\,. \tag{4.4}$$

The QRM means in our case the minimization of the following Tikhonov functional

$$J_\gamma\,(v) = \|v_t - Lv\|^2_{L_2(\Phi_1)} + \gamma\,\|v\|^2_{H^{2,1}(\Phi_1)}\,, \tag{4.5}$$

subject to the boundary conditions (4.4). In (4.5) $\gamma > 0$ is the regularization parameter, which should be chosen in accordance with the level of noise in the data.

While (4.5) is likely good for computations, to prove convergence of the QRM, we need to have zero boundary conditions at Z_1. To do so, we need $\overline{\varphi}_2 \in H^{2,1}\,(Z_1)$ and also $\overline{\psi}_2 \in H^{2,1}\,(Z_1)$. Both of these are ensured by (3.5). Indeed, to establish the second one, we again use Theorem 5.2 of Chapter IV of [45] and obtain the following analog of (3.8) $\left\|\overline{\psi}_2\right\|_{C^{3+\alpha,1+\alpha/2}\left(\overline{P}_1\right)} \leq C_P\,\|\overline{\varphi}_2\|_{C^{4+\alpha,2+\alpha/2}\left(\overline{P}_1\right)}$. An alternative way is to follow the first Remark 3.1. Denote

$$r\,(x,t) = \overline{\varphi}_2\,(x,t) + x_1\overline{\psi}_2\,(x,t) = \overline{\varphi}_2\,(\overline{x},t) + x_1\overline{\psi}_2\,(\overline{x},t)\,,$$

$$\widehat{v}\,(x,t) = v\,(x,t) - r\,(x,t)\,, p\,(x,t) = -\left(\partial_t - L\right)(r)\,,$$

$$\widehat{f}\,(x) = \widehat{v}\,(x,0) = f\,(x) - r\,(x,0)\,.$$

Using (3.2), (3.3) and (4.4), we obtain

$$\widehat{v}_t - L\widehat{v} = p\,(x,t)\,, (x,t) \in \Phi_1, \tag{4.6}$$

$$\widehat{v}\mid_{Z_1} = 0, \widehat{v}_{x_1}\mid_{Z_1} = 0. \tag{4.7}$$

Thus, we have obtained the following

Inverse Problem 3 (IP3). *Find the function $\widehat{f}\,(x)$ for $x \in \Omega$ from conditions (4.6), (4.7).*

To solve IP3 via the QRM, we minimize the following analog of the functional (4.5)

$$\widehat{J}_{\gamma}\left(\widehat{v}\right)=\left\|\widehat{v}_{t}-L\widehat{v}-p\right\|_{L_{2}\left(\Phi_{1}\right)}^{2}+\gamma\left\|\widehat{v}\right\|_{H^{2,1}\left(\Phi_{1}\right)}^{2},\widehat{v}\in H_{0}^{2,1}\left(\Phi_{1}\right), \tag{4.8}$$

$$H_{0}^{2,1}\left(\Phi_{1}\right)=\left\{u\in H^{2,1}\left(\Phi_{1}\right):u\mid_{Z_{1}}=u_{x_{1}}\mid_{Z_{1}}=0.\right\}$$

Let $(,)$ and $[,]$ be scalar products in $L_2\left(\Phi_1\right)$ and $H^{2,1}\left(\Phi_1\right)$ respectively. Let the function $u_\gamma \in H_0^{2,1}\left(\Phi_1\right)$ be a minimizer of the functional (4.8). Then the variational principle implies that

$$\left(\partial_{t}u_{\gamma}-Lu,\partial_{t}w-Lw\right)+\gamma\left[u,w\right]=\left(p,w_{t}-Lw\right),\forall w\in H_{0}^{2,1}\left(\Phi_{1}\right). \tag{4.9}$$

Lemma 4.1 follows immediately from Riesz theorem and (4.9).

Lemma 4.1. *For every function $p\in L_2\left(\Phi_1\right)$ there exists unique minimizer $u_\gamma=u_\gamma\left(p\right)\in H_0^{2,1}\left(\Phi_1\right)$ of the functional (4.8). Furthermore there exists a constant $C>0$ is independent on u,p,γ such that*

$$\left\|u_{\gamma}\right\|_{H^{2,1}\left(\Phi_{1}\right)}\leq\frac{C}{\sqrt{\gamma}}\left\|p\right\|_{L_{2}\left(\Phi_{1}\right)}.$$

The idea now is that if $u_\gamma\left(x,t\right)\in H_0^{2,1}\left(\Phi_1\right)$ is the minimizer mentioned in Lemma 4.1, then the approximate solution of IP1 is

$$f_{\gamma}\left(x\right)=u_{\gamma}\left(x,0\right). \tag{4.10}$$

The question of convergence of minimizers of $\widehat{J}_\gamma$ to the exact solution is more difficult than the question of their existence (Lemma 4.1). To address the question of convergence, we need to introduce the exact solution as well as the noise in the data, just as this is always done in the regularization theory [3, 9, 24, 58]. We assume that there exist an "ideal" noiseless data $p^*\in L_2\left(\Phi_1\right)$. We also assume that there exists the ideal noiseless solution $\widehat{v}^*\in H_0^{2,1}\left(\Phi_1\right)$ of the following problem (see (4.6), (4.7))

$$\widehat{v}_{t}^{*}-L\widehat{v}^{*}=p^{*}\left(x,t\right),\left(x,t\right)\in\Phi_{1}, \tag{4.11}$$

$$\widehat{v}^{*}\mid_{Z_{1}}=0,\widehat{v}_{x_{1}}^{*}\mid_{Z_{1}}=0. \tag{4.12}$$

Note that an upper estimate of the exact solution is often assumed to be known in the regularization theory, also see second Remark 3.2. Let $\eta\in(0,1)$ be a small number, which we regard as the level of the error in the data. We assume that

$$\left\|p-p^{*}\right\|_{L_{2}\left(\Phi_{1}\right)}\leq\eta. \tag{4.13}$$

Theorem 4.1 we establishes the convergence rate of the QRM.

Theorem 4.1. *Let conditions (3.17), (4.1), (4.11)-(4.13) be satisfied and the regularization parameter γ in (4.8) is chosen such that $\gamma=\gamma\left(\eta\right)=\eta\in(0,1)$. Let the function $u_{\gamma(\eta)}\in H_0^{2,1}\left(\Phi_1\right)$ be minimizer of the functional (4.8), which is guaranteed by Lemma 4.1. Let the upper estimate $B=const.>0$ for the exact solution $\widehat{v}^*$ be known, $\left\|\widehat{v}^*\right\|_{H^{2,1}\left(\Phi_1\right)}\leq B$. Then there exists a sufficiently small number $\delta_0=\delta_0\left(L,B\right)\in(0,1)$ such that if η is so small that $\sqrt{\left(B^2+1\right)\eta}\in\left(0,\delta_0\right)$, then the following logarithmic convergence rate takes place*

$$\left\|\widehat{f}^{*}-f_{\gamma(\eta)}\right\|_{L_{2}(\Omega)}\leq\frac{MB}{\sqrt{\ln\left(\eta^{-1}\right)}}, \tag{4.14}$$

where the function $f_{\gamma(\eta)}\left(x\right)$ is defined in (4.10) and $\widehat{f}^*\left(x\right)=\widehat{v}^*\left(x,0\right)$. In addition, for every $\omega\in(0,1/8)$ there exists a number $\rho=\rho\left(L,\Omega,\omega\right)\in(0,1)$ such that for $F\in\left(0,\delta_0\right)$ the following Hölder convergence rate takes place stability estimate is valid

$$\left\|\widehat{v}^{*}-u_{\gamma(\eta)}\right\|_{L_{2}\left(D_{3\omega}\right)}+\left\|\nabla\left(\widehat{v}^{*}-u_{\gamma(\eta)}\right)\right\|_{L_{2}\left(D_{3\omega}\right)}\leq MBF^{\rho}. \tag{4.15}$$

Proof. It follows from (4.11) and (4.12) that the function $\widehat{v}^*$ satisfies the following analog of (4.9)

$$\left(\widehat{v}_t^* - L\widehat{v}^*, w_t - Lw\right) + \gamma\left[\widehat{v}^*, w\right] = \left(p, w_t - Lw\right) + \gamma\left[\widehat{v}^*, w\right], \forall w \in H_0^{2,1}\left(\Phi_1\right). \tag{4.16}$$

Let $\widetilde{v} = u_\gamma - \widehat{v}^* \in H_0^{2,1}\left(\Phi_1\right)$ and $\widetilde{p} = p - p^* \in L_2\left(\Phi_1\right)$. Subtracting (4.16) from (4.9), we obtain

$$\left(\widetilde{v}_t - L\widetilde{v}, w_t - Lw\right) + \gamma\left[\widetilde{v}, w\right] = \left(\widetilde{p}, w_t - Lw\right) - \gamma\left[\widehat{v}^*, w\right], \forall w \in H_0^{2,1}\left(\Phi_1\right).$$

Setting here $w := \widetilde{v}$ and using Cauchy-Schwarz inequality and (4.13), we obtain

$$\int\limits_{\Phi_1} \left(\widetilde{v}_t - L\widetilde{v}\right)^2 dxdt + \gamma \left\|\widetilde{v}\right\|_{H^{2,1}(\Phi_1)}^2 \leq \eta^2 + \gamma \left\|\widehat{v}^*\right\|_{H^{2,1}(\Phi_1)}^2 \leq \eta^2 + \gamma B^2. \tag{4.17}$$

Since $\gamma\left(\eta\right) = \eta \in \left(0,1\right)$, then (4.17) implies that

$$\left\|\widetilde{v}\right\|_{H^{2,1}(\Phi_1)} \leq B + 1, \tag{4.18}$$

$$\int\limits_{\Phi_1} \left(\widetilde{v}_t - L\widetilde{v}\right)^2 dxdt \leq \left(B^2 + 1\right)\eta. \tag{4.19}$$

Now we can apply Theorem 3.3. Comparing (4.19) with (3.32) and (3.33) as well as comparing (4.18) with (3.34), we set

$$K := F := \sqrt{\left(B^2 + 1\right)\eta}, C_3 := B + 1. \tag{4.20}$$

Therefore, (4.14) and (4.15) follow from (4.20), (3.35), (3.36), (4.2) and (4.3). $\square$

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